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Quantified **LP**

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Abstract

An extension, **QLP**, of the propositional logic of explicit proofs, **LP**, is created, allowing quantification over proofs. The resulting logic is given an axiomatization, a Kripke-style semantics, and soundness and completeness are shown. It is shown that **S4** embeds into the logic, when we translate the necessity operator using a quantifier: there exists an explicit proof. And it is shown that the propositional part of **QLP** is exactly **LP**. No connection with arithmetic provability is made.

1 Introduction

There is a long history of using modal logic to investigate provability in arithmetic—the subject originated with Gödel. His published ideas about making $\Box$ correspond to provablity in Peano arithmetic eventually led to the modal logic **GL**, [3, 4]. This is a remarkable logic with beautiful features. But Gödel also made a lesser-known proposal for the development of a logic of *explicit* proofs, as part of a general program to provide a foundation for arithmetic. This was carried through successfully by Artemov, [1, 2]. The resulting logic is called **LP**, for 'logic of proofs.' Instead of a single modal operator, it has infinitely many *proof polynomials*, with natural operations on them. There is an arithmetic completeness theorem for **LP**. There is also a natural relationship with the modal logic **S4**. One can think of **LP** as providing a fine-grain analysis of **S4** validities.

There is also a long history of modal logics of knowledge, going back to [7]. These have had much success, but also some problems, most notably that of logical omniscience. One plausible way of minimizing these problems is to move from *known* to *known for a reason*. Then one might know the consequences of what one knows, but for more complex reasons, and this complexity of reasons provides machinery that might be used to circumscribe an agent's actual knowledge. Of course, mathematical proofs serve as explicit reasons for some of the things we know, and so **LP** might serve as a prototype logic of explicit reasons, where the goal is to introduce reasons that are more general things than formal proofs. This idea was explored in [5].

Besides the arithmetic semantics for **LP**, other approaches have also been created. There is a Kripke-style semantics in [6], which makes use of ideas from a simpler semantics in [8]. The situation is reminiscent of **GL**, which also has both an arithmetic interpretation and a Kripke-style semantics. One feature of the Kripke-style semantics for **LP** is that it makes connections with an **S4** logic of knowledge seem tantalizingly close.

One important result about **LP**, briefly mentioned above, is the *realization* theorem, which connects the logic with **S4**. It says that, if one takes a theorem of **S4**, there is some way of replacing all occurrences of the $\Box$ operator with proof polynomials that will result in a theorem of **LP** (and conversely, though this is the easy direction). The original proof of this is in [2], and is proof-theoretic, making use of a cut-elimination result for a sequent calculus version of **LP**. Alternatively, there is a semantic proof in [6]. Indeed, semantical (non-constructive) proofs of cut elimination can also be given, [9, 6]. A full statement of the realization theorem is more nuanced, involving positive and negative occurrences of $\Box$, and other things, but we need not go into details here.

In effect, the realization theorem involves quantification 'from the outside;' one thinks of the $\Box$ operator as a kind of quantifier, *there is a proof of.* It is reasonable to try and bring this quantification inside. And there has been work on adding quantifiers to **LP**, where quantification is over proofs (proof polynomials). In [10] such a project is carried out, and connections with arithmetic are investigated. Unfortunately, it is shown that this results in a non-axiomatizable logic. But one can come at the problem from another direction. Suppose we simply begin with axiomatically formulated **LP**, add the usual axiomatic machinery for quantification, and perhaps a little more as it seems appropriate. Then perhaps we can find a natural Kripke-style semantics to correspond to this, even if the connection with arithmetic is broken. We can think of what we are doing as part of the project to investigate the logic of explicit reasons, which presumably is broader than that of explicit proofs. Carrying out the approach just outlined constitutes the present paper.

The axiomatization we give is rather obvious except for one item, a uniform Barcan formula, whose exact provence is still unclear. In the corresponding Kripke semantics, quantification is over domains of things we call "reasons," and the proof polynomials of **LP** denote reasons in the same sense that terms of first-order classical logic denote objects in classical domains of quantification. We call the resulting logic **QLP**. As noted, soundness and completeness results are shown. Finally, using the semantic machinery, it is shown that **S4** embeds in **QLP** in the expected way, translating $\Box$ by an explicit existential quantifier, "there exists a proof of." It is also shown that **QLP** is a conservative extension of **LP**—the propositional part of **QLP** is exactly **LP**.

2 Syntax

Following [2], the collection of formulas of **LP** is built up from an infinite list of propositional letters and $\bot$, using $\supset$, and the following additional formation rule. If X is a formula and t is a proof polynomial, then $t{:}X$ is a formula; it should be read "t is a proof of X," or "t is a reason for X." To be neutral here, it will often be said that t *verifies* X. The collection of proof polynomials of **LP** is built up from an infinite list of proof variables (typically x, y, x_1, x_2, ...) and proof constants (typically c, d, c_1, c_2, ...), using the following formation rules. If t and u are proof polynomials, so are $t \cdot u$, $t + u$, and $!t$. Proof constants are meant to be justifications for obvious facts, such as logical axioms. Proof variables are, well, variables. As for the operation symbols, the intuition is as follows. $t \cdot u$ is meant to be the result of joining together the two reasons; typically if t justifies $X \supset Y$ and u justifies X then $t \cdot u$ justifies Y. $t + u$ is a kind of union; it justifies what either t or u justifies. And ! is a verification operator; $!t$ verifies the correctness of an application of t. The axioms in the next section reflect this directly. The language of **LP** will be referred to as $L^{\mathbf{LP}}$. It should be noted that in the formulation of [2] propositional variables are also allowed—they have been omitted here.

The formation rules for **QLP** extend those of **LP**, as follows. First, quantification is added: if $\varphi(x)$ is a formula and x is a proof variable, then $(\forall x)\varphi(x)$ is a formula. The usual definitions covering free and bound occurrences are assumed. We will use the common convention of writing $\varphi(x)$ to indicate a formula with some (possibly no) free occurrences of the variable x, and $\varphi(t)$ to be the result of replacing all free occurrences of x with occurrences of t. The second **QLP** addition is that an additional operation on proof polynomials is allowed: if t is a proof polynomial and x is a proof variable, then $(t \forall x)$ is a proof variable. The occurrence of x in $(t \forall x)$ is considered bound. The intention is that $(t \forall x)$ should serve as a justification of $(\forall x)\varphi(x)$ if t serves as a (uniform) justification of each instance of $\varphi(x)$.

The language of **QLP** thus defined will be referred to as $L^{\mathbf{QLP}}$. Other propositional connectives, and the existential quantifier, are defined symbols as usual.

As noted, constant symbols are intended to be justifications of the 'obvious.' As such, their attachment to formulas is fairly arbitrary, and is handled by the following machinery. A *constant specification* is a mapping $\mathcal{C}$ from proof constants to sets of formulas (possibly empty). A formula X *has a proof constant* with respect to $\mathcal{C}$ if $X \in \mathcal{C}(c)$ for some proof constant c. A proof constant c is *for* a formula X if $X \in \mathcal{C}(c)$. While constant specifications for **LP** can be quite general, for **QLP** we will need to make the following assumption when showing completeness, essentially saying that a constant specification is immune to the renaming of free variables. Some further comments on this condition will be made at the end of Section 3.

Definition 2.1 (Free Variable Condition) Constant specification $\mathcal{C}$ meets the *free variable condition* provided, whenever $\varphi(x_1, \ldots, x_n) \in \mathcal{C}(c)$ and $y_1, \ldots, y_n$ are variables that do not occur in $\varphi(x_1, \ldots, x_n)$, then $\varphi(y_1, \ldots, y_n) \in \mathcal{C}(c)$

3 An Axiom System

The following are **LP** axiom schemas, taken from [2]. They are also axiom schemas of **QLP**.

1. A finite set of classical axiom schemas, sufficient for tautologies.
2. $t{:}(X \supset Y) \supset (s{:}X \supset (t \cdot s){:}Y)$
3. $t{:}X \supset X$
4. $t{:}X \supset\ !t{:}(t{:}X)$
5. $s{:}X \supset (s+t){:}X$ and $t{:}X \supset (s+t){:}X$

We add to these two standard universal quantification axioms.

6. $(\forall x)\varphi(x) \supset \varphi(t)$, for any proof polynomial t that is free for x in $\varphi(x)$.
7. $(\forall x)(\psi \supset \varphi(x)) \supset (\psi \supset (\forall x)\varphi(x))$, where x does not occur free in ψ.

And finally, we add what we call the *uniform Barcan formula*.

8. $(\forall x)t{:}\varphi(x) \supset (t \forall x){:}(\forall x)\varphi(x)$, where x does not occur free in t.

There are three rules of inference. The first two come from **LP**. For starters, we have *modus ponens*.

$$\frac{X, X \supset Y}{Y}$$

Following the terminology of [6], We'll say a constant specification $\mathcal{C}$ is *axiomatically appropriate* if $\mathcal{C}$ provides proof constants for exactly the axioms listed above. This amounts to a requirement that constants serve to justify the obvious, in this case elementary logical truths. The next rule depends on the choice of $\mathcal{C}$, which is assumed to be axiomatically appropriate. Call it the $\mathcal{C}$ *necessitation rule*: if X is an axiom and $X \in \mathcal{C}(c)$, we may conclude the following.

$$\overline{c{:}X}$$

Finally there is the standard universal generalization rule.

$$\frac{\varphi(x)}{(\forall x)\varphi(x)}$$

We'll refer to the system axiomatized above as **QLP** with constant specification $\mathcal{C}$, where $\mathcal{C}$ is the constant specification used in the necessitation rule. As usual, Z is a theorem if it is the last line of a proof.

A note about the axioms chosen. We have deliberately kept them the same as in **LP** as far as possible. Instead of Axiom 1, a quantified version could have been used:

$$(\forall y)(\forall x)[y{:}(X \supset Y) \ \supset (x{:}X \supset (y \cdot x){:}Y)]$$

Of course the Axiom 1 instances follow from this using Axiom 6. Or, using the form of the axiom we did, one instance is

$$y{:}(X \supset Y) \ \supset (x{:}X \supset (y \cdot x){:}Y)$$

from which the quantified version follows using the universal generalization rule.

In [2] an internalization result (Lifting Lemma) is shown for **LP**. That carries over rather easily to **QLP**, and plays an important role later on. Here is the version that we need, though something a bit more general could be shown.

Proposition 3.1 *If X is a theorem of* **QLP***, using axiomatically appropriate constant specification $\mathcal{C}$, then for some proof polynomial p, with no free variables, the formula $p{:}X$ is also a theorem.*

Proof The argument is by induction on proof length. All the **LP** cases from [2] carry over directly, so the only item we need to consider in detail here is the rule of universal generalization.

Suppose we have used the universal generalization rule to conclude $(\forall x)\varphi(x)$ from $\varphi(x)$, and the result is known for $\varphi(x)$, that is, there is a proof polynomial p with no free variables such that $p{:}\varphi(x)$ is provable. Now proceed as follows. From $p{:}\varphi(x)$, conclude $(\forall x)(p{:}\varphi(x))$, using the universal generalization rule. Then by the uniform Barcan formula, axiom 8, and modus ponens, conclude $(p \forall x){:}(\forall x)\varphi(x)$. Note that since x is considered bound in $(p \forall x)$, we again have a proof polynomial with no free variables. ■

Finally, a remark concerning the free variable condition on constant specifications, Definition 2.1. Suppose $\varphi(x_1, \ldots, x_n)$ is one of our axioms and $\mathcal{C}$ is an axiomatically appropriate constant specification, say $\varphi(x_1, \ldots, x_n) \in \mathcal{C}(c)$. Then we can reason formally as follows.

1. $c{:}\varphi(x_1, \ldots, x_n)$, by the $\mathcal{C}$ necessitation rule
2. $(\forall x_1) \ldots (\forall x_n) c{:}\varphi(x_1, \ldots, x_n)$, using the universal generalization rule n times
3. $c{:}\varphi(y_1, \ldots, y_n)$, making use of axiom 6 and modus ponens n times.

Given this, we see that the free variable condition on constant specifications is, in a sense, a recognition that such a derivation can be made, and an incorporation of it directly into the constant specification itself.

4 Semantics

The semantics for **QLP** is, in a sense, a quantified version of the **LP** semantics of [6], and that paper and [5] should be consulted for background and motivation. Actually, there are two versions of the semantics for **LP**, weak and strong. This same split carries over to **QLP** as well. The most notable addition to the **LP** semantics is that each state of the model has a domain of objects called *reasons* associated with it. The remainder of this section is given over to the definition of the semantics.

A *frame* is a structure $\langle \mathcal{G}, \mathcal{R} \rangle$ where $\mathcal{R}$ is a binary relation on the non-empty set $\mathcal{G}$. Members of $\mathcal{G}$ are referred to as *states* or *worlds*, as usual. It will be assumed that $\mathcal{R}$ is reflexive and transitive, that is, the frame is one for **S4**.

A *domain function* on a frame $\langle \mathcal{G}, \mathcal{R} \rangle$ is a mapping $\mathcal{D}$ from members of $\mathcal{G}$ to non-empty sets, whose members are called *reasons*. Think of these however you want—some structure will be placed on the family of reasons below. It will be assumed that domain functions are *monotonic*, that is, for $\Gamma, \Delta \in \mathcal{G}$, $\Gamma \mathcal{R} \Delta$ implies $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Delta)$. Given a frame $\langle \mathcal{G}, \mathcal{R} \rangle$ and a domain function $\mathcal{D}$ on it, $\overline{\mathcal{D}}$ is the *frame domain*, and is defined to be $\cup_{\Gamma \in \mathcal{G}} \mathcal{D}(\Gamma)$. If $\mathcal{D}(\Gamma)$ is the same for every $\Gamma \in \mathcal{G}$, the domain function is *constant domain*. In this case, of course, the domain at each state is the same as the domain of the frame.

Given a frame $\langle \mathcal{G}, \mathcal{R} \rangle$ and a domain function $\mathcal{D}$, an *interpretation* $\mathcal{I}$ is a mapping meeting the following conditions. $\mathcal{I}$ assigns to each proof constant c a member $c^{\mathcal{I}}$ of $\overline{\mathcal{D}}$. $\mathcal{I}$ assigns to the one-place function symbol ! a mapping $!^{\mathcal{I}} : \overline{\mathcal{D}} \to \overline{\mathcal{D}}$. $\mathcal{I}$ assigns to the two-place function symbol $\cdot$ a binary operation $\cdot^{\mathcal{I}} : \overline{\mathcal{D}} \times \overline{\mathcal{D}} \to \overline{\mathcal{D}}$, and to $+$ a binary operation $+^{\mathcal{I}} : \overline{\mathcal{D}} \times \overline{\mathcal{D}} \to \overline{\mathcal{D}}$. And finally, $\mathcal{I}$ assigns to $\forall$ a mapping $\forall^{\mathcal{I}} : \overline{\mathcal{D}} \times V \to \overline{\mathcal{D}}$, where V is the set of variables of $L^{\mathbf{QLP}}$. It will be assumed that, for each $\Gamma \in \mathcal{G}$, $\mathcal{D}(\Gamma)$ contains $c^{\mathcal{I}}$ for every proof constant c, and is closed under $s^{\mathcal{I}}$ for every function and operation symbol s.

Suppose we have a structure $\langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ where $\langle \mathcal{G}, \mathcal{R} \rangle$ is a frame, $\mathcal{D}$ is a domain function, and $\mathcal{I}$ is an interpretation. A *valuation* v is a mapping from proof variables to members of $\overline{\mathcal{D}}$. It is not required that $v(x)$ be in $\mathcal{D}(\Gamma)$ for every $\Gamma \in \mathcal{G}$. As usual, a valuation w is an x-variant of a valuation v if v and w agree on all variables except possibly for x. Given a valuation v, with respect to $\langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ every proof polynomial t is mapped to a member t^v of $\overline{\mathcal{D}}$ by the following rules.

1. $x^v = v(x)$ for x a variable
2. $c^v = c^{\mathcal{I}}$ for c a constant symbol
3. $(t \cdot u)^v = (t^v \cdot^{\mathcal{I}} u^v)$
4. $(t + u)^v = (t^v +^{\mathcal{I}} u^v)$
5. $(!t)^v = !^{\mathcal{I}}(t^v)$
6. $(t \forall x)^v = (t^v \forall^{\mathcal{I}} x)$

An *evidence function* $\mathcal{E}$ is a mapping that assigns to each $\Gamma \in \mathcal{G}$, to each $r \in \mathcal{D}(\Gamma)$, and to each valuation v a set $\mathcal{E}(\Gamma, r, v)$ of formulas of $L^{\mathbf{QLP}}$. Note that the range of $\mathcal{E}$ is syntactic: it consists of sets of formulas. Think of the members of $\mathcal{E}(\Gamma, r, v)$ as the formulas that r provides possible justification for, in state Γ, under circumstances v. It is not assumed that being possibly justified is the same thing as being true, or being known, though it is understood that X cannot be known for reason r unless, at least, r is something that can serve as evidence for X. There is further discussion to be found in [6, 5]

There are special conditions that are imposed on evidence functions, as follows. For all formulas X and Y, for all $\Gamma, \Delta \in \mathcal{G}$, for all reasons r and s in $\mathcal{D}(\Gamma)$, and for all valuations v:

1. $\Gamma\mathcal{R}\Delta$ implies $\mathcal{E}(\Gamma, r, v) \subseteq \mathcal{E}(\Delta, r, v)$ (evidence is also monotonic).
2. $(X \supset Y) \in \mathcal{E}(\Gamma, r, v)$ and $X \in \mathcal{E}(\Gamma, s, v)$ implies $Y \in \mathcal{E}(\Gamma, (r \cdot^{\mathcal{I}} s), v)$ (application).
3. If $X \in \mathcal{E}(\Gamma, r, v)$ and t is a proof polynomial such that $t^v = r$, then $t{:}X \in \mathcal{E}(\Gamma, (!^{\mathcal{I}} r), v)$ (proof checker).
4. $\mathcal{E}(\Gamma, r, v) \cup \mathcal{E}(\Gamma, s, v) \subseteq \mathcal{E}(\Gamma, (r +^{\mathcal{I}} s), v)$ (choice).
5. If $X \in \mathcal{E}(\Gamma, r, w)$ for every w that is an x-variant of v, with $w(x) \in \mathcal{D}(\Gamma)$, then $(\forall x)X \in \mathcal{E}(\Gamma, (r \forall^{\mathcal{I}} x), v)$.
6. If v and w agree on the free variables of X, then $X \in \mathcal{E}(\Gamma, r, v)$ iff $X \in \mathcal{E}(\Gamma, r, w)$.

Finally, consider the structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V} \rangle$ where: $\langle \mathcal{G}, \mathcal{R} \rangle$ is a frame; $\mathcal{D}$ is a domain function, $\mathcal{I}$ is an interpretation, $\mathcal{E}$ is an evidence function, and $\mathcal{V}$ is a mapping of propositional letters to sets of states. Truth at worlds of $\mathcal{M}$, with respect to a valuation v, is evaluated in a way that extends that of [6]. The idea behind clause 5 is that X is known with t as justification at state Γ if X is known at Γ (true in all accessible worlds), and t serves as possible evidence for X at Γ.

1. $\mathcal{M}, \Gamma \Vdash_v P \iff \Gamma \in \mathcal{V}(P)$ for P a propositional letter;
2. $\mathcal{M}, \Gamma \nVdash_v \bot$;
3. $\mathcal{M}, \Gamma \Vdash_v X \supset Y \iff \mathcal{M}, \Gamma \nVdash_v X$ or $\mathcal{M}, \Gamma \Vdash_v Y$;
4. $\mathcal{M}, \Gamma \Vdash_v (\forall x)\varphi \iff \mathcal{M}, \Gamma \Vdash_w \varphi$ for every valuation w that is an x-variant of v and is such that $w(x) \in \mathcal{D}(\Gamma)$.
5. $\mathcal{M}, \Gamma \Vdash_v (t{:}X) \iff X \in \mathcal{E}(\Gamma, t^v, v)$ and $\mathcal{M}, \Delta \Vdash_v X$ for every $\Delta \in \mathcal{G}$ such that $\Gamma\mathcal{R}\Delta$.

We say that X is *meaningful* at state Γ of $\mathcal{M}$ with respect to v provided, for every proof polynomial t that occurs in X, $t^v \in \mathcal{D}(\Gamma)$—equivalently, for each variable x that has a free occurrence in X, $v(x) \in \mathcal{D}(\Gamma)$. We also say X is *true at state* Γ with respect to v if $\mathcal{M}, \Gamma \Vdash_v X$, and otherwise X is *false at* Γ. Finally, X is *valid* in the structure $\mathcal{M}$ if, for every valuation v, X is true at all states Γ of $\mathcal{M}$ at which X is meaningful with respect to v.

A *quasi-model* is a structure meeting all the conditions listed above.

A quasi-model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V} \rangle$ *meets constant specification* $\mathcal{C}$ provided that, for each constant c, each valuation v, and each $\Gamma \in \mathcal{G}$ we have $\mathcal{C}(c) \subseteq \mathcal{E}(\Gamma, c^{\mathcal{I}}, v)$.

A quasi-model is a *weak* **QLP** *model* if all instances of axiom 8 are valid in it.

A quasi-model meets the *fully explanatory* condition provided that, whenever X is meaningful at state Γ with respect to a valuation v, and $\mathcal{M}, \Delta \Vdash_v X$ for every $\Delta \in \mathcal{G}$ such that $\Gamma\mathcal{R}\Delta$, then $X \in \mathcal{E}(\Gamma, r, v)$, for some $r \in \mathcal{D}(\Gamma)$. A weak model that meets the fully explanatory condition is a *strong* **QLP** *model.*

5 Soundness

It will be proved that axiomatic **QLP** is sound with respect to the weak model semantics, and hence also with respect to the strong model semantics. More specifically, what will be proved is this.

Theorem 5.1 *if X has a* **QLP** *proof using the $\mathcal{C}$ necessitation rule, where $\mathcal{C}$ is axiomatically appropriate but does not necessarily meet the free variable condition, then X is valid in all weak* **QLP** *models that meet constant specification $\mathcal{C}$.*

This theorem is shown by induction on the length of the proof of X. Each axiom is easily seen to be valid, with the arguments for the **LP** part being essentially the same as in [6]. This leaves the three rules of inference. Call a rule *sound* provided that, whenever the premises are valid in all weak models meeting constant specification $\mathcal{C}$, so is the conclusion. The soundness of the $\mathcal{C}$ necessitation rule is by the usual argument, and that of the rule of universal generalization is just as in classical logic. This leaves modus ponens.

Proposition 5.2 *Modus ponens is a sound rule.*

Proof Let $\mathcal{M}$ be a weak **QLP** model meeting constant specification $\mathcal{C}$. Suppose both X and $X \supset Y$ are valid in $\mathcal{M}$; We'll show Y is also valid.

Let $\Gamma \in \mathcal{G}$ be a state at which Y is meaningful with respect to v, but $\mathcal{M}, \Gamma \not\Vdash_v Y$; we'll derive a contradiction. The problem we have is that X may have variables that do not occur in Y, and so X and $X \supset Y$ might not be meaningful at Γ with respect to v. Say $x_1, \ldots, x_n$ are all the variables in X that are not in Y. Choose $a_1, \ldots, a_n \in \mathcal{D}(\Gamma)$, and let w be the valuation that agrees with v on all variables, except that $w(x_i) = a_i$, for $i = 1, \ldots, n$. Since $x_1, \ldots, x_n$ do not occur in Y, we have $\mathcal{M}, \Gamma \not\Vdash_w Y$, and of course Y is meaningful at Γ with respect to w. But also, X and $X \supset Y$ are meaningful at Γ with respect to w, so by our validity assumption, $\mathcal{M}, \Gamma \Vdash_w X$ and $\mathcal{M}, \Gamma \Vdash_w X \supset Y$, and a contradiction is immediate. ■

6 Completeness

It will be proved that axiomatic **QLP** is complete with respect to the strong model semantics, and hence also with respect to the weak model semantics. Here is a proper statement of the result. Note that the free variable condition of Definition 2.1 finally comes in.

Theorem 6.1 *Let $\mathcal{C}$ be a constant specification that is axiomatically appropriate and that meets the free variable condition. If a formula X of $L^{\mathbf{QLP}}$ is valid in all strong* **QLP** *models that meet specification $\mathcal{C}$, then X has an axiomatic proof using $\mathcal{C}$.*

The rest of the section is devoted to a proof of this Theorem. Actually the contrapositive will be shown, as is the usual procedure. Let $\mathcal{C}$ be an axiomatically appropriate constant specification meeting the free variable condition. $\mathcal{C}$ is fixed for the rest of the section.

Let $p_1, p_2, \ldots,$ be an infinite list of proof constants that are new to $L^{\mathbf{QLP}}$. We call these *parameters*. Let L^* be the extension of $L^{\mathbf{QLP}}$ in which parameters are allowed to appear in formulas. The expanded language, and sublanguages of it, will be used to construct a model, but it is understood that it is only required to be a model with respect to the original language. For example, in Section 4 we gave as a condition on models that: "for each $\Gamma \in \mathcal{G}$, $\mathcal{D}(\Gamma)$ contains $c^{\mathcal{I}}$ for every proof constant c." In the model we construct, the condition applies only to constants from the language $L^{\mathbf{QLP}}$, and is not required of parameters.

We will be interested in theorems, and proofs, in the extended language L^*, but the constant specification $\mathcal{C}$ is only axiomatically appropriate for the original language L. Consequently we create an axiomatically appropriate constant specification for L^*, which we do as follows, calling the result $\mathcal{C}^*$. Suppose $\varphi(p_1, \ldots, p_k)$ is an instance of one of the axiom schemes for **QLP**, where this is a formula of L^* and $p_1, \ldots, p_k$ are all the parameters it contains. Let $x_1, \ldots, x_k$ be proof variables that do not occur in φ, and consider the formula $\varphi(x_1, \ldots, x_k)$, which is a formula of the original language L, and again an instance of an axiom scheme of **QLP**. Then $\varphi(x_1, \ldots, x_k)$ has one or more proof constants, let c be any one of them. We set $\varphi(p_1, \ldots, p_k) \in \mathcal{C}^*(c)$. The process just described makes use of a choice of proof variables $x_1, \ldots, x_k$, but since $\mathcal{C}$ meets the free variable condition, the actual variables chosen do not matter. The constant specification $\mathcal{C}^*$ extends $\mathcal{C}$, because if a formula contains no parameters, the process just described will assign the same proof constants for the formula that $\mathcal{C}$ itself provides. Obviously $\mathcal{C}^*$ is axiomatically appropriate, with respect to L^*.

The following can now be proved. We'll leave the argument to you—it is here that our formulation of $\mathcal{C}^*$ comes into play.

Lemma 6.2 *Suppose $\varphi(p)$ is a formula of L^*, where p is a parameter (not necessarily the only one appearing in the formula). And suppose $\varphi(p)$ has a* **QLP** *proof, using constant specification $\mathcal{C}^*$. Let x be a proof variable that does not occur in the proof. Replacing occurrences of p throughout the proof by occurrences of x produces another correct* **QLP** *proof with respect to $\mathcal{C}^*$, of $\varphi(x)$.*

Now we continue with the completeness argument. If P is a set of parameters, by $L^*(P)$ is meant the sublanguage of L^* with parameters restricted to the set P. Let S be a set of *closed* formulas from $L^*(P)$. S is *inconsistent* if there is a finite $\{X_1, \ldots, X_n\} \subseteq S$ such that $X_1 \supset (X_2 \supset \cdots (X_n \supset \bot) \cdots)$ has a proof in **QLP** (using the expanded language $L^*(P)$). S is *consistent* if it is not inconsistent. S is *maximally consistent* with respect to $L^*(P)$ if it is consistent, and no proper superset of S consisting of closed formulas from $L^*(P)$ is consistent. Finally, S is $\exists$*-complete* with respect to $L^*(P)$ if, for every formula $\neg(\forall x)\varphi(x) \in S$ there is some closed proof polynomial t in $L^*(P)$ such that $\neg\varphi(t) \in S$.

Proposition 6.3 *Let S be a consistent subset of $L^*(P)$ consisting of closed formulas, and suppose $\neg(\forall x)\varphi(x) \in S$. Also, let p be a parameter not in S. Then $S \cup \{\neg\varphi(p)\}$ is consistent.*

Proof This is a standard argument, with some minor variations. Suppose, under the hypotheses of the Proposition, that $S \cup \{\neg\varphi(p)\}$ is not consistent. Then for some finite subset $X_1, \ldots, X_n$ of S, the set $\{X_1, \ldots, X_n, \neg\varphi(p)\}$ is not consistent, from which it follows that $(X_1 \wedge \ldots \wedge X_n) \supset \varphi(p)$ is provable. Let z be a proof variable that does not appear in this proof. By Lemma 6.2 it follows that $(X_1 \wedge \ldots \wedge X_n) \supset \varphi(z)$ is also provable. Then using universal generalization and axiom 7, $(X_1 \wedge \ldots \wedge X_n) \supset (\forall z)\varphi(z)$ is provable, and hence so is $(X_1 \wedge \ldots \wedge X_n) \supset (\forall x)\varphi(x)$. But this is a contradiction since $X_1, \ldots, X_n \in S$, $\neg(\forall x)\varphi(x) \in S$, and S is consistent. ■

Now a standard Henkin construction gives us the following. We omit the details.

Proposition 6.4 *Suppose S is a consistent subset of $L^*(P)$ consisting of closed formulas, where P omits infinitely many parameters. Then there is an extension, S' of S consisting of closed formulas, and an extension P' of P that also omits infinitely many parameters, such that S' is in the language $L^*(P')$ and is both maximally consistent with respect to $L^*(P')$, and $\exists$-complete.*

We will now describe the *canonical model*. Call a set S of closed formulas of L^* P-*world-like* if S is maximally consistent and $\exists$-complete with respect to $L^*(P)$, where P is a set of parameters that omits infinitely parameters. And call S simply *world-like* if it is P-world-like for some P. Let $\mathcal{G}$ be the collection of all world-like sets of formulas.

For $\Gamma \in \mathcal{G}$, let $\Gamma^\sharp = \{Z \mid t{:}Z \in \Gamma \text{ for some } t\}$. Now, say $\Gamma, \Delta \in \mathcal{G}$, Γ is P-world-like and Δ is Q-world-like. We set $\Gamma\mathcal{R}\Delta$ provided $P \subseteq Q$ and $\Gamma^\sharp \subseteq \Delta$. We now have a frame, $\langle \mathcal{G}, \mathcal{R} \rangle$, and it must be verified that it is reflexive and transitive. For reflexivity, of course $P \subseteq P$; it still must be shown that $\Gamma^\sharp \subseteq \Gamma$. Well, suppose Γ is P-world-like, and $Z \in \Gamma^\sharp$. Then $t{:}Z \in \Gamma$, for some t in the language $L^*(P)$. The axiom $t{:}Z \supset Z$ (an instance of axiom scheme 3) is in the language $L^*(P)$, so it must be in Γ. Since maximal consistent sets are closed under modus ponens, then $Z \in \Gamma$. Thus we have reflexivity. Showing transitivity comes down to showing that $\Gamma^\sharp \subseteq \Gamma^{\sharp\sharp}$, and this is established similarly, using axiom schema 4.

We have a reflexive, transitive frame $\langle \mathcal{G}, \mathcal{R} \rangle$. We next need a domain function $\mathcal{D}$, and this is simple. Suppose $\Gamma \in \mathcal{G}$ is P-world-like; we define $\mathcal{D}(\Gamma)$ to be the set of all *closed proof polynomials* in the language $L^*(P)$. (Remember, the occurrence of x in $(t\forall x)$ is considered to be bound, and thus if t is closed, so is $(t \forall x)$.) It is immediate that we have monotonicity: $\Gamma\mathcal{R}\Delta$ implies $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Delta)$. Note that the frame domain is the set of all closed proof polynomials in the language L^*.

Next we need an interpretation function $\mathcal{I}$, and this too is simple. For a constant c of $L^{\mathbf{QLP}}$, define $c^{\mathcal{I}}$ to be c. For the function symbol $+$, define $+^{\mathcal{I}}$ to be the function that maps the closed proof polynomials t and u to the closed proof polynomial $(t + u)$. And similarly for the other function symbols.

The model we are constructing is a kind of Herbrand model, and so there is a close connection between valuations and substitutions. We introduce some formal machinery to let us make use of this connection. A *substitution* is a mapping from proof variables to proof polynomials (the domain is not required to be finite). The set of proof variables is the same for $L^{\mathbf{QLP}}$ and for L^*, but the set of proof polynomials is different—we'll allow proof polynomials from L^*. The symbol σ, with or without subscripts, will be used for substitutions. The result of applying the substitution σ to the **QLP** formula Z will be denoted by $Z\sigma$. Similarly $t\sigma$ is the result of applying σ to the proof polynomial t. It is, of course, assumed that substitutions only replace free occurrences of variables. As usual, a valuation in the structure $\langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ is a mapping from proof variables to members of $\overline{\mathcal{D}}$, but now these are closed proof polynomials of L^*. Then we can naturally associate a *substitution* with each valuation in the obvious way: for a valuation v, denote the corresponding substitution by σ_v, where σ_v is the substitution that replaces a variable x by the closed term $v(x)$. The following can now be proved by induction on complexity: for each proof polynomial in the language L^* and for each valuation v, with respect to the structure $\langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$, t^v is the closed term $t\sigma_v$.

Now we can define an evidence function $\mathcal{E}$. For $\Gamma \in \mathcal{G}$, for $r \in \mathcal{D}(\Gamma)$, and for a valuation v, define $\mathcal{E}(\Gamma, r, v)$ to be the set of all formulas X of $L^{\mathbf{QLP}}$ such that $r{:}(X\sigma_v) \in \Gamma$.

There are six conditions required for an evidence function. For condition 1, monotonicity, suppose $\Gamma\mathcal{R}\Delta$, and also suppose $X \in \mathcal{E}(\Gamma, r, v)$, which means $r{:}(X\sigma_v) \in \Gamma$. Since members of $\mathcal{G}$ contain all axioms (in the appropriate language), $r{:}(X\sigma_v) \supset\ !r{:}r{:}(X\sigma_v)$ is in Γ, and since members of $\mathcal{G}$ are closed under modus ponens because of maximally consistency, $!r : r : (X\sigma_v) \in \Gamma$. Then $r{:}(X\sigma_v) \in \Gamma^\sharp \subseteq \Delta$, so $X \in \mathcal{E}(\Delta, r, v)$. Conditions 2, 3, 4 on evidence functions are established similarly.

For evidence function condition 5, suppose $X \in \mathcal{E}(\Gamma, t, w)$ for every valuation w that is an x-variant of v with $w(x) \in \mathcal{D}(\Gamma)$. This means the formula $t{:}(X\sigma_w)$ is in Γ for every x-variant w of v with $w(x) \in \mathcal{D}(\Gamma)$. If we had $(\neg(\forall x)t{:}X)\sigma_v \in \Gamma$, by the $\exists$-completeness of Γ we would have $(\neg(t{:}X))\sigma_w \in \Gamma$ for some w that was an x-variant of v, with $w(x) \in \mathcal{D}(\Gamma)$. Since t is a closed proof polynomial, this is equivalent to $\neg(t{:}(X\sigma_w)) \in \Gamma$, and this would contradict the consistency of Γ. Then $(\neg(\forall x)t{:}X)\sigma_v \notin \Gamma$, so by maximality $((\forall x)t{:}X)\sigma_v \in \Gamma$. Suppose we temporarily let σ'_v be like σ_v except that it does not have x in its domain. Then an equivalent formulation of this is: $((\forall x)t{:}X\sigma'_v) \in \Gamma$. Now using Axiom 8 we conclude $(t \forall x){:}((\forall x)X\sigma'_v) \in \Gamma$, or equivalently, $(t \forall x){:}((\forall x)X)\sigma_v \in \Gamma$, and hence $(\forall x)X \in \mathcal{E}(\Gamma, (t \forall x), v)$.

Evidence function condition 6 is straightforward, and verification is omitted.

Finally we define a mapping $\mathcal{V}$. For a propositional letter P, set $\mathcal{V}(P) = \{\Gamma \in \mathcal{G} \mid P \in \Gamma\}$.

We have completed the definition of a structure, $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V}\rangle$ which we will call the *canonical model*, though at this point we have only shown that $\mathcal{M}$ satisfies all the conditions for being a quasi-model that meets constant specification $\mathcal{C}$.

Lemma 6.5 (Truth Lemma) *Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V}\rangle$ be the canonical model. For each $\Gamma \in \mathcal{G}$, for each formula X in the language $L^{\mathbf{QLP}}$, and for each valuation v, if X is meaningful at Γ with respect to v, then $\mathcal{M}, \Gamma \Vdash_v X$ iff $X\sigma_v \in \Gamma$.*

Proof As usual, the argument is by induction on the complexity of X.

1. The atomic case. Let X be P, a propositional letter. This is meaningful at every member of $\mathcal{G}$ with respect to every valuation, and the conclusion is immediate, by definition of $\mathcal{V}$.

2. The propositional connective case. If X is $Y \supset Z$, and the Lemma is known for Y and Z, it follows immediately for X using the familiar properties of maximally consistent sets.

3. The quantifier case. This case too follows the route of standard completeness arguments. It makes use of axiom 6 and the $\exists$-completeness of members of $\mathcal{G}$.

4. The explicit proof polynomial case. Here the argument is much as it is in [6]. Suppose X is $t{:}Y$, and the Lemma is known for Y. For one direction of the Lemma, assume that $(t{:}Y)\sigma_v$ is in Γ. that is, $(t\sigma_v){:}(Y\sigma_v)$. Then, first of all, $Y\sigma_v \in \Gamma^\sharp$, hence $Y\sigma_v \in \Delta$ for every $\Delta \in \mathcal{G}$ such that $\Gamma\mathcal{R}\Delta$, and so by the induction hypothesis, $\mathcal{M}, \Delta \Vdash_v Y$. And second, $Y \in \mathcal{E}(\Gamma, t\sigma_v, v)$, or equivalently, $Y \in \mathcal{E}(\Gamma, t^v, v)$. From these two it follows that $\mathcal{M}, \Gamma \Vdash_v t{:}Y$.

   For the other direction, suppose that $(t:Y)\sigma_v$ is meaningful at Γ with respect to v but $(t:Y)\sigma_v \notin \Gamma$. That is, $(t\sigma_v):(Y\sigma_v) \notin \Gamma$. By definition of $\mathcal{E}$ we have $Y \notin \mathcal{E}(\Gamma, t\sigma_v, v)$ or equivalently, $Y \notin \mathcal{E}(\Gamma, t^v, v)$, and it follows that $\mathcal{M}, \Gamma \not\Vdash_v t{:}Y$.

■

Of course using the Truth Lemma, all instances of axiom 8 are valid in the canonical model, and hence it is a weak **QLP** model. It remains to verify that the canonical model meets the fully explanatory condition, and the proof of this follows the lines of a similar result in [6].

Consider the canonical model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V}\rangle$. Suppose X, in the language $L^{\mathbf{QLP}}$, is meaningful at $\Gamma \in \mathcal{G}$ with respect to valuation v, and for every $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$ we have $\mathcal{M}, \Delta \Vdash_v X$. We'll show $X \in \mathcal{E}(\Gamma, r, v)$ for some $r \in \mathcal{D}(\Gamma)$. More precisely, suppose $X \notin \mathcal{E}(\Gamma, r, v)$ for every $r \in \mathcal{D}(\Gamma)$; we'll derive a contradiction. Our supposition amounts to this: for each $r \in \mathcal{D}(\Gamma)$, $r{:}(X\sigma_v) \notin \Gamma$.

The key item to show is that $\Gamma^\sharp \cup \{\neg X\sigma_v\}$ is consistent. For then we can extend it to a world-like set Δ; we will have $\Gamma\mathcal{R}\Delta$, and by the Truth Lemma, $\mathcal{M}, \Delta \not\Vdash_v X$, which is not the case. So we now concentrate on showing this key item.

Suppose $\Gamma^\sharp \cup \{\neg X\sigma_v\}$ is not consistent. Then for some $Y_1, \ldots, Y_k \in \Gamma^\sharp$, there is a proof in **QLP** of $(Y_1 \wedge \ldots \wedge Y_k \wedge \neg X\sigma_v) \supset \bot$, and hence there is also a proof of $(Y_1 \supset (Y_2 \supset \ldots (Y_k \supset X\sigma_v) \ldots))$.

For each $i = 1, \ldots, k$, since $Y_i \in \Gamma^\sharp$, there is some closed proof polynomial s_i such that $s_i{:}Y_i \in \Gamma$ (recall, maximal consistent sets contain only closed formulas). Using Proposition 3.1, there is a closed proof polynomial p such that in **QLP** we can prove $p{:}(Y_1 \supset (Y_2 \supset \ldots (Y_k \supset X\sigma_v)\ldots))$. Then repeated use of axiom 2 allows us to prove $(s_1{:}Y_1 \wedge \ldots \wedge s_k{:}Y_k) \supset (p \cdot s_1 \cdots s_k){:}X\sigma_v$. Hence $(p \cdot s_1 \cdots s_k){:}X\sigma_v \in \Gamma$, but this contradicts the original assumption that $r{:}X\sigma_v \notin \Gamma$ for each proof polynomial $r \in \mathcal{D}(\Gamma)$.

We have now shown that the canonical model is a strong model. In the usual way, it is a universal counter-model. Let X be a formula in the language $L^{\mathbf{QLP}}$ that does not have an axiomatic **QLP** proof. Since we have both universal generalization and axiom 6, we can assume X is closed. $\{\neg X\}$ is consistent. It can be extended to a world-like set Γ, which will be a state in the canonical model. And the Truth Lemma tells us that $\mathcal{M}, \Gamma \not\Vdash_v X$, for every valuation v, and so X is invalidated in a strong **QLP** model.

7 Relationship With S4

Propositional **S4** embeds exactly into **QLP**, thinking of the necessity operation as represented by the existence of a proof. Of course we need a proper statement of this.

Let $L^{\mathbf{S4}}$ be the language built up from the same propositional letters as $L^{\mathbf{QLP}}$, using $\supset$ and $\perp$, without proof terms or quantifiers, but with the additional formation rule: if X is a formula, so is $\Box X$. That is, $L^{\mathbf{S4}}$ is a standard propositional modal language, whose propositional letters are the same as in $L^{\mathbf{QLP}}$. Now we define an embedding from $L^{\mathbf{S4}}$ into $L^{\mathbf{QLP}}$ as follows.

1. If P is a propositional letter, $P^\exists = P$
2. $\perp^\exists = \perp$
3. $(X \supset Y)^\exists = (X^\exists \supset Y^\exists)$
4. $(\Box X)^\exists = (\exists x)x{:}(X^\exists)$

In the last clause above, the actual choice of variable is not important—we will standardize on x. Note that for every X of $L^{\mathbf{S4}}$, $X^\exists$ is a closed formula of $L^{\mathbf{QLP}}$.

Theorem 7.1 *For each formula X of $L^{\mathbf{S4}}$, X is a theorem of* **S4** *if and only if $X^\exists$ is a theorem of* **QLP**.

The proof of this will occupy the rest of the section. We'll begin with the following direction.

Lemma 7.2 *If X is a theorem of* **S4**, *then $X^\exists$ is a theorem of* **QLP**.

There are two ways of showing Lemma 7.2: proof theoretically and semantically. Each is straightforward, but each has interesting points.

Proof Proceeding proof theoretically first, we show by induction on proof length that theorems of **S4** translate to theorems of **QLP**. If A is an axiom of **S4** (using a standard axiomatization), it is simple to show that $A^\exists$ is a theorem of **QLP**. Here is one case as an example. The axiom $\Box Y \supset \Box\Box Y$ translates to $(\exists x)x{:}X \supset (\exists x)x{:}(\exists x)x{:}X$, where X is $Y^\exists$. Here is a sketch of a proof of this.

1. $x{:}X \supset {!x}{:}x{:}X$. This is an instance of axiom scheme 4.

2. $x{:}X \supset (\exists x)x{:}X$ is provable (it is the dual of an instance of axiom 6)
3. $p{:}(x{:}X \supset (\exists x)x{:}X)$ is provable, for some proof polynomial p, by 2 and Proposition 3.1.
4. $p{:}(x{:}X \supset (\exists x)x{:}X) \supset (!x{:}x{:}X \supset (p{\cdot}!x){:}(\exists x)x{:}X)$, an instance of axiom scheme 1.
5. $!x{:}x{:}X \supset (p{\cdot}!x){:}(\exists x)x{:}X$, from 3 and 4.
6. $!x{:}x{:}X \supset (\exists x)x{:}(\exists x)x{:}X$, from 5.
7. $x{:}X \supset (\exists x)x{:}(\exists x)x{:}X$ from 1 and 6.
8. $(\exists x)x{:}X \supset (\exists x)x{:}(\exists x)x{:}X$, from 7 using the universal generalization rule, and some standard quantifier manipulation.

With axioms out of the way, we turn to the rules of inference. Applications of modus ponens translate to applications of modus ponens. Here is the argument to cover the rule of necessitation case. Suppose, in **S4**, we conclude $\Box Y$ from Y, and in **QLP** we have a proof of $Y^{\exists}$. We proceed as follows. Using Proposition 3.1, we have $p{:}Y^{\exists}$, for some closed proof polynomial p. And from this we easily get $(\exists x)x{:}Y^{\exists}$, which is $(\Box Y)^{\exists}$. ■

Lemma 7.2 has now been proved in one direction, proof theoretically. It is interesting to also give a semantic proof of the result.

Proof Suppose X is a formula of $L^{\mathbf{S4}}$ and $X^{\exists}$ is *not* a theorem of **QLP**; we'll show X is not a theorem of **S4**.

If $X^{\exists}$ is not a theorem of **QLP**, using the completeness result of Section 6 there is a strong **QLP** model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V} \rangle$ in which $X^{\exists}$ is not valid. We'll show how to convert that to an **S4** model in which X is not valid. Very simply: let $\mathcal{N} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$ be the **S4** model with the same possible worlds as $\mathcal{M}$, the same accessibility relation as $\mathcal{M}$, and with propositional letters true at the same worlds as in $\mathcal{M}$. That is, $\mathcal{N}$ is like $\mathcal{M}$ with all the quantificational and proof polynomial structure forgotten. Now we show, by induction on complexity, that for each formula Y of $L^{\mathbf{S4}}$, and for each possible world of $\mathcal{N}$ (or equivalently, of $\mathcal{M}$), that $\mathcal{M}, \Gamma \Vdash_v Y^{\exists}$ iff $\mathcal{N}, \Gamma \Vdash Y$. The notation $\mathcal{N}, \Gamma \Vdash Y$ means that Y is true at world Γ, with truth evaluated in the usual **S4** manner. Note that since $Y^{\exists}$ is always a closed formula, the choice of valuation v is arbitrary.

Establishing this equivalence is a straightforward induction. The only interesting case is where Y is of the form $\Box Z$, and the result has been established for Z. Here is one direction.

1. $\mathcal{M}, \Gamma \Vdash_v (\Box Z)^{\exists}$ is our starting point
2. $\mathcal{M}, \Gamma \Vdash_v (\exists x)x{:}Z^{\exists}$, which is 1 in more detail
3. $\mathcal{M}, \Gamma \Vdash_w x{:}Z^{\exists}$, for some x-variant w of v, where $w(x) \in \mathcal{D}(\Gamma)$
4. $Z^{\exists} \in \mathcal{E}(\Gamma, x^w, w)$ and $\mathcal{M}, \Delta \Vdash_w Z^{\exists}$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$
5. $\mathcal{N}, \Delta \Vdash Z$ for every $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$, by the induction hypothesis
6. $\mathcal{N}, \Gamma \Vdash \Box Z$

The other direction is more interesting.

1. $\mathcal{N}, \Gamma \Vdash \Box Z$

2. $\mathcal{N}, \Delta \Vdash Z$ for every $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$

3. $\mathcal{M}, \Delta \Vdash_w Z^\exists$ for every $\Delta \in \mathcal{G}$ such that $\Gamma\mathcal{R}\Delta$, by the induction hypothesis, where w is an arbitrary valuation

4. $Z^\exists \in \mathcal{E}(\Gamma, r, w)$ for some $r \in \mathcal{D}(\Gamma)$, from 3, using the fact that $\mathcal{M}$ meets the fully explanatory condition

5. Let v be the x-variant of w such that $v(x) = r$. Since x is not free in $Z^\exists$, $Z^\exists \in \mathcal{E}(\Gamma, r, v)$, and $\mathcal{M}, \Delta \Vdash_v Z^\exists$ for every $\Delta \in \mathcal{G}$ such that $\Gamma\mathcal{R}\Delta$

6. It follows that $\mathcal{M}, \Gamma \Vdash_v x{:}Z^\exists$, and hence $\mathcal{M}, \Gamma \Vdash_w (\exists x)x{:}Z^\exists$, that is, $\mathcal{M}, \Gamma \Vdash_w (\Box Z)^\exists$

Note the use of the fully explanatory condition. The proof that it held in the canonical model made use of Proposition 3.1, which played an essential role in the proof-theoretic argument for Lemma 7.2 given above. ■

One direction of Theorem 7.1 has been proved. Now we turn to the other direction.

Lemma 7.3 *If $X^\exists$ is a theorem of* **QLP** *then X is a theorem of* **S4**.

Proof This time the only proof given is semantic, and is of the contrapositive. Suppose X is not a theorem of **S4**. Then there is an **S4** model, $\mathcal{N} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$, in which X is not valid; say $\mathcal{N}, \Gamma \nVdash X$. We'll use $\mathcal{N}$ to construct a **QLP** counter-model for $X^\exists$. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V} \rangle$ be defined as follows. $\mathcal{G}$ is the same set of possible worlds as in $\mathcal{N}$, and $\mathcal{R}$ is the same accessibility relation. Likewise $\mathcal{V}$ assigns the same worlds to propositional letters as in $\mathcal{N}$. Let a be some arbitrary object, and set $\mathcal{D}(\Gamma) = \{a\}$ for every $\Gamma \in \mathcal{G}$. Then define an interpretation $\mathcal{I}$ in the only possible way. For each constant c of $L^{\mathbf{QLP}}$ we set $c^\mathcal{I} = a$. We set $!^\mathcal{I}(a) = a$, $a +^\mathcal{I} a = a$, and so on. Of course only one valuation is possible, mapping every variable to a—call it v. For an evidence function, we set $\mathcal{E}(\Gamma, a, v)$ to be the entire set of formulas of $L^{\mathbf{QLP}}$, for every $\Gamma \in \mathcal{G}$.

We have now fully characterized the structure $\mathcal{M}$. The claim is that it is a strong **QLP** model. Most of the conditions are straightforward. For instance, $\mathcal{M}$ is fully explanatory because we have required that $\mathcal{E}$ include every formula at every world, and for every reason. The only item that really needs checking is that each instance of axiom 8 is valid, but this follows from a result below, Proposition 9.1, since we have a constant domain model.

We now show that for every formula Z of $L^{\mathbf{S4}}$,

$$\mathcal{N}, \Gamma \Vdash Z \Longleftrightarrow \mathcal{M}, \Gamma \Vdash_v Z^\exists$$

In this, Γ is any world in $\mathcal{G}$, and v is the unique valuation.

The proof is by induction on the complexity of Z. The atomic case is by definition. The cases for $\bot$ and $\supset$ are straightforward. Now suppose Z is $\Box W$, and the result is known for W. And suppose $\mathcal{N}, \Gamma \Vdash \Box W$. Then for each $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$ we have $\mathcal{N}, \Delta \Vdash W$, so by the induction hypothesis, $\mathcal{M}, \Delta \Vdash_v W^\exists$. Since $W^\exists \in \mathcal{E}(\Gamma, a, v)$ we have $\mathcal{M}, \Gamma \Vdash_v x{:}W^\exists$ (recall, $v(x) = a$). But then $\mathcal{M}, \Gamma \Vdash_v (\exists x)x{:}W^\exists$, that is, $\mathcal{M}, \Gamma \Vdash_v (\Box W)^\exists$. The other direction is similar, and is omitted.

Since $\mathcal{N}$ is an **S4** counter-model to X it follows that $\mathcal{M}$ is a strong **QLP** counter-model to $X^\exists$, and so $X^\exists$ is not a theorem of **QLP**. ■

8 Relationship With LP

In this section we establish that the logic **QLP** is a conservative extension of the propositional logic **LP**. That it is an extension is trivial—its proof system includes the axioms and the rules of **LP**. That it is a conservative extension requires some work.

Proposition 8.1 *Let X be a formula in the language $L^{\mathbf{LP}}$—equivalently, X is a quantifier-free formula of $L^{\mathbf{QLP}}$. If X is not a theorem of* **LP**, *then X is not a theorem of* **QLP**.

Proof In [6] a semantics for **LP** is given. It will be used here—the reader is referred to that paper for details and terminology.

Suppose X is not a theorem of **LP**. Then it is invalidated in some weak **LP** model. Let us say X is not valid in the weak **LP** model $\mathcal{Q} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$. We'll use this to define a weak **QLP** model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}', \mathcal{V} \rangle$. Note that we already have part of the characterization of $\mathcal{M}$. The underlying frame is the same as in $\mathcal{Q}$. Likewise we are using the same mapping $\mathcal{V}$ of propositional letters to sets of worlds. What remains to specify is $\mathcal{D}$, $\mathcal{I}$, and $\mathcal{E}'$.

For each $\Gamma \in \mathcal{G}$, set $\mathcal{D}(\Gamma)$ to be the set of all proof polynomials in the language $L^{\mathbf{QLP}}$ (thus we are creating a constant domain model). Let $!^{\mathcal{I}}$ be the function that maps the proof polynomial t to the proof polynomial $!t$. Let $+^{\mathcal{I}}$ be the operation that maps the proof polynomials s and t to the proof polynomial $(s + t)$. And so on. And finally we specify the evidence function $\mathcal{E}'$. For each $\Gamma \in \mathcal{G}$ and for each valuation v, proceed as follows. If t does not involve the symbol $\forall$, set $\mathcal{E}'(\Gamma, t, v) = \mathcal{E}(\Gamma, t)$. If t does contain the symbol $\forall$, set $\mathcal{E}'(\Gamma, t, v)$ to be the entire set of formulas of $L^{\mathbf{QLP}}$.

The structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}', \mathcal{V} \rangle$ meets the conditions for being a quasi-model. We'll check a few of the cases.

One condition on the evidence function is: $(X \supset Y) \in \mathcal{E}'(\Gamma, s, v)$ and $X \in \mathcal{E}'(\Gamma, t, v)$ implies $Y \in \mathcal{E}'(\Gamma, s{\cdot}t, v)$. If neither s nor t involves the symbol $\forall$, this condition reduces to $(X \supset Y) \in \mathcal{E}(\Gamma, s)$ and $X \in \mathcal{E}(\Gamma, t)$ implies $Y \in \mathcal{E}(\Gamma, s \cdot t)$, which holds because it is one of the conditions that must be met in a weak **LP** model. If either s or t involves the symbol $\forall$, so does $s \cdot t$, and so $(X \supset Y) \in \mathcal{E}'(\Gamma, s, v)$ and $X \in \mathcal{E}'(\Gamma, t, v)$ implies $Y \in \mathcal{E}'(\Gamma, s \cdot t, v)$ is true since $\mathcal{E}'(\Gamma, s \cdot t, v)$ is the entire set of formulas.

Another condition on the evidence function is: if $X \in \mathcal{E}'(\Gamma, t, w)$ for every w that is an x-variant of v, with $w(x) \in \mathcal{D}(\Gamma)$, then $(\forall x)X \in \mathcal{E}'(\Gamma, (t \forall x), v)$. This is trivially true because $\mathcal{E}'(\Gamma, (t \forall x), v)$ is the set of all formulas.

Further, $\mathcal{M}$ is constant domain, so Proposition 9.1 applies—$\mathcal{M}$ is a weak **QLP** model.

Now it is easy to check that for each formula Z in the propositional language of **LP**, for each world $\Gamma \in \mathcal{G}$, and using the valuation v such that $v(x) = x$, we have

$$\mathcal{M}, \Gamma \Vdash_v Z \Longleftrightarrow \mathcal{Q}, \Gamma \Vdash Z$$

and this is sufficient to establish the **QLP** invalidity of X. ■

9 Conclusions

The role of axiom 8 is not particularly clear. We have not been able to find a good semantic counterpart for it, which is why we simply restricted things to quasi-models in which it is valid—a beneficient form of cheating. We do have the following one-directional result—the converse does not seem to hold, though we have no proof of this.

Proposition 9.1 *Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V} \rangle$ be a quasi-model that is constant domain. Then it is a weak* **QLP** *model—all instances of axiom 8 are valid in it.*

Proof Suppose $\mathcal{M}$ is a constant domain quasi-model, and $\mathcal{M}, \Gamma \Vdash_v (\forall x)t{:}\varphi(x)$, but $\mathcal{M}, \Gamma \not\Vdash_v (t \forall x){:}(\forall x)\varphi(x)$, where x does not occur free in t. By the first, $\mathcal{M}, \Gamma \Vdash_w t{:}\varphi(x)$ for every x-variant w of v (we can omit the condition that $w(x) \in \mathcal{D}(\Gamma)$ since we have constant domains). It follows that $\varphi(x) \in \mathcal{E}(\Gamma, t^w, w)$ for every x-variant w, but since x does not occur free in t, this is the same as $\varphi(x) \in \mathcal{E}(\Gamma, t^v, w)$ for every x-variant w. It follows from condition 5 for evidence functions that $(\forall x)\varphi(x) \in \mathcal{E}(\Gamma, (t^v \forall x), v)$ or equivalently, since the occurrence of x in $(t \forall x)$ is bound, $(\forall x)\varphi(x) \in \mathcal{E}(\Gamma, (t \forall x)^v, v)$. Then, since $\mathcal{M}, \Gamma \not\Vdash_v (t \forall x){:}(\forall x)\varphi(x)$, there must be some $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$, such that $\mathcal{M}, \Delta \not\Vdash_v (\forall x)\varphi(x)$. But then, $\mathcal{M}, \Delta \not\Vdash_w \varphi(x)$ for some x-variant w of v. However, it follows from the fact that $\mathcal{M}, \Gamma \Vdash_w t{:}\varphi(x)$ that $\mathcal{M}, \Delta \Vdash_w \varphi(x)$, and this is a contradiction. ■

A closer look at the completeness proof of Section 6 shows that, if we drop axiom 8, we can establish completeness with respect to quasi-models. But without axiom 8, internalization is not available, and the resulting system does not seem to have any real interest.

Axiom 8 itself is called a *uniform* Barcan formula. Here is the essence of it, stated using quantifiers but not the function symbol $\forall$.

$$(\exists y)(\forall x)y{:}\varphi(x) \supset (\exists y)y{:}(\forall x)\varphi(x)$$

where y does not occur free in $\varphi(x)$. There is a closer analog to the Barcan formula itself, namely the following, where again y does not occur free in $\varphi(x)$.

$$(\forall x)(\exists y)y{:}\varphi(x) \supset (\exists y)y{:}(\forall x)\varphi(x)$$

The difference between the two versions is a little subtle. In the antecedent of the Barcan formula version, $\varphi(x)$ is asserted to have a proof for each x, but that proof might be different for different x. In the antecedent of the uniform version there is a single proof that is asserted to work for $\varphi(x)$ for every x.

Almost certainly the usual completeness arguments for constant domain first-order modal logic carry over, providing a completeness result with respect to constant domain models, if the Barcan formula is taken as an axiom, and not just the uniform version. But this axiom system seems much too strong, though I'm not yet sure why. On the other hand, we don't have a natural semantic condition to correspond to the uniform Barcan formula, which is why the treatment is as it is.

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