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1. INTRODUCTION. The Poisson kernel

$$P_1(x,y) = \frac{1}{2\pi R} \cdot \frac{|R^{\,2} - |x|^2|}{|x-y|^2}\,, \tag{1}$$

where $|y| = R$, is a famous mathematical object. In textbooks on real and complex analysis, this kernel is used usually as a tool to solve the Dirichlet problem for a plane disk of radius R centered at the origin or for its exterior (see, for example, [1], [2]).

In this paper we will see that the same kernel, considered as a function of two variables in the $(k+1)$-dimensional space ($k > 0$), can be useful to obtain a solution of the classical Dirichlet problem in the $(k+1)$-dimensional ball of radius R or for its exterior. The derivation presented here does not involve Green's identity or Green's function and relies on a spherical property of the 2-D Poisson kernel introduced here and expressed by the following identity

$$\alpha + \beta = k \quad \Leftrightarrow \quad \oint\limits_{\mathbb{S}^k} \omega^\alpha \, d\mathbb{S}_y = \oint\limits_{\mathbb{S}^k} \omega^\beta \, d\mathbb{S}_y\,, \tag{2}$$

where α and β are real numbers,

$$\omega = 2\pi R \cdot P_1(x,y) = \frac{|R^2 - |x|^2|}{|x-y|^2} = \frac{||x|^2 - |y|^2|}{|x-y|^2}\,, \tag{3}$$

with $x \in \mathbb{R}^{k+1} \setminus \mathbb{S}^k$ and $y \in \mathbb{S}^k$. $\mathbb{S}^k$ is a k-dimensional sphere of radius R centered at the origin O. From now on, for convenience, all considerations will be carried out in terms of ω.

One of the aims of this paper is to extend the geometric interpretation of ω, originally introduced by Hermann Schwarz (section 2) and then we will obtain several basic results presented in section 3.

Sections 4, 5, 6, 7 give applications of the ideas and results from sections 2 and 3. As the first application, we will see how the geometric interpretation of ω allows us to easily compute certain integrals of the potential type

$$I_\alpha(x) = \oint\limits_{\mathbb{S}^k} \frac{d\mathbb{S}_y}{|x-y|^\alpha}\,. \tag{4}$$

Namely, we obtain explicit formulae for $\alpha = k-1,\ k+1,\ 2k$.

In addition, as a consequence of Theorem 3 (section 7), we shall see that formula (4) for $\alpha = k-1$ implies that the function $1/|x-y|^{k-1}$ is harmonic, i.e. the theorem introduces a new sufficient integral condition for a function, depending only on distance, to be harmonic.

In section 6 we will get some non-trivial inequalities as a consequence of the basic identities presented in section 3.

2. GEOMETRIC INTERPRETATION OF ω.

In this section, we will see the generalization of the geometric interpretation of ω, originally introduced by Hermann Schwarz. He worked with circle and point inside the circle (see [6], p.168).

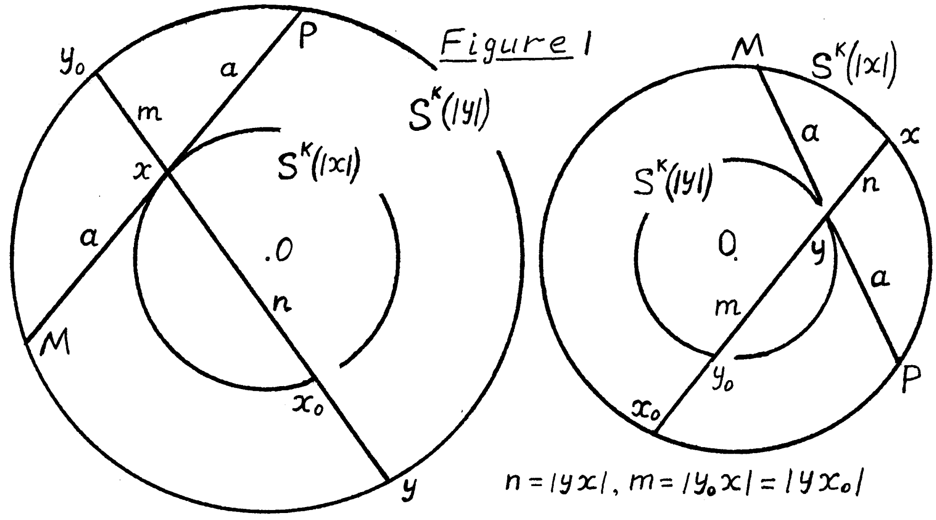
Let x and y be two arbitrary points in $\mathbb{R}^{k+1}$ such that $|x| \neq |y|$. To introduce the geometric interpretation of ω, let us consider two concentric k-dimensional spheres $\mathbb{S}^k(|x|)$ and $\mathbb{S}^k(|y|)$ centered at the origin O with radii $|x|$ and $|y|$ respectively.

Proposition 1. Let l be the line passing through the points x and y. Let us assume that l intersect $\mathbb{S}^k(|y|)$ at y and y_0. And let l intersects $\mathbb{S}^k(|x|)$ at x and x_0. Then

$$\omega = \frac{\big||x|^2 - |y|^2\big|}{|x-y|^2} = \frac{|x-y_0|}{|x-y|} = \frac{|x_0-y|}{|x-y|} = \frac{m}{n} \tag{5}$$

for every $x \in \mathbb{S}^k(|x|)$ and $y \in \mathbb{S}^k(|y|)$. In (5), we denoted $n = |x-y|$ and $m = |x-y_0| = |y-x_0|$.

Proof. Note that to prove this proposition, it is enough to consider the plane determined by the origin O and the points x and y. On this plane we will see the following picture (figure 1).



For now, we consider the case $|x| < |y|$ (figure 1, left). Let MP be the tangent line to the inner circle at point x. Let $|Mx| = a$. By Pythagorean theorem,

$$a^2 = |Mx|^2 = |Px|^2 = \left||x|^2 - |y|^2\right| . \tag{6}$$

Note, also, that $\triangle Mxy_0$ is similar to $\triangle Pxy$. Therefore, $a^2 = m \cdot n$ and then

$$\omega = \frac{\left||x|^2 - |y|^2\right|}{|x-y|^2} = \frac{a^2}{n^2} = \frac{m}{n} . \tag{7}$$

This completes the proof for the case $|x| < |y|$. The same argument may be repeated for the case $|x| > |y|$ (use figure 1, right) ■

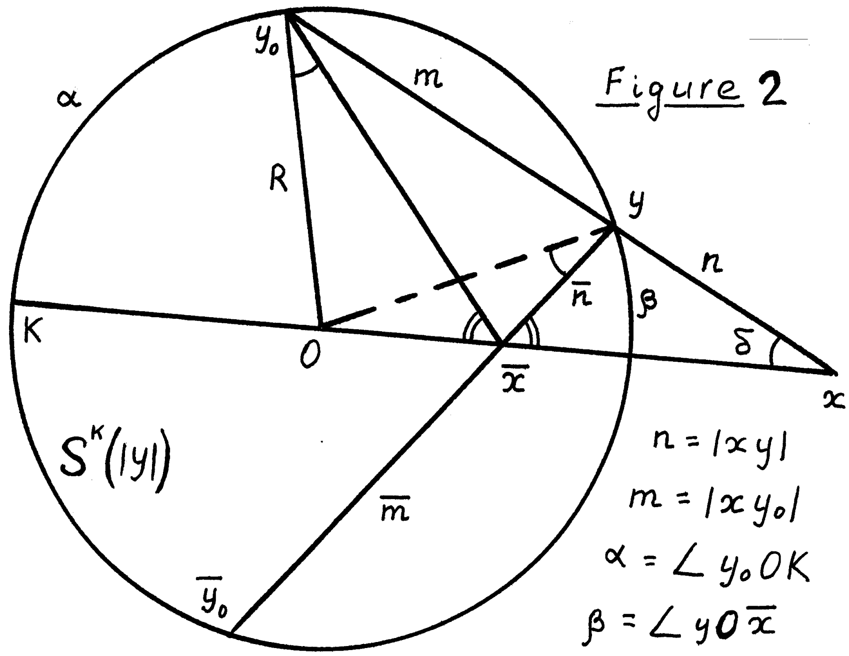
Remark. Using inversion, we can reduce any of the two cases: $|x| < |y|$ and $|x| > |y|$ to the other. The reduction relies on the following proposition. For the notations appearing below, see figure 2.

Proposition 2. Let $|x| > |y| = R$. Let $\overline{x}$ be such a point from the line segment Ox that $|\overline{x}| \cdot |x| = R^2$. Let $\overline{n} = |\overline{x}y|$, $\overline{m} = |\overline{x}\overline{y_0}|$, where $\overline{y_0}$ is the point of the intersection of $S^k(|y|)$ and the line $\overline{x}y$. Let m, n be defined as before. Then $\overline{y_0}$ is symmetric to y_0 with respect to the line Ox,

$$\frac{m}{n} = \frac{\overline{m}}{\overline{n}} \quad \text{and} \quad \omega(x, y) = \omega(\overline{x}, y) , \tag{8}$$

and notice now that $|\overline{x}| < |y| = R$.

Proof. As before, we consider the plane determined by the origin O and the points x and y (figure 2). At first, notice that the two equalities in (8) are equivalent, since by Proposition 1, $m/n = \omega(x, y)$ and $\overline{m}/\overline{n} = \omega(\overline{x}, y)$. The second equality in (8) can be easily checked by direct algebraic computation. Here we will see a geometric proof of the first equality by showing that point $\overline{y_0}$ is symmetric to y_0 with respect to the line Ox.



At first, note that if $\angle y_0OK = \alpha$, $\angle yOx = \beta$, $\angle yxO = \delta$, then

$$\delta = \frac{\alpha - \beta}{2}. \tag{9}$$

Notice, also, that $\triangle Oy\overline{x}$ is similar to $\triangle Oxy$ which yields

$$\angle Oy\overline{x} = \delta \quad \text{and} \quad \frac{\overline{n}}{n} = \frac{R}{|x|}. \tag{10}$$

For the same reason, $\triangle Oy_0\overline{x}$ is similar to $\triangle Oxy_0$ and so

$$\angle Oy_0\overline{x} = \delta \quad \text{and} \quad \frac{|y_0\,\overline{x}|}{m} = \frac{R}{|x|}. \tag{11}$$

Using (10) and (11), it is not hard to see that

$$\angle y\overline{x}x = \beta + \delta = \frac{\alpha + \beta}{2} \tag{12}$$

and

$$\angle y_0\overline{x}O = \alpha - \delta = \frac{\alpha+\beta}{2}\,. \tag{13}$$

Therefore $\angle y\overline{x}x = \angle y_0 x O = \angle O\overline{x}\overline{y}_0 = (\alpha+\beta)/2$, which means that $\overline{y}_0$ is symmetric to y_0 with respect to the line Kx and so $|y_0\overline{x}| = |\overline{y}_0\overline{x}| = \overline{m}$. Using (10) and (11) we can easily see that $m/n = \overline{m}/\overline{n}$ ■.

3. BASIC IDENTITIES.

Theorem 1 (Spherical property). Let $\mathbb{S}^k$ be a k-dimensional sphere of radius R centered at the origin O and

$$\omega(x,y) = \frac{||x|^2 - |y|^2|}{|x-y|^2}\,. \tag{14}$$

If α, β are real, then

$$\alpha + \beta = k \quad \Leftrightarrow \quad \oint\limits_{\mathbb{S}^k} \omega^\alpha \, d\mathbb{S}_y = \oint\limits_{\mathbb{S}^k} \omega^\beta \, d\mathbb{S}_y \tag{15}$$

for every fixed $x \notin \mathbb{S}^k$.

Corollary 1. The identity (15) can be written in the following form

$$\oint\limits_{\mathbb{S}^k} \frac{d\mathbb{S}_y}{|x-y|^\alpha} = |R^2 - |x|^2|^{(\beta-\alpha)/2} \oint\limits_{\mathbb{S}^k} \frac{d\mathbb{S}_y}{|x-y|^\beta}\,, \tag{16}$$

where $\alpha + \beta = 2k$ and $\alpha, \beta \in \mathbb{R}$. In particular,

$$\oint\limits_{\mathbb{S}^k} \frac{d\mathbb{S}_y}{|x-y|^{k-1}} = \oint\limits_{\mathbb{S}^k} \frac{|R^2 - |x|^2|}{|x-y|^{k+1}} \, d\mathbb{S}_y \tag{17}$$

for every $x \notin \mathbb{S}^k$.

Corollary 2. If $k = 1$, identity (15) yields

$$\int\limits_0^{2\pi} \frac{d\theta}{(a - b\,\sin(\theta))^p} = (a^2 - b^2)^{1/2-p} \int\limits_0^{2\pi} (a - b\,\sin(\theta))^{p-1} d\theta \tag{18}$$

for every $p, a, b \in \mathbb{R}$ and $a > b > 0$.

Corollary 3. Most of the results remains true even for α, β, $p \in \mathbb{C}$. The method of extension to complex numbers will be discussed in the proof of Theorem 1 (see remark 4).

At first, let us prove the following lemma introducing a new identity, which is important by itself and will be useful in the proof of the theorem.

Lemma 1. Let $\mathbb{S}^k(r)$ and $\mathbb{S}^k(R)$ be two k-dimensional spheres of radii r and R respectively. Let x be running over $\mathbb{S}^k(r)$ and y be running over $\mathbb{S}^k(R)$. If g is a measurable function on $\overline{\mathbb{R}}$, then

$$R^{\,k} \cdot \oint\limits_{\mathbb{S}^k(r)} g(|x-y|)\, d\,\mathbb{S}_x = r^k \cdot \oint\limits_{\mathbb{S}^k(R)} g(|x-y|)\, d\,\mathbb{S}_y \tag{19}$$

and

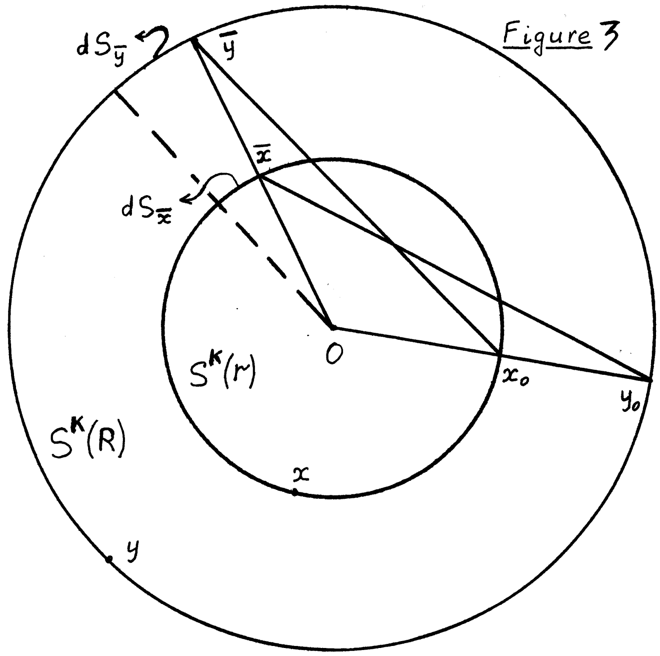
$$R^{\,k} \cdot \oint\limits_{\mathbb{S}^k(r)} g \circ \omega(x,y) d\,\mathbb{S}_x = r^k \cdot \oint\limits_{\mathbb{S}^k(R)} g \circ \omega(x,y) d\,\mathbb{S}_y\,. \tag{20}$$

Moreover, for any continuous function f defined on the unit k-dimensional sphere centered at the origin O we have

$$R^{\,k} \cdot \oint\limits_{\mathbb{S}^k(r)} g(|x-y_0|) \cdot f\left(\frac{x}{|x|}\right) d\,\mathbb{S}_x = r^k \cdot \oint\limits_{\mathbb{S}^k(R)} g(|x_0-y|) \cdot f\left(\frac{y}{|y|}\right) d\,\mathbb{S}_y\,, \tag{21}$$

where $x_0 \in \mathbb{S}^k(r)$, $y_0 \in \mathbb{S}^k(R)$ and O belong to one line.

Proof of Lemma 1. Consider figure 3. Let $x, \overline{x}, x_0 \in \mathbb{S}^k(r)$ and $y, \overline{y}, y_0 \in \mathbb{S}^k(R)$, where $\overline{x}, \overline{y}$ belong to the line segment $O\overline{y}$ and x_0, y_0 belong to the line segment Oy_0. We consider the picture appearing on the cross-sectional plane that contains the points O, x_0, y_0. The points x, y need not be on the cross-sectional plane.



It is not hard to see that for every $\overline{y}, y_0 \in \mathbb{S}^k(R)$ we have

$$|\overline{x} - y_0| = |\overline{y} - x_0| \quad \text{and} \quad d\,\mathbb{S}_{\overline{x}} = \left(\frac{r}{R}\right)^k d\,\mathbb{S}_{\overline{y}}\,. \tag{22}$$

Note also that the integral on the left hand side in (19) does not depend on $x \in \mathbb{S}^k(r)$. The integral on the right hand side in (19) does not depend on $y \in \mathbb{S}^k(R)$. This is why the first and the last identities in the following sequence of equalities hold.

$$\begin{aligned} \oint\limits_{\mathbb{S}^k(r)} g(|x-y|)d\,\mathbb{S}_x &= \oint\limits_{\mathbb{S}^k(r)} g(|\overline{x}-y_0|)d\,\mathbb{S}_{\overline{x}} \\ &= \oint\limits_{\mathbb{S}^k(R)} g(|x_0-\overline{y}|)\left(\frac{|\overline{x}|}{|\overline{y}|}\right)^k d\,\mathbb{S}_{\overline{y}} = \left(\frac{r}{R}\right)^k \oint\limits_{\mathbb{S}^k(R)} g(|x-y|)d\,\mathbb{S}_y \end{aligned} \tag{23}$$

A similar argument could be used to prove (21).

The identity (20) holds, because the function $g \circ \omega(x,y) = \widetilde{g}(|x-y|)$ is also measurable on $\overline{\mathbb{R}}$ since

$$\omega(x,y) = \frac{|R^2 - r^2|}{|x-y|^2} \tag{24}$$

is continuous as a function of $|x-y|$ for $|R-r| < |x-y| < |R+r|$ and therefore, this substitution can add only one point of discontinuity in the case of $R=r$ ■

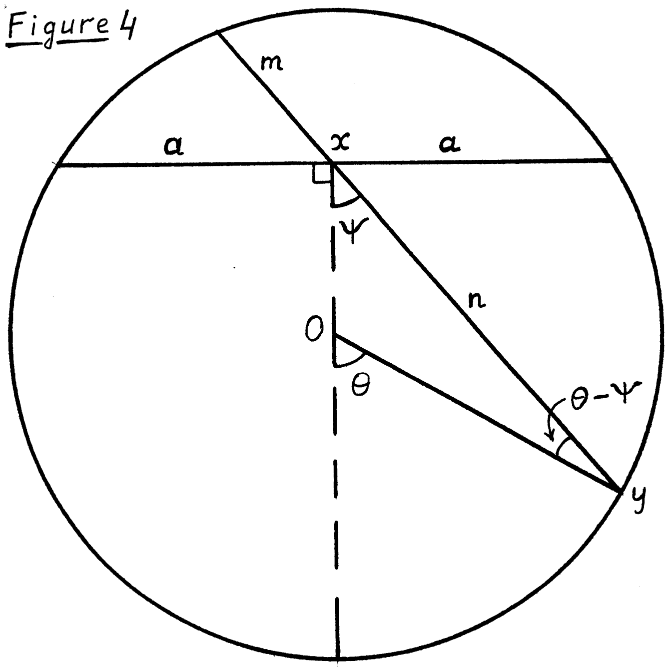
One of the interesting applications of this lemma is presented in the proof of theorem 3 from appendix.

Proof of Theorem 1.

Proof of the right hand side implication " $\Rightarrow$ ". Let, at first, $\alpha\,,\beta \in \mathbb{R}$ and $\alpha+\beta=k$. Then $\beta = k-\alpha$.

The central idea of the proof is related to the geometric interpretation of ω. See figure 4 for definitions of quantities to be introduced bellow.

Figure 4



Let x be inside the sphere $\mathbb{S}^k$. Then

$$\omega = \frac{R^2-|x|^2}{|x-y|^2} = \frac{a^2}{n^2} = \frac{n\cdot m}{n^2} = \frac{m}{n}\ , \tag{25}$$

since $a^2 = n\cdot m$ and, by Pythagorean theorem, $a^2 = R^2-|x|^2$. Thus, $\omega = \frac{m}{n}(\theta)$ is a function depending only on the angle θ or the angle ψ shown in the picture.

Notice that for any continuous function $f(m,n)$ we have

$$f(n(\psi), m(\psi)) = f(m(\pi-\psi), n(\pi-\psi))\,. \tag{26}$$

To use this property, we need to change the variable of integration from θ to ψ and back. From the same picture, using the law of sines, we can easily see that

$$|x|\ \sin(\psi) = R\ \sin(\theta - \psi)\,. \tag{27}$$

Differentiation with respect to θ and ψ gives us

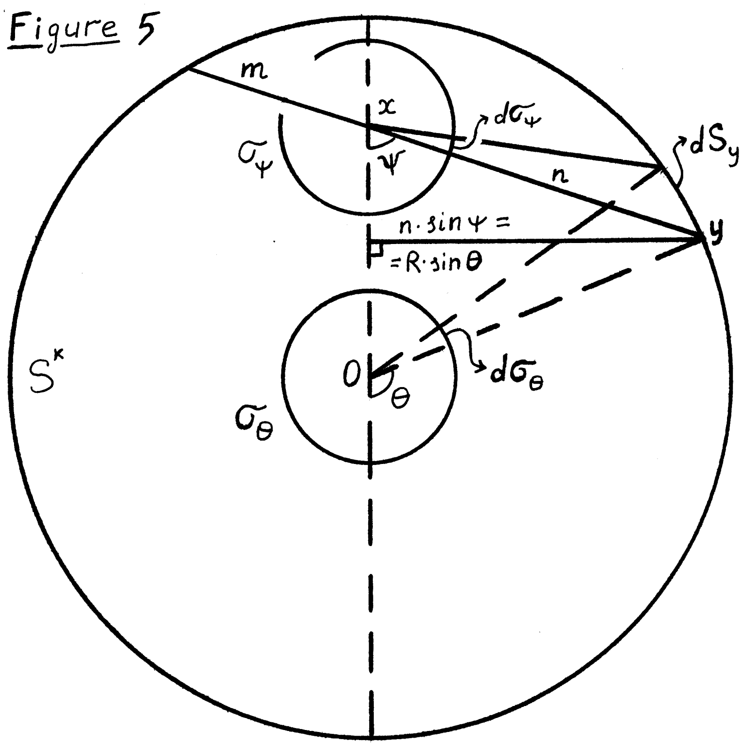
$$d\theta = \frac{|x|\ \cos(\psi) + R\ \cos(\theta - \psi)}{R\ \cos(\theta - \psi)}\, d\psi = \frac{n}{\frac{1}{2}(n+m)}\, d\psi\,. \tag{28}$$

Therefore,

$$d\theta = \frac{2n}{n+m}\, d\psi\,. \tag{29}$$

For the next step we need the next picture (figure 5). Let σ_ψ and σ_θ be the unit k-dimensional spheres centered at the points x and O respectively. Recall that $|x - y| = n$.

Figure 5



It is not hard to see that

$$R\,\sin(\theta) = n\,\sin(\psi)\,. \tag{30}$$

Then, using (29) and (30), we have

$$\begin{aligned} d\mathbb{S}_y &= |\sigma_{k-1}|\,(R\sin(\theta))^{k-1}\,d(R\theta) \\ &= |\sigma_{k-1}|\,(n\,\sin(\psi))^{k-1}\,R\,d\theta \\ &= \frac{2R}{m+n}\,n^k\,(\sin(\psi))^{k-1}|\,\sigma_{k-1}|\,d\psi \\ &= \frac{2R}{m+n}\,n^k\,d\sigma_\psi\,, \end{aligned} \tag{31}$$

where $|\sigma_{k-1}|\,(R\sin(\theta))^{k-1}$ is the volume of $(k-1)$-dimensional sphere of radius $|Ly| = R\sin(\theta)$, centered at the point L (figure 2). L is the orthogonal projection of the point y to the line Ox and we assume here that the $(k-1)$-dimensional sphere is contained in the k-dimensional space orthogonal to the line Ox.

Remark 1. We can use such a representation of $d\mathbb{S}_y$ because all our integrands will depend only on θ or ψ and there will be no dependence on variables in the hyperplane, orthogonal to the line xO.

On the other hand, $d\mathbb{S}_y = R^k\,d\sigma_\theta$ and thus

$$d\mathbb{S}_y = R^k\,d\sigma_\theta = \frac{2R}{m+n}\,n^k\,d\sigma_\psi\,. \tag{32}$$

The last preliminary is to observe that for any continuous function $f(m,n)$, we have

$$\oint_{\sigma_\psi} f(m,n)\,d\sigma_\psi = \oint_{\sigma_\psi} f(n,m)\,d\sigma_\psi\,, \tag{33}$$

which is a simple consequence of (26). In other words, we can interchange m and n when we integrate with respect to σ_ψ. Now we are ready to obtain the basic identity (15). Apply (32), (33) and (32) again to obtain

$$
\begin{aligned}
\oint\limits_{\mathbb{S}^k} \left(\frac{m}{n}\right)^{\alpha} d\mathbb{S} &= \oint\limits_{\sigma_\theta} \left(\frac{m}{n}\right)^{\alpha} R^{\,k}\, d\sigma_\theta \\
&= \oint\limits_{\sigma_\psi} \left(\frac{m}{n}\right)^{\alpha} \frac{2R}{m+n}\, n^k\, d\sigma_\psi \\
&= \oint\limits_{\sigma_\psi} \left(\frac{n}{m}\right)^{\alpha} \frac{2R}{m+n}\, m^k\, d\sigma_\psi \\
&= \oint\limits_{\sigma_\psi} \left(\frac{m}{n}\right)^{-\alpha} \frac{m^k}{n^k}\, \frac{2R}{m+n}\, n^k\, d\sigma_\psi \\
&= \oint\limits_{\sigma_\theta} \left(\frac{m}{n}\right)^{k-\alpha} R^{\,k}\, d\sigma_\theta = \oint\limits_{\mathbb{S}^k} \left(\frac{m}{n}\right)^{k-\alpha} d\mathbb{S}_y\,.
\end{aligned}
\tag{34}
$$

Thus, the proof of the right hand side implication in (15) is complete for the case $|x| < R$. To finish the proof of the implication "$\Rightarrow$", we have to consider the case: $|x| > R$.

Remark 2. A simpler proof for the case $|x| < R$ can be obtained by using an angle θ^* introduced by Schwarz (see [6], p.168, or see figure 6 from section 7). Unfortunately, $\theta^\star$ can not be used to compute integrals (25) for $p = k-1,\ k+1$. For this reason we chose the angle ψ in the proof presented above. An outline of the simpler proof will be given in the appendix.

Let x be outside of the sphere $\mathbb{S}^k$. To use the Lemma 1, we assume that $y \in \mathbb{S}^k = \mathbb{S}^k(R)$, where $R = |y|$ and $x \in \mathbb{S}^k(r)$ is a k-dimensional sphere of radius $r = |x|$ centered at the origin O. Then, using the previous result and the Lemma 1 (see (20)), we have

$$
\oint\limits_{\mathbb{S}^k} \omega^{\alpha} d\,\mathbb{S}_y = \left(\frac{|y|}{|x|}\right)^{k} \oint\limits_{\mathbb{S}^k(r)} \omega^{\alpha} d\,\mathbb{S}_x = \left(\frac{|y|}{|x|}\right)^{k} \oint\limits_{\mathbb{S}^k(r)} \omega^{\beta} d\,\mathbb{S}_x = \oint\limits_{\mathbb{S}^k} \omega^{\beta} d\,\mathbb{S}_y\,. \tag{35}
$$

The above derivation completes the proof of the implication "$\Rightarrow$" of Theorem 1 ■

Remark 3. Similar proof for x outside of the sphere, can be obtain using

Proposition 2 (section 2). I.e.

$$\oint_{\mathbb{S}^k} \left(\frac{m}{n}\right)^{\alpha} d\mathbb{S}_y = \oint_{\mathbb{S}^k} \left(\frac{\overline{m}}{\overline{n}}\right)^{\alpha} d\mathbb{S}_y = \oint_{\mathbb{S}^k} \left(\frac{\overline{m}}{\overline{n}}\right)^{\beta} d\mathbb{S}_y = \oint_{\mathbb{S}^k} \left(\frac{m}{n}\right)^{\beta} d\mathbb{S}_y \,, \tag{36}$$

where $\alpha + \beta = k$, as before ■

Remark 4. The right hand side implication "⇒" in (15) remains true even for complex numbers α and β. To see this, let us consider the following two functions

$$F(\alpha) = \oint_{\mathbb{S}^k} \omega^{\alpha} d\,\mathbb{S}_y \quad \text{and} \quad G(\alpha) = \oint_{\mathbb{S}^k} \omega^{k} \cdot \omega^{-\alpha} d\,\mathbb{S}_y \tag{37}$$

Notice now that

$$F(\alpha) = \int_0^{\pi} \left| \frac{R^2 - |x|^2}{(R^2 + |x|^2 - 2R|x|\cos(\theta))^{1/2}} \right|^{\alpha} R^{\,k} (\sin(\theta))^{k-1} d\theta \tag{38}$$

is analytic in $\mathbb{C}$ by the theorem about interchanging integration and differentiation (see [3], p.159). Similarly, $G(\alpha)$ is also analytic in $\mathbb{C}$.

These two analytic functions coincide on $\mathbb{R}$, i.e. $G(\alpha) = F(\alpha)$ for all $\alpha \in \mathbb{R}$, as it was shown above. Therefore, by the identity theorem (see [2], p.228), $F(\alpha) \equiv G(\alpha)$ for every $\alpha \in \mathbb{C}$ ■

Proof of the left hand side implication " ⇐ " in (15).

Let

$$\oint_{\mathbb{S}^k} \omega^{\alpha} d\,\mathbb{S}_y = \oint_{\mathbb{S}^k} \omega^{\beta} d\,\mathbb{S}_y \,. \tag{39}$$

Assume that α is given and fixed. We have to show that the equation

$$F(\alpha) = \oint_{\mathbb{S}^k} \omega^{\alpha} d\,\mathbb{S}_y = \oint_{\mathbb{S}^k} \omega^{\lambda} d\,\mathbb{S}_y = F(\lambda) \tag{40}$$

has only two solutions with respect to variable λ, i.e. $\lambda = \alpha,\ k - \alpha$. At first, note that, since $\lambda + (k - \lambda) = k$, the function $F(\lambda)$ is symmetric with respect to the point $\lambda = k/2$. Moreover, it is not hard to see that

$$F(\lambda) = \int_0^{\pi} \left| \frac{R^2 - |x|^2}{(R^2 + |x|^2 - 2R|x|\cos(\theta))^{1/2}} \right|^{\lambda} R^{\,k} (\sin(\theta))^{k-1} d\theta \tag{41}$$

is continuous and differentiable with respect to the λ since the integrand, as well as its partial derivative with respect to λ is continuous in the region $0 \leq \theta \leq \pi\,,\;\; \lambda_1 \leq \lambda \leq \lambda_2$ for every $\lambda_1\,,\, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 < \lambda_2$. Therefore, we can apply the Leibnitz rule of differentiation under integral sign (see [4], p. 121). That is

$$\frac{dF(\lambda)}{d\lambda} = \oint\limits_{\mathbb{S}^k} \omega^{\lambda} \cdot (\ln \omega)\, d\,\mathbb{S}_y\,. \tag{42}$$

We can apply the same rule one more time to obtain

$$\frac{d^2F(\lambda)}{(d\lambda)^2} = \oint\limits_{\mathbb{S}^k} \omega^{\lambda} \cdot (\ln \omega)^2\, d\,\mathbb{S}_y > 0 \quad \forall \lambda \in \mathbb{R}\,. \tag{43}$$

Therefore, the graph of our function $\xi = F(\lambda)$, plotted on the plane (λ, ξ) is convex and symmetric with respect to the line $\lambda = k/2$. This means that $\lambda = k/2$ is the minimum point for $F(\lambda)$. Therefore, at every level $\nu \in \mathbb{R}$, such that

$$\nu > \oint\limits_{\mathbb{S}^k} \omega^{k/2} d\,\mathbb{S}_y\,, \tag{44}$$

the equation $F(\lambda) = \nu$ has only two solutions α and β satisfying the condition $\alpha + \beta = k \quad \blacksquare$

Proof of Corollary 2.

Note that the condition: $a > b > 0$ is sufficient to find R and r such that $R > r > 0$ and the equalities $R^2 + r^2 = a\,,\; 2R\,r = b$ hold. Thus, the function under the integral can be reduced to the Poisson kernel which yields

$$
\begin{aligned}
&\int_0^{2\pi} \frac{d\theta}{(a - b\,\sin(\theta))^p} \\
&= (R^2 - r^2)^{-p} \int_0^{2\pi} \left(\frac{R^2 - r^2}{R^2 + r^2 + 2R\,r\,\cos(\theta)} \right)^p d\theta \\
&= (a^2 - b^2)^{-p/2} \int_0^{2\pi} \left(\frac{m}{n} \right)^p d\theta \qquad (45) \\
&= (a^2 - b^2)^{-p/2} \int_0^{2\pi} \left(\frac{m}{n} \right)^{1-p} d\theta \\
&= (a^2 - b^2)^{1/2-p} \int_0^{2\pi} (a - b\,\sin(\theta))^{p-1}\, d\theta\,.
\end{aligned}
$$

Remark 4. When we integrate a 2π-periodic function over the period from 0 to 2π, we can replace $\sin(\theta)$ by $\cos(\theta)$ and vice versa without changing the result of the integration. ■

4. ALGEBRAIC COMPUTATION OF THE ELECTROSTATIC POTENTIAL CREATED BY A CHARGE WITH EVENLY DISTRIBUTED DENSITY

We will see here that the technique developed in the proof of the theorem is found to be useful for the direct computation of some integrals.

Definition 1. Define a point charge in $\mathbb{R}^{k+1}$ as a pair (Q, x), where $Q \in \mathbb{R}$ is called charge measure or, simple, charge and $x \in \mathbb{R}^{k+1}$ is called the location of the charge.

Definition 2. The electrostatic potential in $\mathbb{R}^{k+1}$ created by a point charge (Q, x) is the function of one variable $y \in \mathbb{R}^{k+1} \backslash \{x\}$,

$$
W_k^Q(x, y) = \frac{Q}{|x - y|^{k-1}}\,. \qquad (46)
$$

We will see that for every $y \neq x$, $W_k^Q(x,y)$ is a harmonic function with respect to the variables x, y and $W_k^Q(x,y)$ depends only on the distance between x and y. So, definition 2 is a natural generalization of the electrostatic potential in $\mathbb{R}^3$ created by a point charge (see [5], p. 738).

<u>*Definition* 3.</u> The electrostatic potential created by a k-dimensional sphere $\mathbb{S}^k \in \mathbb{R}^{k+1}$ is the function

$$F_k(x) = \oint\limits_{\mathbb{S}^k} \frac{\rho(y)}{|x-y|^{k-1}}\, d\,\mathbb{S}_y\,, \tag{47}$$

where $y \in \mathbb{S}^k$ and $\rho(y)$ is a continuous function called the density distribution.

Note that this definition is a natural generalization of the potential created by a distributed charge on a surface in $\mathbb{R}^3$ (see [5], p. 738). Note also that the function $F_k(x)$ is well defined for every $x \in \mathbb{R}^{k+1}$, although, we will consider it only for $x \notin \mathbb{S}^k$.

Consider the potential created by the sphere $\mathbb{S}^k \in \mathbb{R}^{k+1}$ for $\rho(y) \equiv \text{const}$, $y \in \mathbb{S}^k$. Without loss of generality, we may assume that $\rho(y) \equiv 1$. Thus, we have

Proposition. Let $\rho(y) \equiv 1$. Then

$$\oint\limits_{\mathbb{S}^k} \frac{d\,\mathbb{S}_y}{|x-y|^{k-1}} = F_k(x) = \frac{|\mathbb{S}^k|}{R^{k-1}}\ , \quad \text{for} \quad |x| < R \tag{48}$$

and

$$\oint\limits_{\mathbb{S}^k} \frac{d\,\mathbb{S}_y}{|x-y|^{k-1}} = F_k(x) = \frac{|\mathbb{S}^k|}{|x|^{k-1}}\ , \quad \text{for} \quad |x| > R \tag{49}$$

Proof of (48)**.** Let x be inside a sphere $\mathbb{S}^k$ (figure 4).

Using (32), (33) and elementary algebra, we have

$$\begin{aligned}
\oint\limits_{\mathbb{S}^k} \frac{d\mathbb{S}_y}{|x-y|^{k-1}} &= \oint\limits_{\sigma_\psi} \frac{1}{n^{k-1}}\, \frac{2R}{m+n}\, n^k\, d\sigma_\psi \\
&= R \oint\limits_{\sigma_\psi} \frac{2\,n}{m+n}\, d\sigma_\psi = R \oint\limits_{\sigma_\psi} \frac{2\,m}{m+n}\, d\sigma_\psi \\
&= R \oint\limits_{\sigma_\psi} \left(2 - \frac{2\,n}{m+n}\right)\, d\sigma_\psi\,.
\end{aligned} \tag{50}$$

Hence

$$F(x) = R \oint\limits_{\sigma_\psi} \frac{2\,n}{m+n}\, d\sigma_\psi = R \oint\limits_{\sigma_\psi} \left(2 - \frac{2\,n}{m+n}\right) d\sigma_\psi\,. \tag{51}$$

Therefore

$$F(x) = \oint\limits_{\mathbb{S}^k} \frac{d\mathbb{S}_y}{|x-y|^{k-1}} = R\,|\sigma_k| = \frac{|\mathbb{S}^k|}{R^{k-1}}\,, \tag{52}$$

where $|\sigma_k|$ is the area of a k-dimensional unit sphere and $|\mathbb{S}^k|$ is the area of a k-dimensional sphere of radius R. Thus, the electrostatic potential inside a sphere created by a charge, evenly distributed on the sphere, is constant.

Proof of (49)**.** Let x be outside of the sphere $\mathbb{S}^k$. Note that identity (49) can be obtained as a simple consequence of Lemma 1 and (48). Indeed, let $r = |x| > |y| = R$ and let $g(|x-y|) = 1/|x-y|^{k-1}$. Then, using (48) and (19), we have

$$|x|^k \oint\limits_{\mathbb{S}^k(R)} g(|x-y|)d\,\mathbb{S}_y = R^{\,k} \oint\limits_{\mathbb{S}^k(r)} g(|x-y|)d\,\mathbb{S}_x = R^{\,k} \cdot \frac{|\mathbb{S}^k(r)|}{|x|^{k-1}}\,, \tag{53}$$

which yields

$$\oint\limits_{\mathbb{S}^k(R)} g(|x-y|)d\,\mathbb{S}_y = \frac{R^{\,k}\cdot|\sigma_k|}{|x|^{k-1}} = \frac{|\mathbb{S}^k(R)|}{|x|^{k-1}}\,, \tag{54}$$

where, as before, $\mathbb{S}^k(r)$ and $\mathbb{S}^k(R) = \mathbb{S}^k$ are 2 concentric k-dimensional spheres of radii r and R respectively centered at the origin O ■

Thus, we see that the potential at point x is equal to the potential created by the point charge with measure $|\mathbb{S}^k|$ and located at O.

Remark 1.

Identity (48) implies that the function $1/|x-y|^{k-1}$ is harmonic. This conclusion follows from Theorem 3 in the Appendix (section 7).

Remark 2.

As a bonus, using cases 1, 2 (see (48), (49)) and identity (17), we can easily see that

1. For x inside a sphere,

$$\oint\limits_{\mathbb{S}^k} \frac{d\mathbb{S}_y}{|x-y|^{k+1}} = \frac{R\,|\sigma_k|}{R^2-|x|^2}\,. \tag{55}$$

2. For x outside of a sphere

$$\oint\limits_{\mathbb{S}^k} \frac{d\mathbb{S}_y}{|x-y|^{k+1}} = \left(\frac{R}{|x|}\right)^{k-1} \frac{R\,|\sigma_k|}{|x|^2 - R^2}\,. \tag{56}$$

3. For every $x \notin \mathbb{S}^k$

$$\oint\limits_{\mathbb{S}^k} \frac{d\mathbb{S}_y}{|x-y|^{2k}} = \frac{|\mathbb{S}^k|}{|R^2 - |x|^2|^k}\,, \tag{57}$$

which is a simple consequence of the basic identity (16) only.

Remark 3. The identity (55) can be obtained directly. Using figure 4, observe that

$$\frac{R^2 - |x|^2}{|x-y|^{k+1}} = \frac{m \cdot n}{n^{k+1}} = \frac{m}{n^k} \tag{58}$$

and repeat the process shown in (50).

5. DIRICHLET PROBLEM.

Let $f(y)$ be a continuous function defined on the k-dimensional sphere $\mathbb{S}^k$ of radius R centered at the origin O. The Dirichlet problem is to find a harmonic function $U(x)$ inside (outside) the sphere $\mathbb{S}^k$ such that $\lim_{x\to x_0} f(x) = f(x_0)$ for every $x_0 \in \mathbb{S}^k$.

Solution.

Let us solve, first, the simplest Dirichlet problem corresponding to $f(y) \equiv 1$. It turns out that the solution is $U(x) \equiv 1$. Note, also, that the solution can be described by means of electrostatic potential function defined in (45), since this function is harmonic. According to (48), we can distribute charge on $\mathbb{S}^k$ in such a way as to make unit potential inside the sphere. It is not hard to see that the density of this distribution should be $\rho = 1/R\,|\sigma_k|$. Therefore, a solution for the simplest Dirichlet problem can be presented in the following form

$$U(x) \equiv 1 = \frac{1}{R\,|\sigma_k|} \oint\limits_{\mathbb{S}^k} \frac{d\,\mathbb{S}_y}{|x-y|^{k-1}} = \frac{1}{R\,|\sigma_k|} \oint\limits_{\mathbb{S}^k} \frac{R^{\,2} - |x|^2}{|x-y|^{k+1}} d\,\mathbb{S}_y\,, \tag{59}$$

where the last equality was obtained by using (17) or the last integral can be computed directly by using the argument described in remark 3 above.

To obtain a solution for an arbitrary Dirichlet problem, as well as to check the boundary condition for the simple one ($f(y) \equiv 1$ for $y \in \mathbb{S}^k$), it is enough to note that the integrand

$$P_k(x,y) = \frac{1}{R\,|\sigma_k|} \cdot \frac{R^2 - |x|^2}{|x-y|^{k+1}}\,, \tag{60}$$

obtained in (59), is harmonic with respect to x, which can be easily checked by direct computation, and

$$\lim_{x \to x_0} \oint\limits_{\mathbb{S}^k} P_k(x,y) \cdot f(y) d\,\mathbb{S}_y = f(x_0) \tag{61}$$

for every $x_0 \in \mathbb{S}^k$ and for every continuous function f defined on $\mathbb{S}^k$. The last property (61) is proved carefully for $k = 1$ in [6] (p.168). One can easily see that the proof for arbitrary k can be obtained as a copy of the proof given in [6]. Note also that both of these properties hold for $|x| < R$, as well as, for $|x| > R$. Therefore, the function

$$W(x) = \frac{1}{R\,|\sigma_k|} \oint\limits_{\mathbb{S}^k} \frac{|R^2 - |x|^2|}{|x-y|^{k+1}} \cdot f(y) d\,\mathbb{S}_y \tag{62}$$

gives a solution for any Dirichlet problem, i.e. $W(x)$ is harmonic inside (outside) the $\mathbb{S}^k$ and $\lim_{x \to x_0} W(x) = f(x_0)$ for every continuous function f defined on $\mathbb{S}^k$ ■

Remark. The traditional way to compute both of the integrals in (59) for x inside $\mathbb{S}^k$ relies on using Stokes' formula (see [7], p. 27, exercises 17, 18).

6. INEQUALITIES

Using identity (16) and elementary algebra, we can obtain some non-trivial inequalities presented in the following theorem:

Theorem 2. Let $x \notin \mathbb{S}^k$ and $y \in \mathbb{S}^k$. Then

$$\frac{|\mathbb{S}^k|}{|R - s|x||^k \cdot |R + s|x||^{\alpha - k}} \leq \oint\limits_{\mathbb{S}^k} \frac{d\,\mathbb{S}_y}{|x-y|^{\alpha}} \leq \frac{|\mathbb{S}^k|}{|R + s|x||^k \cdot |R - s|x||^{\alpha - k}}\,, \tag{63}$$

where $s = sign(\alpha - 2k)$ for $\alpha \neq 2k$ and $s = 1$ for $\alpha = 2k$. The equality holds if and only if $\alpha = 2k$ or $x = 0$.

We will show in the remarks that for $\alpha > k > 1$ these lower and upper bounds are finer then the following obvious bounds C and D

$$C = \frac{|\mathbb{S}|^k}{|R + |x|\,|^\alpha} \le \oint\limits_{\mathbb{S}^k} \frac{d\,\mathbb{S}_y}{|x-y|^\alpha} \le \frac{|\mathbb{S}|^k}{|R - |x|\,|^\alpha} = D\,, \tag{64}$$

where equalities hold only for $|x| = 0$.

Remark 1. For $\alpha > k$ and $x \neq 0$, we have

$$C < \frac{|\mathbb{S}^k|}{|R + s\,|x|\,|^k \cdot |R - s\,|x|\,|^{\alpha-k}} \le \frac{|\mathbb{S}^k|}{|R - s\,|x|\,|^k \cdot |R + s\,|x|\,|^{\alpha-k}} < D \tag{65}$$

where the equality holds only for $\alpha = 2k$.

Remark 2. For $k = 1$, and $a > b > 0$, we can write

$$\begin{aligned} A = \frac{2\pi}{(a - s\,b)^{1/2} \cdot (a + s\,b)^{p-1/2}} &\le \int\limits_0^{2\pi} \frac{d\theta}{(a - b\,\sin(\theta))^p} \\ &\le \frac{2\pi}{(a + s\,b)^{1/2} \cdot (a - s\,b)^{p-1/2}} = B\,, \end{aligned} \tag{66}$$

where $s = sign(p-1)$ for $p \neq 1$ and $s = 1$ for $p = 1$. A and B are just notations for these upper and lower bounds. The equality holds if and only if $p = 1$ or $b = 0$.

For $p > 1/2$, we have

$$\frac{2\pi}{(a+b)^p} < A \le B < \frac{2\pi}{(a-b)^p}\,, \tag{67}$$

where the equality holds only for $p = 1$.

Example 1. If $a > b > 0$, then

$$\int\limits_0^{2\pi} \frac{d\theta}{(a - b\,\sin\theta)^{3/2}} \le \frac{2\pi}{(a-b)\sqrt{a+b}} < \frac{2\pi}{(a-b)^{3/2}}\,. \tag{68}$$

Example 2.

$$\oint\limits_{\mathbb{S}^2} \frac{d\mathbb{S}_y}{|x-y|^5} \le \frac{4\pi R^2}{||x| + R|^2 \cdot ||x| - R|^3} < \frac{4\pi R^2}{||x| - R|^5} \tag{69}$$

where $x \notin \mathbb{S}^2$ and $y \in \mathbb{S}^2$.

Proof of theorem 2. Using the basic inequality (16), we have:

1. For $\alpha \leq 2k$

$$\begin{aligned}
&\oint\limits_{\mathbb{S}^k} \frac{d\,\mathbb{S}_y}{|x-y|^\alpha} = |R^2 - |x|^2|^{k-\alpha} \oint\limits_{\mathbb{S}^k} \frac{d\,\mathbb{S}_y}{|x-y|^{2k-\alpha}} \\
&\leq |\mathbb{S}^k| \frac{|R-|x||^{k-\alpha} \cdot |R+|x||^{k-\alpha}}{|R-|x||^{2k-\alpha}} = \frac{|\mathbb{S}^k|}{|R-|x||^k \cdot |R+|x||^{\alpha-k}} \\
&= \frac{|\mathbb{S}^k|}{|R+s|x||^k \cdot |R-s|x||^{\alpha-k}}\,,
\end{aligned} \tag{70}$$

where $s = \mathrm{sign}(\alpha - 2k)$ for $\alpha \neq 2k$.

2. For $\alpha \geq 2k$

$$\begin{aligned}
&\oint\limits_{\mathbb{S}^k} \frac{d\,\mathbb{S}_y}{|x-y|^\alpha} = |R^2 - |x|^2|^{k-\alpha} \oint\limits_{\mathbb{S}^k} |x-y|^{\alpha-2k} d\,\mathbb{S}_y \\
&\leq |R-|x||^{k-\alpha} \cdot |R+|x||^{k-\alpha} \cdot |R+|x||^{\alpha-2k} \cdot |\mathbb{S}^k| \\
&= \frac{|\mathbb{S}^k|}{|R+|x||^k \cdot |R-|x||^{\alpha-k}} = \frac{|\mathbb{S}^k|}{|R+s|x||^k \cdot |R-s|x||^{\alpha-k}}
\end{aligned} \tag{71}$$

where $s = \mathrm{sign}(\alpha - 2k)$ for $\alpha \neq 2k$.

It is not hard to see that for $\alpha = 2k$ we have to make $s = 1$ or $s = -1$. This gives us equalities in both cases. Thus, combining the results obtained in the cases 1 and 2, we get exactly the upper bound in (63). Similarly, we can obtain the lower bound given in (63). Using the same method we can easily prove the inequalities from remark 2 (see (66)). ■

Let us show how to prove the last inequality from remark 1 (see (65)). The first inequality in (65) is obtained in the same way.

Proof of (65). **1.** Let $k < \alpha < 2k$. Then $s = \mathrm{sign}(\alpha - 2k) = -1$ and therefore

$$\frac{|\mathbb{S}^k|}{|R+s|x||^k \cdot |R-s|x||^{\alpha-k}} = \left|\frac{R-|x|}{R+|x|}\right|^{\alpha-k} \frac{|\mathbb{S}^k|}{|R-|x||^\alpha} < \frac{|\mathbb{S}^k|}{|R-|x||^\alpha} \tag{72}$$

since

$$\left|\frac{R-|x|}{R+|x|}\right|^{\alpha-k} < 1 \quad \text{for} \quad \alpha > k \tag{73}$$

2. For $\alpha \geq 2k$, we have $s = \text{sign}(\alpha - 2k) = 1$ and then

$$\frac{|\mathbb{S}^k|}{|R + s|x||^k \cdot |R - s|x||^{\alpha-k}} = \left|\frac{R - |x|}{R + |x|}\right|^k \frac{|\mathbb{S}^k|}{|R - |x||^\alpha} < \frac{|\mathbb{S}^k|}{|R - |x||^\alpha} \tag{74}$$

since

$$\left|\frac{R - |x|}{R + |x|}\right|^k < 1 \quad \text{for} \quad k > 0 \tag{75}$$

3. For $\alpha = k$, instead of inequality in (72) we will obtain equality.

4. For $\alpha < 2k$, we have $s = \text{sign}(\alpha - 2k) = -1$ and therefore

$$\frac{|\mathbb{S}^k|}{|R + s|x||^k \cdot |R - s|x||^{\alpha-k}} = \left|\frac{R - |x|}{R + |x|}\right|^{\alpha-k} \frac{|\mathbb{S}^k|}{|R - |x||^\alpha} > \frac{|\mathbb{S}^k|}{|R - |x||^\alpha} \tag{76}$$

since

$$\left|\frac{R - |x|}{R + |x|}\right|^{\alpha-k} > 1 \quad \text{for} \quad \alpha < k\,. \quad \blacksquare \tag{77}$$

7. APPENDIX.

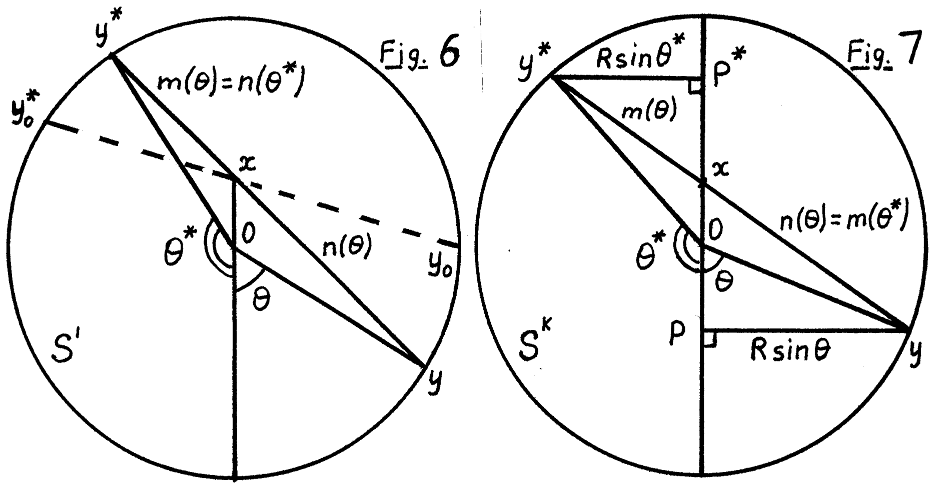
(A). A simpler proof of the right hand side implication "$\Rightarrow$" for the case $x < R$ (Theorem 1).

Proposition.

Let x be inside $\mathbb{S}^k$, where $\mathbb{S}^k$ is a k-dimensional sphere of radius R. Let ω be as defined in (3) and α, β be real numbers such that $\alpha + \beta = k$. Then

$$\oint_{\mathbb{S}^k} \omega^\alpha d\,\mathbb{S}_y = \oint_{\mathbb{S}^k} \omega^\beta d\,\mathbb{S}_y\,. \tag{78}$$

Proof. As it was shown in (25), $\omega = \frac{m}{n}(\theta)$, where m and n were introduced in section 2. For the notations below, see figures 6 and 7. Let y be an arbitrary point from $\mathbb{S}^k$, then $y^\star$ is defined as an intersection of the sphere $\mathbb{S}^k$ and line xy. Angle θ is defined as $\pi - \angle xOy$ and $\theta^\star$ is defined as $\pi - \angle xOy^\star$. A one dimensional sphere $\mathbb{S}^1$ or, just a circle, shown on figure 6 is used to demonstrate the relationship between angles θ and $\theta^\star$. This circle appears on cross-sectional plane defined by three points: O, x and $y \in \mathbb{S}^k$.



Notice, first, that $\triangle y_0xy$ is similar to $\triangle xy_0y$ (figure 6) and then

$$\frac{d\theta^\star}{d\theta} = -\lim_{y_0\to y}\frac{|y_0^\star y^\star|}{|y_0y|} = -\frac{m(\theta)}{n(\theta)}\,. \tag{79}$$

It is not hard to see that

$$m(\theta) = |y^\star x| = n(\theta^\star) \quad \text{and} \quad n(\theta) = |xy| = m(\theta^\star)\,, \tag{80}$$

where all equalities in (80) are also simple consequences of the definition of m and n (see section 2). Therefore, (79) combined with (80) yields

$$d\theta = -\frac{n(\theta)}{m(\theta)}d\theta^\star = -\frac{m(\theta^\star)}{n(\theta^\star)}d\theta^\star\,. \tag{81}$$

Let P and $P^\star$ be orthogonal projections of y and $y^\star$ to line xO respectively (figure 7). Clearly, $\triangle xy^\star P^\star$ is similar to $\triangle xyP$ and then

$$m(\theta)\cdot R\sin\theta = n(\theta)\cdot R\sin(\theta^\star) = m(\theta^\star)\cdot R\sin\theta^\star\,. \tag{82}$$

Now we are ready to get the statement of the proposition given by (78).

Using (25), (81) and (82), we have

$$\begin{aligned}
\oint\limits_{\mathbb{S}^k} \omega^\alpha d\,\mathbb{S}_y &= \int\limits_0^\pi \left(\frac{m(\theta)}{n(\theta)}\right)^\alpha (R\sin\theta))^{k-1}|\sigma_{k-1}|Rd\theta \\
&= -\int\limits_\pi^0 \left(\frac{m(\theta)}{n(\theta)}\right)^\alpha \left(\frac{m(\theta^\star)}{m(\theta)}R\sin\theta^\star\right)^{k-1}|\sigma_{k-1}|R\,\frac{m(\theta^\star)}{n(\theta^\star)}d\theta^\star \\
&= \int\limits_0^\pi \left(\frac{m(\theta^\star)}{n(\theta^\star)}\right)^{k-\alpha}(R\sin\theta^\star)^{k-1}|\sigma_{k-1}|Rd\theta^\star \\
&= \int\limits_0^\pi \left(\frac{m(\theta)}{n(\theta)}\right)^{k-\alpha}(R\sin\theta)^{k-1}|\sigma_{k-1}|Rd\theta = \oint\limits_{\mathbb{S}^k}\omega^{k-\alpha}d\,\mathbb{S}_y\,,
\end{aligned} \tag{83}$$

where $|\sigma_{k-1}|$ is the area of $(k-1)$-dimensional unit sphere. Note also that the representation $d\,\mathbb{S}_y = (R\sin\theta)^{k-1}|\sigma_{k-1}|R\,d\theta$ is possible, since our integrand ω depends only on angle θ when y is running over $\mathbb{S}^k$. For more detailed explanation concerning $d\,\mathbb{S}_y$, see part 1 of theorem 1 ■

(B). Theorem 3.

Let x, y be points in $\mathbb{R}^{k+1}$. Let $g(\xi)$ be a continuous twice differentiable function of one variable $\xi \in \mathbb{R}$ defined for $\xi > 0$. Let us assume that

$$\oint\limits_{\mathbb{S}^k(|y|)} g(|x-y|)d\,\mathbb{S}_y = M(|y|) \tag{84}$$

for every x inside $\mathbb{S}^k(|y|)$, which is a sphere of radius $|y|$ centered at the origin. In other words, the integral depends only on $|y|$ and does not depend on x inside the $\mathbb{S}^k(|y|)$ for all $x, y \in \mathbb{R}^{k+1}$ such that $|x| < |y|$. Then $g(|x-y|)$ is harmonic with respect to x.

Proof. It is not hard to check that

$$\triangle_x g(|x-y|) = \triangle_y g(|x-y|) = \triangle_{(x-y)} g(|x-y|) = \widetilde{g}(|x-y|)\,, \tag{85}$$

where $\widetilde{g}(|x-y|)$ is again some function depending only on the distance between x and y. Note also that $M(|y|) = g(|y|)$, since by the condition, the integral in (84) does not depend on x for $|x| < |y|$ and, therefore, the value of the integral is equal to to the value with $x = 0$. Thus, we have

$$\frac{1}{\sigma_k |y|^k} \oint\limits_{\mathbb{S}^k(|y|)} g(|x-y|)d\,\mathbb{S}_y = g(|y|)\,, \tag{86}$$

where σ_k is the area of k-dimensional unit sphere. Therefore, using Lemma 1 and (85), we can write

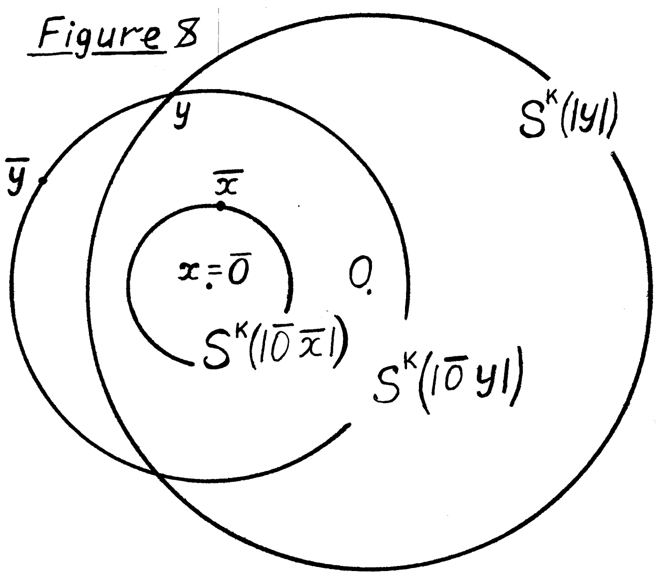
$$\begin{aligned}
0 = \triangle_x g(|y|) &= \triangle_x \left(\frac{1}{\sigma_k |y|^k} \oint\limits_{\mathbb{S}^k(|y|)} g(|x-y|)d\,\mathbb{S}_y \right) \\
&= \frac{1}{\sigma_k |y|^k} \oint\limits_{\mathbb{S}^k(|y|)} \triangle_x g(|x-y|)d\,\mathbb{S}_y = \frac{1}{\sigma_k |y|^k} \oint\limits_{\mathbb{S}^k(|y|)} \widetilde{g}(|x-y|)d\,\mathbb{S}_y \\
&= \frac{1}{\sigma_k |x|^k} \oint\limits_{\mathbb{S}^k(|x|)} \triangle_y g(|x-y|)d\,\mathbb{S}_x = \triangle_y \left(\frac{1}{\sigma_k |x|^k} \oint\limits_{\mathbb{S}^k(|x|)} g(|x-y|)d\,\mathbb{S}_x \right) \\
&= \triangle_y \left(\frac{1}{\sigma_k |y|^k} \oint\limits_{\mathbb{S}^k(|y|)} g(|x-y|)d\,\mathbb{S}_y \right) = \triangle_y g(|y|)\,,
\end{aligned} \tag{87}$$

which, together with (85), yields

$$0 = \triangle_{(x-y)} g(|x-y|) = \triangle_x g(|x-y|)\,. \blacksquare \tag{88}$$

Remark 1. The assumption that $g(\xi)$ is twice differentiable can be replaced by the continuity of $g(\xi)$ and the statement of the theorem remains true by the converse of mean value property (see [7], p.17, theorems 1.24 or 1.25).

Proof. We must only show that the condition (84) implies mean value property for $g(|x-y|)$ with respect to x for every y such that $|y| > |x|$. Indeed, look at figure 8.



Let $x, y \in \mathbb{R}^{k+1}$ such that $|y| > |x|$. Let $\mathbb{S}^k(|y|)$ be the sphere of radius $|y|$ centered at the origin O, $\mathbb{S}^k(|\overline{O}\overline{x}|)$ be a sphere of radius $|\overline{O}\overline{x}|$ centered at $\overline{O}$ and we assume that $\mathbb{S}^k(|\overline{O}\overline{x}|)$ is inside $\mathbb{S}^k(|y|)$. Let $\overline{y} \in \mathbb{S}^k(|\overline{O}y|)$ be a sphere of radius $|\overline{O}y|$ centered at $\overline{O}$. Therefore, $\mathbb{S}^k(|\overline{O}y|) = \mathbb{S}^k(|\overline{O}\overline{y}|)$. Now we are ready to check mean value property for $g(|x-y|)$ with respect to x. Using lemma 1 and (86), we have

$$\begin{aligned}\frac{1}{\mathbb{S}^k(|\overline{O}\overline{x}|)} \oint\limits_{\mathbb{S}^k(|\overline{O}\overline{x}|)} g(|y-\overline{x}|)d\,\mathbb{S}_{\overline{x}} &= \frac{1}{\mathbb{S}^k(|\overline{O}\overline{y}|)} \oint\limits_{\mathbb{S}^k(|\overline{O}\overline{y}|)} g(|\overline{x}-\overline{y}|)d\,\mathbb{S}_{\overline{y}} \\ &= g(|\overline{O}-\overline{y}|) = g(|y-x|)\end{aligned} \tag{89}$$

and so, g satisfies the mean value property. Therefore, $g(|x-y|)$ is harmonic with respect to x for every x inside $\mathbb{S}^k(|y|)$ and for every $y \in \mathbb{S}^k(|y|)$. Using (85), we can easily see that $g(|x-y|)$ is harmonic for all $x, y \in \mathbb{R}^{k+1}$ such that $x \neq y$. ■

Remark 2. The statement of Theorem 3 can not be generalized for every function of two variables $g(x,y)$, i.e., the condition given by

$$\oint\limits_{\mathbb{S}^k(|y|)} g(x,y)d\,\mathbb{S}_y = M(|y|) \tag{90}$$

for all x and y such that $|x| < |y|$ does not imply that $g(x,y)$ is harmonic with respect to x even for all x and y s.t. $|x| < |y|$.

For example, the function

$$\omega^k(x,y) = \left(\frac{|x|^2-|y|^2}{|x-y|^2}\right)^k , \tag{91}$$

according to (15) satisfies (90), but it is harmonic with respect to x for every y such that $|y| > |x|$ only if $x = 0$. This fact is proved in the following proposition.

Proposition.
Let $x, y \in \mathbb{R}^{k+1}$ such that $|x| \neq |y|$ and $k \geq 1$. Then ω^k is harmonic with respect to x for every y only if $x = 0$ or $k = 1$.

Proof. At first, let us consequently obtain the following list of trivial identities.

$$\frac{\partial \omega}{\partial x_1} = \frac{2x_1}{|x-y|^2} - \frac{2(x_1-y_1)}{|x-y|^2}\omega , \tag{92}$$

$$\frac{\partial^2 \omega}{(\partial x_1)^2} = \frac{2}{|x-y|^2} - \frac{8x_1(x_1-y_1)}{|x-y|^4} + \frac{8(x_1-y_1)^2\omega}{|x-y|^4} - \frac{2\omega}{|x-y|^2} , \tag{93}$$

$$(x, x-y) = \frac{|x|^2-|y|^2+|x-y|^2}{2} = \frac{1}{2}(\omega+1)\cdot|x-y|^2 . \tag{94}$$

Using (93), (94) and (92), (94) respectively, we obtain

$$\triangle_x \omega = \frac{2(k-1)(1-\omega)}{|x-y|^2} \quad \text{and} \quad |\nabla_x \omega|^2 = \frac{4|y|^2}{|x-y|^4} . \tag{95}$$

Thus, ω^k is harmonic for $k = 1$ and $x, y \in \mathbb{R}^2$.

To proof the statement of the proposition, notice that

$$\triangle_x \omega^k = k(k-1)\omega^{k-2}|\nabla_x \omega|^2 + k\omega^{k-1} \triangle_x \omega \tag{96}$$

and one can easily check that for $k > 1$,

$$\triangle_x \omega^k = 0 \quad \Leftrightarrow \quad |x-y|^2 = \frac{(|x|^2-|y|^2)^2}{|x|^2+|y|^2} , \tag{97}$$

where the last equality holds for every $y \in \mathbb{R}^{k+1}$ only if $x = 0$ ∎.

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