TR-2006008: Every P-Recursion Category Has an Index Composer

Florian Lengyel
EVERY \( P \)-RECURATION CATEGORY HAS AN INDEX COMPOSER

FLORIAN LENGYEL

Abstract. In a \( P \)-recursion category, composition of morphisms is definable uniformly in the collection of indices with respect to a Turing morphism. This fact follows from the coassociativity of the near-product of a \( P \)-category.

MSC: 03D75; 03G30

1. Introduction

In [Ste93], Stefano Stefani defined the notion of a composer, which was intended as a reflection principle for \( P \)-recursion categories. A composer in a \( P \)-recursion category enables composition of morphisms to be expressed internally. We show that every \( P \)-recursion category contains a morphism called an index composer, closely related to a composer in Stefani’s sense, which enables a \( P \)-category to express composition of any two of its morphisms \( \varphi \) and \( \psi \), uniformly in terms of a fixed expression in the indices of \( \varphi \) and \( \psi \). The existence of an index composer in a \( P \)-recursion category follows from the coassociativity of the near-product of a \( P \)-category.

A recursive function \( f : \omega \to \omega \) is an index morphism if, with respect to some standard numbering of the partial computable functions, \( f \) is a recursive enumeration of indices of some fixed partial computable function. In contrast to Stefani’s notion of a composer, the definition of which involves quantification over all total morphisms in a \( P \)-recursion category, the notion of an index composer reflects that in classical recursion theory, not every recursive function is an index morphism.

2. \( P \)- and \( P \)-recursion categories

Let \( C \) be a category; for our purposes, categories may be taken to be small. The diagonal functor \( \delta_C : C \to C \times C \) is given on objects by \( A \mapsto (A, A) \) and on morphisms by \( f \mapsto (f, f) \).

A \( P \)-category consists of a category \( C \) together with a bifunctor \( \times : C \times C \to C \), called a near product, a natural transformation \( \Delta : 1_C \to \times \circ \delta_C \), and for each object \( X \) of \( C \), natural transformations

\[
p_{0(\_),X} : (\_ \times X) \to 1_C, \quad p_{1X(\_)} : X \times (\_) \to 1_C
\]

such that the conditions i), ii) and iii) below are satisfied [Ros86].

Email address: flengyel@gc.cuny.edu.
i) The following equations hold.

\[
p_{0X,X}\Delta_X = 1_X = p_{1X,X}\Delta_X, \quad (p_{0X,Y} \times p_{1X,Y})\Delta_{X \times Y} = 1_{X \times Y},
p_{0X,Y}(1_X \times p_{0Y,Z}) = p_{0X,Y} \times p_{1Y,Z}, \quad p_{0X,Z}(1_X \times p_{1Y,Z}) = p_{0X,Y} \times p_{1Z},
p_{1X,Z}(p_{0X,Y} \times 1_Z) = p_{1X,Y} \times p_{1Y,Z}, \quad p_{1Y,Z}(p_{1X,Y} \times 1_Z) = p_{1X,Y} \times Z.
\]

ii) There is an isomorphism

\[
\text{ass}_\times : (-(\times -) \times -) \to (\times (\times -))
\]

of functors \(C^3 \to C\), whose component \(\text{ass}_{X,Y,Z} \times \times \) is given by

\[
\langle p_{0X,Y} p_{0X \times Y,Z}, p_{1X,Y} p_{0X \times Y,Z}, p_{1X \times Y,Z} \rangle : (X \times Y) \times Z \to X \times (Y \times Z),
\]
in which, for morphisms \(f : X \to Y\), \(g : X \to Z\) of \(C\), we set

\[
\langle f, g \rangle = (f \times g)\Delta_X : X \to Y \times Z.
\]

iii) Let \(\text{tr}_\times\) be the endofunctor on \(C \times C\) given by \((X, Y) \mapsto (Y, X)\). There is an isomorphism

\[
\text{tr}_\times : \times \to \times \circ \text{tr}_\times
\]

of functors \(C^2 \to C^2\) whose component \(\text{tr}_{X,Y} \times \times \) is given by

\[
\langle p_{1X,Y}, p_{0X,Y} \rangle : X \times Y \to Y \times X.
\]

The isomorphisms \(\text{ass}_\times\) and \(\text{tr}_\times\) are natural in each of their component variables.

One might say that the projections of a \(P\)-category are “half natural”–while \(p_0(\varphi \times 1_X) = \varphi p_0\) holds, the companion equation \(p_1(\varphi \times 1_X) = 1_X p_1\) fails in general (the right side is always total, the left may not be). In general, the projections \(p_{i,X,Y}\) of a \(P\)-category \(C\) are natural in both variables if and only if for any morphisms \(\varphi_i : Y \to X_i\) (\(i = 0, 1\)) in \(C\), \(p_i(\varphi_0, \varphi_1) = \varphi_i\). Cockett and Lack observe that \(P\)-categories are “symmetric monoidal categories in which each object has a monoidal natural cocommutative coassociative comultiplication (and possibly an unnatural counit)” [CL02].

A category is isotypical if any two objects are isomorphic. A system of zero morphisms is a collection of morphisms \(0_{X,Y} : X \to Y\) for each pair of objects \(X\) and \(Y\) of \(C\) such that for objects \(W, Z\) and morphisms \(f : W \to X\) and \(g : Y \to Z\) of \(C\), one has \(g 0_{X,Y} f = 0_{W,Z}\). A system of zero morphisms is unique if it exists. A \(P\)-category is prodominical if it contains a system of zero morphisms and, for any \(\phi : A \to B, \phi \times 0_{C,D} = 0_{A \times C,B \times D}\). We write \(0_X\) for \(0_{X,X}\) [Hel90]. We will always assume that our \(P\)-categories are prodominical.

2.1. Domains and total morphisms. We recall the definition of the domain of a morphism in a \(P\)-category [DPH87, Ros86].

The domain of a morphism \(\varphi : X \to Y\) in a \(P\)-category is a morphism \(\partial(\varphi) : X \to X\), defined by

\[
\partial(\varphi) = p_0(1_X \times \varphi)\Delta_X = p_1(\varphi \times 1_X)\Delta_X.
\]

The morphism \(\varphi\) is total if \(\partial(\varphi) = 1_X\). We will use the following properties of domains:

\[
\varphi = \varphi \partial(\varphi)
\]

(2)

\[
\partial(\psi \varphi) = \partial(\partial(\psi))\varphi
\]

(3)

\[
\partial(\psi(\partial(\varphi))) = \partial(\psi) \partial(\varphi)
\]

(4)
EVERY $P$-RECURSION CATEGORY HAS AN INDEX COMPOSER

For any object $X$ of $C$, the collection of domains $\partial \varphi$ for some $\varphi : X \to Y$ is denoted by $\text{Dom}(X)$.

The set $\text{Dom}(X)$ forms a commutative meet-semilattice under composition, with least element $0_{X,X}$ and greatest element $1_X$.

The collection $C_T$ of total morphisms of $C$ forms a subcategory [Ros86]; the near product of a $P$-category becomes a categorical product on this subcategory.

In general, a $P$-category is equivalent to one obtained from a category with products, following a procedure analogous to that by which the category $\text{Pfn}$ of sets and partial functions can be obtained from the category $\text{Set}$ of sets [Ros88, Theorem 1.2].

2.2. Turing morphisms and $P$-recursion categories. A Turing morphism in a prodominical isotype is a morphism $\tau : W \times X \to Y$ such that for any $\phi : V \times X \to Y$ there exists a total $g : V \to W$, such that the following diagram commutes.

$$
\begin{array}{ccc}
V \times X & \xrightarrow{g \times 1_X} & W \times X \\
\phi \downarrow & & \tau \downarrow \\
& V \times X & \\
\end{array}
$$

A $P$-recursion category (or, simply, a recursion category) is a prodominical isotype which contains a Turing morphism.

Examples of Turing morphisms and recursion categories are given in [DPH87, Mon89, Hel90, DPM91, Ste93, Len04].

A morphism $e : X \to X$ is idempotent if $e^2 = e$. A morphism $t : X \times X \to X$ in a prodominical isotype $C$ is an (idempotent) weak Turing morphism if for every morphism $f : X \to X$, there exists a total (idempotent) morphism $e : X \to X$ such that the following diagram commutes.

$$
\begin{array}{ccc}
X \times X & \xrightarrow{e \times 1_X} & X \times X \\
\downarrow p_{1,X} & & \downarrow t \\
X \times X & \xrightarrow{f} & X \\
\end{array}
$$

We call the morphism $e$ in (5) a $t$-index of $f$. More generally, a $t$-index is a total morphism $e$ such that for some morphism $f$, the diagram (5) commutes. If $t$ is a Turing morphism, then in the situation of diagram (5), $t$ is also a weak Turing morphism, and we say that $e$ is a $t$-index of $f$.

Every Turing morphism in a $P$-recursion category is a weak Turing morphism; conversely, in a prodominical isotype, a Turing morphism can be obtained from a weak Turing morphism [Hel90, Lemma 9.1].

Lemma 2.0.1. If a prodominical isotype has a weak Turing morphism, then it has a Turing morphism.

3. INDEX COMPOSERS

A total morphism $s : X \times X \to X$ in a $P$-recursion category $C$ is a composer relative to a morphism $t : X \times X \to X$ if for each pair of total morphisms $f, g :
X → X, the following identity holds [Ste93].

\[(6) \quad t(s(f \times g)\Delta_X \times 1_X)\Delta_X = t(f \times 1_X)\Delta_X t(g \times 1_X)\Delta_X.\]

The apparent motivation, from classical recursion theory, of this definition is that there exists a recursive function \( s \) of two variables such that for all natural numbers \( f \) and \( g \),

\[\varphi_{s(f,g)} = \varphi_{f} \varphi_{g}.\]

This assertion is translated in \( P \)-categories by equation \((6)\) if one assumes that index morphisms behave somewhat like constants; however, constant morphisms need not exist in a \( P \)-category.

Let \( t : X \times X \to X \) be a Turing morphism in a \( P \)-category \( C \). Recall that \( t \) is also a weak Turing morphism. In a \( P \)-recursion category \( C \) with Turing morphism \( t : X \times X \to X \), a \textit{t-index composer} is a total morphism \( s : X \times X \to X \) such that for each weak \( t \)-index \( f : X \to X \), and for each weak \( t \)-index \( g : X \to X \), the identity \((6)\) holds.

In the notation of the preceding, let \( \varphi, \psi : X \to X \) be morphisms in the \( P \)-recursion category \( C \), and suppose that \( s \) is a \( t \)-index composer. If \( f \) is a \( t \)-index of \( \varphi \) and if \( g \) is a \( t \)-index of \( \psi \), then equation \((6)\) immediately yields

\[(7) \quad t(s(f \times g)\Delta_X \times 1_X)\Delta_X = \varphi \psi.\]

This shows that composition in the monoid \( C(X, X) \) is definable uniformly in terms of the left hand expression in \((7)\) and the collection of \( t \)-indices.

**Theorem 3.1.** Every \( P \)-recursion category \( C \) contains an index composer.

This fact follows from the coassociativity of the near-product of a \( P \)-category; for completeness, we include a statement and proof of coassociativity.

**Proposition 3.1.1.** In a \( P \)-category, the following diagram commutes.

\[\begin{array}{ccc}
X \times X & \xrightarrow{\Delta_X \times 1_X} & (X \times X) \times X \\
\downarrow & & \downarrow \text{ass}_{X, X, X} \\
X & \xrightarrow{1_X \times \Delta_X} & X \times (X \times X)
\end{array}\]

**Proof.**

\[
\text{ass}_{X, X, X}(\Delta_X \times 1_X)\Delta_X = \langle p_{0,X}p_0 X \times X, X, \langle p_{1,X} X, p_0 X \times X, X, p_{1,X} X, X, X \rangle \rangle (\Delta_X \times 1_X)\Delta_X
\]

\[
= \langle p_{0,X}p_0 X \times X, X, (\Delta_X \times 1_X) \langle p_{1,X} X, p_0 X \times X, X, p_{1,X} X, X, X \rangle \rangle (\Delta_X \times 1_X)\Delta_X
\]

\[
= \langle p_{0,X} X, \Delta_X X, p_0 X, X, \langle p_{1,X} X, X, p_0 X \times X, X, p_{1,X} X, X, X \rangle \rangle (\Delta_X \times 1_X)\Delta_X
\]

\[
= \langle p_{0,X} X, \Delta_X X, p_0 X, X, X \rangle \Delta_X = \langle p_{0,X} X, X, X \rangle \Delta_X \times \Delta_X
\]

\[
= \langle p_{0,X} X \times 1_X, X \rangle (\Delta_X \times \Delta_X) \Delta_X = (p_{0,X} X \Delta_X \times \Delta_X) \Delta_X
\]

\[
= (1_X \times \Delta_X) \Delta_X
\]

\(\square\)

We let \( t : X \times X \to X \) be a Turing morphism in \( C \). A \( t \)-index composer will be exhibited. We make the following observation.
Proposition 3.1.2. If \( e : X \to X \) is a \( t' \)-index of a morphism \( \psi : X \to X \), where \( t' : X \times X \to X \) is a weak Turing morphism, then for any morphism \( \varphi : X \to X \),

\[
   t'(e \times \varphi) \Delta_X = \psi \varphi = t(e \times 1_X) \Delta_X \varphi.
\]

Proof. The first identity follows from the diagram below.

The second identity follows from the diagram below.

\[
\]

3.1. Proof of Theorem 3.1.

Proof. Since \( t \) is a Turing morphism, there exists a total morphism \( s : X \times X \to X \) such that

\[
   (X \times X) \times X \xrightarrow{s \times 1_X} X \times X
\]

Compute as follows; the third and fourth lines use coassociativity, and the last line uses Proposition 3.1.2 with \( e = f \) and \( \varphi = t(g \times 1_X) \Delta_X \).

\[
   t(s(f \times g) \Delta_X \times 1_X) \Delta_X = t(s \times 1_X)((f \times g) \times 1_X)(\Delta_X \times 1_X) \Delta_X
   = t(1_X \times t)(f \times (g \times 1_X)) \Delta_X (\Delta_X \times 1_X) \Delta_X
   = t(f \times (g \times 1_X))(1_X \times \Delta_X) \Delta_X
   = t(f \times (g \times 1_X)) \Delta_X (\Delta_X \times 1_X) \Delta_X
   = t(f \times 1_X) \Delta_X (g \times 1_X) \Delta_X.
\]

Note that the property that \( g \) is a weak \( t \)-index was not used.

Acknowledgements. I thank Sergei Artemov, Alex Heller, Alphonse Vasquez, and Noson Yanofsky for their comments.
References


The Graduate School and University Center, Ph.D Program in Computer Science, The City University of New York, 365 Fifth Avenue, New York, NY 10016