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LOCALLY CONNECTED RECURSION CATEGORIES

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Abstract. A recursion category is locally connected if connected domains are jointly epimorphic. New proofs of the existence of non-complemented and recursively inseparable domains are given in a locally connected category. The use of local connectedness to produce categorical analogs of undecidable problems is new; the approach allows us to relax the hypotheses under which the results were originally proved. The results are generalized to non-locally connected recursion categories by transporting the range restriction category structure of a non-locally connected recursion category to a locally connected restriction category by means of a *range* functor; i.e., a functor that preserves coproducts, restrictions and ranges; a range functor need not preserve the near-product.

MSC: 03D75; 03G30

1. Introduction

A recursion category is locally connected if connected domains are jointly epimorphic. Proofs of the existence of non-complemented and recursively inseparable domains are given for locally connected recursion categories. The results are generalized to non-locally connected recursion categories by transporting the range restriction category structure of a non-locally connected recursion category to a locally connected restriction category by means of a *range* functor $F : \mathbf{C} \to \mathbf{D}$; i.e., a functor that preserves coproducts, restrictions and ranges; the functor F need not preserve the near-product of $\mathbf{C}$.

The use of local connectedness to produce categorical analogs of undecidable problems is new; the approach allows us to relax the hypotheses under which the results were originally proved [DPH87, Ros86, Mon89, Hel90, DPM91, Ste93].

Di Paola and Heller proved the existence of recursively inseparable domains in a dominical recursion category $\mathbf{C}$ under the assumption that $\mathbf{C}$ satisfies the axiom of choice [DPH87, Theorem 8.15]. Rosolini later generalized this to a P-category satisfying the axiom of choice [Ros88a]. Here we produce a pair of recursively inseparable domains in a locally connected P-recursion category $\mathbf{C}$ which satisfies the weak axiom of choice, as well as a criterion, valid in classical recursion theory, for a partial morphism to be a partial monomorphism; provided an appropriate range functor $F : \mathbf{C} \to \mathbf{D}$ exists, the additional hypotheses on $\mathbf{C}$ can be transferred to $\mathbf{D}$.

Recursion categories were originally developed with the (as yet unrealized) goal of formulating and proving an algebraic statement (which mentions no notion of logic) equivalent to one of the generalized Gödel incompleteness theorems.

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Somewhat against the point-free philosophy of category theory, researchers in the subject have defined and used various categorical surrogates for points, such as atoms, constants, and connected domains [DPH87, Hel90, DPM91, Len04]. Connected domains arose in connection with Heller's existence theorem for recursion categories, which applied to locally connected categories [Hel90]. Di Paola and Montagna produced examples of non-locally connected recursion categories [DPM91]; they raised the question of generalizing Heller's existence theorem to handle this case; this question was addressed in [Len04]. Here we follow the approach of [Len04], in which range functors were used.

The dominical categories of Di Paola and Heller were the original categorical setting for recursion categories; the setting promptly changed to that of Rosolini's more general P-categories after their introduction [Ros86, Mon89, Hel90, DPM91, Ste93]. While P-categories superceded the dominical categories, notions of classical computability theory, as these were originally defined for dominical categories, led to pathologies in P-categories: weakly total morphisms are always total in a dominical category, but need not be total in a P-category; Myhill's theorem that creative domains are m-complete may fail in a P-category; and point-like morphisms, such as constants and connected domains, may fail to exist [Mon89, Ros86, DPM91].

Most recently, Cockett and Lack defined the restriction categories; under an equivalent axiomatization, Stefano Stefani defined the RDP-categories. These are categories with the extra structure of a unary operation on morphisms; the domain of a partial morphism can be defined directly using this operator, without the additional structure of a near product [CL02, CL03, Ste02].

Restriction categories are precisely the full subcategories of categories of partial maps (with the "induced" restriction structure). The language and axiomatization of restriction categories is convenient; however, we rely on the near-product structure of a P-category for the definiability of Turing morphisms and the existence of diagonals for diagonalization arguments, so our results will be stated in this context.

2. Restriction categories, P-categories and P-recursion categories

2.1. **Restriction categories.** A *restriction category* is a category $\mathbf{C}$ with a unary operation on morphisms called *restriction*, which sends each morphism $f : X \to Y$ in $\mathbf{C}$ to the morphism $\overline{f} : X \to X$, called a *restriction idempotent*, subject to the following four axioms, which hold whenever the indicated compositions are defined [CL02, CL03].

R1. $f\overline{f} = f$.
R2. $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$.
R3. $\overline{g\overline{f}} = \overline{g}\,\overline{f}$.
R4. $\overline{g}f = f\overline{gf}$.

These axioms imply a frequently used identity.

$$\overline{gf} = \overline{\overline{g}f}. \tag{1}$$

A restriction category $\mathbf{C}$ is a *split restriction category* if every restriction idempotent $f : X \to X$ in $\mathbf{C}$ splits; i.e., there is a commutative diagram with m mono, as below.

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "\overline{f}"'] & Z \arrow[d, "m"] \arrow[dr, equal, "1_Z"] & \\ & X \arrow[r, "e"] & Z \end{tikzcd}$$

We will be concerned with a special class of restriction categories: the P-categories.

2.2. **P-categories.** Let $\mathbf{C}$ be a category. The *diagonal functor* $\Delta_{\mathbf{C}} : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ is given on objects by $A \mapsto (A, A)$ and on morphisms by $f \mapsto (f, f)$.

A *P-category* consists of a category $\mathbf{C}$ together with a bifunctor $\times : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, called a *near product*, a natural transformation $\Delta : 1_{\mathbf{C}} \to \times \circ \Delta_{\mathbf{C}}$, and for each object X of $\mathbf{C}$, natural transformations

$$p_{0(\ ),X} : (\ ) \times X \to 1_{\mathbf{C}}, \quad p_{1X,(\ )} : X \times (\ ) \to 1_{\mathbf{C}}$$

such that the conditions i), ii) and iii) below are satisfied [Ros86].

i) The following equations hold.

$$\begin{array}{ll} p_{0X,X}\Delta_X = 1_X = p_{1X,X}\Delta_X, & (p_{0X,Y} \times p_{1X,Y})\Delta_{X\times Y} = 1_{X\times Y}, \\ p_{0X,Y}(1_X \times p_{0Y,Z}) = p_{0X,Y\times Z}, & p_{0X,Z}(1_X \times p_{1Y,Z}) = p_{0X,Y\times Z}, \\ p_{1X,Z}(p_{0X,Y} \times 1_Z) = p_{1X\times Y,Z}, & p_{1Y,Z}(p_{1X,Y} \times 1_Z) = p_{1X\times Y,Z}. \end{array}$$

ii) There is an isomorphism

$$\mathrm{ass}_\times : ((- \times -) \times -) \to (- \times (- \times -))$$

of functors $\mathbf{C}^3 \to \mathbf{C}$, whose component $\mathrm{ass}_{\times X,Y,Z}$ is given by

$$\langle p_{0X,Y}p_{0X\times Y,Z}, \langle p_{1X,Y}p_{0X\times Y,Z}, p_{1X\times Y,Z}\rangle\rangle : (X \times Y) \times Z \to X \times (Y \times Z),$$

in which, for morphisms $f : X \to Y$, $g : X \to Z$ of $\mathbf{C}$, we set

$$\langle f,\ g\rangle = (f \times g)\Delta_X : X \to Y \times Z.$$

iii) Let $\mathrm{tr}_\times$ be the endofunctor on $\mathbf{C} \times \mathbf{C}$ given by $(X, Y) \mapsto (Y, X)$. There is an isomorphism

$$\mathrm{tr}_\times : \times \to \times \circ \mathrm{tr}_\times$$

of functors $\mathbf{C}^2 \to \mathbf{C}^2$ whose component $\mathrm{tr}_{\times X,Y}$ is given by

$$\langle p_{1X,Y},\ p_{0X,Y}\rangle : X \times Y \to Y \times X.$$

The isomorphisms $\mathrm{ass}_\times$ and $\mathrm{tr}_\times$ are natural in each of their component variables.

One might say that the projections of a P-category are "half natural"–while $p_0(\varphi \times 1_X) = \varphi p_0$ holds, the companion equation $p_1(\varphi \times 1_X) = 1_X p_1$ fails in general (the right side is always total, the left may not be). In general, the projections $p_{i,X,Y}$ of a P-category $\mathbf{C}$ are natural in both variables if and only if for any morphisms $\varphi_i : Y \to X_i$ $(i = 0, 1)$ in $\mathbf{C}$, $p_i\langle\varphi_0, \varphi_1\rangle = \varphi_i$. Cockett and Lack observe that P-categories are "symmetric monoidal categories in which each object has a a monoidal natural cocommutative coassociative comultiplication (and possibly an unnatural counit)" [CL02].

Dually, one may speak of a category with a binary coproduct $+$, together with natural transformations $i_0, i_1, \nabla, \mathrm{ass}_+, \mathrm{tr}_+$ satisfying the duals of conditions i), ii) and iii), where the *injections* i_0, i_1 replace the *projections* p_0, p_1, the *codiagonal* ∇ replaces the *diagonal* Δ, where ass_+ and tr_+ replace $\mathrm{ass}_\times$ and $\mathrm{tr}_\times$, respectively, and where $+$ replaces $\times$.

A P-category *has coproducts* if it has a binary coproduct as above, in which the coproduct injections i_j and the codiagonals ∇_X are required to be natural.

A category is *isotypical* if any two objects are isomorphic. A *system of zero morphisms* is a collection of morphisms $0_{X,Y} : X \to Y$ for each pair of objects X and Y of $\mathbf{C}$ such that for objects W, Z and morphisms $f : W \to X$ and $g : Y \to Z$ of $\mathbf{C}$, one has $g0_{X,Y}f = 0_{W,Z}$. A system of zero morphisms is unique if it exists. A P-category is *prodominical* if it contains a system of zero morphisms and, for any $\phi : A \to B$, $\phi \times 0_{C,D} = 0_{A\times C,B\times D}$. We write 0_X for $0_{X,X}$ [Hel90]. We will always assume that our P-categories are prodominical.

2.3. **Restrictions in P-categories.** The unary operation of restriction is defined in a P-category as follows. A morphism $\varphi : X \to Y$ in a P-category goes to the restriction idempotent $\overline{\varphi} : X \to X$, defined by the following [DPH87, Ros86].

$$\overline{\varphi} = p_0(1_X \times \varphi)\Delta_X = p_1(\varphi \times 1_X)\Delta_X. \tag{2}$$

This definition satisfies the axioms R1 through R4 of a restriction category. The restriction idempotents were originally termed domains; we may revert to this older usage when it is clear in context that a restriction idempotent, and not the source object of a morphism, is intended.

For any object X of $\mathbf{C}$, the collection of restriction idempotents $\overline{\varphi}$ for some $\varphi : X \to Y$ is denoted by $\mathrm{Dom}(X)$. Since $\mathrm{Dom}(X) \subseteq \mathbf{C}(X, X)$, $\mathrm{Dom}(X)$ is a set; it forms a commutative meet-semilattice under composition, with least element 0_X and greatest element 1_X.

The morphism φ is *total* if $\overline{\varphi} = 1_X$. The collection $\mathbf{C}_T$ of total morphisms of $\mathbf{C}$ forms a subcategory [Ros86]; the near product of a P-category becomes a categorical product on this subcategory.

In general, a P-category is equivalent to one obtained from a category with products, following a procedure analogous to that by which the category **Pfn** of sets and partial functions can be obtained from the category **Set** of sets [Ros88b, Theorem 1.2].

2.4. **Turing morphisms and P-recursion categories.** A *Turing morphism* in a prodominical isotype is a morphism $\tau : W \times X \to Y$ such that for any $\phi : V \times X \to Y$ there exists a total $g : V \to W$, such that the following diagram commutes.

$$\begin{tikzcd} V \times X \arrow[r, "g\times 1_X"] \arrow[dr, "\phi"'] & W \times X \arrow[d, "\tau"] \\ & Y \end{tikzcd}$$

A *P-recursion category* (or, simply, a *recursion category*) is a prodominical isotype which contains a Turing morphism. Previously, the term *recursion category* was used for dominical P-recursion categories; for us a recursion category is always a P-recursion category. The prototypical example of a Turing morphism $t : X \times X \to X$ is the universal partial function of classical recursion theory [Odi89, Theorem II.1.8, page 132]. Examples of Turing morphisms and recursion categories are given in [DPH87, Mon89, Hel90, DPM91, Ste93, Len04].

A morphism $e : X \to X$ is *idempotent* if $e^2 = e$. A morphism $t : X \times X \to X$ in $\mathbf{C}$ is an *(idempotent) weak Turing morphism* if for every morphism $f : X \to X$, there exists a total (idempotent) morphism $e : X \to X$ such that the following diagram commutes.

$$\begin{tikzcd} X \times X \arrow[r, "e\times 1_X"] \arrow[d, "p_{1X}"] & X \times X \arrow[d, "t"] \\ X \arrow[r, "f"] & X \end{tikzcd} \tag{3}$$

We call the morphism e in (3) a *t-index of f*. More generally, a *t-index* is a total morphism e such that for some morphism f, the diagram (3) commutes. If t is a Turing morphism, then in the situation of diagram (3), t is also a weak Turing morphism, and we say that e is a t-index of f. Conversely, a Turing morphism can be obtained from a weak Turing morphism [Hel90, Lemma 9.1].

It can simplify calculations to observe that any P-recursion category has a weak idempotent Turing morphism. We state the following without proof.

Proposition 2.0.1. *Every P-recursion category $\mathbf{C}$ has an idempotent weak Turing morphism t, together with a mapping F such that for every t-index e of φ in $\mathbf{C}$, $F(e)$ is an idempotent t-index of φ.*

3. A domain with no complement

We show that if t is a weak Turing morphism in a locally connected recursion category $\mathbf{C}$, the domain $\overline{t\Delta_X}$ has no complement. The same conclusion holds more generally if $\mathbf{C}$ is a non-locally connected recursion category with binary coproducts and ranges, provided there exists a range functor $F : \mathbf{C} \to \mathbf{D}$ to a locally connected P-category $\mathbf{D}$.

Di Paola and Heller showed that in a dominical recursion category, the domain $\kappa = \overline{t\Delta_X}$ has no quasicomplement [DPH87, Corollary 8.6]. A domain $\delta \in \mathrm{Dom}(X)$ is a *quasicomplement* of $\varepsilon \in \mathrm{Dom}(X)$ if $\delta\epsilon = 0$ and for each morphism ϕ with codomain X, $\varepsilon\phi = \delta\phi$ implies $\phi = 0$. Complements and quasicomplements coincide in a dominical category. More can be said. A morphism ϕ is *weakly total* if for each morphism ψ, $\phi\psi = 0$ implies ψ=0, whenever the composition makes sense.

Proposition 3.0.2. *In a P-category with a system of zeros, binary coproducts and ranges, the following assertions are equivalent.*

(i) Weakly total morphisms are total.

(ii) [DPH87, Proposition 5.3] *If $\varepsilon \in \mathrm{Dom}(X)$ satisfies $\varepsilon \neq 1_X$, there exists $\delta \in \mathrm{Dom}(X)$ with $\delta \neq 0$ and $\varepsilon\delta = 0$.*

(iii) [DPH87, Proposition 7.3] *Complements and quasicomplements coincide.*

Rosolini has written in [Ros88a, page 313] that in view of his representation Theorem 2.8 (*loc. cit.*), all the results of section 5 of [DPH87] "...can be generalized to P-categories with ranges stable under products by checking that they hold in a P-category of the form $\mathcal{M}$–Ptl($\mathbf{A}$)." However, the preceding proposition shows this is false for Proposition 5.3 of [DPH87]. In any case, if a domain has no quasicomplement, it has no complement, but we prove that the domain $\overline{t\Delta_X}$ has no complement in P-categories more general than dominical categories; also no appeal is made to criteria of creativeness, as in [DPH87, Corollary 8.6].

3.1. **Ranges.** If $\varepsilon \in \mathrm{Dom}(Y)$, we write $\phi \prec \varepsilon$ if and only if $\varepsilon\phi = \phi$; we say that ε *receives* ϕ. If ε receives ϕ, and in addition, ε satisfies for all appropriate ψ, ψ', $\psi\phi = \psi'\phi$ implies $\psi\varepsilon = \psi'\varepsilon$, then ε is the least domain in $\mathrm{Dom}(Y)$ receiving ϕ. In this case we say that ϕ has range ε and we write $\mathrm{ran}\phi = \varepsilon$.

The receives relation $\prec$, which holds between a morphism and a domain, is to be distinguished from the partial order $\leq$ on domains, where they coincide. We record the following seemingly innocuous fact.

Proposition 3.0.3. *If $\phi : X \to Y$ is a morphism and β, γ are domains in Y such that $\phi \prec \beta$ and $\beta \leq \gamma$, then $\phi \prec \gamma$.*

Proof. We have $\phi = \beta\phi = (\beta\gamma)\phi = \gamma(\beta\phi) = \gamma\phi$, so that $\phi \prec \gamma$. □

If every morphism of the P-category $\mathbf{C}$ has a range, and if for morphisms ϕ, ψ of $\mathbf{C}$, $\mathrm{ran}(\phi \times \psi) = \mathrm{ran}\phi \times \mathrm{ran}\psi$, then one says that $\mathbf{C}$ *has ranges*; such a category has a calculus of ranges [Hel90, DPH87].

In a P-category with coproducts, if f and g are morphisms with the same codomain Y, then we define $[f,\ g] = \nabla_Y(f + g)$. In an P-category with coproducts and ranges, the meet semilattice $\mathrm{Dom}(X)$ becomes a distributive lattice if one defines the join by $\varepsilon \vee \delta = \mathrm{ran}[\varepsilon,\ \delta]$ for $\varepsilon, \delta \in \mathrm{Dom}(X)$ [DPH87, Hel90]. The next statement follows immediately from the definitions [DPH87, Proposition 5.4].

Proposition 3.0.4. *If ε is a domain in $\mathrm{Dom}(Y)$ and $\varphi : X \to Y$ is any morphism, then $\mathrm{ran}(\varepsilon\varphi) = \varepsilon\mathrm{ran}(\varphi)$.*

The next fact follows from the preceding.

Proposition 3.0.5. *For any morphisms $\varphi : X \to Y, \psi : Y \to Z$, $\psi\varphi = 0$ if and only if $\overline{\psi}\mathrm{ran}(\varphi) = 0$.*

Proof. If $\overline{\psi}\mathrm{ran}(\varphi) = 0$, then $\psi\varphi = \psi\overline{\psi}\mathrm{ran}(\varphi)\varphi = 0$. Conversely, if $\psi\varphi = 0$, then $0 = \overline{\psi\varphi} = \overline{\overline{\psi}\varphi}$ by equation (1), which implies that $\overline{\psi}\varphi = 0$. Take ranges and apply the preceding proposition. □

Let $f : X \to Y$ be a morphism in $\mathbf{C}$, and let $\beta \in \mathrm{Dom}(Y)$ be a domain in Y. Define the *pre-image* $\phi^* : \mathrm{Dom}(Y) \to \mathrm{Dom}(X)$ by $f^*\beta = \overline{\beta f}$. If the P-category $\mathbf{C}$ has has ranges, then for a morphism $\phi : X \to Y$ we define the image $\phi_* : \mathrm{Dom}(X) \to \mathrm{Dom}(Y)$ by $\phi_*\alpha = \mathrm{ran}(\phi\alpha)$ for $\alpha \in \mathrm{Dom}(X)$. The pre-image ϕ^* preserves meets and joins; the image ϕ_* preserves joins. The following fact will be used below [DPH87, Cor. 5.6 (iii)].

Proposition 3.0.6. *Let $\phi : X \to Y$ be a morphism, and let $\alpha \in \mathrm{Dom}(X)$ and $\beta \in \mathrm{Dom}(Y)$, then $\phi_*(\alpha)\beta = \phi_*(\alpha\phi^*\beta)$.*

For ϕ total we have that ϕ_* is left adjoint to ϕ^* [Hel90].

Proposition 3.0.7. *If $\phi : X \to Y$ is total in a P-category with ranges, then for each $\alpha \in \mathrm{Dom}(X)$ and for each $\beta \in \mathrm{Dom}(Y)$, $\phi_*\alpha \leq \beta$ if and only if $\alpha \leq \phi^*\beta$.*

Adjointness is a consequence of the next proposition, which will be used in the proof of Theorem 4.1 below.

Proposition 3.0.8. *If $\phi : X \to Y$ is a morphism in a P-category $\mathbf{C}$ with ranges, then for each $\alpha \in \mathrm{Dom}(X)$ and for each $\beta \in \mathrm{Dom}(Y)$, then*
a) $\phi_*\alpha \leq \beta$ implies $\alpha\phi^*\beta = \alpha\overline{\phi}$; and
b) $\alpha \leq \phi^*\beta$ implies $\phi_*\alpha \leq \beta$.

Proof. For a), the condition implies $\beta\mathrm{ran}(\phi\alpha) = \mathrm{ran}(\phi\alpha)$ Composing on the right with $\phi\alpha$, we have $\beta\phi\alpha = \phi\alpha$. Taking domains, we have

$$\overline{\beta\phi}\alpha = \overline{\beta\phi\alpha} = \overline{\phi\alpha} = \overline{\phi}\alpha.$$

Assertion b) follows from Proposition 3.0.6. □

3.2. **A calculus of connected domains.** Connected domains are a surrogate in P-categories for points; they first appeared in [Hel90] and subsequently in [DPM91, Len04]. A domain α is *connected* if $\alpha = \beta \vee \gamma$ for domains β, γ with $\beta \wedge \gamma = 0$, then either $\beta = 0$ or $\gamma = 0$. The zero domain is always connected. A P-category is *locally connected* if it has ranges and binary coproducts, and for all morphisms $f, g : X \to Y$, $f = g$ if and only if for every connected domain α in X, $f\alpha = g\alpha$ [Hel90]. With respect to connected domains, total morphisms are analogous to continuous functions: the image under a total morphism of a connected domain is connected. This fact leads to an algebraic device for distinguishing among mutually exclusive alternatives.

Proposition 3.0.9. *Suppose that $\alpha \in \mathrm{Dom}(X)$ is connected.*
(i) If $f : X \to Y$ is total, then $f_*\alpha$ is connected.
(ii) If $f : X \to Y + Z$ is total, then $f_*\alpha \leq 1_Y + 0_Z$ or $f_*\alpha \leq 0_Y + 1_Z$.
(iii) If $f : X \to Y + Z$ is total, then $f\alpha \prec 1_Y + 0_Z$ or $f\alpha \prec 0_Y + 1_Z$.

Proof. The proof of (i) uses adjointness and Proposition 3.0.6. Suppose that $\phi_*\alpha = \beta \vee \gamma$. Since ϕ is total, by adjointness, $\alpha \leq \phi^*(\beta \vee \gamma) = \phi^*\beta \vee \phi^*\gamma$; equivalently, $\alpha = \alpha\phi^*\beta \vee \alpha\phi^*\gamma$. We have that $\alpha(\phi^*\beta)\alpha(\phi^*\gamma) = \alpha\phi^*(\beta\gamma) = \alpha\overline{0\phi} = 0$. Therefore, either $\alpha\phi^*\beta = 0$ or $\alpha\phi^*\gamma = 0$. If $\alpha\phi^*\beta = 0$, then $0 = \phi_*(\alpha\phi^*\beta) = \phi_*(\alpha)\beta = (\beta \vee \gamma)\beta = \beta$. Similarly, if $\alpha\phi^*\gamma = 0$, $\gamma = 0$.

For (ii), note that $f_*\alpha = f_*\alpha \wedge (1_Y + 1_Z) = f_*\alpha \wedge ((1_Y + 0) \vee (0 + 1_Z)) = (f_*\alpha \wedge (1_Y + 0)) \vee (f_*\alpha \wedge (0 + 1_Z))$; the conclusion follows from (i).

For (iii), note that $f_*\alpha$ is a range, so that $f\alpha \prec f_*\alpha$; i.e., $(f_*\alpha)f\alpha = f\alpha$. The result follows from (ii) and Proposition 3.0.3. □

3.3. **The domain $\overline{t\Delta_X}$.** A domain $\gamma \in \mathrm{Dom}(X)$ is *complemented* if there exists a domain $\gamma^\perp \in \mathrm{Dom}(X)$ such that such that $\gamma \vee \gamma^\perp = 1_X$ and $\gamma \wedge \gamma^\perp = 0$. If γ is complemented, $\gamma^\perp$ denotes the unique complement of γ in $\mathrm{Dom}(X)$.

Let $\mathbf{C}, \mathbf{D}$ be P-categories with coproducts and ranges. A functor $F : \mathbf{C} \to \mathbf{D}$ that preserves the restriction, coproducts, ranges, and zero-morphisms is called a *range* functor. A range functor preserves total morphisms; and it preserves joins, since it commutes with coproducts and ranges. A range functor need not preserve the near-product; examples of such range functors, involving P-categories of coalgebras for an appropriate **Set** endofunctor, were given in [Len04]. Range functors will be used to relate a recursion category which may not have point-like morphisms to a P-category which has them. Let $\mathbf{C}$ and $\mathbf{D}$ be P-categories each with a system of zeros (e.g., prodominical categories) and let $F : \mathbf{C} \to \mathbf{D}$ be a range functor. We define $\ker F$, the *kernel* of F, by

$$\ker F := \{f : X \to Y \text{ in } \mathbf{C} : F(f) = 0_{FXFY}\}$$

The kernel is closed under composition.

Proposition 3.0.10. *If $F : \mathbf{C} \to \mathbf{D}$ is a range functor and if $\mathbf{D}$ is locally connected, then for each object X in $\mathbf{C}$ such that $1_X \notin \ker F$, there exists a nonzero connected domain in $\mathrm{Dom}(FX)$.*

Proof. Suppose that 0 is the only connected domain in $\mathrm{Dom}(FX)$. Then $1_{FX}0 = 0$, which implies that $F1_X = 1_{FX} = 0$, since $\mathbf{D}$ is locally connected. But then $1_X \in \ker F$, contradiction. $\square$

Theorem 3.1. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be prodominical categories with binary coproducts and ranges, suppose that* $\mathbf{C}$ *is a* P*-recursion category, suppose that* $\mathbf{D}$ *is locally connected and there exists a range functor* $F : \mathbf{C} \to \mathbf{D}$. *Let* $t : X \times X \to X$ *be a weak Turing morphism in* $\mathbf{C}$ *with* $1_X \notin \ker F$. *Then* $\overline{t\Delta_X}$ *is not complemented.*

Proof. Suppose otherwise that there exists a domain ε such that

$$\overline{t\Delta_X} \vee \varepsilon = 1_X \text{ and } \overline{t\Delta_X} \wedge \varepsilon = 0. \tag{4}$$

Apply the functor F to obtain the following; recall that the meet $\wedge$ is composition.

$$\overline{F(t\Delta_X)} \vee F\varepsilon = 1_{FX} \text{ and } \overline{F(t\Delta_X)} \wedge F\varepsilon = 0. \tag{5}$$

Let $e : X \to X$ be a t-index (in $\mathbf{C}$) of ε; we may assume that e is idempotent. Since $1_X \notin \ker F$, by Proposition 3.0.10 there exists a connected domain ψ in FX. Since range functors preserve total morphisms, Fe is total, so by Proposition 3.0.9, $\mathrm{ran}((Fe)\psi)$ is a connected domain in $\mathbf{D}$. We compute as follows.

$$\begin{aligned} F(t\Delta_X e)\psi &= F(t(e\times e)\Delta_X)\psi = F(t(e^2\times e)\Delta_X)\psi \\ &= F(t(e\times 1_X)(e\times e)\Delta_X)\psi = F(t(e\times 1_X)\Delta_X e)\psi \\ &= F(\varepsilon e)\psi. \end{aligned} \tag{6}$$

Since $\mathrm{ran}((Fe)\psi)$ is connected, either

$$\mathrm{ran}((Fe)\psi) \le \overline{F(t\Delta_X)} \text{ or } \mathrm{ran}((Fe)\psi) \le F\varepsilon.$$

Suppose that $\mathrm{ran}((Fe)\psi) \le \overline{F(t\Delta_X)}$. We have by (5) that

$$(F\varepsilon)\mathrm{ran}((Fe)\psi) \le (F\varepsilon)\overline{F(t\Delta_X)} = 0,$$

so by (6),

$$\begin{aligned} 0 &= (F\varepsilon)\mathrm{ran}((Fe)\psi) = (F\varepsilon)\mathrm{ran}((Fe)\psi)(Fe)\psi \\ &= (F\varepsilon)(Fe)\psi = F(\varepsilon e)\psi = F(t\Delta_X e)\psi = F(t\Delta_X)(Fe)\psi. \end{aligned}$$

By Proposition 3.0.5, $\overline{F(t\Delta_X)}\mathrm{ran}((Fe)\psi) = 0$; by assumption the left-hand-side is $\mathrm{ran}((Fe)\psi)$, which is nonzero by assumption; this is a contradiction.

Now, suppose that $\mathrm{ran}((Fe)\psi) \le F\varepsilon$. Observe that

$$\overline{F(t\Delta_X)}\mathrm{ran}((Fe)\psi) \le F\overline{t\Delta_X}F\varepsilon = F(\overline{t\Delta_X}\varepsilon) = F0 = 0,$$

which, by Proposition 3.0.5 and by (6), implies that

$$0 = F(t\Delta_X)F(e)\psi = F(t\Delta_X e)\psi = F(\varepsilon e)\psi = F(\varepsilon)(Fe)\psi.$$

By Proposition 3.0.5, $\overline{F\varepsilon}\mathrm{ran}((Fe)\psi) = 0$. However, F preserves domains, and so $F\varepsilon = \overline{F\varepsilon}$; moreover, by assumption, $\overline{F\varepsilon}\mathrm{ran}((Fe)\psi) = \mathrm{ran}((Fe)\psi)$, which is a nonzero connected domain. This is a contradiction; it follows that $\overline{t\Delta_X}$ cannot have a complement in 1_X. $\square$

4. Recursively inseparable domains and local connectedness.

Di Paola and Heller proved the existence of recursively inseparable domains in a dominical recursion category **C** with coproducts, under the assumption that **C** satisfies the axiom of choice [DPH87, Theorem 8.15]. Rosolini removed the requirement that **C** be dominical and that products distribute over coproducts, provided domains split in **C** [Ros88a].

We prove the existence of recursively inseparable domains in a P-recursion category **C** for which there exists a range functor $F : \mathbf{C} \to \mathbf{D}$ to a locally connected split P-recursion category **D** which satisfies the weak axiom of choice.

4.1. **The weak axiom of choice and a test for partial monomorphisms.** In a P-category, a *section* of a morphism $\phi : X \to Y$ is a morphism $\sigma : Y \to X$ such that $\phi\sigma = \overline{\sigma}$ and $\phi\sigma\phi = \phi$. It follows that $\overline{\sigma} = \mathrm{ran}(\phi)$. A P-category satisfies the *axiom of choice* if every morphism has a section. A morphism $\phi : X \to Y$ is a *partial monomorphism* if for all morphisms θ, θ', $\phi\theta = \phi\theta'$ implies $\overline{\phi}\theta = \overline{\phi}\theta'$. A P-category **C** satisfies the *weak axiom of choice* if every partial monomorphism has a section.

In classical recursion theory, to check that the partial recursive function ϕ is a partial monomorphism, it suffices to check that for all (total) recursive functions θ, θ', $\phi\theta = \phi\theta'$ implies $\overline{\phi}\theta = \overline{\phi}\theta'$. This follows from the fact that a recursively enumerable set is either empty or the image of a total recursive function. The following is a generalization.

Proposition 4.0.1. *Let* **C** *be a split restriction category, and let $f : X \to Y$ b a morphism in* **C**. *Suppose that for all morphisms $g, h \in \mathbf{C}_T$, $fg = fh$ implies $\overline{f}g = \overline{f}h$. Then f is a partial monomorphism.*

Proof. Let $g, h : W \to X$ in **C** satisfy $fg = fh$ and let $e : X \to Z$, $m : Z \to X$ be a splitting of $\overline{fg} = \overline{fh}$, with m mono, as in the following diagram.

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, "\overline{fg}=\overline{fh}"'] & Z \arrow[d, "m"] \arrow[dr, equal, "1_Z"] & \\
& X \arrow[r, "e"] & Z
\end{tikzcd}$$

Since m is mono, $\overline{m} = 1_Z$; consequently fgm and fhm are total:

$$\overline{fgm} = \overline{\overline{fg}m} = \overline{mem} = \overline{m} = 1_Z.$$

Therefore gm and hm are total. By the hypothesis, $\overline{f}gm = \overline{f}hm$. Moreover,

$$\overline{f}g = \overline{f}g\overline{\overline{f}g} = \overline{f}g\overline{fg} = \overline{f}gme = \overline{f}hme = \overline{f}h\overline{fh} = \overline{f}h\overline{\overline{f}h} = \overline{f}h.$$

Therefore, f is a partial monomorphism. □

Proposition 4.0.2. *Suppose that* **C** *is a locally connected P-category. Suppose that α and β are domains in X such that $\alpha\beta = 0_X$. Then for all total morphisms $\theta, \theta' : Y \to X + X$,*

$$\nabla_X(\alpha + \beta)\theta = \nabla_X(\alpha + \beta)\theta' \text{ implies } (\alpha + \beta)\theta = (\alpha + \beta)\theta'.$$

Proof. First observe that since $\alpha\beta = 0$,

$$\nabla_X(\alpha + 0)\theta = \alpha\nabla_X(\alpha + \beta)\theta = \alpha\nabla_X(\alpha + \beta)\theta' = \nabla_X(\alpha + 0); \tag{7}$$

similarly,

$$\nabla_X(0+\beta)\theta = \nabla_X(0+\beta)\theta'. \tag{8}$$

Let γ be a connected domain in Y. Up to symmetry, there are two cases.
CASE ONE: $\theta\gamma \prec 1_X + 0$ and $\theta'\gamma \prec 0 + 1_X$. By equation (7),

$$\nabla_X(\alpha+\beta)\theta\gamma = \nabla_X(\alpha+0)\theta\gamma = \nabla_X(\alpha+0)\theta'\gamma = \nabla_X(0+0)\theta'\gamma = 0.$$

By equation (8),

$$\nabla_X(\alpha+\beta)\theta'\gamma = \nabla_X(0+\beta)\theta'\gamma = \nabla_X(0+\beta)\theta\gamma = \nabla_X(0+0)\theta'\gamma = 0.$$

It follows that

$$\begin{aligned}(\alpha+\beta)\theta\gamma &= (\alpha+0)\theta\gamma = i_0\nabla_X(\alpha+0)\theta\gamma = 0\\ &= i_1\nabla_X(0+\beta)\theta'\gamma = (0+\beta)\theta'\gamma = (\alpha+\beta)\theta'\gamma.\end{aligned}$$

CASE TWO: $\theta\gamma \prec 1_X + 0$ and $\theta'\gamma \prec 1_X + 0$. By equation (7),

$$\begin{aligned}(\alpha+\beta)\theta\gamma &= (\alpha+0)\theta\gamma = i_0\nabla_X(\alpha+0)\theta\gamma\\ &= i_0\nabla_X(\alpha+0)\theta'\gamma = (\alpha+0)\theta'\gamma = (\alpha+\beta)\theta'\gamma.\end{aligned}$$

Since γ is an arbitrary connected domain in Y and $\mathbf{C}$ is locally connected, $(\alpha+\beta)\theta = (\alpha+\beta)\theta'$. $\square$

Proposition 4.0.3. *Suppose that $\mathbf{C}$ is a locally connected split P-category satisfying the weak axiom of choice. Suppose that α and β are domains in X such that $\alpha\beta = 0_X$. Then the morphism*

$$X+Y \xrightarrow{\alpha+\beta} X+Y \xrightarrow{\nabla_X} X$$

has a unique section $f : X \to X + X$. Moreover

$$\nabla_X(\alpha+\beta)f = \overline{f} = \operatorname{ran}(\nabla_X(\alpha+\beta)) = \alpha\vee\beta.$$

If $\mathbf{C}$ were to satisfy the axiom of choice, then the conclusion would hold without the assumption of local connectedness.[1]

Proposition 4.0.4. *Let $\mathbf{C}$ be a locally connected connected split P-category satisfying the weak axiom of choice. Let α be a complemented domain in X, with complement $\alpha^\perp$, and let $f : X \to X + X$ be the unique section of the morphism*

$$X+Y \xrightarrow{\alpha^\perp+\alpha} X+Y \xrightarrow{\nabla_X} X\ .$$

Then

$$f\alpha \prec 0_X + 1_X \text{ and } f\alpha^\perp \prec 1_X + 0_X.$$

Proof. We have that $\overline{f} = \alpha^\perp \vee \alpha = 1_X$, so that f is total. Claim: for any connected domain γ in X,

$$f\alpha\gamma \prec 1_X + 0_X \text{ or } f\alpha\gamma \prec 0_X + 1_X. \tag{9}$$

Since γ is connected, either $\gamma \le \alpha^\perp$ or $\gamma \le \alpha$. Both conditions of (9) hold trivially when $\gamma \le \alpha^\perp$. Otherwise, (9) reduces to $f\gamma \prec 1_X + 0_X$ or $f\gamma \prec 0_X + 1_X$, which holds since f is total and γ is connected.

[1]This follows from [DPH87, Proposition 7.6], which holds in a P-category with coproducts in which the axiom of choice holds [Ros88a].

Let γ be a connected domain in X; we may assume that $\gamma \neq 0$. If $\gamma \leq \alpha$ and the first alternative in (9) holds, then equivalently $(1+0)f\alpha\gamma = f\alpha\gamma$. We have the following equation.

$$\nabla_X(\alpha^\perp + 0)f\alpha\gamma = \nabla_X(\alpha^\perp + \alpha)(1_X + 0_X)f\alpha\gamma = \nabla_X(\alpha^\perp + \alpha)f\alpha\gamma = \alpha\gamma.$$

Composing with α on the left, we have

$$\alpha\gamma = \alpha\nabla_X(\alpha^\perp + 0)f\alpha\gamma = \nabla_X(\alpha\alpha^\perp + \alpha 0)f\alpha\gamma = 0.$$

This contradicts $\gamma \neq 0$ since $\gamma \leq \alpha$. We conclude that $f\alpha\gamma \prec 0_X + 1_X$ if $\gamma \leq \alpha$. If $\gamma \leq \alpha^\perp$, then $f\alpha\gamma \prec 0_X + 1_X$ holds trivially. Since γ is an arbitrary connected domain, we have that $f\alpha \prec 0_X + 1_X$.

A similar argument yields $f\alpha^\perp \prec 1_X + 0_X$. □

4.2. **Recursively inseparable domains.** Let C be a P-recursion category. The pair (α, β) of domains in $\mathrm{Dom}(X)$ is *recursively separable* if $\alpha\beta = 0$ and there exists a complemented domain $\gamma \in \mathrm{Dom}(X)$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma^\perp$, the complement of γ; the pair (α, β) is *recursively inseparable* otherwise.

Theorem 4.1. *Let* $\mathbf{C}$ *be a* P*-recursion category with coproducts and ranges and let* $\mathbf{D}$ *be a locally connected split range restriction with coproducts and a system of zeros, and which satisfies the weak axiom of choice. Suppose that there is a range functor* $F : \mathbf{C} \to \mathbf{D}$*. Let* $t : X \times X \to X$ *be a weak Turing morphism in* $\mathbf{C}$ *such that* $1_X \notin \ker F$*, and let* $b : X \to X + X$ *be an isomorphism in* $\mathbf{C}$*. Then the pair*

$$(\alpha, \beta) = ((bt\Delta_X)^*(1_X + 0), (bt\Delta_X)^*(0 + 1_X))$$

is a recursively inseparable pair in $\mathrm{Dom}(X)$*.*

Proof. We note first that since the pre-image operation preserves meets,

$$\alpha\beta = (bt\Delta_X)^*(1_X + 0 \circ 0 + 1_X) = (bt\Delta_X)^*(0_{X+X}) = \overline{0bt\Delta_X} = 0_X.$$

Suppose that the pair (α, β) is separated by the complemented domain $\gamma \in \mathrm{Dom}(X)$, such that $\alpha \leq \gamma$ and $\beta \leq \gamma^\perp$, where $\gamma^\perp$ is the complement of γ.

We may assume that $\mathbf{C} = \mathbf{D}$ and that F is the identity functor, since the argument will depend only on manipulations in $\mathbf{D}$ of terms in which products, diagonals and projections occur only within the scope of the functor F; in particular, we will never require F to commute with the near-product, the diagonal or the projections.

We let $f : X \to X + X$ be the unique section of the morphism

$$X + Y \xrightarrow{\gamma^\perp + \gamma} X + Y \xrightarrow{\nabla_X} X$$

as in Proposition 4.0.3; in particular,

$$\nabla_X(\gamma^\perp + \gamma)f = \overline{f} = \mathrm{ran}\left(\nabla_X(\gamma^\perp + \gamma)\right) = \gamma^\perp \vee \gamma = 1_X.$$

Let e be an idempotent t-index of $b^{-1}f$. We compute as follows.

$$bt\Delta_X e = bt(e \times 1_X)\Delta_X e = bb^{-1}fe = fe. \tag{10}$$

By assumption, there exists a connected domain δ in X such that $\mathrm{ran}(e\delta)$ is a nonzero connected domain in $\mathrm{Dom}(X)$; we may assume that $\mathrm{ran}(e)$ is connected. Since $\mathrm{ran}(e)$ is connected, either $\mathrm{ran}(e) \leq \gamma$ or $\mathrm{ran}(e) \leq \gamma^\perp$.

Suppose that $\mathrm{ran}(e) \leq \gamma$. By Proposition 4.0.4, $f\gamma \prec 0 + 1$; composing with $\mathrm{ran}(e)$ and taking ranges, we have $f_*\mathrm{ran}(e) \leq 0 + 1$.

Take ranges in equation (10), and recall that $\mathrm{ran}(fg) = \mathrm{ran}(f\mathrm{ran}(g))$.

$$\mathrm{ran}(bt\Delta_X e) = (bt\Delta_X)_* \mathrm{ran}(e) = f_* \mathrm{ran}(e) \leq 0 + 1.$$

Therefore $(bt\Delta_X)_* \mathrm{ran}(e) \leq 0_X + 1_X$. Note that adjointness cannot be applied, since $bt\Delta_X$ is not total. Instead, we apply Proposition 3.0.8 a).

$$\mathrm{ran}(e)\overline{bt\Delta_X} = \mathrm{ran}(e)(bt\Delta_X)^*(0_X + 1_X) \leq \gamma\beta \leq \gamma\gamma^\perp = 0.$$

It follows that $\overline{bt\Delta_X}\mathrm{ran}(e) = 0$; by Proposition 3.0.5 and the previous calculation, $0 = bt\Delta_X e = fe$. Again by Proposition 3.0.5, $0 = \overline{f}\mathrm{ran}(e) = \mathrm{ran}(e)$. This is a contradiction, since $\mathrm{ran}(e)$ is nonzero connected domain, by assumption.

The case of $\mathrm{ran}(e) \leq \gamma^\perp$ is similar. It follows that there exists no such complemented domain γ, and thus the pair (α, β) is recursively inseparable. □

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