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EVERY P -RECURSION CATEGORY HAS AN INDEX COMPOSER

FLORIAN LENGYEL

ABSTRACT. In a P -recursion category, composition of morphisms is definable uniformly in the collection of indices with respect to a Turing morphism. This fact follows from the coassociativity of the near-product of a P -category.

MSC: 03D75; 03G30

1. INTRODUCTION

In [Ste93], Stefano Stefani defined the notion of a composer, which was intended as a reflection principle for P -recursion categories. A composer in a P -recursion category enables composition of morphisms to be expressed internally. We show that every P -recursion category contains a morphism called an index composer, closely related to a composer in Stefani's sense, which enables a P -category to express composition of any two of its morphisms φ and ψ , uniformly in terms of a fixed expression in the indices of φ and ψ . The existence of an index composer in a P -recursion category follows from the coassociativity of the near-product of a P -category.

A recursive function $f : \omega \rightarrow \omega$ is an *index morphism* if, with respect to some standard numbering of the partial computable functions, f is a recursive enumeration of indices of some fixed partial computable function. In contrast to Stefani's notion of a composer, the definition of which involves quantification over all total morphisms in a P -recursion category, the notion of an index composer reflects that in classical recursion theory, not every recursive function is an index morphism.

2. P - AND P -RECURSION CATEGORIES

Let \mathbf{C} be a category; for our purposes, categories may be taken to be small. The *diagonal functor* $\delta_{\mathbf{C}} : \mathbf{C} \rightarrow \mathbf{C} \times \mathbf{C}$ is given on objects by $A \mapsto (A, A)$ and on morphisms by $f \mapsto (f, f)$.

A P -category consists of a category \mathbf{C} together with a bifunctor $\times : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$, called a *near product*, a natural transformation $\Delta : 1_{\mathbf{C}} \rightarrow \times \circ \delta_{\mathbf{C}}$, and for each object X of \mathbf{C} , natural transformations

$$p_{0(\cdot), X} : (\cdot) \times X \rightarrow 1_{\mathbf{C}}, \quad p_{1X, (\cdot)} : X \times (\cdot) \rightarrow 1_{\mathbf{C}}$$

such that the conditions i), ii) and iii) below are satisfied [Ros86].

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i) The following equations hold.

$$\begin{aligned} p_{0X,X}\Delta_X &= 1_X = p_{1X,X}\Delta_X, & (p_{0X,Y} \times p_{1X,Y})\Delta_{X \times Y} &= 1_{X \times Y}, \\ p_{0X,Y}(1_X \times p_{0Y,Z}) &= p_{0X,Y \times Z}, & p_{0X,Z}(1_X \times p_{1Y,Z}) &= p_{0X,Y \times Z}, \\ p_{1X,Z}(p_{0X,Y} \times 1_Z) &= p_{1X \times Y,Z}, & p_{1Y,Z}(p_{1X,Y} \times 1_Z) &= p_{1X \times Y,Z}. \end{aligned}$$

ii) There is an isomorphism

$$\text{ass}_\times : ((- \times -) \times -) \rightarrow (- \times (- \times -))$$

of functors $\mathbf{C}^3 \rightarrow \mathbf{C}$, whose component $\text{ass}_{\times X,Y,Z}$ is given by

$$\langle p_{0X,Y}p_{0X \times Y,Z}, \langle p_{1X,Y}p_{0X \times Y,Z}, p_{1X \times Y,Z} \rangle \rangle : (X \times Y) \times Z \rightarrow X \times (Y \times Z),$$

in which, for morphisms $f : X \rightarrow Y$, $g : X \rightarrow Z$ of \mathbf{C} , we set

$$\langle f, g \rangle = (f \times g)\Delta_X : X \rightarrow Y \times Z.$$

iii) Let tr_\times be the endofunctor on $\mathbf{C} \times \mathbf{C}$ given by $(X, Y) \mapsto (Y, X)$. There is an isomorphism

$$\text{tr}_\times : \times \rightarrow \times \circ \text{tr}_\times$$

of functors $\mathbf{C}^2 \rightarrow \mathbf{C}^2$ whose component $\text{tr}_{\times X,Y}$ is given by

$$\langle p_{1X,Y}, p_{0X,Y} \rangle : X \times Y \rightarrow Y \times X.$$

The isomorphisms ass_\times and tr_\times are natural in each of their component variables.

One might say that the projections of a P -category are “half natural”–while $p_0(\varphi \times 1_X) = \varphi p_0$ holds, the companion equation $p_1(\varphi \times 1_X) = 1_X p_1$ fails in general (the right side is always total, the left may not be). In general, the projections $p_{i,X,Y}$ of a P -category \mathbf{C} are natural in both variables if and only if for any morphisms $\varphi_i : Y \rightarrow X_i$ ($i = 0, 1$) in \mathbf{C} , $p_i \langle \varphi_0, \varphi_1 \rangle = \varphi_i$. Cockett and Lack observe that P -categories are “symmetric monoidal categories in which each object has a monoidal natural cocommutative coassociative comultiplication (and possibly an unnatural counit)” [CL02].

A category is *isotypical* if any two objects are isomorphic. A *system of zero morphisms* is a collection of morphisms $0_{X,Y} : X \rightarrow Y$ for each pair of objects X and Y of \mathbf{C} such that for objects W, Z and morphisms $f : W \rightarrow X$ and $g : Y \rightarrow Z$ of \mathbf{C} , one has $g0_{X,Y}f = 0_{W,Z}$. A system of zero morphisms is unique if it exists. A P -category is *prodominical* if it contains a system of zero morphisms and, for any $\phi : A \rightarrow B$, $\phi \times 0_{C,D} = 0_{A \times C, B \times D}$. We write 0_X for $0_{X,X}$ [Hel90]. We will always assume that our P -categories are prodominical.

2.1. Domains and total morphisms. We recall the definition of the domain of a morphism in a P -category [DPH87, Ros86].

The domain of a morphism $\varphi : X \rightarrow Y$ in a P -category is a morphism $\partial(\varphi) : X \rightarrow X$, defined by

$$(1) \quad \partial(\varphi) = p_0(1_X \times \varphi)\Delta_X = p_1(\varphi \times 1_X)\Delta_X.$$

The morphism φ is *total* if $\partial(\varphi) = 1_X$. We will use the following properties of domains:

$$\begin{aligned} (2) \quad & \varphi = \varphi \partial(\varphi) \\ (3) \quad & \partial(\psi \varphi) = \partial(\partial(\psi) \varphi) \\ (4) \quad & \partial(\psi(\partial \varphi)) = \partial \psi \partial \varphi \end{aligned}$$

For any object X of \mathbf{C} , the collection of domains $\partial\varphi$ for some $\varphi : X \rightarrow Y$ is denoted by $\text{Dom}(X)$.¹ The set $\text{Dom}(X)$ forms a commutative meet-semilattice under composition, with least element $0_{X,X}$ and greatest element 1_X .

The collection \mathbf{C}_T of total morphisms of \mathbf{C} forms a subcategory [Ros86]; the near product of a P -category becomes a categorical product on this subcategory.

In general, a P -category is equivalent to one obtained from a category with products, following a procedure analogous to that by which the category \mathbf{Pfn} of sets and partial functions can be obtained from the category \mathbf{Set} of sets [Ros88, Theorem 1.2].

2.2. Turing morphisms and P -recursion categories. A *Turing morphism* in a prodominical isotype is a morphism $\tau : W \times X \rightarrow Y$ such that for any $\phi : V \times X \rightarrow Y$ there exists a total $g : V \rightarrow W$, such that the following diagram commutes.

$$\begin{array}{ccc} V \times X & \xrightarrow{g \times 1_X} & W \times X \\ & \searrow \phi & \downarrow \tau \\ & & Y \end{array}$$

A *P -recursion category* (or, simply, a *recursion category*) is a prodominical isotype which contains a Turing morphism.² Examples of Turing morphisms and recursion categories are given in [DPH87, Mon89, Hel90, DPM91, Ste93, Len04].

A morphism $e : X \rightarrow X$ is *idempotent* if $e^2 = e$. A morphism $t : X \times X \rightarrow X$ in a prodominical isotype \mathbf{C} is an (*idempotent*) *weak Turing morphism* if for every morphism $f : X \rightarrow X$, there exists a total (idempotent) morphism $e : X \rightarrow X$ such that the following diagram commutes.

$$(5) \quad \begin{array}{ccc} X \times X & \xrightarrow{e \times 1_X} & X \times X \\ \downarrow p_{1X} & & \downarrow t \\ X & \xrightarrow{f} & X \end{array}$$

We call the morphism e in (5) a *t -index of f* . More generally, a *t -index* is a total morphism e such that for some morphism f , the diagram (5) commutes. If t is a Turing morphism, then in the situation of diagram (5), t is also a weak Turing morphism, and we say that e is a *t -index of f* .

Every Turing morphism in a P -recursion category is a weak Turing morphism; conversely, in a prodominical isotype, a Turing morphism can be obtained from a weak Turing morphism [Hel90, Lemma 9.1].

Lemma 2.0.1. *If a prodominical isotype has a weak Turing morphism, then it has a Turing morphism.*

3. INDEX COMPOSERS

A total morphism $s : X \times X \rightarrow X$ in a P -recursion category \mathbf{C} is a *composer relative to a morphism $t : X \times X \rightarrow X$* if for each pair of total morphisms $f, g :$

¹Since our categories are assumed to be small, $\text{Dom}(X)$ will be a set.

²Previously, the term *recursion category* was used for dominical P -recursion categories.

$X \rightarrow X$, the following identity holds [Ste93].

$$(6) \quad t(s(f \times g)\Delta_X \times 1_X)\Delta_X = t(f \times 1_X)\Delta_X t(g \times 1_X)\Delta_X.$$

The apparent motivation, from classical recursion theory, of this definition is that there exists a recursive function s of two variables such that for all natural numbers f and g ,

$$\varphi_{s(f,g)} = \varphi_f \varphi_g.$$

This assertion is translated in P -categories by equation (6) if one assumes that index morphisms behave somewhat like constants; however, constant morphisms need not exist in a P -category.

Let $t : X \times X \rightarrow X$ be a Turing morphism in a P -category \mathbf{C} . Recall that t is also a weak Turing morphism. In a P -recursion category \mathbf{C} with Turing morphism $t : X \times X \rightarrow X$, a t -index composer is a total morphism $s : X \times X \rightarrow X$ such that for each weak t -index $f : X \rightarrow X$, and for each weak t -index $g : X \rightarrow X$, the identity (6) holds.

In the notation of the preceding, let $\varphi, \psi : X \rightarrow X$ be morphisms in the P -recursion category \mathbf{C} , and suppose that s is a t -index composer. If f is a t -index of φ and if g is a t -index of ψ , then equation (6) immediately yields

$$(7) \quad t(s(f \times g)\Delta_X \times 1_X)\Delta_X = \varphi\psi.$$

This shows that composition in the monoid $\mathbf{C}(X, X)$ is definable uniformly in terms of the left hand expression in (7) and the collection of t -indices.

Theorem 3.1. *Every P -recursion category \mathbf{C} contains an index composer.*

This fact follows from the coassociativity of the near-product of a P -category; for completeness, we include a statement and proof of coassociativity.

Proposition 3.1.1. *In a P -category, the following diagram commutes.*

$$\begin{array}{ccc} & X \times X & \xrightarrow{\Delta_X \times 1_X} & (X \times X) \times X \\ & \nearrow \Delta_X & & \downarrow \text{ass}_{X \times X, X, X} \\ X & & & \\ & \searrow \Delta_X & & \\ & X \times X & \xrightarrow{1_X \times \Delta_X} & X \times (X \times X) \end{array}$$

Proof.

$$\begin{aligned} & \text{ass}_{X \times X, X, X}(\Delta_X \times 1_X)\Delta_X \\ &= \langle p_{0X, X} p_{0X \times X, X}, \langle p_{1X, X} p_{0X \times X, X}, p_{1X \times X, X} \rangle \rangle (\Delta_X \times 1_X)\Delta_X \\ &= \langle p_{0X, X} p_{0X \times X, X}(\Delta_X \times 1_X), \langle p_{1X, X} p_{0X \times X, X}, p_{1X \times X, X} \rangle (\Delta_X \times 1_X) \rangle \Delta_X \\ &= \langle p_{0X, X} \Delta_X p_{0X, X}, \langle p_{1X, X} p_{0X \times X, X}(\Delta_X \times 1_X), p_{1X \times X, X}(\Delta_X \times 1_X) \rangle \rangle \Delta_X \\ &= \langle p_{0X, X}, \langle p_{0X, X}, p_{1X, X} \rangle \rangle \Delta_X = (p_{0X, X} \times 1_{X \times X})\Delta_{X \times X} \Delta_X \\ &= (p_{0X, X} \times 1_{X \times X})(\Delta_X \times \Delta_X)\Delta_X = (p_{0X, X} \Delta_X \times \Delta_X)\Delta_X \\ &= (1_X \times \Delta_X)\Delta_X \end{aligned}$$

□

We let $t : X \times X \rightarrow X$ be a Turing morphism in \mathbf{C} . A t -index composer will be exhibited. We make the following observation.

Proposition 3.1.2. *If $e : X \rightarrow X$ is a t' -index of a morphism $\psi : X \rightarrow X$, where $t' : X \times X \rightarrow X$ is a weak Turing morphism, then for any morphism $\varphi : X \rightarrow X$, $t'(e \times \varphi)\Delta_X = \psi\varphi = t(e \times 1_X)\Delta_X\varphi$.*

Proof. The first identity follows from the diagram below.

$$\begin{array}{ccccccc}
 X & \xrightarrow{\Delta_X} & X \times X & \xrightarrow{1_X \times \varphi} & X \times X & \xrightarrow{e \times 1_X} & X \times X \\
 & \searrow^{1_X} & \downarrow p_{1X} & & \downarrow p_{1X} & & \downarrow t' \\
 & & X & \xrightarrow{\varphi} & X & \xrightarrow{\psi} & X
 \end{array}$$

The second identity follows from the diagram below.

$$\begin{array}{ccccccc}
 X & \xrightarrow{\Delta_X} & X \times X & \xrightarrow{e \times 1_X} & X \times X & & \\
 & \searrow^{1_X} & \downarrow p_{1X} & & \downarrow t' & & \\
 & & X & \xrightarrow{\psi} & X & &
 \end{array}$$

□

3.1. Proof of Theorem 3.1.

Proof. Since t is a Turing morphism, there exists a total morphism $s : X \times X \rightarrow X$ such that

$$\begin{array}{ccc}
 (X \times X) \times X & \xrightarrow{s \times 1_X} & X \times X \\
 \text{ass}_\times \downarrow & & \downarrow t \\
 X \times (X \times X) & \xrightarrow{t(1_X \times t)} & X
 \end{array}$$

Here ass_\times is the natural associativity isomorphism.

Compute as follows; the third and fourth lines use coassociativity, and the last line uses Proposition 3.1.2 with $e = f$ and $\varphi = t(g \times 1_X)\Delta_X$.

$$\begin{aligned}
 t(s(f \times g)\Delta_X \times 1_X)\Delta_X &= t(s \times 1_X)((f \times g) \times 1_X)(\Delta_X \times 1_X)\Delta_X \\
 &= t(1_X \times t)\text{ass}_\times((f \times g) \times 1_X)(\Delta_X \times 1_X)\Delta_X \\
 &= t(1_X \times t)(f \times (g \times 1_X))\text{ass}_\times(\Delta_X \times 1_X)\Delta_X \\
 &= t(f \times t(g \times 1_X))(1_X \times \Delta_X)\Delta_X \\
 &= t(f \times t(g \times 1_X)\Delta_X)\Delta_X \\
 &= t(f \times 1_X)\Delta_X t(g \times 1_X)\Delta_X.
 \end{aligned}$$

□

Note that the property that g is a weak t -index was not used.

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