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Initial conditions in high-energy collisions

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THE CITY UNIVERSITY OF NEW YORK
INITIAL CONDITIONS IN HIGH-ENERGY COLLISIONS

by

Elena Petreska

Advisor: Professor Adrian Dumitru

This thesis is focused on the initial stages of high-energy collisions in the saturation regime. We start by extending the McLerran-Venugopalan distribution of color sources in the initial wave-function of nuclei in heavy-ion collisions. We derive a fourth-order operator in the action and discuss its relevance for the description of color charge distributions in protons in high-energy experiments. We calculate the dipole scattering amplitude in proton-proton collisions with the quartic action and find an agreement with experimental data. We also obtain a modification to the fluctuation parameter of the negative binomial distribution of particle multiplicities in proton-proton experiments. The result implies an advancement of the fourth-order action towards Gaussian when the energy is increased. Finally, we calculate perturbatively the expectation value of the magnetic Wilson loop operator in the first moments of heavy-ion collisions. For the magnetic flux we obtain a first non-trivial term that is proportional to the square of the area of the loop. The result is close to numerical calculations for small area loops.
To my parents,

Marica and Milan
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Chapter 1

Introduction

Elementary particles and their interactions are described by the Standard Model. The Standard Model theory specifies the electromagnetic, weak and strong forces. In this thesis we concentrate on the strong interactions of elementary particles.

The Standard Models is a quantum field theory. Quantum electrodynamics (QED) describes the electromagnetic forces between electrically charged elementary particles, leptons and quarks, and the mediator of these interactions, the photon. Unlike leptons, quarks carry additional degree of freedom; they have a color charge. Quantum chromodynamics (QCD) studies the strong interactions between colored particles realized through an exchange of a gluon. At last, the mediators of the weak interactions, that influence all the fermions in the Standard Model and the Higgs boson, are the gauge bosons $W^\pm$ and $Z$.

In this chapter we explain the concepts underlying quantum chromodynamics. We start with the basic ideas in QCD and concentrate on the unique features that distinguish it from the theories describing the rest of the natural forces. We write down the QCD Lagrangian and the classical Yang-Mills equations of motion. We describe the phenomenon of running coupling constant, asymptotic freedom and confinement. The Bjorken scaling and the Callan-Gross relation are emphasized as an evidence of quark existence.
1.1 The QCD Lagrangian

A fundamental characteristic of the strong forces is that they exist only between particles carrying a color charge. ¹ Color is an extra degree of freedom attributed to quarks and gluons. Quarks are spin 1/2 particles (fermions) with a fractional electric charge. They come in six flavors, divided in three families of pairs: up and down, charm and strange, and top and bottom. Each quark has its own antiquark. The quarks and antiquarks belong to one of three color states: blue, green or red. Color-neutral bound states composed of a quark and an antiquark are called mesons, and those made of three quarks are called baryons.

The mediators of the strong force are the gluons. They are massless vector bosons that carry one unit of color and one of anticolor. From group theory, with three colors and three anticolors we can make nine states, a gluon octet and a gluon singlet. However only the eight gluons from the octet exist in nature.

In quantum field theory elementary particles arise as quanta of the corresponding fields. QCD is a quantum field theory describing the strong interactions between quarks and gluons. It contains a Yang-Mills theory with a symmetry group $SU(3)$ [1]. The quarks are quanta of the Dirac fields $q^f$, and the gluons are excitations of the gauge fields $A_\mu$. The QCD Lagrangian density is:

$$L_{QCD} = \sum_{\text{flavors } f} \bar{q}^f (i \gamma_\mu D^\mu - m_f) q^f - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu}_a, \quad (1.1)$$

where $m$ is the quark mass, $\gamma_\mu$ are the Dirac matrices, and $D_\mu$ is the covariant derivative:

$$D_\mu = \partial_\mu - ig t^a A^a_\mu. \quad (1.2)$$

We work in natural units, with $\hbar = c = 1$.

¹The strong force between nucleons (which have zero total color charge) inside a nucleus is referred to as nuclear force or residual strong force.
The first term in the Lagrangian density stands for the free Dirac fields. The second term in the covariant derivative gives the strong interaction between these fields and the gauge fields $A_\mu$, with the coupling $g$ representing its strength $^2$. The Dirac fields generate currents:

$$J^a_\mu = \bar{q} \gamma_\mu t^a q ,$$  \hspace{1cm} (1.3)

which act as sources for the gauge fields. The matrices $t^a$ are the generators of the $SU(3)$ group and they obey the commutation relations:

$$[t^a, t^b] = if^{abc}t^c .$$  \hspace{1cm} (1.4)

The numbers $f^{abc}$ are called antisymmetric structure constants. The gauge fields can be written in terms of the generators of the $SU(3)$ group as $A_\mu = A_\mu^a t^a$, where the index $a$ gets values from 1 to 8, for the eight gluon fields.

The last term in the Lagrangian density represents the gauge fields and their self-interactions. The gluon field strength tensor is:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu] .$$  \hspace{1cm} (1.5)

Since gluons carry color charge they interact strongly, not only with quarks, but with themselves too (unlike in electrodynamics). This is expressed through the last term of the above expression and it is a consequence of the fact that the generators of the theory do not commute. These theories are called non-Abelian gauge theories. This fact allows for the existence of hypothetical glueball, a purely gluon bound state.

The Euler-Lagrange equations of motion that follow from the QCD Lagrangian give a

$^2g$ expresses the color charge, the analogue of the electric charge $e$ in QED.
classical Yang-Mills theory. The classical Yang-Mills equations of motion have the form:

\[ [D_\mu, F_{\mu\nu}^a] = J_\nu^a. \]  

(1.6)

Although this theory does not have a realization in nature, it is a useful concept that can be applied to studying realistic strong interactions. We shall exploit the classical Yang-Mills theory for studying ultra-relativistic collisions.

1.2 The running coupling, asymptotic freedom and confinement

The strength of the coupling in QCD is not a constant, but it depends on the momentum transfer in the interaction. The phenomenon is called running of the coupling constant.

In Quantum Electrodynamics (QED), the strength of attraction or repulsion between two electric charges increases with decreasing the distance. Technically, this is due to the screening of the electric charge, expressed as vacuum polarization diagrams in Feynman language.

In QCD, the presence of gauge field self-interactions changes the behaviour of the coupling completely. It allows for the presence of transverse and longitudinal gluon loops in the vacuum polarization diagrams in addition to the fermionic ones. The contribution from the longitudinal gluon loops has an opposite effect on the strength of the strong interactions; we can say they produce an anti-screening. Moreover, the anti-screening effect dominates over the screening one, and as a result, the gauge coupling parameter decreases at small distances and increases in the infrared. Quantitatively, the dependence of the strong coupling constant
on the momentum transfer $|q^2|$ (in the leading logarithmic approximation) is expressed as:

$$\alpha_s(|q^2|) = \frac{\alpha_s(\mu^2)}{1 + \frac{\alpha_s(\mu^2)}{12\pi} (11N_c - 2N_f) \ln \frac{|q^2|}{\mu^2}}.$$  \hfill (1.7)

In this expression $N_c$ is the number of colors and $N_f$ the number of flavors. Since in QCD, $N_c = 3$ and $N_f = 6$, and therefore $11N_c - 2N_f = 21$, the coupling constant decreases with increasing $|q^2|$ and at short distances the strong force becomes relatively weak. On the right hand side of the above equation $\alpha_s$ is defined at some random reference point (or renormalization point) $\mu$ that renders an ultraviolet cut-off. The common way is to introduce the QCD scale $\Lambda_{QCD}$ at which the coupling becomes strong as $|q^2|$ decreases:

$$\ln (\Lambda_{QCD}) = \ln (\mu^2) - \frac{12\pi}{(11N_c - 2N_f) \alpha_s(\mu^2)}.$$  \hfill (1.8)

With this definition, the final form of the strong running coupling constant is:

$$\alpha_s(|q^2|) = \frac{12\pi}{(11N_c - 2N_f) \ln \frac{|q^2|}{\Lambda_{QCD}^2}}.$$  \hfill (1.9)

From the 2012 report of the particle data group, the value of the QCD scale is $\Lambda_{QCD} = 216^{+25}_{-24} \text{MeV}$.

The running of the coupling constant has significant consequences. At very short distances the coupling goes asymptotically to zero and the quarks are essentially free particles. This feature of QCD is called asymptotic freedom [2–4]. At large momentum transfer, the coupling is weak and a use of a perturbation theory is fairly justified. The Feynman diagrams representing the higher order terms in the perturbation series involve higher powers of $\alpha_s$ and can be safely neglected. QCD perturbation theory is valid for momentum transfers larger than $\Lambda_{QCD}$ for which $\alpha_s < 1$. 
However, at larger distances the coupling becomes strong and perturbative computations are not reliable any more. Non-perturbative effects come into account. A promising non-perturbative calculation tool is lattice gauge theory. The space-time is discretized and QCD quantities are computed numerically.

An essential non-perturbative effect is confinement. It represents the fact that no colored particles are observed in nature. All the particles that can be observed in experiments are color singlets, no isolated quarks or gluons can be found. Only their bound states mesons and baryons are detected. The potential energy between quarks increases with their separation (in contrast with Coulomb-like interaction) and at some point the potential energy would be enough to produce a quark - anti-quark pair. Forming a $q\bar{q}$ pair is more energetically favorable than isolating the initial quarks, and hence they remain in a confined state.

1.3 Partonic content of hadrons

Collisions provide a successful way for exploring the dynamics of elementary particles. Since one can not perform experiments with individual quark and gluons, one must rely on indirect conclusions drawn from collisions of bound states. The composite particles baryons and mesons collectively are called hadrons. The simplest way of analysing the structure of hadrons is deep inelastic scattering (DIS) of leptons on these compound states.

The quantitative formulation of scattering processes (in perturbative QCD) consists of evaluation of the Feynman diagrams relevant for the problem in question, and calculating the relativistic scattering amplitude or scattering cross section. The scattering amplitude contains information about the structure and the dynamics of the participants in the interaction. A basic problem for any theory of hadronic structure is to reproduce the scattering cross sections observed in experiments. QCD is focused on the hadronic part of the DIS amplitude.
CHAPTER 1. INTRODUCTION

Here, we shall assume that the leptonic segment of the DIS amplitude is known and concentrate only on the element describing the hadron involved (for simplicity we take a proton as an example). A typical DIS process is sketched in fig.1. The scattering between the lepton and the proton is realized by an exchange of a virtual photon. The process is inelastic, and the proton degrades into a number of final-state particles. Our ignorance of the structure of the proton is represented by a gray circle.

\[ Q^2 \equiv -q^2 \] (the virtuality of the photon), \hspace{1cm} \text{(1.10)}
\[ x_B \equiv \frac{Q^2}{2 p \cdot q} \] (the Bjorken-\( x \) variable), and \hspace{1cm} \text{(1.11)}
\[ y \equiv \frac{p \cdot q}{p \cdot k} \] . \hspace{1cm} \text{(1.12)}

The differential cross section can be factorized into a leptonic \( L_{\mu \nu} \) and a hadronic part \( W^{\mu \nu} \):

\[
\frac{d\sigma}{dk d^3k'} = \frac{\alpha_{EM}^2}{EE'Q^4} L_{\mu \nu} W^{\mu \nu} . \hspace{1cm} \text{(1.13)}
\]

Here, \( E \) and \( E' \) are the energies of the incoming and outgoing leptons respectively. Since we
do not know the form of the hadronic input, the strategy is to construct the most general form of the tensor $W_{\mu\nu}$:

$$\begin{align*}
W_{\mu\nu} &= W_1(x_B, Q^2) \left( -g_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{q^2} \right) + W_2(x_B, Q^2) \frac{1}{M^2} \left( p_{\mu} - \frac{p \cdot q}{q^2}q_{\mu} \right) \left( p_{\nu} - \frac{p \cdot q}{q^2}q_{\nu} \right) .
\end{align*}$$

(1.14)

The functions $W_1$ and $W_2$ are called structure functions. They depend on two independent variables, $Q^2$ and $x_B$. Substituting eq. (1.14) in the cross section, eq (1.13), we get:

$$\frac{d\sigma}{dE'd\Omega} = \frac{\alpha_{EM}^2}{4E^2 \sin^4 \frac{\theta}{2}} \left[ W_2(x_B, Q^2) \cos^2 \frac{\theta}{2} + 2W_1(x_B, Q^2) \sin^2 \frac{\theta}{2} \right] .$$

(1.15)

In the above expression $\theta$ is the lepton’s scattering angle and $\Omega$ is the solid scattering angle.

It is more common and convenient to define dimensionless structure functions $F_1(x_B, Q^2)$ and $F_2(x_B, Q^2)$:

$$\begin{align*}
F_1(x_B, Q^2) &= MW_1(x_B, Q^2) , \\
F_2(x_B, Q^2) &= \nu W_2(x_B, Q^2) ,
\end{align*}$$

(1.16) \hspace{1cm} (1.17)

where $\nu = p \cdot q/M$.

The structure functions can be obtained experimentally by measuring the DIS cross section. They give an indirect information about the proton’s structure. An objective of QCD is to provide a theoretical calculation of these functions.

An important conclusion following from the structure functions is the Bjorken scaling. In the limit where $Q^2 \to \infty$ and $x_B$ is fixed, the inelastic structure functions become independent of $Q^2$ and depend only on $x_B$ alone: $F_1(x_B)$ and $F_2(x_B)$. The Bjorken scaling law follows from the assumption that the proton is a system composed of elementary particles and that the virtual photon interacts with one of the point-like particles in the hadron. This
is the parton model of hadrons [5]. The structure independence on this resolution scale implies point-like elements inside the proton. In addition, the Callan-Gross relation:

$$F_2(x_B) = 2x_B F_1(x_B)$$

(1.18)

expresses the fact that the charged elements of the proton carry spin 1/2. Both, the Bjorken scaling and the Callan-Gross relation have been confirmed experimentally and provide an evidence of the existence of quarks.

The form of the structures functions depends on the construction of the hadrons. All of the proton elements carry a fraction of its total momentum $p$ and energy $E$. We can define a parton momentum distribution $f_i(x)$ as the probability that the parton $i$ that interacts with the virtual photon carries a fraction $x$ of the proton’s total momentum. The probability functions are called parton distribution functions (PDFs). By this definition, all the fractions need to add up to one, and the sum over all the partons gives:

$$\sum_i \int dx \, x \, f_i(x) = 1 .$$

(1.19)

In the infinite momentum frame (IMF) in which the proton is ultra-relativistic, the structure functions can be expressed in terms of the PDF’s:

$$F_2(x_B) = \sum_i \int dx \, e_i^2 \, f_i(x) x \delta(x - x_B) ,$$

(1.20)

$$F_1(x_B) = \frac{1}{2x_B} F_2(x_B) .$$

(1.21)

Here, $e_i^2$ is the charge of the $i$ parton. In the large $Q^2$ limit, the Bjorken $x$ has the role of
the momentum fraction of the struck quark. In terms of $x$ the structure functions are:

\begin{align}
F_2(x) &= \sum_i e_i^2 x f_i(x), \\
F_1(x) &= \frac{1}{2x} F_2(x).
\end{align}

In the IMF, because of relativistic time dilation, the quarks inside the proton are essentially free and the inelastic scattering lepton-proton becomes an elastic scattering lepton-quark. When $Q^2 \gg \Lambda_{QCD}^2$ perturbation theory can be applied.

### 1.4 Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equations

The parton model assumes free quarks, antiquarks and gluons in the wave-function of the proton. In reality, elementary particles’ strong interactions are also present. The interactions bring higher order corrections and the Bjorken scaling is violated.

One possible correction comes from gluon radiation diagrams. The quark that interacts with the virtual photon may radiate a gluon before or after the interaction. The modified structure function is then:

\begin{equation}
F_2(x, Q^2) = \sum_i e_i^2 x \int_x^1 \frac{dy}{y} \left[ \delta \left( 1 - \frac{x}{y} \right) + \frac{\alpha_s}{2\pi} \frac{Q^2}{\kappa^2} P_{qq}(x/y) \right] f_q(y).
\end{equation}

We have introduced the notation $f_q$ for the quark distribution function. The function $P_{qq}(x/y)$ is called a splitting function and it represents the probability for a quark to emit a gluon and consequently its momentum fraction to change from $y$ to $x$. The gluon emission is proportional to $\alpha_s k_{2,\perp}/k_{1,\perp}^2$. Its integral over $k_{\perp}$ has a singularity, and $\kappa$ is introduced as a lower cut-off on $k_{\perp}$ to regulate this divergence. The presence of $\ln Q^2$ violates the Bjorken scaling and the structure functions become $Q^2$ dependent.
The above expression for $F_2(x, Q^2)$ may be considered as a power expansion in terms of the coupling $\alpha_s(Q^2)$. However, $\alpha_s(Q^2) \ln(Q^2/\kappa^2)$ is not small at large $Q^2$:

$$\alpha_s(Q^2) \ln \frac{Q^2}{\kappa^2} \sim 1.$$  \hfill (1.25)

Hence, $\alpha_s(Q^2) \ln(Q^2/\kappa^2)$ is taken to be the resummation parameter of the perturbation series. The resummation is called the leading logarithmic approximation (LLA). The soft gluon emission contributions are absorbed in the quark distribution function. The renormalized distribution now depends on $\ln(Q^2/\kappa^2)$ and we define it as:

$$f_q(x, Q^2) = f_q(x) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\kappa^2} P_{qq}(x/y) f_q(y) .$$  \hfill (1.26)

To remove the singularity from the renormalized parton distribution, we can introduce a factorization scale $\mu^2$ so that $\kappa^2 < \mu^2 < Q^2$ and the divergence can be included in:

$$f_q(x, \mu^2) = f_q(x) + \frac{\alpha_s}{2\pi} \ln \frac{\mu^2}{\kappa^2} P_{qq}(x/y) f_q(y) .$$  \hfill (1.27)

So, finally:

$$f_q(x, Q^2) = f_q(x, \mu^2) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2} P_{qq}(x/y) f_q(x, \mu^2) .$$  \hfill (1.28)

If the quarks were non-interacting, the Bjorken scaling would be exact. However, with increasing $Q^2$ the number of resolved partons that share the momentum of the proton increases. The evolution of the parton distributions with $Q^2$ is determined by the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations [6–8]. DGLAP equations resum the perturbation series in terms of $\alpha_s(Q^2) \ln(Q^2/\kappa^2)$. Taking into account the gluon distribution functions, the evolution equations for the quark, antiquark, and gluon densities to leading logarithmic
level are:

\[
\frac{\partial f_q(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} \left[ P_{qq}(x/y) f_q(y,Q^2) + P_{qg}(x/y) f_g(y,Q^2) \right]; \quad (1.29)
\]

\[
\frac{\partial f_g(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} \left[ P_{qg}(x/y) \left( f_q(y,Q^2) + f_g(y,Q^2) \right) + P_{gg}(x/y) f_g(y,Q^2) \right]; \quad (1.30)
\]

\[
\frac{\partial f_{\bar{q}}(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} \left[ P_{qq}(x/y) f_{\bar{q}}(y,Q^2) + P_{qg}(x/y) f_g(y,Q^2) \right]. \quad (1.31)
\]

The first of this equations express the fact that the quark with momentum fraction \( x \) on the left hand side could have come from a quark with momentum fraction \( y \) which has emitted a gluon, or from a gluon which turned into a quark-antiquark pair. The probability for the first process is proportional to \( \alpha_s P_{qq}(x/y) \), and for the second one to \( \alpha_s P_{qg}(x/y) \). Similar explanations hold for the second and third equations.

The resolution of the process of a virtual photon scattering off a hadron is proportional to \( 1/Q \). The photon can resolve all the partons that have transverse momenta less than \( Q \). Therefore, the density distributions contains the partons with transverse momenta \( k^2_\perp < Q^2 \).

We can define an unintegrated gluon distribution \( \phi(x,k^2_\perp) \) as:

\[
x f_g(x,Q^2) \equiv \int_{k^2_\perp}^{Q^2} dk^2_\perp \phi(x,k^2_\perp).
\]  

The evolution equations need an initial condition given by the parton distributions at some initial point \( Q^2_0 \), \( f(x,Q^2_0) \). The initial condition for small \( Q^2_0 \) cannot be calculated perturbatively and is taken from experimental data. Having the initial value of the parton distributions, PDFs can be calculated for any other virtuality \( Q^2 > Q^2_0 \), using the evolution equations.

The DGLAP evolution equations change the behavior of the PDFs with changing the Bjorken \( x \) as well. The number of probed partons in the hadron increases with increasing
CHAPTER 1. INTRODUCTION

$Q^2$ and there is also an increased probability of finding a quark at small $x$ since high-momentum quarks lose their momentum by radiating gluons. For large $Q^2$, in the small $x$ regime, which is equivalent to a high-energy limit, the DGLAP evolution predicts rise of the gluon distributions with decreasing $x$. Taking the limit $x \to 0$ of the splitting functions shows that gluon distributions increase much faster than quark distributions. In this limit, the evolution equation for the gluon densities is:

$$\frac{\partial^2 x f_g(x, Q^2)}{\partial \ln(1/x) \partial \ln(Q^2/Q_0^2)} = \frac{\alpha_s(Q^2) N_c}{\pi} x f_g(x, Q^2).$$

(1.33)

The resummation parameter for the solution of this equation in the large $Q^2$ and small $x$ regime is:

$$\alpha_s \ln \frac{1}{x} \ln \frac{Q^2}{Q_0^2}. \tag{1.34}$$

Resummation of the perturbation series with this parameter is called the double logarithmic approximation (DLA). The solution for the gluon distribution in DLA is proportional to:

$$x f_g(x, Q^2) \sim \exp \left[ 2 \sqrt{\frac{N_c}{\pi \beta_2}} \ln \frac{\ln(Q^2/\Lambda_{QCD}^2)}{\ln(Q_0^2/\Lambda_{QCD}^2)} \ln \frac{1}{x} \right]. \tag{1.35}$$

Evidently, the gluon density increases as $x$ decreases.
Chapter 2

Gluon saturation

In the previous chapter we introduced the DGLAP evolution equations for the parton distribution functions. In what follows, we shall explain briefly the parton evolution with \( x \) at fixed \( Q^2 \) given by the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation. At small values of \( x \) the BFKL evolution violates unitarity and requires a non-linear modification. The unitarity is restored when gluon saturation is taken into account. We also introduce the dipole picture of high-energy scattering and how saturation emerges in this description.

2.1 Balitsky-Fadin-Kuraev-Lipatov equation

The high-energy limit of QCD processes is equivalent to the small-\( x \) limit. DIS experiments are investigated by taking \( x \to 0 \) while keeping \( Q^2 \) fixed.

In the double logarithmic approximation of the DGLAP equations, the resummation parameter is \( \alpha_s \ln(1/x) \ln(Q^2/Q_0^2) \) for \( Q^2 \) large and \( x \) small. However, if one keeps \( Q^2 \) fixed, the term \( \ln(Q^2/Q_0^2) \) is not longer large and can be removed. The resummation parameter
for the limit $x \to 0$ and $Q^2$ fixed is then:

$$\alpha_s \ln \frac{1}{x} \sim 1. \quad (2.1)$$

The evolution equation that resums leading logarithms of $1/x$ is the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [9–11]. This equation gives the evolution of parton distributions with $x$. The resummation is called leading logarithmic approximation (LLA) in $1/x$. If the parton densities are known at some initial value $x_0$, the BFKL equation provides the density distributions at any other value of $x$.

The BFKL evolution equation for the unintegrated gluon distribution is:

$$\frac{\partial \phi(x, k_{\perp}^2)}{\partial \ln(1/x)} = \frac{\alpha_s N_c}{\pi^2} \int \frac{d^2q_{\perp}}{(k_{\perp} - q_{\perp})^2} \left[ \phi(x, q_{\perp}^2) - \frac{k_{\perp}^2}{2q_{\perp}^2} \phi(x, k_{\perp}^2) \right]. \quad (2.2)$$

An important observation coming from the solution of the BFKL equation is the fast growth of the gluon distribution with decreasing $x$. In the LLA in $1/x$, the dependence on $x$ is:

$$\phi(x, k_{\perp}^2) \sim \left( \frac{1}{x} \right)^\lambda, \quad (2.3)$$

where:

$$\lambda = \frac{4\alpha_s N_c}{\pi} \ln 2. \quad (2.4)$$

This rise is much faster than rise of gluon distributions at small $x$ coming from the DLA DGLAP equations.

In the double logarithmic approximation the BFKL equation gives the same solution as the DGLAP equations. Since BFKL resums orders of $\alpha_s \ln(1/x)$ while keeping $Q^2$ fixed, and DGLAP resums powers of $\alpha_s \ln Q^2$ while keeping $x$ fixed, the DLA solutions have to consist of the same power of the resummation parameter $\alpha_s \ln(1/x) \ln Q^2$. 

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Besides the much faster increase of gluon densities at small $x$ predicted by BFKL, the two evolution equations differ also in the fashion in which the partons in the evolution are ordered with respect to each other. The partons in the DGLAP evolution are strongly ordered in their transverse momenta. On the other hand, the partons evolved with the BFKL equation are ordered in their longitudinal momenta, or equivalently in $x$.

In addition, the partons in the BFKL evolution have similar transverse momenta. The typical transverse size of a parton is $x_\perp \approx 1/k_\perp$, and the partons have similar transverse sizes as well. Hence, in the small-$x$ evolution the number of gluon rises rapidly but their sizes stay the same. At some point this will inevitably lead to an overlap of the gluons in the transverse plane and to gluon interactions. Gluon interactions at small $x$ are not described by the BFKL formalism.

### 2.2 Gluon saturation

The BFKL evolution equation explained in the previous section violates the principle of unitarity. Namely, total cross sections at high energy can not grow faster than the logarithm squared of the energy $s$. This is known as a Froissart bound for QCD cross sections [12, 13] and it is mathematically expressed as:

$$\sigma_{tot} \leq \frac{\pi}{m_\pi^2} \ln^2 \frac{s}{s_0}.$$  \hspace{1cm} (2.5)

The mass of the pion is used as the smallest inverse length scale of the strong force, and $s_0$ is an unknown scale. On the other hand, the BFKL dominated total cross section grows as a positive power of the energy:

$$\sigma \sim s^\lambda.$$  \hspace{1cm} (2.6)
where $\lambda > 0$. This obviously violates the Froissart bound and the BFKL evolution equation requires a modification in the high-energy limit. As described in the previous section this is due to the fact that the gluon density increases as the negative power of $x$ when $x \to 0$. At some critical point, the gluons will start overlapping and gluon interactions should be included in the modification of the BFKL formalism.

We can estimate the scale at which the gluons start to interact. The gluon number rises as $(1/x)^\lambda$, so for a nucleus with $A$ number of nucleons, the number of gluons in the wave-function is proportional to $A(1/x)^\lambda$. The transverse size of the partons is inversely proportional to the momentum scale at which they are probed: $x_\perp \sim 1/Q^2$. Hence, the transverse area occupied by the gluons is proportional to:

$$\frac{A}{Q^2} \left( \frac{1}{x} \right)^\lambda . \quad (2.7)$$

The gluons start to overlap when this area gets on the order of the transverse size of the nucleus $S_\perp$, which in turn is $S_\perp \sim A^{2/3}$. Having in mind that the probability for gluons’ interactions is proportional to $\alpha_s$, we finally get:

$$Q_s^2 \sim \alpha_s A^{1/3} \left( \frac{1}{x} \right)^\lambda . \quad (2.8)$$

The momentum $Q_s^2$ is called saturation scale. It is a new dimensional scale in the problem and violation of unitarity occurs for $Q^2 < Q_s^2$. From the above expression it follows that for small $x$ the saturation scale is large, $Q_s^2 \gg \Lambda_{QCD}$, and perturbation theory is applicable. The goal is then, using perturbative QCD, to develop new evolution equations that will include non-linear gluon interactions and lead to saturation of gluon densities for $Q^2 < Q_s^2$.

First evolution equation taking into account the non-linear effects at high energy is the Gribov-Levin-Ryskin (GLR) equation for the unintegrated gluon distributions [14, 15]. The
CHAPTER 2. GLUON SATURATION

GLR equation consists of a linear term that is equivalent to the BFKL equation, but has an additional quadratic damping term that slows down the rise of gluon densities with energy and leads to gluon saturation. A non-linear evolution equation in the double leading-logarithmic approximation for the integrated gluon distribution was derived by Mueller and Qiu [16].

At the end of this section we mention an important property of the saturation scale. From eq.(2.8) it follows that $Q_s^2$ increases with increasing nucleon number $A$. The saturation scale for DIS on a nucleus is larger than the saturation scale for DIS on a proton and therefore perturbative calculations are more justified in the first case.

2.3 Dipole picture of gluon saturation

Since in Chapter 5 we present some new results for the dipole scattering amplitude, in this section we explain the concepts behind the dipole picture of high-energy scatterings. First, we introduce the light cone coordinate system which shall be used throughout this thesis.

2.3.1 Light cone coordinates

In describing high-energy collisions one usually works in the light cone coordinate system. When a scattering process involves ultra-relativistic particles and a preferred axis the use of light cone coordinates makes the description clearer.

A four-vector in the light cone coordinate system has the form:

$$w^\mu = (w^+, w^-, w_\perp) .$$ \hspace{1cm} (2.9)
The components of the light cone four-vector are connected to the usual four-vector:

\[ w^\mu = (w^0, w^1, w^2, w^3) \]  \hspace{1cm} (2.10)

through the following relations:

\[ w^+ = \frac{w^0 + w^3}{\sqrt{2}}, \quad w^- = \frac{w^0 - w^3}{\sqrt{2}}, \quad w_\perp = (w^1, w^2) . \]  \hspace{1cm} (2.11)

In these coordinates the preferred axis is the \( z \)-axis which in high-energy scattering is taken to be the collision axis. The light cone coordinates have simple transformation relations under boosts along the \( z \) axes. The four-vector transforms as:

\[ w'^+ = w^+ e^\psi, \quad w'^- = w^- e^{-\psi}, \quad w'_\perp = w_\perp , \]  \hspace{1cm} (2.12)

where the velocity \( v \) is \( v = \tanh \psi \), so that:

\[ \psi = \frac{1}{2} \ln \frac{1 + v}{1 - v} . \]  \hspace{1cm} (2.13)

Under consecutive boosts, \( \psi_1 \) and \( \psi_2 \), the resulting boost is \( \psi_1 + \psi_2 \). The boosted four-momentum of a particle of mass \( m \) is:

\[ p^\mu = \left( \frac{m}{\sqrt{2}} e^\psi, \frac{m}{\sqrt{2}} e^{-\psi}, 0_\perp \right) . \]  \hspace{1cm} (2.14)

It is clear that under large boosts along the \( z \)-axes, the light-cone momentum four-vector consists of explicitly large and small momentum components.
Finally, we define a quantity called rapidity:

\[ y = \frac{1}{2} \ln \frac{p^+}{p^-} = \frac{1}{2} \ln \frac{E + p^z}{E - p^z}. \]  

(2.15)

Under boosts the rapidity transforms as:

\[ y' = y + \psi. \]  

(2.16)

Therefore, when dealing with boosts from the rest frame it is convenient to express the four-momentum of a particle in terms of rapidity:

\[ p^\mu = \left( \sqrt{\frac{m^2 + p^2}{2}} e^y, \sqrt{\frac{m^2 + p^2}{2}} e^{-y}, p_\perp \right). \]  

(2.17)

### 2.3.2 Saturation of the dipole scattering amplitude

Here, we consider deep inelastic scattering of a lepton off a nucleus in the rest frame of the nucleus. The process is realized by an exchange of a virtual photon. In the dipole picture of DIS [18–21] the interaction consists of two parts. First, the photon fluctuates into a quark-antiquark pair forming a color dipole. In the second part of the process the dipole interacts with the color field of the nucleons. At small \( x \), the dipole probes the gluon field of the hadron. A diagrammatic representation of the scattering is given in fig. 2.1.

![Diagram of DIS](image)

**Figure 2.1:** Dipole picture of DIS.
Using this scheme, the scattering cross section for the hadronic part of the interaction, photon-nucleus ($\gamma^*A$), factorizes into a product of the square of the light cone wave function, $\Psi_{\gamma^*\rightarrow q\bar{q}}(x_\perp, z)$, representing the splitting of the virtual photon into a $q\bar{q}$ pair, and of the total cross section for the interaction of the dipole with the nucleus, $\sigma^{q\bar{q}A}(x_\perp, y)$:

$$\sigma^{\gamma^*A}(x, Q^2) = \int \frac{d^2 x_\perp}{4\pi} \int_0^1 \frac{dz}{z(1-z)} |\Psi_{\gamma^*\rightarrow q\bar{q}}(x_\perp, z)|^2 \sigma^{q\bar{q}A}(x_\perp, y). \quad (2.18)$$

In the above expression $q^+$ is the light cone momentum of a virtual photon with four-momentum:

$$q^\mu = \left(q^+, -\frac{Q^2}{q^+}, 0_\perp \right). \quad (2.19)$$

The longitudinal momentum fraction carried by the quark in the dipole is $z = k^+ / q^+$. The distance between the quark and the antiquark is $x_\perp$. Finally, $y$ is the rapidity, $y \approx \ln 1/x$.

The advantage of the dipole approach is that the process separates into a purely QED part and a QCD dynamics contained in the cross section of the dipole scattering off the nucleus. In addition, the transverse size of the dipole ($x_\perp$) in high-energy collisions is invariant; it does not change during the interaction [18, 21–23]. The $S$-matrix of the scattering is diagonal with respect to $x_\perp$. Color dipoles represent degrees of freedom in high-energy scatterings.

The scattering cross section $\sigma^{\gamma^*A}(x, Q^2)$ can then be used for calculating the structure functions $F_1(x, Q^2)$ and $F_2(x, Q^2)$:

$$F_2(x, Q^2) = \frac{Q^2}{4\pi^2\alpha} \left(\sigma_T^{\gamma^*A} + \sigma_L^{\gamma^*A}\right), \quad (2.20)$$

and

$$2xF_1(x, Q^2) = \frac{Q^2}{4\pi^2\alpha} \sigma_T^{\gamma^*A}, \quad (2.21)$$

where $\sigma_T^{\gamma^*A}$ and $\sigma_L^{\gamma^*A}$ correspond to cross sections that involve transversely and longitudinally polarized virtual photons respectively.

The dipole-nucleus scattering cross section can be written in terms of the imaginary part
of the forward scattering amplitude $N(x_\perp, b_\perp, y)$:

$$\sigma^{qqA}(x_\perp, y) = 2 \int d^2bN(x_\perp, b_\perp, y).$$  \hfill (2.22)

The dipole scattering amplitude is given at some impact parameter $b_\perp$. The problem of finding the cross section reduces to finding the dipole amplitude $N(x_\perp, b_\perp, y)$.

As a beginning step, one first considers scattering of a dipole on a single nucleon in the target. The lowest order approximation includes two-gluon interaction, i.e. the dipole interacts with a nucleon by exchanging two gluons. The result for the single nucleon cross section is:

$$\sigma^{qqN} \approx \frac{2\pi\alpha_s^2 C_F}{N_c} x_\perp^2 \ln \frac{1}{x_\perp^2 \Lambda^2}.$$  \hfill (2.23)

The above expression (where $C_F = (N_c^2 - 1) / 2N_c$ and $\Lambda$ is an infrared cutoff) is obtained for a perturbatively small dipole; it is assumed that $x_\perp \ll 1/\Lambda_{QCD}$.

For a toy model of a nucleon, consisting of a single quark, the unintegrated gluon distribution, to the lowest order considered here, is [17]:

$$x g_N(x, Q^2) = \frac{\alpha_s C_F}{\pi} \ln \frac{Q^2}{\Lambda^2},$$  \hfill (2.24)

from which it follows:

$$\sigma^{qqN} \approx 2\frac{\alpha_s \pi^2}{N_c} x_\perp^2 x g_N(x, Q^2 = 1/x_\perp^2).$$  \hfill (2.25)

At high energies, the probability of interaction with one nucleon becomes large and scatterings off multiple nucleons need to be considered. To find the dipole nucleus cross section with multiple rescatterings it is convenient to use the Glauber approximation of a nucleus. In the Glauber model the nucleus is assumed to be large and dilute, it is composed of $A \gg 1$ nucleons whose correlations are suppressed and can be viewed as independent.
If the nucleus moves in the $x^+$ direction, then the nucleons are taken to be ordered in $x^-$. In the Lorenz gauge ($\partial_\mu A^\mu = 0$) the dipole interacts with each nucleon independently. If we restrict the calculation to first order in perturbation theory, i.e. to two-gluon exchange per nucleon, it can be shown that the dipole scattering off two nucleons factorizes into a product of two independent dipole scatterings off a single nucleon. The cross section for dipole scattering on any number of nucleons can be then obtained by iteration.

The result for the forward scattering amplitude of a color dipole off a nucleus including multiple rescatterings in this model is given by the Glauber-Gribov-Mueller (GGM) formula:

$$N(x_\perp, y = 0) = 1 - \exp \left[ -\frac{\alpha_s \pi^2}{2N_c} \frac{A}{S_\perp} x_\perp^2 x g_N \left( x, \frac{1}{x_\perp^2} \right) \right].$$  \hspace{1cm} (2.26)

The transverse size of the nucleus is $S_\perp \sim A^{2/3}$. When the size of the dipole times the density is small, the amplitude is approximately equal to:

$$N(x_\perp, y = 0) \approx \frac{\alpha_s \pi^2}{2N_c} \frac{A}{S_\perp} x_\perp^2 x g_N \left( x, \frac{1}{x_\perp^2} \right).$$  \hspace{1cm} (2.27)

Using expression (2.3.2) the dipole scattering amplitude becomes:

$$N(x_\perp, y = 0) = 1 - \exp \left[ -\frac{\alpha_s^2 C_F \pi}{N_c} \frac{A}{S_\perp} x_\perp^2 x g_N \left( x, \frac{1}{x_\perp^2} \ln \frac{1}{x_\perp \Lambda} \right) \right].$$  \hspace{1cm} (2.28)

If we define the quark saturation scale as:

$$Q_s^2 \equiv \frac{4\pi \alpha_s^2 C_F}{N_c} \frac{A}{S_\perp},$$  \hspace{1cm} (2.29)

the scattering amplitude gets the form:

$$N(x_\perp, y = 0) = 1 - \exp \left[ -\frac{x_\perp^2 Q_s^2}{4} \ln \frac{1}{x_\perp \Lambda} \right].$$  \hspace{1cm} (2.30)
Multiple nucleon scatterings become important when the dipole size is of order \( x_\perp \sim 1/Q_s \). For very small transverse separations \( x_\perp \ll 1/Q_s \), the amplitude grows as \( \sim x_\perp^2 \). However, for \( x_\perp \gtrsim 1/Q_s \) the amplitude saturates, \( N \to 1 \). The transition occurs for dipole sizes on the order of the inverse saturation scale \( x_\perp \sim 1/Q_s \). Since the structure functions \( F_1 \) and \( F_2 \) are given in terms of the dipole scattering amplitude (or dipole scattering cross section), saturation of \( N \) implies saturation of parton distributions.

An important property of the scattering amplitude is color transparency. When the dipole size is zero, \( x_\perp = 0 \), the amplitude is zero as well. A dipole of zero size is color neutral and does not interact strongly with the target, therefore it produces zero scattering cross section.

At the lowest order of perturbation theory considered here (two gluon exchange) the gluon density does not depend on \( x \), i.e. energy. The dependence on energy in the scattering amplitude enters through quantum corrections leading to evolution equations for \( N \) with decreasing \( x \).
Chapter 3

The McLerran-Venugopalan model

This chapter addresses the McLerran-Venugopalan (MV) model of the wave-function of nuclei at high energies. We present the ideas behind the classical description of the gluon fields radiated by the large-\(x\) color sources in ultra-relativistic nuclei. The gluons emerge from non-Abelian Weizsäcker-Williams fields. In the MV model, the distribution of color sources is Gaussian. We also give a description of the classical field resulting from a collision of two nuclei.

3.1 Classical gluon fields

The McLerran-Venugopalan model considers the nuclear wave function of a nucleus in the small-\(x\) saturation regime. The essential idea in the theory is that the gluon fields in a nucleus at high energies can be treated classically.

In the MV model the high-energy nucleus is assumed to be large, i.e. it consists of a large number of nucleons \(A\). Since the nucleus moves with a speed close to the speed of light, it is Lorentz contracted and it is modelled as a sheet of color charges, infinite in the transverse directions, and with a uniform valence quark density in the transverse plane. The
average color charge of the valence quarks is zero and the way of generating color sources is by fluctuations. The color charge of the valence distribution fluctuates around zero from one point to the next in the transverse plane. A large number of quarks from different nucleons contribute to the effective color charge on a transverse size scale that is much larger than the typical quark separation, but much smaller than the size of a nucleon. At this resolution, the valence quarks are uncorrelated and the problem of finding the effective color charge reduces to a random walk in color space. Therefore, the typical value of fluctuating color charge at a given point in the transverse direction is \( \sim g \sqrt{N} \), where \( N \) is the number of quarks at that point. The density of color charge fluctuations in the transverse plane is obtained by squaring \( g \sqrt{N} \) and dividing by the transverse area of the nucleus:

\[
\frac{g^2 N}{S_\perp} \sim \frac{g^2 A}{A^{2/3}} \sim \Lambda_{QCD}^2 A^{2/3}.
\]

The density of partons given above \( (\rho \sim A^{1/3}) \) defines a dimensionful scale \( \mu^2 \):

\[
\mu^2 \sim \Lambda_{QCD}^2 A^{1/3}.
\]

The factor \( \mu^2 \) sets the scale of the running coupling constant, \( \alpha_s(\mu^2) \). In the MV model for a large nucleus \( A^{1/3} \to \infty \). Hence, \( \mu^2 \gg \Lambda_{QCD}^2 \) and the coupling is small \( \alpha(\mu^2) \ll 1 \). This allows for a weak coupling computational methods for the quark and gluon distribution functions.

Since for a large nucleus \( A^{1/3} \to \infty \), the number of color sources is large, they belong to a high dimensional representation of the color algebra and can be treated classically. Because the coupling is small, the MV theory is dominated by classical fields and the quantum corrections are suppressed by extra powers of \( \alpha_s(\mu^2) \).

The leading contribution to the production of gluon field around the valence quarks at weak coupling is bremsstrahlung. For small \( \alpha_s \), the transferred momentum in the process
is small and we can assume that the quarks do not lose momentum. The valence quarks carrying a large longitudinal momentum fraction \( x \) act as recoilless sources for the small-\( x \) gluon field. Since the nucleus is Lorentz contracted, the sources are modelled as a delta function along the light cone.

The gluon density in a nucleus at small \( x \) is proportional to:

\[
\frac{Axg_N}{S_\perp} \sim A^{3/2}.
\]  

At large \( A \) the occupation number of gluons is large and the gluon field is classical. The form of the gauge field is obtained as a solution to the classical Yang-Mills equations of motion with the large-\( x \) quarks as color sources. We introduce this solution in the next section.

Because of time dilation, the distribution of large-\( x \) sources in the nucleus and the field they emit appears frozen to a probe. However, the color charge distribution varies between collision events. Hence, to find the gluon distribution function of a nucleus one needs to average over all possible configurations of large-\( x \) sources. If we label the weight functional defining the distribution of partons \( \rho \) with \( W[\rho] \), then the expectation value of any physical observable (given by the operator \( O \)) is:

\[
\langle O \rangle_\rho = \frac{\int D\rho \ O W[\rho]}{\int D\rho \ W[\rho]}.
\]  

For a large nucleus and a zero average color charge density, the distribution of color charge fluctuations is Gaussian:

\[
W[\rho] = \exp \left[ \int d^2x_\perp \frac{\delta^{ab} \rho^a(x_\perp) \rho^b(x_\perp)}{2\mu^2} \right],
\]  

where \( \mu^2 \) is the average charge density squared per unit area. Higher-order corrections in \( \rho \) will be derived in the next chapter.
3.2 Non-Abelian Weizsäcker-Williams field of a single nucleus

The Abelian Weizsäcker-Williams field is a classical field radiated by an ultra-relativistic electron. The field of an electron moving in the $x^+$ direction is of the following form:

$$ A^\mu(x) = \delta^{\mu+}(x^-) \Lambda(x_\perp) . $$

(3.6)

The electromagnetic potential is localized at $x = x^-$ due to Lorentz contraction. In the above expression $\Lambda(x_\perp)$ represents a two-dimensional Coulomb potential of a point charge in the transverse plane.

Similarly, in the MV model one can consider a color charge representing valence quarks moving along the light cone. As discussed in the previous section, the field radiated by this charge is classical and its form can be obtained as a solution to the classical Yang-Mills equations of motion:

$$ [D_\mu, F^{\mu\nu}] = J^\nu . $$

(3.7)

Since the valence charges generating the current $J^\nu$ move along the $x^+$ direction, the current runs in the $x^+$ direction as well:

$$ J^\nu = \delta^{\nu+} J^+ . $$

(3.8)

Along with the classical equations of motion, the problem requires conservation of $J^\nu$:

$$ [D_\mu, J^\mu] = 0 . $$

(3.9)

The above condition allows for one to find the $x^+$ dependence of the current. The current is
given in terms of the color charge density $\rho$ and at a particular initial point $x_0^+$ we can write:

$$J^+(x^+ = x_0^+) = g\rho(x^-, x_-) .$$  \hspace{1cm} (3.10)

We can extend the current along the $x^+$ direction by assuming translational invariance ($\partial^- \rho = 0$) and by using a path-ordered exponential for linking the points along the $x^+$ axis:

$$S(x^+) = P \exp \left\{ ig \int_{x_0^+}^{x^+} dx^- A^- (x^-) \right\} .$$  \hspace{1cm} (3.11)

The current’s dependence on $x^+$ is then given by:

$$J^+(x^+, x^-, x_-) = gS(x^+)\rho(x^-, x_-)S^\dagger (x^+) .$$  \hspace{1cm} (3.12)

The above expression satisfies the current conservation condition. The Yang-Mills equations become:

$$[D_\mu, F^{\mu\nu}] = g\delta^{\nu+} S\rho S^\dagger .$$  \hspace{1cm} (3.13)

A convenient gauge choice for solving the equations of motion is the light cone gauge $A^+ = 0$. There is a particular solution in this gauge for which $A^- = 0$. Hence, the only non-zero component is the transverse component of the field. Under these considerations, the Yang-Mills equations reduce to:

$$[D_i, F^{ij}] = 0 ;$$  \hspace{1cm} (3.14)

$$[D_\perp, F^{\perp+}] = g\rho .$$  \hspace{1cm} (3.15)
From the first equation it follows that the transverse potential is a pure gauge:

\[ \alpha_\perp = i \frac{g}{\mathcal{A}} \partial_\perp U^\dagger, \]  

(3.16)

where \( U \) is a gauge transformation and where we have defined \( \alpha_\perp \) so that:

\[ A_\perp = \theta(x^-)\alpha_\perp(x_\perp) . \]  

(3.17)

The \( x_\perp \) dependence of \( \alpha_\perp(x_\perp) \) can be obtained from the second equation. This gives:

\[ \nabla \cdot \alpha = g \rho(x_\perp) . \]  

(3.18)

The field \( A^\mu \) is now fully defined.

In order to compare the MV result to the Abelian Weizsäcker-Williams field we consider the form of the field in the \( A^- = 0 \) gauge. In this gauge choice the field \( A^\mu \) is:

\[ A^\mu = \frac{i}{g} \delta^{\mu+}\delta(x^-)U\partial^+U^\dagger. \]  

(3.19)

It is clear that the above potential represents a non-Abelian analogue of the Weizsäcker-Williams field.

In conclusion, the field of an ultra-relativistic nucleus is a two dimensional gauge transformation of the vacuum. We can choose the field in the front of the nucleus to be zero and behind the nucleus to be given by eq.(3.16). Gluons emerge from a non-Abelian Weizsäcker-Williams fields generated by the color sources at large \( x \) represented as a delta function along the light cone.
3.3 Classical color field in a collision of two nuclei

In the previous section we introduced the classical non-Abelian Weizsäcker-Williams field of a single high-energy nucleus. Next, we will present the color field produced in a collision of two ultra-relativistic nuclei following the treatment in [77]. We focus only on the solution at proper time zero, which is relevant for the work presented in this thesis.

We consider two nuclei moving along the $x^+$ and the $x^−$ light cone directions respectively. The currents produced by the color sources, $ρ_1$ and $ρ_2$, of the two nuclei are:

$J^+ = \delta(x^-)ρ_1(x_\perp)$, \hspace{1cm} (3.20)

$J^- = \delta(x^+)ρ_2(x_\perp)$, \hspace{1cm} (3.21)

$J^i = 0$. \hspace{1cm} (3.22)

The solution to the classical Yang-Mills equations of motion gives a vanishing gauge potential in the backward light cone (fig.3.1). Before the collision, in regions 1 and 2, presented in fig.

![High-energy collision of two nuclei in a light cone coordinate system.](image)

Figure 3.1: High-energy collision of two nuclei in a light cone coordinate system.
3.1, the solution is:

\[ A^+ = 0 \, , \quad (3.23) \]
\[ A^- = 0 \, , \quad (3.24) \]
\[ A^i = \theta(x^-)(-x^+)\alpha_1^i(x_\perp) + \theta(x^+)\theta(-x^-)\alpha_2^i(x_\perp) \, , \quad (3.25) \]

where, similarly to the single nucleus case:

\[ \nabla \cdot \alpha_{1,2} = g\rho_{1,2}(x_\perp) \, . \quad (3.26) \]

After the collision, the equations of motion in the forward light cone have the form:

\[ [D_\mu, F^{\mu\nu}] = 0 \, . \quad (3.27) \]

An ansatz that solves the above equations can be written as:

\[ A^+ = x^+\alpha(\tau, x_\perp) ; \quad (3.28) \]
\[ A^- = x^-\beta(\tau, x_\perp) ; \quad (3.29) \]
\[ A^i = \alpha^i_3(\tau, x_\perp) \, , \quad (3.30) \]

where \( \tau \) is the proper time:

\[ \tau = \sqrt{t^2 - z^2} = \sqrt{2x^+ x^-} \, . \quad (3.31) \]

A choice of an axial gauge, \( x^+ A^- + x^- A^+ = 0 \), implies:

\[ \beta(\tau, x_\perp) = -\alpha(\tau, x_\perp) \, . \quad (3.32) \]
In this gauge the solution of the equations of motion covering all the regions of the light cone is:

\[ A^i(\tau, x_\perp) = \alpha_3^i \theta(x^-) \theta(x^+) + \alpha_1^i \theta(x^-) \theta(-x^+) + \alpha_2^i \theta(-x^-) \theta(x^+), \]  
\( (3.33) \)

\[ A^+(\tau, x_\perp) = x^+ \alpha \theta(x^-) \theta(x^+) , \]  
\( (3.34) \)

\[ A^-(\tau, x_\perp) = -x^- \alpha \theta(x^-) \theta(x^+) . \]  
\( (3.35) \)

To find the fields in the very first moment of the collision (the solution we are interested in) we need the boundary conditions along the light cone. The requirement is for the gauge fields to be regular at \( \tau = 0 \). Removing the singular pieces from the equations of motion gives the fields at proper time zero:

\[ \alpha_3^i(\tau = 0, x_\perp) = \alpha_1^i(x_\perp) + \alpha_2^i(x_\perp) , \]  
\( (3.36) \)

\[ \alpha(\tau = 0, x_\perp) = -\frac{ig}{2} [\alpha_1^i(x_\perp), \alpha_2^i(x_\perp)] . \]  
\( (3.37) \)

The above solution will be used in Chapter 7.

### 3.4 Scattering off an MV nucleus

In this section we analyze scatterings off a field of a nucleus in the McLerran-Venugopalan description. The result incorporates multiple scatterings and saturation. First, the scattering \( S \)-matrix is expressed in terms of the gauge field and then one performs averaging over all possible configurations of color charges.

In the previous chapter we explained the dipole picture of high-energy scatterings. In the same way as the scattering of a virtual photon is replaced by a scattering of a color dipole, the scattering of a gluon is replaced by a scattering of a quark-antiquark or a gluon
The dipole then undergoes multiple scatterings off the target. In scatterings with high-energy probes it is common to use the eikonal approximation. In this approximation the fast moving parton is taken to be recoilless. The transverse position of the parton is invariant under multiple scatterings off the gluon field. Under these assumptions, the series of interactions with the scattering centers can be resummed into a path ordered exponential. The propagator of an eikonal quark moving along the light cone $x^-$ axis can be written as a Wilson line:

$$V(x_\perp) = \mathcal{P} \exp \left\{ ig \int dx^- A^+_a(x^-, x_\perp) t^a \right\}.$$

(3.38)

Since the Wilson line is a pure phase, the eikonal parton only picks up a color phase from the background field. The propagator of an antiquark is represented by the conjugate Wilson line $V^\dagger(x_\perp)$. Similarly, an eikonal gluon is given by an adjoint Wilson line:

$$U(x_\perp) = \mathcal{P} \exp \left\{ ig \int dx^- A^+_a(x^-, x_\perp) t^a \right\}.$$

(3.39)

The scatterings of quark and gluon dipoles can then be expressed in terms of Wilson line correlators. The $S$-matrix of a color dipole scattering on a nucleus is:

$$S(x_\perp, y_\perp) = \frac{1}{N_c} \left\langle \operatorname{tr} V(x_\perp) V^\dagger(y_\perp) \right\rangle_\rho.$$

(3.40)

The scattering $S$-matrix for a gluon dipole is:

$$S(x_\perp, y_\perp) = \frac{1}{N_c^2 - 1} \left\langle \operatorname{tr} U(x_\perp) U^\dagger(y_\perp) \right\rangle_\rho.$$

(3.41)

The expectation value in the above expressions refers to averaging over all possible distributions of color large-$x$ sources. In the Gaussian MV model the averaged dipole scattering
amplitude, \(N(x_\perp, y_\perp) = 1 - S(x_\perp, y_\perp)\), is:

\[
N(r_\perp) = 1 - \exp\left(-\frac{Q_s^2 r_\perp^2}{4} \ln \frac{1}{r_\perp \Lambda}\right),
\]

(3.42)

where \(r \equiv |x_\perp - y_\perp|\) and \(\Lambda\) is an infrared cutoff on the order of the inverse nucleon radius. The scale \(Q_s\) denotes the saturation momentum given with:

\[
Q_s^2 \equiv \frac{g^4}{2\pi^2} C_F \int_{-\infty}^{\infty} dz^- \mu^2(z^-) .
\]

(3.43)

For small transverse dipole sizes \((r_\perp \ll 1/Q_s)\), the amplitude grows as \(\sim r_\perp^2\). However, multiple scatterings lead to a scattering amplitude \(N(r_\perp)\) that preserves unitarity of the theory. The dipole amplitude saturates for \(r_\perp \geq 1/Q_s\).

The non-Abelian Weizsäcker-Williams gluon distribution for a large nucleus can be defined through the dipole amplitude as:

\[
\Phi_{WW}(k_\perp^2) = \frac{C_F}{2\pi^3 \alpha_s} \int d^2 r_\perp e^{i k_\perp \cdot r_\perp} \frac{1}{r_\perp^2} N(r_\perp) .
\]

(3.44)

The unintegrated gluon distribution function in the MV model shows a power-like behavior for momenta larger than the saturation momentum:

\[
\Phi_{WW}(k_\perp^2) \sim \frac{Q_s^2}{k_\perp^2},
\]

(3.45)

and it saturates for momenta smaller than the saturation momentum \((k_\perp \ll Q_s)\):

\[
\Phi_{WW}(k_\perp^2) \sim \ln \frac{k_\perp^2}{Q_s^2} .
\]

(3.46)
At the end of this chapter we point out again that the McLerran-Venugopalan model is a classical theory. Quantum corrections enter through small-$x$ evolution. Evolution can be incorporated using the Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner (JIMWLK) equation [24–29]. The JIMWLK renormalization group equation gives the small-$x$ evolution of the weight functional $W[\rho]$. Through $W_x[\rho]$ all observables become $x$ dependent. The MV action can serve as an initial condition for the evolution equations. In the next chapter we introduce corrections to the Gaussian distribution at higher orders in $\rho$. 
Chapter 4

Initial conditions beyond the McLerran-Venugopalan model

The following four chapters are dedicated to the research done towards this thesis. The material presented is based on the work and results published in several papers. We start by introducing corrections to the MV model.

In the previous chapter we described the McLerran-Venugopalan model as an initial condition for high-energy evolution equations. We pointed out that the model assumes high parton densities in the hadron’s wave-function and it is valid in the limit of large nucleon number, $A^{1/3} \rightarrow \infty$. However for protons and for peripheral nucleus-nucleus collisions, this assumption might not be well justified. In this chapter we analyse the contributions of higher order operators in the action, which are relevant for processes with finite $A$. We start by deriving the fourth-order operator in the action following the methods introduced by McLerran and Venugopalan. In order to investigate the effect of the quartic operator on physical observables we calculate the two-point and four-point functions. The results shall be used in the following chapters.
CHAPTER 4. INITIAL CONDITIONS BEYOND THE McLERRAN-VENUGOPALAN MODEL

4.1 Quartic action

In the MV model, the large-\(x\) valence charges act as recoilless sources for the soft, small-\(x\) gluon fields. As \(A \to \infty\), the variance of the “valence” charge density \(\sim A^{1/3}\) grows large and the distribution of color charges should be Gaussian,

\[
S_{MV} = \int d^2x_\perp \frac{\rho^2(x_\perp)}{\mu^2} = \int d^2x_\perp \frac{\delta^{ab}\rho^a(x_\perp)\rho^b(x_\perp)}{2\mu^2}, \quad \mu^2 \sim \frac{g^2A}{\pi R^2}. \tag{4.1}
\]

In reality the mass number \(A\) resp. the number of valence charges is finite, in particular in (high-multiplicity) \(pp\) as well as peripheral \(AA\) collisions. It is therefore interesting to consider extensions of the MV-model action involving higher powers of the color charge density. In fact, Jeon and Venugopalan have derived [30] an “odderon” operator contribution

\[-d_{abc}\rho^a\rho^b\rho^c/\kappa_3,\]

where the cubic coupling \(\kappa_3 \sim g^3(A/\pi R^2)^2\). Below, we shall show that at quartic order in \(\rho\) a contribution \(\delta^{ab}\delta^{cd}\rho^a\rho^b\rho^c/\kappa_4\) to the effective action arises, with \(\kappa_4 \sim g^4(A/\pi R^2)^3\).

For a proton, the saturation scale \(Q_s(x)\) is expected to be on the order of 1 GeV at \(x \approx \) a few times \(10^{-4}\), on average; high-multiplicity collisions should correspond to configurations with significantly higher parton densities. For nuclei, the valence charge density should be boosted by a factor of \(\sim A^{1/3}\).

In this section we derive the form of the effective action for a system of \(k\) quarks in \(SU(3)\) following the methods developed in [30]. The probability to find this system in a representation labelled by the integers \((m, n)\), also denoted as Dynkin labels, is given by

\[
e^{-S} \equiv d_{mn}N_{m,n}^{(k)} = d_{mn}C_{m,n}^{(k)} + G_{m+3,n}^{(k)} + G_{m,n+3}^{(k)} - G_{m+2,n-1}^{(k)} - G_{m-1,n+2}^{(k)} - G_{m+2,n+2}^{(k)} \tag{4.2}
\]
CHAPTER 4. INITIAL CONDITIONS BEYOND THE MCLERRAN-VENUGOPALAN MODEL

with

\[ G_{m,n}^{(k)} = \frac{k!}{(\frac{k+2m+n}{3})! (\frac{k-m+n}{3})! (\frac{k-m-2n}{3})!}. \]  

(4.3)

We are interested in the representation with the largest weight and so consider the limit \( k \gg m, n \gg 1 \). Thus, we expand the factorials in each \( G \) in powers of \( 1/k \) up to and including order \( \sim 1/k^3 \). The leading terms of the form \( \sim m^j n^{l+1} j / k^l \) can be written as

\[ S(m,n; k) \simeq \frac{N_c}{k} C_2(m,n) - \frac{1}{3} \left( \frac{N_c}{k} \right)^2 C_3(m,n) + \frac{1}{6} \left( \frac{N_c}{k} \right)^3 C_4(m,n), \]  

(4.4)

where \( C_2, C_3, C_4 \) are the Casimir operators for the representation \((m,n)\) given in the appendix. We have verified eq. (4.4) also by explicit calculation for \( N_c = 4 \) and \( N_c = 5 \) but refrain from listing the corresponding expressions for the Casimirs.

Before we rewrite the action \( S(m,n; k) \) in terms of the color charges \( \rho \), it is perhaps useful to remind the reader of this derivation and to illustrate it in the simpler case of SU(2) spins [30]. In this case, the representation \( R \) is labelled by one index, \( l \), and there is only one independent Casimir defined as \( C_2(R) = \delta^{ab} L_a^a R L_b^b R = L_R^2 \), where \( L_R^a \) are the generators of the representation \( R \). For a large representation we have \( C_2 = l(l+1) \simeq \ell^2 \) as the quadratic Casimir. One can then write \( \ell^2 = \ell_i \ell_i \) where \( \ell_i \) is a vector describing spin and express the action in terms of invariants formed by multiplying \( \ell_i \)'s (in this example, there is only one invariant, \( \ell^2 \)) as \( S \simeq N_c \ell^2 / k \). For large \( \ell \), one introduces a classical charge density per unit transverse area, \( \rho^i(x) = g \ell^i / \Delta^2 x \). Next, \( k \) is expressed in terms of number of valence quarks in a nucleus as \( k = N_c A \Delta^2 x / \pi R^2 \) which then leads to

\[ S = \frac{N_c}{k} \ell^2 \simeq \int d^2 x \frac{\rho^a \rho^a}{2 \mu^2}, \]

(4.5)

with \( \mu^2 \equiv \frac{g^2 A}{2 \pi R^2} \).

We start by defining the Casimirs for general \( N_c \) in terms of the generators \( T^{ai}_R \) of the
representation $R$,

$$C_n(R) \equiv F_R^{a_1,\ldots,a_n} T_{R}^{a_1} \cdots T_{R}^{a_n},$$

(4.6)

where $I_R$ is the unit matrix in the representation $R$. Taking the trace of both sides gives $C_n(R)$, the $n^{th}$ Casimir of the representation $R$:

$$C_n(R) = \frac{1}{d_R} F_R^{a_1,\ldots,a_n} \text{tr} T_{R}^{a_1} \cdots T_{R}^{a_n}.$$  

(4.7)

The color tensors $F_R^{a_1,\ldots,a_n}$ are the most general color invariant tensors one can construct out of the SU($N_c$) orthonormal basis tensors. For example, $F_R^{a_1,a_2} = \delta^{a_1,a_2}$ for $n = 2$, while for $n = 4$ these basis tensors are given by $\delta^{ab} \delta^{cd}$ plus permutations, $a^{abc} d^{cde}$ plus permutations, and $d^{abc} f^{cde}$ plus permutations. More explicitly,

$$C_2(R) = \frac{1}{d_R} \delta^{ab} \text{tr} T_{R}^{a} T_{R}^{b}$$

(4.8)

$$C_3(R) = \frac{1}{d_R} a^{abc} \text{tr} T_{R}^{a} T_{R}^{b} T_{R}^{c}$$

(4.9)

$$C_4(R) = \frac{1}{d_R} \left[ \alpha (\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}) + \times \right.$$

$$\left. \beta (d^{abc} d^{cde} + d^{ace} d^{bde} + d^{ade} d^{bce}) \right] \text{tr} T_{R}^{a} T_{R}^{b} T_{R}^{c} T_{R}^{d}$$

(4.10)

Since the number of independent Casimirs of SU($N_c$) is equal to its rank, $N_c - 1$, there are only 2 independent Casimirs for SU(3). These are usually taken to be $C_2$ and $C_3$. This can be seen explicitly by noting that for $N_c = 3$ one has the constraint

$$d^{abe} d^{cde} + d^{ace} d^{bde} + d^{ade} d^{bce} = \frac{1}{3} (\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc})$$

(4.11)

and $3\alpha + \beta = 1$ so that the quartic Casimir can be written as the square of the quadratic invariant. To proceed, we write the action in terms of $Q^a$, a $N_c^2 - 1$ dimensional vector
related to the second Casimir via $|Q| = \sqrt{Q^aQ^a} \equiv \sqrt{C_2}$; the vector $Q^a$ is analogous to the angular momentum vector $\ell^a$ in our SU(2) example from above:

$$S \approx \left[ \frac{N_c}{k} Q^a Q^a - \left( \frac{N_c}{k} \right)^2 \frac{d^{abc}}{3} Q^a Q^b Q^c + \left( \frac{N_c}{k} \right)^3 \frac{\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}}{18} Q^a Q^b Q^c Q^d \right].$$

(4.12)

We then follow McLerran-Venugopalan and define the color charge per unit area $\rho^a \equiv g Q^a/\Delta^2 x$ to finally arrive at

$$S[\rho(x)] \approx \int d^2 x \left[ \frac{\delta^{ab} \rho^a \rho^b}{2 \mu^2} - \frac{d^{abc} \rho^a \rho^b \rho^c}{\kappa_3} + \frac{\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}}{\kappa_4} \rho^a \rho^b \rho^c \rho^d \right].$$

(4.13)

To write the action in this form we have used eq. (4.11) and $k = N_c A \Delta^2 x / \pi R^2$. This assumes that the large-$x$ sources correspond to $N_c A$ valence quarks. On the other hand, if initial conditions for small-$x$ evolution are set at, say, $x_0 = 0.01$ then the number of “valence” charges would be bigger although still parametrically proportional to $A$ and to $N_c$.

The couplings in this action are given by, parametrically,

$$\mu^2 \equiv \frac{g^2 A}{2 \pi R^2} \sim O(g^2 A^{1/3}),$$

(4.15)

$$\kappa_3 \equiv 3 \frac{g^3 A^2}{(\pi R^2)^2} \sim O(g^3 A^{2/3}),$$

(4.16)

$$\kappa_4 \equiv 6 \frac{g^4 A^3}{(\pi R^2)^3} \sim O(g^4 A).$$

(4.17)

The non-Gaussian terms in the action thus involve additional inverse powers of $gA^{1/3}$. Also, the expressions from above are obtained by averaging over the transverse (impact parameter) plane; at a fixed distance $b$ from the center of a nucleus one should replace $A/\pi R^2$ by the thickness function $T(b)$.
In what follows we restrict to the action \((4.14)\) as appropriate for \(N_c = 3\) colors. This action is coupled to the soft gauge fields \([24–29]\). At leading order and in the classical approximation this leads to the relation \((4.22)\) below.

### 4.2 Two-point function and unintegrated gluon distribution

In order to compute the contribution of the quartic term to a physical observable, such as single-inclusive hadron production in the forward region \([31]\), one needs to perform the color averaging of the dipole operator with this new action: within the Glauber-Mueller approach the expectation value of the full dipole operator \(\langle \text{tr} \ V_x \ V_y^\dagger \rangle\), where \(V\) denotes a Wilson line, resums multiple scattering effects on a dense target \([32, 33]\). This can in principle be done numerically using lattice gauge theory methods \([34, 35]\). Nevertheless, it is useful to consider a limit where this expectation value can be evaluated analytically, such as the dilute limit of the dipole cross section given by the two-point function of color charges. This limit is relevant for Deep-Inelastic Scattering (DIS) at high \(Q^2 \gg Q_s^2(x)\) and for high transverse momentum particle production in hadronic collisions \(p_\perp \gg Q_s(x)\).

Even in the dilute limit, where the Wilson lines are expanded to linear order in the charge density \(\rho\), one can not perform the integration over \(\rho\) analytically when the action is not quadratic. We therefore resort to a perturbative expansion in \(1/\kappa_4\) and keep only the first term in the expansion of the exponential of the quartic term. We then compute the two-point function of the color charge density of hard sources which is related to the unintegrated gluon distribution. To first non-trivial order in \(1/\kappa_4\),

\[
\langle O[\rho] \rangle \equiv \frac{\int \mathcal{D}\rho \ O[\rho] \ e^{-S_G[\rho]} \left[ 1 - \frac{1}{\kappa_4} \int d^2u \ \rho_u^a \rho_u^a \rho_u^b \rho_u^b \right]}{\int \mathcal{D}\rho \ e^{-S_G[\rho]} \left[ 1 - \frac{1}{\kappa_4} \int d^2u \ \rho_u^a \rho_u^a \rho_u^b \rho_u^b \right]}.
\]  
(4.18)
Here, $S_G$ denotes the Gaussian action. For the two-point function $O = \rho^a_x \rho^b_y$, the possible contractions are shown diagrammatically in fig. 4.1; for the denominator of eq. (4.18) one amputates the points $x$ and $y$. We compute the functional integral in lattice regularization, i.e. we approximate the two-dimensional transverse space by a lattice with $N_s$ sites of area $\Delta^2 x$. The two-point function becomes

$$\langle \rho^a_x \rho^b_y \rangle = \mu^2 \frac{\delta^{ab} \delta_{xy}}{\Delta^2 x} \frac{1 - \frac{\mu^4}{\kappa^4} (N_c^2 + 1)(N_c^2 - 1) \frac{N_s}{\Delta^2 x} - 4 \frac{\mu^4}{\kappa^4} (N_c^2 + 1) \frac{1}{\Delta^2 x}}{1 - \frac{\mu^4}{\kappa^4} (N_c^2 + 1)(N_c^2 - 1) \frac{N_s}{\Delta^2 x}}. \quad (4.19)$$

The disconnected contribution exhibits both a UV ($\Delta^2 x \to 0$) as well as a IR ($N_s \to \infty$) divergence but appears both in the numerator and in the denominator and so cancels as usual. The third term in the numerator is due to the tadpole diagram from fig. 4.1 which renormalizes the “bare” parameter $\mu^2$:

$$\tilde{\mu}^2 \equiv \mu^2 \left(1 - 4 \frac{\mu^4}{\kappa^4} \frac{N_c^2 + 1}{\Delta^2 x}\right). \quad (4.20)$$

This renormalization of $\mu^2$ will be applied to all diagrams to absorb the insertion of a tadpole into any line.

---

1 The right-hand-side of this expression is to be understood in lattice notation: $x$ and $y$ are discrete points and $\delta_{xy}$ is a Kronecker symbol.
Thus, the two-point function now reads (in continuum notation)

\[
\langle \rho^a_x \rho^b_y \rangle = \tilde{\mu}_2 \delta^{ab} \delta(x - y) \quad \leftrightarrow \quad \langle \rho^*^a(k) \rho^b(k') \rangle = \tilde{\mu}_2^2 \delta^{ab} (2\pi)^2 \delta(k - k') .
\] (4.21)

To relate (4.21) to the unintegrated gluon distribution we note that at leading order and in covariant gauge the field generated by a fast particle moving in the positive \( z \)-direction is related to the charge density by

\[
A^\mu(x) \equiv \delta^{\mu+}(x) = g \delta^{\mu+}(x^-) \frac{1}{\nabla^2} \rho(x) ,
\] (4.22)

where again \( x \) denotes a transverse coordinate. This field also satisfies \( A^- = 0 \) and so the only non-vanishing field-strength is \( F^{+i} = -\partial^i \alpha \). In momentum space we have the relation \( k^2 \alpha(k) = g\rho(k) \). Thus, we define the unintegrated gluon distribution \( \Phi(k^2) \sim k^2 \langle F^{+i} F^{+i} \rangle \) via

\[
\langle \rho^*_a(k) \rho^b(k') \rangle = \frac{1}{\alpha_s N_c^2 - 1} (2\pi)^3 \delta(k - k') \Phi(k^2) .
\] (4.23)

With this convention, \( \Phi(k^2) = \alpha_s(N_c^2 - 1) \tilde{\mu}_2^2/(2\pi) \).

### 4.3 Four-point function

![Four-point function diagram](image)

Figure 4.2: Four-point function at leading order in \( 1/\kappa_4 \).
The four-point function to first non-trivial order in $1/\kappa_4$ is given by

$$\langle \rho^a_x \rho^b_y \rho^c_u \rho^d_v \rangle \equiv \int \mathcal{D}\rho e^{-S_G[\rho]} \rho^a_x \rho^b_y \rho^c_u \rho^d_v \left[ 1 - \frac{1}{\kappa_4} \int d^2w \rho^a_w \rho^b_w \rho^c_w \rho^d_w \right] \frac{1}{1 - \frac{1}{\kappa_4} \int d^2u \rho^a_u \rho^b_u \rho^c_u \rho^d_u} \right].$$

(4.24)

The different type of contractions which arise at this order are shown in Fig. 4.2. Eq. (4.24) then gives

$$\langle \rho^a_x \rho^b_y \rho^c_u \rho^d_v \rangle = \frac{1}{1 - \frac{\mu^4}{\kappa_4} (N^2_c + 1)(N^2_c - 1) \frac{N_s}{\Delta^2 x}} \times$$

$$\left\{ \frac{\mu^4}{(\Delta^2 x)^2} \left( \delta^{ab} \delta_{xy} \delta^{cd} \delta_{uv} + \delta^{ac} \delta_{xu} \delta^{bd} \delta_{yv} + \delta^{ad} \delta_{xy} \delta^{bc} \delta_{uv} \right) \right\} \frac{1}{1 - \frac{\mu^4}{\kappa_4} (N^2_c + 1)(N^2_c - 1) \frac{N_s}{\Delta^2 x} - \frac{8\mu^4}{\kappa_4} (N^2_c + 1) \frac{1}{\Delta^2 x}} - \frac{8\mu^4}{\kappa_4} (\Delta^2 x)^3 \left( \delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \right) \delta_{xy} \delta_{xu} \delta_{uv} \right\}.$$

(4.25)

The second line in this expression corresponds to the sum of the first three diagrams shown in Fig. 4.2: the second term due to the disconnected diagram again cancels against the overall normalization factor once the latter is expanded to leading order in $1/\kappa_4$. Also, the third term from the second line is again absorbed into the renormalized value of $\mu^2$ as in (4.20).

Figure 4.3: Diagram at order $1/\kappa_4^2$ which renormalizes $\mu^2$ in the connected contribution to the four-point function.

In fact such a factor of $1 - 16\frac{\mu^4}{\kappa_4} (N^2_c + 1) \frac{1}{\Delta^2 x}$ appears at order $1/\kappa_4^2$ from diagrams of the type...
shown in Fig. 4.3 which induce the shift $\mu^8 \rightarrow \tilde{\mu}^8$ in the last line of eq. (4.25). Thus, the four-point function to order $1/\kappa_4$ finally becomes (switching again from lattice to continuum notation)

$$
\langle \rho^a_x \rho^b_y \rho^c_u \rho^d_v \rangle = \tilde{\mu}^4 \left[ \delta^{ab} \delta^{cd} \delta(x-y) \delta(u-v) \left( 1 - 8 \frac{\tilde{\mu}^4}{\kappa_4} \delta(x-u) \right) 
+ \delta^{ac} \delta^{bd} \delta(x-u) \delta(y-v) \left( 1 - 8 \frac{\tilde{\mu}^4}{\kappa_4} \delta(x-y) \right) 
+ \delta^{ad} \delta^{bc} \delta(x-v) \delta(y-u) \left( 1 - 8 \frac{\tilde{\mu}^4}{\kappa_4} \delta(x-y) \right) \right].
$$

(4.26)

In momentum space,

$$
\langle \rho^*^a(k_1) \rho^*^b(k_2) \rho^c(k_3) \rho^d(k_4) \rangle = (2\pi)^4 \tilde{\mu}^4 \left[ \delta^{ab} \delta^{cd} \delta(k_1 + k_2) \delta(k_3 + k_4) 
+ \delta^{ac} \delta^{bd} \delta(k_1 - k_3) \delta(k_2 - k_4) + \delta^{ad} \delta^{bc} \delta(k_1 - k_4) \delta(k_2 - k_3) 
- 2 \frac{\tilde{\mu}^4}{\pi^2 \kappa_4} \left( \delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \right) \delta(k_1 + k_2 - k_3 - k_4) \right].
$$

(4.27)

In summary, we have computed corrections to the quadratic MV-model action up to quartic order in the density of hard (large-$x$) sources. Such terms are accompanied by additional powers of $1/gA^{1/3}$. This action could be employed to determine initial conditions for the JIMWLK high-energy evolution of various $n$-point operator expectation values.

4.4 Appendix

In this appendix we illustrate how one can express the values of various Casimirs in terms of the Dynkin labels $n$ and $m$ of SU(3). The representation $(m, n)$ is reached by multiplying the $(1, 0)$ fundamental representation $k$ times. The corresponding Young tableau thus has $k$ boxes in $N = 3$ rows. The first row has $h_1 = m + n + (k - m - 2n)/3$ boxes; the second row
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has $h_2 = n + (k - m - 2n)/3$ boxes; and the last row has $h_3 = (k - m - 2n)/3$ boxes.

It is useful to define the set $n'_i$ for the $U(N)$ group from these “row lengths” $h_i$ via $n'_i = h_i + (N + 1)/2 - i$:

\[
\begin{align*}
n'_1 &= m + n + \frac{k - m - 2n}{3} + 2 - 1 = \frac{k + 2m + n}{3} + 1 \\
n'_2 &= n + \frac{k - m - 2n}{3} + 2 - 2 = \frac{k - m + n}{3} \\
n'_3 &= \frac{k - m - 2n}{3} + 2 - 3 = \frac{k - m - 2n}{3} - 1
\end{align*}
\]

\[
\sum_i n'_i = k.
\]

To pass from $U(3)$ to $SU(3)$ one needs to shift $n_i = n'_i - \sum_j n'_j/N$ to get

\[
\begin{align*}
n_1 &= \frac{2m + n}{3} + 1 \\
n_2 &= \frac{-m + n}{3} \\
n_3 &= \frac{-m - 2n}{3} - 1.
\end{align*}
\]

Therefore,

\[
\begin{align*}
\sum_i n_i^2 &= \frac{2}{3} (m^2 + n^2 + mn) + 2(m + n + 1) \\
\sum_i n_i^4 &= \frac{2}{9} (3 + m^2 + n^2 + 3n + 3m + mn)^2.
\end{align*}
\]

The quadratic casimir is constructed from $n_i$ via $[36]^2$

\[
C_2(m, n) = \sum_i n_i^2 - \frac{N(N^2 - 1)}{12} = \frac{1}{3} (m^2 + n^2 + mn) + (m + n).
\]

\[\text{Note that their normalization of the generators is different from ours so that their expressions for the Casimirs need to be divided by 2.}\]
For a single quark corresponding to the fundamental representation ($m = 1$, $n = 0$) of QCD, we get $C_2 = (N_c^2 - 1)/(2N_c) = 4/3$.

To compute the cubic Casimir we require the set of $l_i = n_i + (N - 1)/2$,

\[
\begin{align*}
l_1 &= \frac{2m + n}{3} + 2 \quad (4.38) \\
l_2 &= \frac{-m + n}{3} + 1 \quad (4.39) \\
l_3 &= \frac{-m - 2n}{3} \quad (4.40)
\end{align*}
\]

\[
\sum_i l_i = N \quad (4.41)
\]

\[
\sum_i l_i^2 = N + \sum_i n_i^2 = \frac{2}{3} (m^2 + n^2 + mn) + 2(m + n) + 5 \quad (4.42)
\]

\[
\sum_{i<j} l_il_j = l_1(l_2 + l_3) + l_2l_3 = 2 - \frac{2m + n}{3} - \frac{4m^2 + n^2 + 4mn}{9} - \frac{3m + 6n - nm + 2n^2 - m^2}{9} \quad (4.43)
\]

\[
\sum_i l_i^3 = \frac{2m^3 - 2n^3 + 3m^2n - 3mn^2}{9} + 3m^2 + n^2 + 2mn + 7m + 5n + 9. \quad (4.44)
\]

These can then be used to construct the cubic Casimir following [36] and lead to

\[
C_3(m, n) \equiv \frac{1}{18} (m + 2n + 3) (n + 2m + 3) (m - n). \quad (4.45)
\]

Constructing $C_4(m, n)$ requires the vectors $n_i^4$ and can be done similarly to $C_2(m, n)$; this leads to

\[
C_4(m, n) \equiv \frac{1}{9} \left( m^4 + n^4 + 2mn^3 + 2m^3n + 3m^2n^2 \right) + \frac{2}{3} \left( m^3 + n^3 + 2m^2n + 2mn^2 \right) + \frac{1}{6} (5m^2 + 5n^2 + 11mn) - \frac{1}{2} (m + n). \quad (4.46)
\]
Chapter 5

Initial conditions for dipole evolution beyond the McLerran-Venugopalan model

In this chapter we will use the results obtained previously to derive the scattering amplitude for a QCD dipole on a dense target in the semi-classical approximation. We restrict to a perturbative expansion of the Wilson lines valid at short distances $r$ but include the first subleading (in density) correction arising from $\sim \rho^4$ operators in the effective action for the large-$x$ valence charges. The result for $N(r)$ can be matched to a phenomenological proton fit by Albacete et al. over a broad range of dipole sizes $r$ and provides a definite prediction for the $A$-dependence for heavy-ion targets. We find a suppression of $N(r)$ for finite $A$ for dipole sizes a few times smaller than the inverse saturation scale, corresponding to a suppression of the classical bremsstrahlung tail. Our result may prove useful for a better theoretical understanding of the $A$-dependence of the initial condition for high-energy evolution of $N(r)$, as discussed in more detail below.
5.1 Quartic action correction to the dipole scattering amplitude

The form of the McLerran-Venugopalan action when the longitudinal direction is taken into account is:

\[ S_{MV}[\rho] = \int d^2x_\perp \int_{-\infty}^{\infty} dx^- \frac{\rho^a(x^-, x_\perp) \rho^b(x^-, x_\perp)}{2\mu^2(x^-)} . \] (5.1)

Here, \( \mu^2(x^-) dx^- \) is the density of color sources per unit transverse area in the longitudinal slice between \( x^- \) and \( x^- + dx^- \) and

\[ \int_{-\infty}^{\infty} dx^- \mu^2(x^-) \sim g^2 A^{1/3} \] (5.2)

is proportional to the thickness \( \sim A^{1/3} \) of the target nucleus. The expectation value of the dipole operator in the MV model is:

\[ D(r) \equiv \frac{1}{N_c} \langle \text{tr} V(x_\perp)V^\dagger(y_\perp) \rangle \] (5.3)

\[ = \exp \left( - \frac{g^4 C_F}{8\pi} \int dx^- \mu^2(x^-) r^2 \ln \frac{1}{r\Lambda} \right) \] (5.4)

\[ = \exp \left( - \frac{Q_s^2 r^2}{4} \ln \frac{1}{r\Lambda} \right) , \] (5.5)

where \( r \equiv |x_\perp - y_\perp| \) and \( \Lambda \) is an infrared cutoff on the order of the inverse nucleon radius; the explicit \( r \)-dependence has been obtained in the limit \( \ln 1/(r\Lambda) \gg 1 \). The scale \( Q_s \) denotes the saturation momentum at the rapidity of the sources. For details of the calculation we refer to refs. [24, 37–39] and to the appendix below.

The effective action describing color fluctuations then involves a sum of the quadratic,
cubic, and quartic Casimirs which can be written in the form [40]

\[ S[\rho] = \int d^2 \mathbf{v}_\perp \int_{-\infty}^{\infty} dv^-_1 \left\{ \frac{\rho^a(v^-_1, \mathbf{v}_\perp) \rho^a(v^-_1, \mathbf{v}_\perp)}{2\mu^2(v^-_1)} - \frac{d^{abc} \rho^a(v^-_1, \mathbf{v}_\perp) \rho^b(v^-_1, \mathbf{v}_\perp) \rho^c(v^-_1, \mathbf{v}_\perp)}{\kappa_3} \right. \\
+ \left. \int_{-\infty}^{\infty} dv^-_2 \frac{\rho^a(v^-_1, \mathbf{v}_\perp) \rho^a(v^-_1, \mathbf{v}_\perp) \rho^b(v^-_2, \mathbf{v}_\perp) \rho^b(v^-_2, \mathbf{v}_\perp)}{\kappa_4} \right\} \right. \ . \quad (5.6) \]

The coefficients of the higher dimensional operators are

\[ \kappa_3 \sim g^3 A^{2/3} \ , \quad (5.7) \]
\[ \kappa_4 \sim g^4 A \ , \quad (5.8) \]

and so involve higher powers of \( gA^{1/3} \). In what follows we restrict to leading order in \( 1/\kappa_4 \) and drop the “odderon” operator \( \sim \rho^3 \) from (7.29) since it does not contribute to the expectation value of the dipole operator at leading order in \( 1/\kappa_3 \). The details of the calculation are shown in the appendix, here we just quote the final result for the \( T \)-matrix

\[ N(r) \equiv 1 - D(r) = \frac{Q_s^2 r^2}{4} \ln \frac{1}{r\Lambda} - \frac{C_F^2 g^8}{6\pi^3 \kappa_4} \left[ \int_{-\infty}^{\infty} dz^- \mu^4(z^-) \right]^2 r^2 \ln^2 \frac{1}{r\Lambda} \ , \quad (r^2 Q_s^2 < 1) \ . \quad (5.9) \]

This now involves a new moment of the valence color charge distribution, namely \( \int dx^- \mu^4(x^-) \).

We have calculated the \( \mathcal{O}(1/\kappa_4) \) correction analytically only in the “short distance” regime up to order \( \sim r^2 \). Recall, however, that the effective theory (6.3) or (7.29) does not apply to the DGLAP regime at asymptotically short distances. Our result could in principle be extended into the saturation region \( r \gtrsim 1/Q_s \) by generating the color charge configurations \( \rho^a(x^-, \mathbf{x}_\perp) \) non-perturbatively, numerically [41].

The dipole scattering amplitude for a proton target has been fitted in ref. [42] to deep-inelastic scattering data. The Albacete-Armesto-Milhano-Quiroga-Salgado (AAMQS) model
for the initial condition for small-$x$ evolution is given by

$$N_{\text{AAMQS}}(r, x_0 = 0.01) = 1 - \exp \left[ -\frac{1}{4} (r^2 Q_s^2(x_0)) \gamma \ln \left( e + \frac{1}{rA} \right) \right], \quad (5.10)$$

with $\gamma \approx 1.119$. This model *simultaneously* provides a good description of charged hadron transverse momentum distributions in $p+p$ collisions at 7 TeV center of mass energy [44, 45]. This is a rather non-trivial cross-check: the MV model initial condition (5.5) overshoots the LHC data by roughly an order of magnitude at $p_\perp \gtrsim 6\, \text{GeV}$ [44, 45].

However, since the model (5.10) was introduced essentially “by hand” it is an open question how it extends to nuclei. This is a crucial issue for predicting the nuclear modification factor $R_{pA}$ for $p+Pb$ collisions at LHC, and for heavy-ion structure functions which could be measured at a future electron-ion eIC collider [46]. One possibility is that the AAMQS modification of the MV model dipole is due to some unknown $A$-independent non-perturbative effect. Here, we explore another option, namely that for protons the effects due to the $\sim \rho^4$ operators may not be negligible.

### 5.2 Comparison of results

We can match our result (5.9) approximately to the AAMQS model by choosing

$$\beta \equiv \frac{C_F}{6\pi^2} \frac{g'^8}{Q_s^2 \kappa_4} \left[ \int_{-\infty}^{\infty} dz^- \mu^4(z^-) \right]^2 \approx \frac{1}{100}, \quad (A = 1). \quad (5.11)$$

For nuclei, $\beta_A \sim A^{-2/3}$ since each longitudinal integration over $z^-$ is proportional to the thickness $\sim A^{1/3}$ of the nucleus while $\mu^2(z^-)$ and $\mu^4(z^-)$ are $A$-independent. The scattering amplitude for a dipole in the adjoint representation with this $\beta$ is shown in fig. 5.1. \footnote{There are actually several fits, we refer to the AAMQS paper [42] for a more detailed discussion.} One \footnote{To regularize the behavior at large $r$, for this figure we have replaced $\ln \frac{1}{rA} \to \ln(e + \frac{1}{rA})$ and assumed exponentiation of the $O(r^2)$ expression. This does not affect the behavior at $rQ_s < 1$.}
observes that the dipole scattering amplitude derived from the quartic action is similar to the AAMQS model over a broad range, \( r Q_s \gtrsim 0.04 \). Discrepancies appear at very short distances where none of the above can be trusted. A more careful and quantitative matching of \( \beta \) to the AAMQS fit beyond the leading \( \ln 1/r \Lambda \gg 1 \) approximation should be performed in the future.

On the right, we plot the scattering amplitude for a nucleus with \( A = 200 \) nucleons, assuming that \( Q_s^2 \sim A^{1/3} \) while \( \beta_A \sim A^{-2/3} \). This illustrates how expectation values obtained with the quartic action converge to those from the MV model when the valence charge density is high (i.e., at large \( A^{1/3} \)). If our idea that the \( \rho^4 \) term in the action provides the explanation for the AAMQS model is indeed correct then their modification of the MV model should vanish like \( \beta_A \sim A^{-2/3} \). This should be observable via \( R_{pA} \) at the LHC.
CHAPTER 5. INITIAL CONDITIONS FOR DIPOLE EVOLUTION BEYOND THE MCLERRAN-VENUGOPALAN MODEL

5.3 Appendix

Expectation values of operators $O[\rho]$ are computed as

$$\langle O[\rho] \rangle \equiv \int D\rho \ O[\rho] \ e^{-S[\rho]} / \int D\rho \ e^{-S[\rho]} .$$

We work perturbatively in $1/\kappa_4$ and keep only the first term in the expansion. Then,

$$\langle O[\rho] \rangle \equiv \int D\rho \ O[\rho] \ e^{-S_G[\rho]} \left[ 1 - \frac{1}{\kappa_4} \int d^2 v_\perp \int dv_1^- dv_2^- \rho_{v_1}^a \rho_{v_2}^b \rho_{v_1}^a \rho_{v_2}^b \right]$$

$$= \langle O[\rho] \left( 1 - \frac{1}{\kappa_4} \int d^2 v_\perp \int dv_1^- dv_2^- \rho_{v_1}^a \rho_{v_2}^b \rho_{v_1}^a \rho_{v_2}^b \right) \rangle_G .$$

(5.12)

In lattice regularization the denominator evaluates to

$$\langle 1 - \frac{1}{\kappa_4} \int d^2 v_\perp \int dv_1^- dv_2^- \rho_{v_1}^a \rho_{v_2}^b \rho_{v_1}^a \rho_{v_2}^b \rangle_G$$

$$= 1 - \frac{1}{\kappa_4} \frac{N_s}{\Delta v_\perp} \left\{ (N_c^2 - 1)^2 \left[ \int_{-\infty}^{\infty} dv^- \mu^2(v^-) \right] + 2 (N_c^2 - 1) \int_{-\infty}^{\infty} dv^- \mu^4(v^-) \right\} .$$

(5.13)

where $N_s$ denotes the number of lattice sites (the volume) and $\Delta x_\perp$ the transverse area of a single lattice site (square of lattice spacing). Also, we have used a local $\langle \rho \rho \rangle$ correlation function:

$$\langle \rho^a(x^-, \mathbf{x}_\perp) \rho^b(y^-, \mathbf{y}_\perp) \rangle = \delta^{ab} \mu^2(x^-) \delta(x^- - y^-) \delta(\mathbf{x}_\perp - \mathbf{y}_\perp) .$$

(5.14)

5.3.1 Dipole Operator

We are interested in the expectation value of the dipole operator defined as

$$\hat{D}(\mathbf{x}_\perp, \mathbf{y}_\perp) \equiv \frac{1}{N_c} \mathbf{tr} \ V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp) .$$

(5.15)
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Here, $V$ denotes a Wilson line

$$V(x_\perp) = \mathcal{P} \exp \left\{ -ig^2 \int_{-\infty}^{\infty} dz^- \frac{1}{\nabla^2_{\perp}} \rho_a(z^-, x_\perp) t^a \right\} ,$$

(5.16)

where

$$\frac{1}{\nabla^2_{\perp}} \rho_a(z^-, x_\perp) = \int d^2 z_\perp G_0(x_\perp - z_\perp) \rho_a(z^-, z_\perp) = -\frac{1}{g} A^+ ,$$

(5.17)

is proportional to the gauge potential in covariant gauge. The matrices $t^a$ are the generators of the fundamental representation of SU(3), normalized according to $\text{tr} t^a t^b = \frac{1}{2} \delta^{ab}$.

$G_0$ is the static propagator which inverts the 2-dimensional Laplacian:

$$\frac{\partial^2}{\partial z^2_{\perp}} G_0(x_{\perp} - z_{\perp}) = \delta(x_{\perp} - z_{\perp}) ;$$

(5.18)

$$G_0(x_{\perp} - z_{\perp}) = \int \frac{d^2 k_{\perp}}{(2\pi)^2} \frac{e^{i k_{\perp} \cdot (x_{\perp} - z_{\perp})}}{k^2_{\perp}} .$$

(5.19)

With this propagator we can write the Wilson line as

$$V(x_{\perp}) = \mathcal{P} \exp \left\{ -ig^2 \int_{-\infty}^{\infty} dz^- \int d^2 z_{\perp} G_0(x_{\perp} - z_{\perp}) \rho_a(z^-, z_{\perp}) t^a \right\} .$$

(5.20)

The correlator $\langle V(x_{\perp}) V^\dagger(y_{\perp}) \rangle$ for a Gaussian (MV) action has already been calculated before, see for example ref. [39]. The result is:

$$\langle V(x_{\perp}) V^\dagger(y_{\perp}) \rangle_G = \exp \left\{ -\frac{g^4}{2} (t^a t_a) \left[ \int_{-\infty}^{\infty} dz^- \mu^2(z^-) \right] \int d^2 z_{\perp} \left[ G_0(x_{\perp} - z_{\perp}) - G_0(y_{\perp} - z_{\perp}) \right]^2 \right\}$$

(5.21)

Note that this is diagonal in color (proportional to $\mathbb{1}_{3 \times 3}$). To calculate the expectation value of the dipole operator with the new action, we first expand the Wilson lines order by order.
in the gauge coupling $g^2$,

$$V(x_\perp) = 1 - ig^2 \int d^2 z_{\perp 1} G_0(x_\perp - z_{\perp 1}) \int_0^\infty dz_1^\perp \rho^a(z_1)t^a$$

$$-g^4 \int d^2 z_{\perp 1} d^2 z_{\perp 2} G_0(x_\perp - z_{\perp 1}) G_0(x_\perp - z_{\perp 2}) \int_0^\infty dz_1^\perp \int_0^\infty dz_2^\perp \rho^a(z_1) \rho^b(z_2) t^at^b$$

$$+ \ldots$$

\hspace{1em} (5.22)

$$V^\dagger(y_\perp) = 1 + ig^2 \int d^2 u_{\perp 1} G_0(y_\perp - u_{\perp 1}) \int_0^\infty du_1^\perp \rho^a(u_1)t^a$$

$$-g^4 \int d^2 u_{\perp 1} d^2 u_{\perp 2} G_0(y_\perp - u_{\perp 1}) G_0(y_\perp - u_{\perp 2}) \int_0^\infty du_1^\perp \int_0^\infty du_2^\perp \rho^a(u_1) \rho^b(u_2) t^at^b$$

$$+ \ldots$$

\hspace{1em} (5.23)

For brevity we only write the terms up to $O(g^4)$ but below we shall actually require terms up to $O(g^8)$.

To zeroth order in $g$, the expectation value of the correlator is just 1:

$$O(g^0) = 1 .$$

The order $g^2$ contribution is zero because $\langle \rho^a(z) \rangle = 0$:

$$O(g^2) = 0 .$$
5.3.2 Order $g^4$

The first non-trivial contribution arises at $O(g^4)$ and is given by the sum of expectation values of three terms (two from each Wilson line and one mixed term):

$$-g^4 \int d^2 z_{\perp 1} d^2 z_{\perp 2} G_0(x_{\perp} - z_{\perp 1}) G_0(x_{\perp} - z_{\perp 2}) \int_{-\infty}^{\infty} dz_{\perp 1} \int_{\frac{z_{\perp 1}}{z_{\perp 2}}}^{\infty} d z_{\perp 2} (\rho^a(z_1) \rho^b(z_2)) t^a t^b , \quad (5.24)$$

$$-g^4 \int d^2 u_{\perp 1} d^2 u_{\perp 2} G_0(y_{\perp} - u_{\perp 1}) G_0(y_{\perp} - u_{\perp 2}) \int_{-\infty}^{\infty} du_{\perp 1} \int_{\frac{u_{\perp 1}}{u_{\perp 2}}}^{\infty} d u_{\perp 2} (\rho^a(u_1) \rho^b(u_2)) t^a t^b , \quad (5.25)$$

$$g^4 \int d^2 z_{\perp 1} \int d^2 u_{\perp 1} G_0(x_{\perp} - z_{\perp 1}) G_0(y_{\perp} - u_{\perp 1}) \int_{-\infty}^{\infty} dz_{\perp 1} \int_{-\infty}^{\infty} du_{\perp 1} (\rho^a(z_1) \rho^b(u_1)) t^a t^b . \quad (5.26)$$

Using (5.12) and (5.21) the first term becomes (we will divide by the normalization factor later):

$$- \frac{g^4}{2} t^a t_a \int_{-\infty}^{\infty} dz^- \mu^2(z^-) \int d^2 z_{\perp} G_0^2(x_{\perp} - z_{\perp})$$

$$+ \frac{g^4}{k_4} \int d^2 z_{\perp 1} d^2 z_{\perp 2} d^2 v_{\perp} G_0(x_{\perp} - z_{\perp 1}) G_0(x_{\perp} - z_{\perp 2})$$

$$\times \int_{-\infty}^{\infty} dz_{\perp 1} \int_{\frac{z_{\perp 1}}{z_{\perp 2}}}^{\infty} d z_{\perp 2} \int_{-\infty}^{\infty} d u_{\perp 1} d v_{\perp} (\rho^a(z_1) \rho^b(u_1) \rho^c(v_1) \rho^d(v_2)) t^a t^b . \quad (5.27)$$

All possible contractions for the second term in the above expression are

$$\langle \rho^a_{z_1} \rho^b_{z_2} \rho^c_{v_1} \rho^d_{v_2} \rangle = \langle \rho^a_{z_1} \rho^b_{z_2} \rangle \langle \rho^c_{v_1} \rangle \langle \rho^d_{v_2} \rangle + 2 \langle \rho^c_{v_1} \rangle \langle \rho^d_{v_2} \rangle + 2 \langle \rho^c_{v_1} \rangle \langle \rho^d_{v_2} \rangle + 4 \langle \rho^a_{z_1} \rangle \langle \rho^b_{z_2} \rangle \langle \rho^d_{v_2} \rangle \langle \rho^d_{v_2} \rangle , \quad (5.28)$$

and are shown diagrammatically in fig. 5.2. Using (5.14) to perform the contractions and
dividing also by the normalization factor (5.13) leads us to\(^3\)

\[
1 - \frac{1}{\kappa_4} \frac{N_s}{\Delta v_\perp} \left\{ \left( N_c^2 - 1 \right)^2 \left[ \int_{-\infty}^{\infty} dv^- \mu^2(v^-) \right]^2 + 2 \left( N_c^2 - 1 \right) \int_{-\infty}^{\infty} dv^- \mu^4(v^-) \right\} \\
\times \left\{ -\frac{g^4}{2} t^at_a \int_{-\infty}^{\infty} dz^- \mu^2(z^-) \int d^2z_\perp G_0^2(x_\perp - z_\perp) \times \right\}
\frac{1}{\kappa_4} \frac{N_s}{\Delta v_\perp} \left[ \left( N_c^2 - 1 \right)^2 \left[ \int_{-\infty}^{\infty} dv^- \mu^2(v^-) \right]^2 + 2 \left( N_c^2 - 1 \right) \int_{-\infty}^{\infty} dv^- \mu^4(v^-) \right] + \\
\frac{2 g^4}{\kappa_4 \Delta v_\perp} t^at_a \int d^2z_\perp G_0^2(x_\perp - z_\perp) \left[ \left( N_c^2 - 1 \right) \int_{-\infty}^{\infty} dz^- \mu^2(z^-) \int_{-\infty}^{\infty} dv^- \mu^4(v^-) \right. \\
+ \left. 2 \int_{-\infty}^{\infty} dz^- \mu^6(z^-) \right\}.
\]

The third line in the above expression cancels\(^4\) the normalization factor once the latter is expanded to leading order in \(1/\kappa_4\) so that the previous expression simplifies to

\[
- \frac{g^4}{2} t^at_a \int d^2z_\perp G_0^2(x_\perp - z_\perp) \\
\times \left\{ \int_{-\infty}^{\infty} dz^- \mu^2(z^-) \\
- \frac{4}{\kappa_4 \Delta v_\perp} \left[ \left( N_c^2 - 1 \right) \int_{-\infty}^{\infty} dz^- \mu^2(z^-) \int_{-\infty}^{\infty} dv^- \mu^4(v^-) + 2 \int_{-\infty}^{\infty} dz^- \mu^6(z^-) \right] \right\}
\]

The correction is absorbed into a renormalization of \(\int_{-\infty}^{\infty} dz^- \mu^2(z^-)\) in order that the two-

\(^3\)One has to be careful with performing the integration over \(z_\perp^-\): \(\int_{z_1^-}^{\infty} dz_\perp^- \delta(z_1^- - z_\perp^-) = 1/2\). If \(x^-\) is discretized, \(z_\perp^-\) should be placed ahead of \(z_1^-\) by at least half a lattice spacing \(\Delta x^-\). Similarly, when expanding a single Wilson lines to order \(g^6\): \(z_1^- \geq z_2^- + \Delta x^-/2 \geq z_1^- + \Delta x^-\).

\(^4\)This is the standard cancellation of disconnected diagrams.
point function $\langle \rho\rho \rangle$ remains unaffected by the quartic term in the action:

$$
\int_{-\infty}^{\infty} dz^- \tilde{\mu}^2(z^-) = \\
\int_{-\infty}^{\infty} dz^- \mu^2(z^-) - \frac{4}{\kappa_4 \Delta v_\perp} \left[ (N_c^2 - 1) \int_{-\infty}^{\infty} dz^- \mu^2(z^-) \int_{-\infty}^{\infty} dv^- \mu^4(v^-) + 2 \int_{-\infty}^{\infty} dz^- \mu^6(z^-) \right].
$$

Finally, the expectation value of $V(x_\perp)$ to order $g^4$ can simply be written as

$$
-\frac{g^4}{2} t^a t_a \int_{-\infty}^{\infty} dz^- \tilde{\mu}^2(z^-) \int d^2 z_\perp G_0^2(x_\perp - z_\perp). \tag{5.31}
$$

Similarly, $\langle V^\dagger(y_\perp) \rangle$:

$$
-\frac{g^4}{2} t^a t_a \int_{-\infty}^{\infty} dz^- \tilde{\mu}^2(z^-) \int d^2 u_\perp G_0^2(y_\perp - u_\perp). \tag{5.32}
$$

The mixed term will be the same as the previous terms except with a positive sign and without the factor of 1/2 which originated from the path ordering ($z^-_1$ and $u^-_1$ are not ordered relative to each other):

$$
g^4 t^a t_a \int_{-\infty}^{\infty} dz^- \tilde{\mu}^2(z^-) \int d^2 z_\perp G_0(x_\perp - z_\perp) G_0(y_\perp - z_\perp). \tag{5.33}
$$

Summing (5.31), (5.32) and (5.33), we obtain the complete result at order $g^4$:

$$
\mathcal{O}(g^4) = -\frac{g^4}{2} t^a t_a \int_{-\infty}^{\infty} dz^- \tilde{\mu}^2(z^-) \int d^2 z_\perp [G_0(x_\perp - z_\perp) - G_0(y_\perp - z_\perp)]^2. \tag{5.34}
$$

This is identical to the result obtained in the Gaussian theory once the two-point function $\langle \rho\rho \rangle \sim \mu^2$ has been matched. This was to be expected, of course, since only two-point functions of $\rho$ arise at $\mathcal{O}(g^4)$. Note, also, that (5.34) vanishes as $y_\perp \to x_\perp$. 

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5.3.3 Order $g^8$

Next, we consider order $g^8$. There are all in all five terms: two of order $g^8$ from the expansion of a single Wilson line, two mixed terms (order $g^2$ from the first line and $g^6$ from the second line and vice versa), and one term from multiplying $g^4$ terms from both Wilson lines.

$g^8$ from $V(x_\perp)$

First, we calculate $\langle V(x_\perp) \rangle$ at order $g^8$. In the Gaussian theory

$$
\frac{g^8}{8} (t^a t_a)^2 \left[ \int_{-\infty}^{\infty} dz^- \mu^2 (z^-) \right]^2 \left[ \int d^2 z_\perp G_0^2 (x_\perp - z_\perp) \right]^2.
$$

Again, using (5.12), the correction is

$$
-\frac{g^8}{\kappa_4} \int d^2 z_{11} d^2 z_{12} d^2 z_{13} d^2 z_{14} \int d^2 v_\perp G_0 (x_\perp - z_{11}) G_0 (x_\perp - z_{12})
\times G_0 (x_\perp - z_{13}) G_0 (x_\perp - z_{14})
\times \int_{-\infty}^{\infty} dz_1^- \int_{z_1^-}^{\infty} dz_2^- \int_{z_2^-}^{\infty} dz_3^- \int_{z_3^-}^{\infty} dz_4^- \int_{-\infty}^{\infty} dv_1^- \int_{-\infty}^{\infty} dv_2^-
\times \langle \rho_{z_1}^b \rho_{z_2}^c \rho_{z_3}^d \rho_{e_1}^e \rho_{e_2}^f \rho_{f_1}^l \rangle t^a t^b t^c t^d.
$$

(5.35)

All possible contractions for $\langle \rho_{z_1}^b \rho_{z_2}^c \rho_{z_3}^d \rho_{e_1}^e \rho_{e_2}^f \rho_{f_1}^l \rangle$ are shown diagrammatically in fig. 5.3.

The disconnected diagrams 5.3a give the following contribution:
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\[-\frac{g^8}{8} (t^a t_a)^2 \left[ \int_{-\infty}^{\infty} dz^- \mu^2(z^-) \right]^2 \left[ \int d^2z_\perp G_0^2(x_\perp - z_\perp) \right]^2 \]

\times \frac{1}{\kappa_4} \frac{N_s}{\Delta v_\perp} \left[ (N_c^2 - 1)^2 \left[ \int_{-\infty}^{\infty} dv^- \mu^2(v^-) \right]^2 + 2 (N_c^2 - 1) \int_{-\infty}^{\infty} dv^- \mu^4(v^-) \right], \quad (5.36)

which will cancel the normalization factor.

The connected diagrams from fig. 5.3b again renormalize \( \mu^2(z^-) \) to \( \tilde{\mu}^2(z^-) \):

\[ \left[ \int_{-\infty}^{\infty} dz^- \tilde{\mu}^2(z^-) \right]^2 = \left[ \int_{-\infty}^{\infty} dz^- \mu^2(z^-) \right]^2 - \frac{8}{\kappa_4 \Delta v_\perp} \left[ (N_c^2 - 1) \left[ \int_{-\infty}^{\infty} dz^- \mu^2(z^-) \right]^2 \int_{-\infty}^{\infty} dv^- \mu^4(v^-) + 2 \int_{-\infty}^{\infty} dz^- \mu^2(z^-) \int_{-\infty}^{\infty} dv^- \mu^0(v^-) \right]. \quad (5.37)

Finally, the diagrams 5.3c give the correction

\[ -\frac{g^8}{8} (t^a t_a)^2 \left[ \int_{-\infty}^{\infty} dz^- \mu^4(z^-) \right]^2 \left[ \int d^2z_\perp G_0^4(x_\perp - z_\perp) \right]. \quad (5.38) \]

That leads us to the complete \( \mathcal{O}(g^8) \) contribution to \( \langle V(x_\perp) \rangle \):

\[ \frac{g^8}{8} (t^a t_a)^2 \left[ \int_{-\infty}^{\infty} dz^- \tilde{\mu}^2(z^-) \right]^2 \left[ \int d^2z_\perp G_0^2(x_\perp - z_\perp) \right]^2 \]

\[-\frac{g^8}{8} (t^a t_a)^2 \left[ \int_{-\infty}^{\infty} dz^- \mu^4(z^-) \right]^2 \left[ \int d^2z_\perp G_0^4(x_\perp - z_\perp) \right]. \quad (5.39) \]

The expectation value of \( V^\dagger(y_\perp) \) is obtained from the above by substituting \( x_\perp \to y_\perp \).

\( g^6 \) from \( V(x_\perp) \times g^2 \) from \( V^\dagger(y_\perp) \)

Next is the mixed term obtained when multiplying the \( g^6 \) term from the \( x \) Wilson line with the \( g^2 \) term from the \( y \) Wilson line. In the Gaussian theory,

\[-\frac{g^8}{2} (t^a t_a)^2 \left[ \int_{-\infty}^{\infty} dz^- \mu^2(z^-) \right]^2 \left[ \int d^2z_\perp G_0^2(x_\perp - z_\perp) \right] \left[ \int d^2u_\perp G_0(x_\perp - u_\perp) G_0(y_\perp - u_\perp) \right]. \quad (5.40) \]
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Figure 5.4: $V(x_\perp)V^\dagger(y_\perp)$ at order $g^8/\kappa_4$ ($g^6$ from $V(x_\perp) \times g^2$ from $V^\dagger(y_\perp)$).

The correction is:

$$
\begin{align*}
&+ \frac{g^8}{\kappa_4} \int d^2 z_{\perp 1} d^2 z_{\perp 2} d^2 z_{\perp 3} G_0(x_\perp - z_{\perp 1}) G_0(x_\perp - z_{\perp 2}) G_0(x_\perp - z_{\perp 3}) \int d^2 u_{\perp 1} G_0(y_\perp - u_{\perp 1}) \\
&\times \int d^2 v_{\perp 1} \int_{-\infty}^{\infty} d\bar{z}_{\perp 1} \int_{-\infty}^{\infty} d\bar{z}_{\perp 2} \int_{-\infty}^{\infty} d\bar{z}_{\perp 3} \int_{-\infty}^{\infty} d\bar{u}_{\perp 1} \int_{-\infty}^{\infty} d\bar{v}_{\perp 1} \int_{-\infty}^{\infty} d\bar{v}_{\perp 2} \\
&\times \langle \rho_{z_{1}}^{a} \rho_{z_{2}}^{b} \rho_{z_{3}}^{c} \rho_{u_{1}}^{d} \rho_{v_{1}}^{e} \rho_{v_{2}}^{f} \rangle \delta^{a} \delta^{b} \delta^{c} \delta^{d}.
\end{align*}
\tag{5.41}
$$

The possible contractions are given in fig. 5.4. As before, the disconnected diagrams 5.4a cancel against the normalization, while the diagrams in 5.4b renormalize $\mu^2$ to $\tilde{\mu}^2$; finally the diagrams 5.4c will give the correction. Summing all these diagrams plus the Gaussian
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part, we get:

\[-\frac{g^8}{2}(t^at_a)^2 \left[ \int_{-\infty}^{\infty} dz_- \bar{\mu}^2(z_-) \right]^2 \int d^2z_1 G_0^2(x_\perp - z_\perp) \int d^2u_\perp G_0(x_\perp - u_\perp) G_0(y_\perp - u_\perp) + 4 \frac{g^8}{\kappa_4} (t^at_a)^2 \left[ \int_{-\infty}^{\infty} dz_- \mu^4(z_-) \right]^2 \int d^2z_1 G_0^3(x_\perp - z_\perp) G_0(y_\perp - z_\perp) \right] \] (5.42)

Once again, a similar contribution (with \(x_\perp \leftrightarrow y_\perp\)) arises from \(V(x_\perp)\) at \(O(g^2)\) times \(V^\dagger(y_\perp)\) at order \(g^6\).

\(g^4\) from \(V(x_\perp) \times g^4\) from \(V^\dagger(y_\perp)\)

The last term to consider is the one obtained when multiplying \(O(g^4)\) from the \(x\) Wilson line with \(O(g^4)\) from the \(y\) Wilson line. The Gaussian contribution is:

\[ \frac{g^8}{4} (t^at_a)^2 \left[ \int_{-\infty}^{\infty} dz_- \mu^2(z_-) \right]^2 \int d^2z_1 G_0^2(x_\perp - z_\perp) \int d^2u_\perp G_0^2(y_\perp - u_\perp) + \frac{g^8}{2} (t^at_a)^2 \left[ \int_{-\infty}^{\infty} dz_- \mu^2(z_-) \right]^2 \left[ \int d^2z_1 G_0(x_\perp - z_\perp) G_0(y_\perp - u_\perp) \right]^2 . \] (5.43)

For the quartic action we have to add

\[-\frac{g^8}{\kappa_4} \int d^2z_\perp d^2z_\perp G_0(x_\perp - z_\perp) G_0(x_\perp - z_\perp) \int d^2u_\perp d^2u_\perp G_0(y_\perp - u_\perp) G_0(y_\perp - u_\perp) \times \int d^2v_\perp \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 \int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} du_2 \int_{-\infty}^{\infty} dv_1 \int_{-\infty}^{\infty} dv_2 \times \langle \rho^{a}_{z_1} \rho^{b}_{z_2} \rho^{c}_{u_1} \rho^{d}_{u_2} \rho^{e}_{v_1} \rho^{f}_{v_2} \rho^{l}_{u_2} \rangle t^{a} t^{b} t^{c} t^{d} . \] (5.44)

Following the same procedure as before, the diagrams from fig. 5.5 give:
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Figure 5.5: Expectation value of $V(x_\perp)$ at order $g^4$ times $V(y_\perp)$ at order $g^4$. 
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\[ \frac{g^8}{4} (t^a t_a)^2 \left[ \int_{-\infty}^{\infty} dz^- \tilde{\mu}^2(z^-) \right]^2 \int d^2z_\perp G_0^2(\mathbf{x}_\perp - \mathbf{z}_\perp) \int d^2u_\perp G_0^2(\mathbf{y}_\perp - u_\perp) \]

\[ + \frac{g^8}{2} (t^a t_a)^2 \left[ \int_{-\infty}^{\infty} dz^- \tilde{\mu}^2(z^-) \right]^2 \left[ \int d^2z_\perp G_0(\mathbf{x}_\perp - \mathbf{z}_\perp) G_0(\mathbf{y}_\perp - u_\perp) \right]^2 \]

\[ - 6 \frac{g^8}{\kappa_4} (t^a t_a)^2 \left[ \int_{-\infty}^{\infty} dz^- \mu^4(z^-) \right]^2 \int d^2z_\perp [G_0(\mathbf{x}_\perp - \mathbf{z}_\perp) G_0^2(\mathbf{y}_\perp - \mathbf{z}_\perp) - G_0(\mathbf{y}_\perp - \mathbf{z}_\perp)]^2 \].

(5.45)

**Complete order** \( g^8 \)

Combining eqs. (5.39+\( x_\perp \leftrightarrow y_\perp \)), (5.42+\( x_\perp \leftrightarrow y_\perp \)), and (5.45) we get the complete contribution to the expectation value of the dipole at order \( g^8 \):

\[ \frac{1}{2} \left( \frac{g^4 (t^a t_a)}{2} \right)^2 \left[ \int_{-\infty}^{\infty} dz^- \tilde{\mu}^2(z^-) \right]^2 \left[ \int d^2z_\perp [G_0(\mathbf{x}_\perp - \mathbf{z}_\perp) - G_0(\mathbf{y}_\perp - \mathbf{z}_\perp)]^2 \right]^2 \]

\[ - \frac{g^8}{\kappa_4} (t^a t_a)^2 \left[ \int_{-\infty}^{\infty} dz^- \mu^4(z^-) \right]^2 \int d^2z_\perp [G_0(\mathbf{x}_\perp - \mathbf{z}_\perp) - G_0(\mathbf{y}_\perp - \mathbf{z}_\perp)]^4 \].

Note that this again vanishes as \( y_\perp \to x_\perp \).

**5.3.4 Complete expectation value of the dipole operator**

Adding together the terms of order 1, \( g^4 \) and \( g^8 \), the expectation value of the dipole operator becomes

\[ D(x_\perp - y_\perp) \equiv \left\langle \frac{1}{N_c} \text{tr} \, V(\mathbf{x}_\perp) V^\dagger(\mathbf{y}_\perp) \right\rangle = \]

\[ 1 - \frac{g^4}{2} C_F \int_{-\infty}^{\infty} dz^- \tilde{\mu}^2(z^-) \int d^2z_\perp [G_0(\mathbf{x}_\perp - \mathbf{z}_\perp) - G_0(\mathbf{y}_\perp - \mathbf{z}_\perp)]^2 \]

\[ + \frac{1}{2} \left( \frac{g^4 C_F}{2} \right)^2 \left[ \int_{-\infty}^{\infty} dz^- \tilde{\mu}^2(z^-) \right]^2 \left[ \int d^2z_\perp [G_0(\mathbf{x}_\perp - \mathbf{z}_\perp) - G_0(\mathbf{y}_\perp - \mathbf{z}_\perp)]^2 \right]^2 \]

\[ - \frac{g^8}{\kappa_4} C_F^2 \left[ \int_{-\infty}^{\infty} dz^- \mu^4(z^-) \right]^2 \int d^2z_\perp [G_0(\mathbf{x}_\perp - \mathbf{z}_\perp) - G_0(\mathbf{y}_\perp - \mathbf{z}_\perp)]^4 + \cdots \] (5.46)
where

\[ C_F = \frac{N_c^2 - 1}{2N_c}. \]

To write this in a more compact form we introduce the saturation scale

\[ Q_s^2 \equiv \frac{g^4}{2\pi} C_F \int_{-\infty}^{\infty} dz^{-} \mu^2(z^{-}), \quad (5.47) \]

so that

\[
D(r) = 1 - \pi Q_s^2 \int d^2 \mathbf{z}_\perp \left[ G_0(\mathbf{x}_\perp - \mathbf{z}_\perp) - G_0(\mathbf{y}_\perp - \mathbf{z}_\perp) \right]^2 \\
+ \frac{\pi^2}{2} Q_s^4 \left[ \int d^2 \mathbf{z}_\perp \left[ G_0(\mathbf{x}_\perp - \mathbf{z}_\perp) - G_0(\mathbf{y}_\perp - \mathbf{z}_\perp) \right]^2 \right]^2 \\
- \frac{g^8}{\kappa_4} C_F \left[ \int_{-\infty}^{\infty} dz^{-} \mu^4(z^{-})^2 \right]^2 \int d^2 \mathbf{z}_\perp \left[ G_0(\mathbf{x}_\perp - \mathbf{z}_\perp) - G_0(\mathbf{y}_\perp - \mathbf{z}_\perp) \right]^4. \quad (5.48)
\]

Here, \( r = |\mathbf{x}_\perp - \mathbf{y}_\perp| \).

### 5.3.5 Explicit evaluation of \( D(r) \) to leading \( \ln 1/r \) accuracy

It is useful to obtain an explicit expression for \( D(r) \) in the limit \( \ln 1/r \Lambda \gg 1 \), where \( \Lambda \) is an infrared cutoff on the order of the inverse nucleon radius.

The first non-trivial term in eq. (5.48) gives

\[
\pi Q_s^2 \int d^2 \mathbf{z}_\perp \left[ G_0(\mathbf{x}_\perp - \mathbf{z}_\perp) - G_0(\mathbf{y}_\perp - \mathbf{z}_\perp) \right]^2 \\
= Q_s^2 \int_0^\infty \frac{dk}{k^3} \left[ 1 - J_0(kr) \right] \simeq \frac{1}{4} r^2 Q_s^2 \ln \frac{1}{r \Lambda}, \quad (5.49)
\]

in the leading \( \ln 1/r \Lambda \gg 1 \) approximation.
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Next, we need to compute the integral

$$
\int d^2 z_\perp \left[ G_0(x_\perp - z_\perp) - G_0(y_\perp - z_\perp) \right]^4.
$$

From eq. (5.19) for the propagator,

$$
\int d^2 z_\perp G_4^0(x_\perp - z_\perp) = \frac{1}{(2\pi)^8} \int d^2 z_\perp \int d^2 k_1 d^2 k_2 d^2 k_3 d^2 k_4 \frac{e^{i(k_1+k_2+k_3+k_4)\cdot(x_\perp - z_\perp)}}{k_1^2 k_2^2 k_3^2 k_4^2}.
$$

We regularize the integral in the square brackets by introducing a cutoff $\Lambda$:

$$
\int d^2 k \frac{1}{\left( \frac{k}{2} + k \right)^2 \left( \frac{k}{2} - k \right)^2} = \frac{2\pi}{K^2} \ln \frac{K^2}{\Lambda^2}.
$$

Then,

$$
\int d^2 z_\perp G_4^0(x_\perp - z_\perp) = \frac{1}{(2\pi)^4} \int \frac{d^2 K}{K^4} \ln^2 \frac{K^2}{\Lambda^2}.
$$

Following the same procedure for $\int d^2 z_\perp G_0^2(x_\perp - z_\perp) G_0^2(y_\perp - z_\perp)$ we arrive at:

$$
\int d^2 z_\perp G_0^2(x_\perp - z_\perp) G_0^2(y_\perp - z_\perp) = \frac{1}{(2\pi)^4} \int \frac{d^2 K}{K^4} e^{iK\cdot(x_\perp - y_\perp)} \ln^2 \frac{K^2}{\Lambda^2}
$$

$$
= \frac{1}{(2\pi)^3} \int_{\Lambda}^{\infty} \frac{d^2 K}{K^3} J_0(Kr) \ln^2 \frac{K^2}{\Lambda^2} = \frac{1}{(2\pi)^3} \left( \frac{1}{\Lambda^2} + \frac{1}{3} r^2 \ln^3(r\Lambda) \right) + O \left[ r^2 \ln^2(r\Lambda) \right].
$$
Similarly,

\[
\int d^2z_\perp G_0^3(x_\perp - z_\perp)G_0(y_\perp - z_\perp) = \frac{1}{(2\pi)^5} \int \frac{d^2K}{K^2} e^{i\frac{K}{2}(x_\perp - y_\perp)} \ln \frac{K^2}{\Lambda^2} \int d^2q \frac{e^{i\frac{q}{2}(x_\perp - y_\perp)}}{\left(\frac{K}{2} + q\right)^2 \left(\frac{K}{2} - q\right)^2}.
\]

Using:

\[
\int d^2q \frac{1 + iq \cdot r}{\left(\frac{K}{2} + q\right)^2 \left(\frac{K}{2} - q\right)^2} \approx \frac{2\pi}{K^2} \ln \frac{K^2}{\Lambda^2} + O(Kr),
\]

we get

\[
\int d^2z_\perp G_0^3(x_\perp - z_\perp)G_0(y_\perp - z_\perp) = \frac{1}{(2\pi)^3} \int_{\Lambda}^{\infty} d^2K \frac{K^3}{K^3} J_0\left(\frac{1}{2}Kr\right) \ln^2 \frac{K^2}{\Lambda^2} = \frac{1}{4} \frac{1}{(2\pi)^3} \left(\frac{4}{\Lambda^2} + \frac{1}{3} r^2 \ln^3 (r\Lambda)\right),
\]

so that, finally,

\[
D(r) = 1 - \frac{r^2Q_s^2}{4} \ln \frac{1}{r\Lambda} + \frac{1}{6\pi^3} \kappa_4 C_F^2 \left[\int_{-\infty}^{\infty} dz^- \mu_4^2(z^-)\right]^2 r^2 \ln^3 \frac{1}{r\Lambda}
\]

\[
= 1 - \frac{r^2Q_s^2}{4} \ln \frac{1}{r\Lambda} + \beta r^2Q_s^2 \ln^3 \frac{1}{r\Lambda}.
\]

\(\beta\) has been defined in eq. (6.19).

Performing a Fourier transform we obtain the transverse momentum dependence of the
dipole for $k_\perp \gg Q_s \gg \Lambda$:

$$D(k_\perp) \approx 2\pi \frac{Q_s^2}{k_\perp^4} - \frac{g_s^4 \alpha_s^2}{\pi^2 k_4^4 C_F} \left[ \int_{-\infty}^{\infty} dz^- \mu^4(z^-) \right]^2 \frac{1}{k_\perp^4} \ln^2 \frac{k_\perp^2}{\Lambda^2} \quad (5.61)$$

$$= 2\pi \frac{Q_s^2}{k_\perp^4} - 6\pi \beta \frac{Q_s^2}{k_\perp^4} \ln^2 \frac{k_\perp^2}{\Lambda^2}. \quad (5.62)$$

The first term was taken from appendix B of ref. [39]. The second term provides a correction to the classical \textit{bremsstrahlung} tail for finite valence parton density.

### 5.3.6 Gluon density

One can define [37] a Weizsäcker-Williams like gluon density of a nucleus, in the limit of small $x_\perp^2$, as

$$xG_A(x, x_\perp^2) = \frac{(N_c^2 - 1) \pi R^2}{\alpha_s \pi^2} \frac{1}{x_\perp^2} N(x_\perp^2) \quad (5.63)$$

For the quartic action

$$N(x_\perp^2) = \frac{Q_s^2 x_\perp^2}{4} \ln \frac{1}{x_\perp \Lambda} - \frac{\beta Q_s^2}{x_\perp^2} \ln^3 \frac{1}{x_\perp \Lambda}. \quad (5.64)$$

The first term, via eq. (5.63), gives $xG_A \sim A$ [37]. The second term gives a correction

$$- \frac{(N_c^2 - 1) \pi R^2}{\alpha_s \pi^2 N_c} \beta Q_s^2 \ln^3 \frac{1}{x_\perp \Lambda} \sim - A^{3/2}. \quad (5.65)$$

### 5.3.7 Form of the quartic term in the action

In this section we explain why the $\rho$’s in the quartic term of the action should sit at two different points in the longitudinal direction in order that $N(r)$ vanishes as $\sim r^2$ as required by color transparency.

First, let us note that the correction to the dipole scattering amplitude due to the quartic
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The analogue of diagram 5.3c for $\langle V \rangle$ at order $g^8/\kappa_4$ for this action is shown in fig. 5.6. However, this diagram vanishes due to the longitudinal path ordering in the Wilson line.

The same reasoning applies to the analogue of 5.4c shown in fig. 5.7. In this case the three points $z_1^-$, $z_2^-$ and $z_3^-$, can not be connected simultaneously to one $v^-$ point. The delta functions coming from this kind of contractions, $\delta(z_1^- - v^-)\delta(z_2^- - v^-)\delta(z_3^- - v^-)$, imply overlap of $z_1^-$ and $z_3^-$ which is not there in a path ordered integral.

That leaves us with only one type of diagram proportional to $1/\kappa_4$, shown in fig. 5.8. This diagram is not zero since there is no relative ordering between the points $z^-$ and $u^-$ on the two lines. This diagram has a constant $r$-independent contribution which does not cancel because of the missing diagrams 5.6 and 5.7. This is (only terms at order $1/\kappa_4$ are

$$S[\rho] = \int d^2v_\perp \int_{-\infty}^{\infty} dv^- \left\{ \frac{\rho^a(v^-, v_\perp)\rho^a(v^-, v_\perp)}{2\mu^2(v^-)} \right. + \left. \frac{\rho^a(v^-, v_\perp)\rho^a(v^-, v_\perp)\rho^b(v^-, v_\perp)}{\kappa_4} \right\}.$$ 

(5.66)
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Figure 5.8: Order $g^4$ from $V(x_\perp)$ times order $g^4$ from $V^\dagger(y_\perp)$ at order $1/\kappa_4$.

$$D(r) \sim 1 - \frac{3g^8}{\kappa_4} \left( C_F^2 - \frac{2}{N_c} \right) \int_{-\infty}^{\infty} dz^- \mu^8(z^-) \int d^2z_\perp G_0^2(x_\perp - z_\perp) G_0^2(y_\perp - z_\perp)$$
$$= 1 - \frac{3g^8}{(2\pi)^3\kappa_4} \left( C_F^2 - \frac{2}{N_c} \right) \int_{-\infty}^{\infty} dz^- \mu^8(z^-) \left( \frac{1}{\Lambda^2} - \frac{1}{3} r^2 \ln^3 \left( \frac{1}{r\Lambda} \right) \right). \quad (5.67)$$

Hence we see that for the action (5.66) that $N(r) = 1 - D(r)$ approaches a constant as $r \to 0$, in violation of color transparency.

For numerical (lattice gauge) computations of expectation values (5.12) it may be easier to integrate $\rho(x^-)$ and to drop the longitudinal path ordering,

$$\bar{\rho}(x_\perp) \equiv \int_{-\infty}^{\infty} dx^- \rho(x^-, x_\perp) \quad (5.68)$$
$$V(x_\perp) = e^{\frac{ig}{2\kappa_4} \frac{1}{x_\perp} \bar{\rho}(x_\perp)}. \quad (5.69)$$

For a detailed discussion, we refer to ref. [41]. One would then consider the two-dimensional action

$$S[\bar{\rho}] = \int d^2x_\perp \left\{ \frac{\bar{\rho}^a(x_\perp)\bar{\rho}^a(x_\perp)}{2\mu^2} + \frac{\bar{\rho}^a(x_\perp)\bar{\rho}^a(x_\perp)\bar{\rho}^b(x_\perp)\bar{\rho}^b(x_\perp)}{\kappa_4} \right\}. \quad (5.70)$$

The diagrams from figs. 5.6 and 5.7 then do exist and cancel the $r$-independent contribution from fig. 5.8 so that again $N(r) \sim r^2$ at $r \to 0$. 

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Chapter 6

KNO scaling from a nearly Gaussian action for small-x gluons

Data from proton-proton experiments at RHIC and LHC shows that particle multiplicities follow a Negative Binomial Distribution (NBD). This has been theoretically reproduced within the MV model. In this chapter we obtain the quartic action modification to the fluctuation parameter $k$ which gives the variance of the distribution. From the form of the modification we conclude that NBD fits to proton-proton data might illustrate that the action progresses towards Gaussian when the energy is increased. In addition, we were able to connect the Koba-Nielsen-Olesen (KNO) law observed in transverse momentum integrated multiplicity distributions to properties of soft gluons around the saturation scale.

6.1 Koba-Nielsen-Olesen scaling

The color fields of hadrons boosted to the light cone are thought to grow very strong, parametrically of order $A^+ \sim 1/g$ where $g$ is the coupling [38]. The fields of nuclei are enhanced further by the high density of valence charges per unit transverse area, which is
proportional to the thickness $A^{1/3}$ of a nucleus [47].

In collisions of such strong color fields a large number of soft gluons is released. Due to the genuinely non-perturbative dynamics of the strong color fields a semi-hard “saturation scale” $Q_s$ emerges; it corresponds to the transverse momentum where the phase space density of produced gluons is of order $1/\alpha_s$. The mean multiplicity per unit rapidity in high-energy collisions is then $\bar{n} \equiv \langle dN/dy \rangle \sim 1/\alpha_s$. Below we argue that a semi-classical effective theory of valence color charge fluctuations predicts that the variance of the multiplicity distribution is of order $k^{-1} \sim \mathcal{O}(\alpha_s^0)$ so that the perturbative expansion of $\bar{n}/k$ begins at order $1/\alpha_s \gg 1$. We show that in the strong field limit then a Gaussian effective theory leads to Koba-Nielsen-Olesen (KNO) scaling [49]. This relates the emergence of KNO scaling in $p_\perp$-integrated multiplicity distributions from high-energy collisions to properties of soft gluons around the saturation scale.

![Figure 6.1: KNO scaling of charged particle multiplicity distributions in non-single diffractive $pp$/$p\bar{p}$ collisions at various energies as measured by the UA5 [50], ALICE [51] and CMS [52] collaborations, respectively. Note that we restrict to the bulk of the distributions up to 3.5 times the mean multiplicity.](image)

The KNO scaling conjecture refers to the fact that the particle multiplicity distribution in high-energy hadronic collisions is universal (i.e., energy independent) if expressed in terms of the fractional multiplicity $z \equiv n/\bar{n}$. This is satisfied to a good approximation in the central
(pseudo-) rapidity region at center of mass energies of 900 GeV and above [51, 52] as shown in fig. 6.1. On the other hand, UA5 data [50] taken at $\sqrt{s} = 200$ GeV appears to show a slightly distorted multiplicity distribution. This is in line with the observation that at lower energies higher-order factorial moments $G_q$ of the distribution are energy dependent and significantly different from the reduced moments $C_q$ [53]:

$$G_q \equiv \frac{\langle n(n-1)\cdots(n-q+1)\rangle}{\bar{n}^q}, \quad C_q \equiv \frac{\langle n^q \rangle}{\bar{n}^q}. \quad (6.1)$$

In fact, since the difference of $G_q$ and $C_q$ is subleading in the density of valence charges one may interpret this finding to indicate that the high density approximation is less accurate for $\sqrt{s} = 200$ GeV $pp$ collisions. Approximate KNO scaling has been predicted to persist also for min-bias $p+Pb$ collisions (at LHC energies) in spite of additional Glauber fluctuations of the number of participants and binary collisions [54]. A more detailed discussion of multiplicity distributions at TeV energies is given in refs. [55–57], and references therein.

Transverse momentum integrated multiplicities in inelastic hadronic collisions are not governed by an external hard scale, unlike say multiplicity distributions in $e^+e^-$ annihilation or in jets [58–61]. Hence, the explanation for the experimental observation should relate to properties of the distribution of produced gluons around the saturation scale $Q_s$.

### 6.2 KNO scaling from a Gaussian action in the classical limit

We shall first discuss the multiplicity distribution of small-$x$ gluons obtained from a Gaussian effective action for the color charge fluctuations of the valence charge densities $\rho$ [47],
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\[
Z = \int \mathcal{D}\rho \, e^{-S_{MV}[\rho]}, \quad (6.2)
\]

\[
S_{MV}[\rho] = \int d^2 x_\perp \int_{-\infty}^{\infty} dx^- \, \frac{\rho^\alpha(x^-, x_\perp) \rho^\alpha(x^-, x_\perp)}{2\mu^2(x^-)}. \quad (6.3)
\]

In the strong field limit a semi-classical approximation is appropriate and the soft gluon field (in covariant gauge) can be obtained in the Weizsäcker-Williams approximation as

\[
A^+(z^-, x_\perp) = -g \frac{1}{\nabla^2_{\perp}} \rho^\alpha(z^-, x_\perp) = g \int d^2 z_\perp \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{e^{i k_\perp \cdot (x_\perp - z_\perp)}}{\mathbf{k}^2} \rho^\alpha(z^-, z_\perp). \quad (6.4)
\]

Parametrically, the mean multiplicity obtained from the action (6.3) is then

\[
\bar{n} \sim \frac{N_c (N_c^2 - 1)}{\alpha_s} Q_s^2 S_\perp, \quad (6.5)
\]

where \(S_\perp\) denotes a transverse area and \(Q_s \sim g^2 \mu\). The prefactor in (6.5) can be determined numerically [62–66] but is not required for our present considerations.

One can similarly calculate the probability to produce \(q\) particles by considering fully connected diagrams with \(q\) valence sources \(\rho\) in the amplitude and \(q\) sources \(\rho^\ast\) in the conjugate amplitude (for both projectile and target, respectively). These can be expressed as [67]

\[
\left\langle \frac{d^q N}{dy_1 \cdots dy_q} \right\rangle_{\text{conn.}} = C_q \left\langle \frac{dN}{dy_1} \right\rangle \cdots \left\langle \frac{dN}{dy_q} \right\rangle, \quad (6.6)
\]

where the reduced moments

\[
C_q = \frac{(q - 1)!}{k_q^{-1}}. \quad (6.7)
\]

This expression is valid with logarithmic accuracy and was derived under the assumption

\footnote{The rapidities \(y_1 \cdots y_q\) of the \(q\) particles should be similar. Here we assume that all particles are in the same rapidity bin.}
that all transverse momentum integrals over $p_{T,1} \cdots p_{T,q}$ are effectively cut off in the infrared at a scale $\sim Q_s$ due to non-linear effects.

The fluctuation parameter $k$ in eq. (6.7) is of order

$$k \sim (N_c^2 - 1) Q_s^2 S_{\perp}.$$  \hfill (6.8)

Once again, the precise numerical prefactor (in the classical approximation) has been determined by a numerical computation to all orders in the valence charge density $\rho$ [68].

The multiplicity distribution is therefore a negative binomial distribution (NBD) [67],

$$P(n) = \frac{\Gamma(k + n)}{\Gamma(k) \Gamma(n + 1)} \frac{\bar{n}^n k^k}{(\bar{n} + k)^{n+k}}.$$  \hfill (6.9)

Indeed, multiplicity distributions observed in high-energy $pp$ collisions (in the central region) can be described quite well by a NBD, see for example refs. [54, 69]. The parameter $k^{-1}$ determines the variance of the distribution\footnote{More precisely, the width is given by $\bar{n} \sqrt{k^{-1} + \bar{n}^{-1}} \sim \bar{n} \sqrt{k}$; the latter approximation applies in the limit $\bar{n}/k \gg 1$, see below.} and can be obtained from the (inclusive) double-gluon multiplicity:

$$\left< \frac{d^2N}{dy_1 dy_2} \right>_{\text{conn}} = \frac{1}{k} \left< \frac{dN}{dy_1} \right> \left< \frac{dN}{dy_2} \right>.$$  \hfill (6.10)

From this expression it is straightforward to see that the perturbative expansion of $k^{-1}$ starts at $O(\alpha_s^0)$ since the connected diagrams on the lhs of eq. (6.10) involve the same number of sources and vertices as the disconnected diagrams on the rhs of that equation (also see appendix). This observation is important since in general the NBD (6.9) exhibits KNO scaling only when $\bar{n}/k \gg 1$, and if $k$ is not strongly energy dependent. A numerical analysis of the multiplicity distribution at 2360 GeV, for example, achieves a good fit to the data for $\bar{n}/k \simeq 6 - 7$ [54], which we confirm below. Such values for $\bar{n}/k$ have also been found
for peripheral collisions of heavy ions from ab initio solutions of the classical Yang-Mills equations [70]; furthermore those solutions predict that $\bar{n}/k < 1$ for central collisions of $A \sim 200$ nuclei.

### 6.3 Deviations from KNO scaling

To illustrate how deviations from KNO scaling arise it is instructive to consider a “deformed” theory with an additional contribution to the quadratic action. We shall add a quartic operator [71],

$$S_Q[\rho] = \int d^2 v_\perp \int_{-\infty}^{\infty} dv_1^- \left\{ \frac{\rho^a(v_1^-, v_\perp)\rho^a(v_1^-, v_\perp)}{2\tilde{\mu}^2(v_1^-)} + \int_{-\infty}^{\infty} dv_2^- \frac{\rho^a(v_1^-, v_\perp)\rho^a(v_2^-, v_\perp)\rho^b(v_2^-, v_\perp)\rho^b(v_2^-, v_\perp)}{\kappa_4} \right\}.$$  \hspace{1cm} (6.11)

We assume that the contribution from the quartic operator is a small perturbation since $\kappa_4 \sim g(gA^{1/3})^3$ while $\tilde{\mu}^2 \sim g(gA^{1/3})$. In the classical approximation the mean multiplicity is unaffected by the correction as it involves only two-point functions$^3$. On the other hand, $k^{-1}$ as defined in (6.10) now becomes

$$\frac{N_c^2 - 1}{2\pi} Q_s^2 S_{\perp} \frac{1}{k} = 1 - 3\beta (N_c^2 + 1) \left( \text{with } \beta \equiv \frac{C_F^2}{6\pi^3} \frac{g^8}{Q_s^2 \kappa_4} \left[ \int_{-\infty}^{\infty} dz^- \mu^4(z^-) \right]^2 \right). \hspace{1cm} (6.12)$$

Therefore, in the classical approximation

$$\frac{\bar{n}}{k} \sim \frac{N_c}{\alpha_s} \left( 1 - 3\beta (N_c^2 + 1) \right). \hspace{1cm} (6.13)$$

$^3$The two-point functions $\langle \rho \rho \rangle$ in the theories (6.3) and (7.29) need to be matched. Thus, the “bare” parameters $\mu^2$ in (6.3) and $\tilde{\mu}^2$ in (7.29) are different as the latter absorbs some self-energy corrections. We refer to ref. [71] for details.
Figure 6.2: KNO scaling plots of charged particle multiplicity distributions at $|\eta| < 0.5$ in NSD collisions at various energies and NBD fits; $z \equiv N_{ch}/\langle N_{ch} \rangle$ and $\Psi(z) \equiv \langle N_{ch} \rangle P(N_{ch})$. Note that the mean multiplicity quoted for the fits has been rescaled by $1.5$ to include neutral particles; also, that here $k$ is integrated over the transverse plane of the collision.

This result illustrates that $\bar{n}/k$ decreases as the contribution of the $\sim \rho^4$ operator increases. We repeat that the derivation assumed that the correction is small so that (6.13) does not apply for large values of $\beta N_c^2$.

Ref. [71] estimated by entirely different considerations that for protons $\beta \simeq 0.01$ at $x = 10^{-2}$. That would correspond to a smaller value of $\bar{n}/k$ by a factor of $1.43$ than for the Gaussian theory. Assuming that RG flow with energy approaches a Gaussian action [72, 73], $\bar{n}/k$ should increase by about this factor. NBD fits to the data shown in fig. 6.2 confirm that $\bar{n}/k$ indeed increases with energy, which might indicate flow towards a Gaussian action; however, the observed increase from $\sqrt{s} = 200$ GeV to 7 TeV is much stronger: a factor of about 3. This apparent discrepancy could be resolved at least partially by running of the coupling in eqs. (6.5, 6.13) with $Q_s$ but this requires more careful analysis\footnote{In fact, the running of $\alpha_s$ at the effective scale $Q_s$ is taken into account if the mean multiplicity is computed with energy evolved unintegrated gluon distributions like e.g. in refs. [54, 64–66].}.

4In fact, the running of $\alpha_s$ at the effective scale $Q_s$ is taken into account if the mean multiplicity is computed with energy evolved unintegrated gluon distributions like e.g. in refs. [54, 64–66].
6.4 Appendix

6.4.1 The moment $C_2 = k^{-1}$ with the quartic action

We can obtain the fluctuation parameter $k$ by calculating the inclusive double gluon multiplicity and expressing it in terms of the single inclusive or mean multiplicity. The connected two particle production cross section for gluons with rapidity $y_1$ and $y_2$ has the form:

$$N_2(p, q) \equiv \left\langle \frac{d^2N}{dy_1dy_2} \right\rangle - \left\langle \frac{dN}{dy_1} \right\rangle \left\langle \frac{dN}{dy_2} \right\rangle \equiv \left\langle \frac{d^2N}{dy_1dy_2} \right\rangle_{\text{conn.}}. \quad (6.14)$$

$\left\langle \frac{dN}{dy} \right\rangle$ is the mean multiplicity and the brackets denote an average over events. $N_2(p, q)$ is given by:

$$N_2(p, q) = \frac{g^{12}}{4(2\pi)^6} f_{gaa} f_{gbb} f_{gcc} f_{g'dd} \int \prod_{i=1}^{4} d^2k_i \left[ L_\mu(p, k_1)L_\nu(p, k_2)L_\lambda(q, k_3)L_\sigma(q, k_4) \times \right. \left. \langle \rho^*_a(k_2)\rho^*_b(k_4)\rho^*_c(k_1)\rho^*_d(k_3) \rangle \langle \rho^*_{a'}(p - k_2)\rho^*_{b'}(q - k_4)\rho^*_{c'}(p - k_1)\rho^*_{d'}(q - k_3) \rangle \right].$$

$L^\mu$ denotes the Lipatov vertex, for which:

$$L_\mu(p, k)L_\mu(p, k) = -\frac{4k^2}{p^2}(p - k)^2. \quad (6.15)$$

For the four-point function in the target and projectile fields we use [71]

$$\langle \rho^*_{a'}(p - k_2)\rho^*_{b'}(q - k_4)\rho^*_c(p - k_1)\rho^*_d(q - k_3) \rangle =$$

$$\left(2\pi\right)^4 \left[ \int dz \hat{\Lambda}^2(z^-) \right] \left[ \delta^{a'b'} \delta^{c'd'} \delta(p + q - k_2 - k_4) \delta(p + q - k_1 - k_3) + \delta^{a'd'} \delta^{b'c'} \delta(p - q - k_2 + k_3) \delta(p - q - k_1 + k_4) \right]$$

$$- \left(2\pi\right)^4 \frac{2}{\pi^2\kappa_4} \left[ \int dz \hat{\Lambda}^4(z^-) \right] \left( \delta^{a'b'} \delta^{c'd'} + \delta^{a'c'} \delta^{b'd'} + \delta^{a'd'} \delta^{b'c'} \right) \delta(k_1 + k_3 - k_2 - k_4).$$
The first two lines on the rhs of the above equation originate from the quadratic part of the action while the third line is due to the quartic operator. The product of the Gaussian parts of the two four-point functions gives nine terms, one of which ($\sim \delta^{ac}\delta^{bd}\delta^{a'c'}\delta^{b'd'}$) corresponds to a disconnected contribution. It exactly cancels the second term in eq. (6.14).

Figure 6.3: One of eight connected diagrams for two-gluon production with the quadratic MV action.

Four of the other eight terms ($\sim \delta^{ac}\delta^{bd}$ or $\sim \delta^{a'c'}\delta^{b'd'}$) give identical leading contributions to double gluon production. They correspond to a “rainbow” diagram like the one shown in Fig. 6.3. In the “rainbow” diagram, on one side (target or projectile), the $\rho$’s corresponding to the same gluon momentum are contracted with each other. The remaining four “non-rainbow” diagrams are suppressed relative to the terms we keep at large $p$ and $q$ [67]. Hence, the leading Gaussian contribution is:

$$\sim \frac{g^{12}}{8\pi^4} \left[ \int dz^{-}\bar{\mu}^2(z^{-}) \right]^4 \frac{S_{\perp}}{Q_s^2} \frac{N_c^2(N_c^2 - 1)}{p^4q^4}. \tag{6.16}$$

The same reasoning applies also for the additional quartic contribution and only “rainbow” diagrams are considered, like the one in Fig. 6.4. There are two of them (one for the projectile
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and one for the target) to first order in $\kappa^{-1}_4$, and their contribution is:

$$
\sim - \frac{g^{12}}{2(2\pi)^6} f_{gaa'} f_{gbb'} f_{gce'} f_{g'd'd'} \int \prod_{i=1}^{4} d^2 k_i \left( \frac{L_\mu(p, k_1)L_\mu(p, k_2)L_\nu(q, k_3)L_\nu(q, k_4)}{(2\pi)^2 k_i^2(p-k_1)^2(p-k_2)^2(q-k_3)^2(q-k_4)^2} \right) \times
\frac{2(2\pi)^8}{\pi^2 \kappa_4^2} \left[ \int dz^- \tilde{\mu}^2(z^-) \right]^2 \left[ \int dz^- \tilde{\mu}^4(z^-) \right]^2 \delta^{ac} \delta^{bd} \delta(k_1 - k_2)\delta(k_3 - k_4) \times
\left( \delta^{d'd'} \delta^c c' + \delta^{d'd'} \delta^{b'd'} + \delta^{a'd'} \delta^{b'c'} \right) \delta(k_1 + k_3 - k_2 - k_4).
$$

The color factor evaluates to

$$
\mathcal{F}_{gaa'} f_{gbb'} f_{gce'} f_{g'd'd'} \delta^{ac} \delta^{bd} \left( \delta^{d'd'} \delta^c c' + \delta^{d'd'} \delta^{b'd'} + \delta^{a'd'} \delta^{b'c'} \right) = 2 N_c^2(N_c^2 - 1) + N_c^2(N_c^2 - 1)^2.
$$

Figure 6.4: Connected diagram for two-gluon production from the quartic operator in the action [71].

Using eq. (6.15) we get:

$$
- \frac{16g^{12}}{(2\pi)^8 \pi^2 \kappa_4^2} \left[ \int dz^- \tilde{\mu}^2(z^-) \right]^2 \left[ \int dz^- \tilde{\mu}^4(z^-) \right]^2 \left[ 2 N_c^2(N_c^2 - 1) + N_c^2(N_c^2 - 1)^2 \right] \times
\frac{S_\perp}{p^2 q^2} \int \frac{d^2 k_1}{k_1^2(p-k_1)^2} \int \frac{d^2 k_3}{k_3^2(q-k_3)^2}. \right.
$$
The integral over the ladder momentum is again cut off at the saturation scale $Q_s$:

$$
\int \frac{d^2 k_1}{k_1 (p - k_1)^2} \approx \frac{2\pi}{p^2} \log \frac{p}{Q_s} .
$$

Then, the quartic contribution to connected two gluon production becomes

$$
- \frac{g^{12}}{4\pi^8 \kappa_4} \left[ \int dz^- \tilde{\mu}^2(z^-) \right]^2 \left[ \int dz^- \tilde{\mu}^4(z^-) \right]^2 \left[ 2N_c^2(N_c^2 - 1) + N_c^2(N_c^2 - 1)^2 \right] \frac{S_1}{p^4 q^4} \log \frac{p}{Q_s} \log \frac{q}{Q_s} .
$$

(6.17)

The last step is to express the fully connected diagrams in terms of the single inclusive cross section:

$$
\left\langle \frac{dN}{dy} \right\rangle = \frac{g^6}{4\pi^4} \left[ \int dz^- \tilde{\mu}^2(z^-) \right]^2 N_c(N_c^2 - 1) \frac{S_1}{p^4} \log \frac{p}{Q_s} .
$$

(6.18)

Summing eq. (6.16) and eq. (6.17) and using eq. (6.18) we get:

$$
\left\langle \frac{d^2 N}{dy_1 dy_2} \right\rangle_{\text{conn.}} = \left[ \frac{2\pi}{Q_s^2(N_c^2 - 1) S_1} - \frac{4(N_c^2 + 1)}{\kappa_4 S_1(N_c^2 - 1) \left[ \int dz^- \tilde{\mu}^4(z^-) \right]^2} \right] \left\langle \frac{dN}{dy_1} \right\rangle \left\langle \frac{dN}{dy_2} \right\rangle .
$$

The fluctuation parameter $k^{-1}$ is now identified with the expression in the square brackets.

We rewrite it in terms of

$$
\beta \equiv \frac{C_F^2}{6\pi^3 Q_s^2 \kappa_4} \left[ \int dz^- \tilde{\mu}^4(z^-) \right]^2 ,
$$

(6.19)

and use

$$
Q_s^2 = \frac{g^4 C_F}{2\pi} \int dz^- \tilde{\mu}^2(z^-) ,
$$

(6.20)

to arrive at the final expression

$$
\frac{N_c^2 - 1}{2\pi} Q_s^2 S_1 \frac{1}{k} = 1 - 3\beta(N_c^2 + 1) .
$$

(6.21)
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6.4.2 The moment $C_3$ with the quartic action

In this section we are going to calculate the connected diagrams for three-gluon production to obtain the correction to the reduced moment $C_3$ at order $1/\kappa_4 \sim 1/[g(gA^{1/3})^3]$, assuming as before that $gA^{1/3} > 1$. At the end of this section we also outline corrections suppressed by higher powers of $gA^{1/3}$.

We are looking for the contribution of the connected diagrams to the following expression [75]:

$$
\langle \frac{d^3N}{dy_1 dy_2 dy_3} \rangle = \langle \frac{(-ig^3)^6}{8(2\pi)^9} f_{gaaf} f_{gbbe} f_{g'ce'} f_{g'f'} f_{g'ee'} f_{g'd'd'} \times
\prod_{i=1}^{6} \frac{d^2k_i}{(2\pi)^2} L_\alpha(p, k_1) L^\alpha(p, k_2) L_\beta(q, k_3) L^\beta(q, k_4) L_\gamma(l, k_5) L^\gamma(l, k_6) \times
\langle \rho_1^s f(p - k_2) \rho_1^s e(q - k_4) \rho_1^d (l - k_6) \rho_1^a (p - k_1) \rho_1^b (q - k_3) \rho_1^c (l - k_5) \rangle \times
\langle \rho_2^s f(k_2) \rho_2^s e' (k_4) \rho_2^d (k_6) \rho_2^a (k_1) \rho_2^b (k_3) \rho_2^c (k_5) \rangle \rangle .
$$

(6.22)

As before, the $\rho$ correlators of the target and the projectile consist of two parts, one from the quadratic operator in the action and another from the additional $\rho^4$ operator:

$$
\langle \rho_1^s f \rho_1^e \rho_1^d \rho_1^a \rho_1^b \rho_1^c \rangle = \langle \rho_1^s f \rho_1^e \rho_1^d \rho_1^a \rho_1^b \rho_1^c \rangle_{\text{Gaussian}} + \langle \rho_1^s f \rho_1^e \rho_1^d \rho_1^a \rho_1^b \rho_1^c \rangle_{\text{Correction}}.
$$

The product of the two Gaussian contributions from the target and the projectile, to leading order in the gluon momenta, gives rise to 16 “rainbow” diagrams. The result has been obtained previously [67] and reads (expressed in terms of the mean multiplicity):

$$
\left\langle \frac{d^3N}{dy_1 dy_2 dy_3} \right\rangle_{\text{Conn. Gaussian}} = \frac{8\pi^2}{Q_s^2 S_{L}(N_c^2 - 1)^2} \left\langle \frac{dN}{dy_1} \right\rangle \left\langle \frac{dN}{dy_2} \right\rangle \left\langle \frac{dN}{dy_3} \right\rangle .
$$

(6.23)
The correction, to first order in $\kappa_{4}^{-1}$ is

$$\sim 2\langle \rho^{*f} \rho^{*e} \rho^{*d} \rho^{a} \rho^{b} \rho^{c} \rangle_{\text{Gaussian}} \langle \rho^{*f} \rho^{*e} \rho^{*d} \rho^{a} \rho^{b} \rho^{c} \rangle_{\text{Correction}} \ . \quad (6.24)$$

Again, we are considering only rainbow diagrams, so for the Gaussian six-point function in the above expression, from all possible contractions, we keep only the term

$$(2\pi)^6 \int dz^- \mu^2(z^-)^3 \delta^{ef} \delta^{ef} \delta^{cd} \delta(k_1 - k_2) \delta(k_3 - k_4) \delta(k_5 - k_6) \ .$$

To calculate the correction to the six-point function to first order in $\kappa_{4}^{-1}$ we factorize it into a product of two- and four-point functions. There are fifteen possible factorizations of that kind. Three of them are disconnected diagrams and the remaining twelve give identical contributions. We consider, for example, the following combination:

$$\langle \rho^{*f}_1(p - k_2) \rho^{*e}_1(q - k_4) \rho^{*d}_1(l - k_6) \rho^{a}_1(p - k_1) \rho^{b}_1(q - k_3) \rho^{c}_1(l - k_5) \rangle$$

$$= \langle \rho^{a}_1(p - k_1) \rho^{b}_1(q - k_3) \rangle \langle \rho^{*f}_1(p - k_2) \rho^{*e}_1(q - k_4) \rho^{*d}_1(l - k_6) \rho^{c}_1(l - k_5) \rangle \ .$$

The two point function is

$$\langle \rho^{a}_1(p - k_1) \rho^{b}_1(q - k_3) \rangle = (2\pi)^2 \int dz^- \mu^2(z^-) \delta^{ab} \delta(p + q - k_1 - k_3) \ ,$$

and for the correction to the four-point function we use the last line from eq. (6.16).

The color factor is

$$f_{gaa'} f_{g'b'b} f_{g'e'c} f_{g'f'f} f_{g'e'e} f_{g'd'd} \delta^{ab} \left( \delta^{cd} \delta^{ef} + \delta^{ce} \delta^{df} + \delta^{cf} \delta^{de} \right) \delta^{ef} \delta^{e'f'} \delta^{c'd'}$$

$$= 2N_c^3(N_c^2 - 1) + N_c^3(N_c^2 - 1)^2 \ .$$
Putting everything together into eq. (6.22) and multiplying by two [because of (6.24)] and by twelve (which is the number of possible diagrams) we get:

\[
\left\langle \frac{d^3N}{dy_1dy_2dy_3} \right\rangle_{\text{Conn. Correction}} = \frac{48g^{18}\pi}{\kappa_4} \left[ \int dz^- \bar{\mu}^2(z^-) \right]^4 \left[ \int dz^- \bar{\mu}^4(z^-) \right]^2 \left[ 2N_c^3(N_c^2 - 1) + N_c^3(N_c^2 - 1)^2 \right] \times \\
\prod_{i=1}^{6} (2\pi)^2 k_i^2 \left( \frac{p - k_1}{p - k_2} \right)^2 \left( \frac{q - k_3}{q - k_4} \right)^2 \left( \frac{L_\alpha(l, k_5)}{L_\beta(l, k_6)} \right)^2 \times \\
\delta(k_1 - k_2) \delta(k_3 - k_4) \delta(k_5 - k_6) \delta(p + q - k_1 - k_3) \delta(p + q - k_2 - k_4 - k_6) \\
\left( p^2q^2/l^2 \right)
\]

\[
\left( \frac{d^3N}{dy_1dy_2dy_3} \right)_{\text{Conn. Correction}} = -\frac{3g^{18}}{16\kappa_4 \pi^{13} p^2q^2l^2} \left[ \int dz^- \bar{\mu}^2(z^-) \right]^4 \left[ \int dz^- \bar{\mu}^4(z^-) \right]^2 \left[ 2N_c^3(N_c^2 - 1) + N_c^3(N_c^2 - 1)^2 \right] \times \\
\int \frac{d^2k_1}{k_1^2(p + q - k_1)^2(p - k_1)^4} \int \frac{d^2k_2}{k_2^2(l - k_2)^2}.
\]

Again, we regularize the ladder integrals at the saturation scale,

\[
\int \frac{d^2k}{k^2(p + q - k)^2(p - k)^4} \approx \frac{2\pi}{p^2q^2 Q_s^2}.
\]

Finally, using expression (6.18) for the mean multiplicity the \(\rho^4\) contribution to three-gluon production becomes

\[
\left\langle \frac{d^3N}{dy_1dy_2dy_3} \right\rangle_{\text{Conn. Correction}} = -\frac{48\pi(N_c^2 + 1)}{\kappa_4 Q_s^2 S_e^2(N_c^2 - 1)^2} \left[ \int dz^- \bar{\mu}^4(z^-) \right]^2 \left( \frac{dN}{dy_1} \right) \left( \frac{dN}{dy_2} \right) \left( \frac{dN}{dy_3} \right).
\]

Summing (6.23) and (6.25),

85
\[
\left\langle \frac{d^3N}{dy_1dy_2dy_3} \right\rangle_{\text{Conn.}} = \left[ \frac{8\pi^2}{Q_s^4S_\perp^2(N_c^2 - 1)^2} - \frac{48\pi(N_c^2 + 1)}{\kappa_4Q_s^2S_\perp^2(N_c^2 - 1)^2} \left[ \int dz^- \tilde{\mu}^4(z^-) \right]^2 \right] \left\langle \frac{dN}{dy_1} \right\rangle \left\langle \frac{dN}{dy_2} \right\rangle \left\langle \frac{dN}{dy_3} \right\rangle.
\]

From the above equation the third reduced moment is:
\[
C_3 = \frac{8\pi^2}{Q_s^4S_\perp^2(N_c^2 - 1)^2} - \frac{48\pi(N_c^2 + 1)}{\kappa_4Q_s^2S_\perp^2(N_c^2 - 1)^2} \left[ \int dz^- \tilde{\mu}^4(z^-) \right]^2,
\]
or
\[
\frac{(N_c^2 - 1)^2}{4\pi^2} \frac{C_3}{Q_s^4S_\perp^2} = 1 - \frac{9}{\beta}(N_c^2 + 1),
\]
where we have used expressions (6.19) and (7.18).

For a NBD we have that \( C_3 = 2/k^2 \) but if we compare (6.26) to the square of eq. (6.21), which is
\[
\frac{(N_c^2 - 1)^2}{4\pi^2} \frac{1}{k^2} = 1 - \frac{6}{\beta}(N_c^2 + 1),
\]
we see that the coefficients of the corrections at order \( O(\beta) \) differ. That means that the \( \rho^4 \) operator in the action provides a correction to the negative binomial distribution.

In fact, such deviation from a NBD is more obvious if even higher order operators are added to the action. Dropping the longitudinal dependence of the operators for simplicity, such an action would have the form
\[
S \simeq \int d^2v_\perp \left[ \frac{\delta^{ab}\rho^a\rho^b}{2\tilde{\mu}^2} - \frac{\delta^{abc}\rho^a\rho^b\rho^c}{\kappa_3} + \frac{\delta^{ab}\delta^{cd} + \text{perm.}}{\kappa_4} \rho^a\rho^b\rho^c\rho^d - \frac{\delta^{ab}d^{cde} + \text{perm.}}{\kappa_5} \rho^a\rho^b\rho^c\rho^d\rho^e
\]
\[
+ \frac{(\delta^{ab}\delta^{cd}\delta^{ef} + \text{perm.}) + (d^{abc}d^{def} + \text{perm.})}{\kappa_6} \rho^a\rho^b\rho^c\rho^d\rho^e\rho^f + \ldots \right].
\]
The additional terms are suppressed by powers of $gA^{1/3}$ [30, 40]:

$$\bar{\mu}^2 \sim g(gA^{1/3}), \quad \kappa_3 \sim g(gA^{1/3})^2, \quad \kappa_4 \sim g(gA^{1/3})^3,$$

$$\kappa_5 \sim g(gA^{1/3})^4, \quad \kappa_6 \sim g(gA^{1/3})^5.$$  

The cubic operator gives a correction to the six-point function, i.e. to $C_3$ at order $1/\kappa_3^2$ but does not correct the four-point function, i.e. $C_2 = 1/k$ (it only renormalizes $\mu^2$). The same applies to the $\rho^6$ operator: $C_3$ will contain a term $\sim 1/\kappa_6$ but $1/k$ does not. Hence, beyond a quadratic action the relation $C_3 = 2/k^2$ is not exact.
Chapter 7

Spatial Wilson loops in the classical field of high-energy heavy-ion collisions

The field configuration in the first moments after high-energy heavy-ion collisions is represented by strong longitudinal chromo-electric and chromo-magnetic fields. The form of the fields is obtained by solving the classical Yang-Mills equations of motion for two color charge sheets passing through each other. Here, we find the magnetic flux fluctuations accompanying the fields by calculating the expectation value of the magnetic Wilson loop operator. The perturbative calculation involves an expansion of the classical gauge field in terms of the coupling constant. We obtain a flux that is proportional to the square of the area of the loop. We compare the analytic result to lattice computation of the loop. Numerical calculations give an area law behavior of the magnetic Wilson loop which indicates existence of independent magnetic vortices over distance scales up to few times the inverse saturation scale.
7.1 Wilson loop operator from lattice calculations

In high-energy collisions the target and the projectile are represented as Lorentz-contracted sheets of valence color charges moving along recoilless trajectories along the light cone. These charges act as sources of a purely transverse gluon field that has a small fraction $x$ of the total longitudinal momentum of the nucleus. Since the charge density $\rho(x)$ is large, the sources belong to a higher dimensional representation of the color algebra and the gauge field they emit can be computed classically [47, 48]. After the two sheets of color charge have passed through each other, longitudinal chromo-electric and chromo-magnetic fields are produced [76].

The gluon field of the target and the projectile is obtained by solving the classical Yang-Mills equations of motion. Before the collision, the solution corresponds to a non-Abelian analogue of the Weizsäcker-Williams field. In light-cone gauge its form is:

$$\alpha_i^m = i g U_m \partial^i U_m^\dagger, \quad \partial^i \alpha_i^m = g \rho_m .$$  \hspace{1cm} (7.1)

The subscript $m$, with values 1 and 2, denotes the projectile and the target respectively. Introducing the gauge potential as

$$\Phi_m = - \frac{g}{\sqrt{2}} \rho_m , \hspace{1cm} (7.2)$$

the solution to (7.1) can be written as [77]:

$$\alpha_i^m = \frac{i}{g} e^{-ig\Phi_m} \partial^i e^{ig\Phi_m} .$$  \hspace{1cm} (7.3)

The Yang-Mills equations for scattering of two ultra-relativistic nuclei with appropriate boundary conditions on the light cone give the classical field after the collision [77]. At
proper time $\tau = \sqrt{t^2 - z^2} = 0$, the resulting field is a sum of two pure gauge fields:

$$A^i = \alpha_1^i + \alpha_2^i .$$  \hfill (7.4)

The sum of two pure gauges is not a pure gauge and strong longitudinal chromo-electric and chromo-magnetic fields are produced in the collision [76]:

$$E_z = ig[\alpha_1^i, \alpha_2^j] ,$$
$$B_z = ig\epsilon^{ij}[\alpha_1^i, \alpha_2^j] , \quad (i, j = 1, 2) .$$  \hfill (7.5)

$\epsilon^{ij}$ is the antisymmetric tensor. The transverse components of the field strength are zero.

The non-Abelian Wilson loop operator is defined as a path order exponential of the gauge field:

$$M(R) = \mathcal{P} \exp \left( ig \oint dx^i A^i \right) = \mathcal{P} \exp \left[ ig \oint dx^i (\alpha_1^i + \alpha_2^i) \right] ,$$  \hfill (7.6)

with $R$ the radius of the loop. Note that $M(R) \equiv 1$ if evaluated in the field of a single nucleus ($\alpha_1^i$ or $\alpha_2^i$) as those are pure gauges. We want to find the expectation value of the loop:

$$W_M(R) = \frac{1}{N_c} \langle \text{tr} M(R) \rangle .$$  \hfill (7.7)

We compare, also, to the expectation value of the $Z(N_c)$ part of the loop; for a magnetic field configuration corresponding simply to a superposition of independent vortices the loop should equal $\exp(2\pi i n/N_c)$, with $n$ the total vortex charge piercing the loop. Thus, for two colors we compute

$$W_M^{Z(2)}(R) = \langle \text{sgn} \text{ tr} M(R) \rangle$$  \hfill (7.8)

where $\text{sgn}()$ denotes the sign function. Comparing (7.7) to (7.8) tests the interpretation that
the drop-off of $W_M(R)$ is due to $Z(N_c)$ vortices, without requiring gauge fixing of the SU$(N_c)$ links [78].

The expectation value in (7.7) refers to averaging over the color charge distributions in each nucleus. For large nuclei the color sources are treated as random variables with Gaussian probability distribution. Physical observables are then averaged with a Gaussian (McLerran-Venugopalan) action:

$$S_{\text{eff}}[\rho^a] = \frac{1}{2} \int d^2 x \left[ \frac{\rho^a_1(x) \rho^a_1(x)}{\mu_1^2} + \frac{\rho^a_2(x) \rho^a_2(x)}{\mu_2^2} \right],$$  \quad (7.9)

where $\mu^2$ is the color charge squared per unit area, related to the saturation scale via $Q_s^2 \sim g^4 \mu^2$.

Fig. 7.1 shows numerical results for $W_M$ immediately after a collision. It exhibits area law behavior for loops larger than $A \gtrsim 2/Q_s^2$. The corresponding “magnetic string tension” is $\sigma_M/Q_s^2 = 0.12(1)$. The area law indicates uncorrelated magnetic flux fluctuations through the Wilson loop and that the area of magnetic vortices is rather small, their radius being on the order of $R_{\text{vtx}} \sim 0.8/Q_s$. A breakdown of the area law is not observed up to $A \sim 4/Q_s^2$, implying that vortex correlations are small at such distance scales. Also, restricting to the Z(2) part reduces the magnetic flux through small loops but $\sigma_M$ is comparable to the full SU(2) result, if somewhat smaller. The numerically small vortex size that is parametrically consistent with the classical Gaussian approximation at weak coupling which, as already mentioned above, applies for areas $\Delta A_\perp \gg g^4/Q_s^2$.

To estimate the density of vortices one can consider a simple combinatorial model whereby the area $A$ of the loop is covered by patches of size $1/Q_s^2$ containing a Z(2) vortex with probability $p$. Averaging over random, uncorrelated vortex fluctuations leads to [79]

$$W_M(A) \sim \exp \left( \frac{-\pi^2}{4} p(1-p)AQ_s^2 \right),$$  \quad (7.10)
Figure 7.1: Expectation value of the magnetic flux loop right after a collision of two nuclei (time $\tau = +0$) as a function of its area $A' \equiv A Q_s^2$. We define $Q_s^2 = (C_F/2\pi) g^4 \mu^2$. Symbols show numerical results for $SU(2)$ Yang-Mills on a $4096^2$ lattice; the lattice spacing is set by $g^2 \mu L = 0.0661$. The solid and dashed lines represent fits over the range $4 \geq A' \geq 2$. The short dotted line shows $\cos 2A'$ for $A' < 0.3$.

or $\sigma_M = (\pi^2/4) p(1 - p) Q_s^2$. From this relation we estimate that the probability of finding a vortex within an area $1/Q_s^2$ is $p \simeq 1/20$.

Fig. 7.2 shows the time evolution of the magnetic flux loop after a collision. The magnetic field strength decreases due to longitudinal expansion and so $W_M$ approaches unity. On the other hand, the onset of area law behavior is pushed to smaller loops, implying that the size of elementary flux excitations or “vortices” decreases; by the time $\tau \sim 1/Q_s$ area law behavior is satisfied even for rather small loops. Since long wavelength magnetic fields remain even at times $\sim 1/Q_s$, it will be important in the future to understand the transition of $W_M$ to behavior expected in thermal QCD where $\sigma_M \sim (g^2 T)^2$ [80–83]. In the context of late-time behavior much beyond $t \sim 1/Q_s$ we refer to ref. [84] where area law scaling of spatial loops has been observed for classical field configurations emerging from unstable plasma evolution.
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Figure 7.2: Time evolution of the magnetic flux loop after a collision of two nuclei \( (4096^2 \text{ lattice}, g^2\mu_L = 0.05) \). From bottom to top, the curves correspond to time \( \tau \times g^2\mu = 0, 1, 2, 3 \), where \( g^2\mu \approx 3Q_s \) so that \( \tau = 3/(g^2\mu) \) corresponds to about \( \tau \approx 1/Q_s \) in physical units.

We have also investigated the dependence of the magnetic flux loop in the adjoint representation on its area,

\[
W_M^{\text{adj}} = \frac{1}{N_c^2 - 1} \langle |\text{tr} M|^2 - 1 \rangle ,
\]  

(7.11)

and found behavior similar to fig. 7.2. The adjoint magnetic string tension is about two times larger, as expected from (7.11).

The third color component of the longitudinal magnetic field is shown in fig. 7.3, using a random residual gauge for \( A^i \). Domain-like structures where the magnetic field is either positive or negative are clearly visible; they lead to the above-mentioned area law of the Wilson loop. Also, one can see that in time the magnetic fields become weaker and smoother.

Thus far we have not addressed the longitudinal structure of the initial fields. The
Figure 7.3: Color-3 component of the magnetic field $F_{xy}^3(x, y)$ in the transverse plane at time $\tau = +0$ (top) and $1/g^2\mu \sim 1/3Q_s$ (bottom) for a single configuration of color charge sources $\rho$.

solution of eqs. (7.1) is boost invariant and so, naively, the two-dimensional vortex structures mentioned above would form boost invariant strings. However, this simple picture could be modified by longitudinal smearing of the valence charge distributions [85] and therefore
requires more detailed consideration.

### 7.2 Perturbative result

To obtain $W_M(R)$ perturbatively we need to determine the deviation of $A^i$ from a pure gauge. From the Baker-Campbell-Hausdorff formula [86]

$$W_M(R) \simeq \frac{1}{N_c} \left\langle \text{tr} \exp \left( -\frac{1}{2} [X_1, X_2] \right) \right\rangle \simeq 1 - \frac{1}{2N_c} \langle g^2 h^2 \rangle , \quad (7.12)$$

where:

$$g^2 h^2 = \frac{1}{8} f^{abc} f^{\dot{a}\dot{b}c} X^a_1 X^{\dot{a}}_1 X^b_2 X^\dot{b}_2 . \quad (7.13)$$

with:

$$X_m = ig \oint dx^i \alpha^a_i m^a . \quad (7.14)$$

$f^{abc}$ are the structure constants of the special unitary group and $h^2$ corresponds to a four gluon vertex of the fields.

To calculate this expectation value analytically, we expand the fields $\alpha^i$ from eq. (7.3) perturbatively in terms of the coupling constant:

$$\alpha^i_m = -\partial^i \Phi_m + \frac{ig}{2} \left( \delta^{ij} - \partial^i \frac{1}{\nabla^2_{\perp}} \partial^j \right) [\Phi^j_m, \partial^j \Phi^i_m] + O(\Phi^3_m) . \quad (7.15)$$

The loop integral over the first term in (7.15) vanishes. Therefore, the leading term in the expansion of the field $\alpha^i$ in terms of the gauge potential $\Phi$ does not contribute to the Wilson loop $W_M$ and it is required to go to quadratic order: $\alpha^i_a \sim g f^{abc} \Phi^b \partial^i \Phi^c$:

$$X_m = -\frac{g^2}{2} \oint dx^i \left[ \Phi^j_m, \partial^i \Phi^j_m \right] . \quad (7.16)$$
The Feynman diagram representation of the expansion of the field $\alpha^i$ is given in fig. 7.4. At this order the field is still dominated by classical diagrams [87].

The expectation value $\langle h^2 \rangle$ that enters in the expression for the magnetic loop involves the fields of both nuclei. The corresponding classical diagram is shown in fig. 7.5a. A quantum correction at the same order is given in fig. 7.5b. We defer an analysis of quantum corrections to future work since the primary goal of this paper is to provide a point of comparison to resummed classical lattice gauge computations of the loop [86] which consider strong fields. However, our present analysis should not be taken to imply that small loops in weak fields can be obtained in the classical approximation.

The final result we obtain for the expectation value of the magnetic Wilson loop for classical fields $\alpha^i$ to second order in the gauge potential is:

$$W_M(R) \simeq 1 - \frac{\pi^2 N_c^6}{64(N_c^2 - 1)^3} \frac{Q_{s1}^4 Q_{s2}^4}{\Lambda^4} A^2. \quad (7.17)$$

Details of the calculation are given in the appendix. $Q_{s1}$ and $Q_{s2}$ are the saturation scales of the projectile and the target, respectively. They are determined by the variance of the color...
charge distribution. We use the relation:

\[ Q_s^2 = \frac{g^4 C_F}{2\pi} \mu^2, \quad (7.18) \]

where:

\[ C_F = \frac{N_c^2 - 1}{2N_c}. \quad (7.19) \]

The cut-off \( \Lambda \) regulates the infrared divergence of the integrals over the gluon momentum \( k \) shown in diagram 7.5a. It sets the mass scale for the gluon propagator.

From a fit to the lattice data for the Wilson loop for small areas, it was estimated that \( W_M(R) \simeq 1 - 2 (AQ_s^2)^2 \) [86]. By comparing this expression to the result (7.17) for \( SU(2) \) we can extract \( Q_s^4/\Lambda^4 \approx 5.477 \).

We have calculated the expectation value of the magnetic loop with a Gaussian action. For a finite nuclear thickness, higher order corrections in the charge density of cubic [30] and quartic [40, 71] order arise. As shown in the appendix, the calculation consists of averaging four-point functions and therefore the cubic part of the effective action does not contribute.
On the other hand, one would expect the fourth order term to bring a correction to the four-point function. However, the correction to the Wilson loop from the quartic action vanishes because of its vanishing color factor (see appendix).

We now turn to a discussion of the final result (7.17). The perturbative result for the expectation value of the magnetic Wilson loop gives a leading non-trivial contribution that is proportional to the square of the area. A term proportional to the area of the loop would involve single powers of the target’s and projectile’s saturation scales: $\sim A Q_{s1} Q_{s2}$ [86]. This has been verified numerically as shown in fig. 7.6.

![Figure 7.6: Expectation value of the magnetic flux loop for asymmetric projectile and target saturation momenta.](image)

Gaussian contractions can only give powers of $Q_{s1}^2$ and $Q_{s2}^2$:

$$\langle \rho^a_m(x) \rho^b_m(y) \rangle = \mu^2 \delta^{ab} \delta(x - y) \sim Q_{s_m}^2,$$  \hspace{1cm} (7.20)

and therefore a contribution $\sim A^2$. Area law scaling of the Wilson loop presumably requires resummation of screening effects and of condensation.

In reference [88] numerical calculations of the expectation value of the Wilson operator have been performed for small area loops. The lattice result is similar to the perturbative
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calculation; it gives an area power close to two.

7.3 Corrections to the free propagator

The magnetic fields modify the propagation of semi-hard modes with $p_T$ not too far above $Q_s$. Quantum mechanically, the transition amplitude from a state $|x_i, t_i\rangle$ to $|x_f, t_f\rangle$ is given by a Feynman sum over paths,

$$
\int_0^\infty ds \int Dx^\mu \left\langle \exp i \int_0^s d\tau \left( m\dot{x}^2 + gA_\mu \dot{x}^\mu \right) \right\rangle \sim \int_0^\infty ds \int Dx^\mu \exp \left( i \int_0^s d\tau m\dot{x}^2 \right) \exp(-\sigma_M A), \tag{7.21}
$$

where $x^\mu(\tau)$ is a parametrization of the path with the given boundary conditions and length $s$; and $\dot{x}^\mu = dx^\mu/d\tau$. Here, the area $A$ is that enclosed by a quantum mechanical path from the initial to the final point returning to $x_i$ via the classical path; see fig. 7.7. The classical path is obtained by extremizing the action but a single path is a set of measure zero. Semi-classical paths can dominate the integral only if there is constructive interference among neighboring paths from within a de Broglie distance. On the other hand, destructive interference of such paths leads to Anderson localization of the wave function. Hence, up to a numerical factor, the area in eq. (7.21) should be given by $A \sim s/p_T$. Integrating over the Schwinger parameter then leads to the propagator

$$
\frac{i}{p^2 + i\sigma_M \frac{m}{p_T}}, \tag{7.22}
$$
where \( \sigma_M = 0.12 Q_s^2 \) from above and \( m \) is the mass (time-like virtuality) of the particle. This expression accounts for corrections to free propagation and could be useful for studies of the dynamics of the very early stage of a heavy-ion collision.

### 7.4 Appendix

In the appendix we list some steps of the calculation leading to the final result (7.17). To calculate the expectation value of the Wilson loop, we need the average:

\[
\langle g^2 h^2 \rangle = \frac{1}{8} f^{abc} f^{\bar{a} \bar{b} \bar{c}} \langle X_1^a X_1^{\bar{a}} \rangle_{\rho_1} \langle X_2^b X_2^{\bar{b}} \rangle_{\rho_2} . \tag{7.23}
\]

Using the second term in the expression for the fields \( \alpha_m \) in eq. (7.15) we have:

\[
X_m^a = -\frac{ig^2}{2} f^{ade} \int dx^i \Phi_m^a \partial_i \Phi_m^e , \tag{7.24}
\]
or, in momentum space:

\[ X^a_m = -\frac{ig^2}{2(2\pi)^3} f^{a\cdots e} R \times \int d^2k d^2p |k| J_1(R|p|) \sin(\alpha - \theta) \Phi^d_m(k) \Phi^e_m(p - k). \] (7.25)

In the above expression, \( R \) is the radius of the loop, \( k \) and \( p \) are the momenta of the gluons shown in fig. 7.5a, and \( \alpha \) and \( \theta \) are their corresponding azimuthal angles. \( J_1(R|p|) \) is a Bessel function of the first kind. Then:

\[
\langle X^a_m X^{\bar{a}}_{\bar{m}} \rangle_{\rho_m} = -\frac{g^4}{4(2\pi)^6} f^{a\cdots \bar{e} \bar{d}} R^2 \int d^2k \, d^2\bar{p} \, d^2k \, d^2\bar{p} \, |k| |\bar{k}| \, J_1(R|p|) J_1(R|\bar{p}|) \times \sin(\alpha - \theta) \sin(\bar{\alpha} - \bar{\theta}) \left\langle \Phi^d_{\bar{m}}(k) \, \Phi^e_{\bar{m}}(p - k) \, \Phi^d_{\bar{m}}(\bar{k}) \, \Phi^e_{\bar{m}}(\bar{p} - \bar{k}) \right\rangle_{\rho_m}. \] (7.26)

The gauge potential and the two-point function in momentum space are

\[
\Phi^a(k) = -\frac{g}{k^2} \rho^a(k) \quad \text{and} \quad \langle \rho^a(k) \, \rho^b(p) \rangle = \mu^2 \delta^{ab} (2\pi)^2 \delta(k + p). \] (7.27) (7.28)

The four point-function in (7.26) receives a contribution from the fourth order term in the extended Gaussian action [40, 71]:

\[
S_Q[\rho] = \int d^2x \left[ \frac{\rho^a(x) \rho^a(x)}{2\mu^2} + \frac{\rho^a(x) \rho^a(x) \rho^b(x) \rho^b(x)}{\kappa_4} \right]. \] (7.29)

The correction due to the \( \rho^4 \) operator is:

\[
-32\pi^2 \frac{\mu^8}{\kappa_4} \left( \delta^{de} \delta^{\bar{d}\bar{e}} + \delta^{dd} \delta^{ee} + \delta^{d\bar{d}} \delta^{e\bar{e}} \right) \delta(p + \bar{p}). \] (7.30)
But, the total color factor of this correction to the expectation value \( \langle X^a_m \bar{X}^\alpha_m \rangle_{\rho_m} \) in (7.26) is equal to zero:

\[
f_{a\bar{d}e} f_{\bar{a}d\bar{e}} \left( \delta^{de} \delta^{\bar{d}\bar{e}} + \delta^{d\bar{d}} \delta^{e\bar{e}} + \delta^{d\bar{e}} \delta^{\bar{d}e} \right) = 0 ,
\]

and does not bring a modification to the expectation value of the Wilson loop.

With the Gaussian contractions (7.28) the expectation value \( \langle X^a_m \bar{X}^\alpha_m \rangle_{\rho_m} \) becomes:

\[
\langle X^a_m \bar{X}^\alpha_m \rangle_{\rho_m} = -\frac{g^8 \mu_m^4}{16\pi^2} f_{a\bar{d}e} f_{\bar{a}d\bar{e}} R^2 \int d^2k \ d^2\bar{p} \ d^2\bar{k} \ d^2p \ J_1(R|\bar{p}|) J_1(R|\bar{p}|) \left| \frac{1}{|k||\bar{k}|(p-k)^2(\bar{p} - \bar{k})^2} \right| \sin(\alpha - \theta) \sin(\bar{\alpha} - \bar{\theta}) \left[ \delta(k + \bar{k}) \delta(p - k + \bar{p} - \bar{k}) - \delta(k + \bar{p} - \bar{k}) \delta(p - k + \bar{k}) \right] \]

After performing two of the integrals using the delta functions in (7.32), we get:

\[
\langle X^a_m X^\alpha_m \rangle_{\rho_m} = \frac{g^8 \mu_m^4}{8} N_c \delta^{a\alpha} R^2 \int \frac{dk}{k^3} \int dp \frac{J_1^2(R|p|)}{|p|} . \]

The integral over the momentum \( p \) is convergent and equal to \( 1/2 \). The integral over \( k \) is infrared divergent and we introduce a cut-off \( \Lambda \) to regulate this divergence:

\[
\int_\Lambda^\infty \frac{dk}{k^3} = \frac{1}{2\Lambda^2} .
\]

So, finally:

\[
\langle X^a_m X^\alpha_m \rangle_{\rho_m} = \frac{g^8 \mu_m^4}{32\Lambda^2} N_c \delta^{a\alpha} R^2 .
\]

The cut-off \( \Lambda \) can be thought of as due to screening of the gauge potential. Introducing screened propagators in (7.27):

\[
\Phi^a(k) = -\frac{g}{k^2 + m^2} \rho^a(k) ,
\]

reproduces the result (7.35) with \( \Lambda^2 \) replaced by \( m^2 \). A self-consistent resummation of
screening effects is beyond the purpose of the present analysis.

In terms of the saturation scale (7.18) the final result is:

$$\langle g^2 h^2 \rangle = \frac{\pi^2 N_c^7}{32(N_c^2 - 1)^3} \frac{Q_{s1}^4 Q_{s2}^4}{A^4} A^2, \quad (7.37)$$

and

$$W_M(R) \approx 1 - \frac{\pi^2 N_c^6}{64(N_c^2 - 1)^3} \frac{Q_{s1}^4 Q_{s2}^4}{A^4} A^2. \quad (7.38)$$
Chapter 8

Summary

To conclude this thesis we briefly summarize the new results we presented in Chapters 4 through 7 and we discuss some possible future projects.

Our first result is the correction to the Gaussian distribution of classical color sources in an ultrarelativistic nucleus. We derived a fourth order term in the density of color charges $\rho$. The correction is suppressed by a negative power of the nucleon number: $A^{-2/3}$. The MV model is relevant for an infinitely large nucleus, for which it is assumed that $A^{1/3} \to \infty$. For more realistic cases the quartic term is not highly suppressed and the correction can be especially relevant for distributions of large-$x$ sources in a proton, where $A = 1$. The quartic action was then used for a computation of the two-point ($\langle \rho_x \rho_y \rangle$) and four-point ($\langle \rho_x \rho_y \rho_z \rho_u \rangle$) functions. The $\rho^4$ term simply renormalizes the average charge density squared per unit area ($\bar{\mu}^2$) in the result for the $\langle \rho_x \rho_y \rangle$ correlator, but brings correction $\sim 1/\kappa_4$ to $\langle \rho_x \rho_y \rho_z \rho_u \rangle$. Quartic action corrections to physical observables enter through the modification of the four-point function.

In Chapter 5 we used the extended action to derive the dipole scattering amplitude on a proton and a nucleus. In the proton case we obtained a dipole operator that is in agreement with the AAMQS collaboration’s fits of the dipole amplitude to DIS data. The MV model
does not follow the AAMQS fits. We conclude that the problem might be solved by including the fourth order operator in the theory.

The result for the dipole amplitude was found perturbatively to first order in $\frac{1}{\kappa_4}$ and at the end assumed that it exponentiates. It might be possible to prove explicitly that the lowest order correction we obtained is the first non-trivial term in an expansion in $\frac{1}{\kappa_4}$ of an exponential. Higher orders in perturbation theory would then also involve six-point functions which would receive a correction proportional to $\frac{1}{\kappa_3^2}$ from the "oddron" operator. Hence, higher orders of $\frac{1}{\kappa_3}$ should also be included.

We also considered the Negative Binomial Distribution (NBD) of particle multiplicities in proton-proton experiments. By evaluating the correction to the double-gluon production diagrams we obtained a quartic action modification to the fluctuation parameter $k$. The modification and the associated NBD fits to proton-proton data illustrate the possibility that the fourth-order action progresses towards Gaussian when the energy is increased. We were also able to connect the KNO scaling law observed in transverse momentum integrated multiplicity distributions to properties of soft gluons around the saturation scale.

We can extend our analysis of NBD from proton-proton to nucleus-nucleus collisions. Particularly, we can examine whether KNO scaling still holds. When going from proton-proton to nucleus-nucleus collisions, one would expect the fourth order correction to the MV action to be suppressed, and the initial conditions to become Gaussian. We can also examine proton-nucleus collisions. In this asymmetric case, the initial configuration should be different on the two sides of the collision: Gaussian on the nucleus side and quartic on the proton side. KNO scaling was predicted to hold for p-Pb collisions at LHC at 4.4 TeV [54].

Finally, we presented a perturbative result for the spatial Wilson loop operator in the classical field created in a collision of two high-energy nuclei. The calculation involves an expansion of the classical gauge field in terms of the coupling constant. We showed that
the leading diagram corresponds to two sources, for both projectile and target, whose field is evaluated at second order in the gauge potential. The result to lowest order in $A$ is proportional to area squared. Screening effects are not present in the analytical calculation. A numerical result for larger loops gives an area law behavior of the flux, which indicates existence of independent magnetic vortices.

As described before, the analytical computation of the magnetic Wilson loop involves inspection of quantum corrections to the gluon field in higher orders of perturbation theory. An analysis of whether their contribution can be consistently included in the calculation may be a part of a future work.

Recently, in reference [88], numerical calculations of the expectation value of the magnetic Wilson loop were performed in the forward light cone. The authors fit the lattice data to:

$$W(A) = \exp \left[- (\sigma A)^\gamma \right] ,$$  \hspace{1cm} (8.1)

for two regions of interest:

$$\text{IR} : \quad e^{0.5} < AQ_s^2 < e^5 ,$$  \hspace{1cm} (8.2)

$$\text{UV} : \quad e^{-3.5} < AQ_s^2 < e^{-0.5} .$$  \hspace{1cm} (8.3)

Perturbative calculations are valid in the region of small areas, i.e. in the UV region. For small areas and at proper time zero, in the classical MV approximation, reference [88] found $\gamma \approx 1.6$. For later times ($\tau \sim 1/Q_s$) the power $\gamma$ becomes independent of $\tau$ and its value is approximately 1.8. These results are close to the area power, $\gamma = 2$, that we have obtained analytically. It would be interesting to analyse the origin of the small discrepancy between these results, especially at $\tau = 0$. Corrections that have been neglected in the perturbative calculation may bring the analytical result close to the numerical observations.
Reference [88] provides a numerical computation of the two point correlator of the longitudinal magnetic field as well. At time zero, the authors observe a change of the sign of the correlator around distances of order $1/Q_s$. They interpret the result as a possible evidence of a domain structure of the magnetic field. The magnetic field correlator can be computed analytically following the same procedure as for the calculation of the expectation value of the Wilson loop. It would be interesting to see whether the flip of the sign can be found perturbatively. The anti-correlation in the lattice result is observed close to the region of applicability of the perturbative expansion.
Bibliography


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[79] see, for example, J. Preskill, “Lecture Notes on Quantum Field Theory”, http://www.theory.caltech.edu/~preskill/notes.html


