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EVERY P-RECURSION CATEGORY HAS AN INDEX COMPOSER

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ABSTRACT. In a P-recursion category, composition of morphisms is definable uniformly in the collection of indices with respect to a Turing morphism. This fact follows from the coassociativity of the near-product of a P-category.

MSC: 03D75; 03G30

1. Introduction

In [Ste93], Stefano Stefani defined the notion of a composer, which was intended as a reflection principle for P-recursion categories. A composer in a P-recursion category enables composition of morphisms to be expressed internally. We show that every P-recursion category contains a morphism called an index composer, closely related to a composer in Stefani's sense, which enables a P-category to express composition of any two of its morphisms φ and ψ, uniformly in terms of a fixed expression in the indices of φ and ψ. The existence of an index composer in a P-recursion category follows from the coassociativity of the near-product of a P-category.

A recursive function $f : \omega \to \omega$ is an *index morphism* if, with respect to some standard numbering of the partial computable functions, f is a recursive enumeration of indices of some fixed partial computable function. In contrast to Stefani's notion of a composer, the definition of which involves quantification over all total morphisms in a P-recursion category, the notion of an index composer reflects that in classical recursion theory, not every recursive function is an index morphism.

2. P- and P-recursion categories

Let $\mathbf{C}$ be a category; for our purposes, categories may be taken to be small. The *diagonal functor* $\delta_{\mathbf{C}} : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ is given on objects by $A \mapsto (A, A)$ and on morphisms by $f \mapsto (f, f)$.

A *P-category* consists of a category $\mathbf{C}$ together with a bifunctor $\times : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, called a *near product*, a natural transformation $\Delta : 1_{\mathbf{C}} \to \times \circ \delta_{\mathbf{C}}$, and for each object X of $\mathbf{C}$, natural transformations

$$p_{0(\ ),X} : (\ ) \times X \to 1_{\mathbf{C}}, \quad p_{1X,(\ )} : X \times (\ ) \to 1_{\mathbf{C}}$$

such that the conditions i), ii) and iii) below are satisfied [Ros86].

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i) The following equations hold.

$$\begin{array}{ll} p_{0X,X}\Delta_X = 1_X = p_{1X,X}\Delta_X, & (p_{0X,Y} \times p_{1X,Y})\Delta_{X\times Y} = 1_{X\times Y}, \\ p_{0X,Y}(1_X \times p_{0Y,Z}) = p_{0X,Y\times Z}, & p_{0X,Z}(1_X \times p_{1Y,Z}) = p_{0X,Y\times Z}, \\ p_{1X,Z}(p_{0X,Y} \times 1_Z) = p_{1X\times Y,Z}, & p_{1Y,Z}(p_{1X,Y} \times 1_Z) = p_{1X\times Y,Z}. \end{array}$$

ii) There is an isomorphism

$$\mathrm{ass}_\times : ((- \times -) \times -) \to (- \times (- \times -))$$

of functors $\mathbf{C}^3 \to \mathbf{C}$, whose component $\mathrm{ass}_{\times X,Y,Z}$ is given by

$$\langle p_{0X,Y} p_{0X\times Y,Z}, \langle p_{1X,Y} p_{0X\times Y,Z}, p_{1X\times Y,Z}\rangle\rangle : (X \times Y) \times Z \to X \times (Y \times Z),$$

in which, for morphisms $f : X \to Y$, $g : X \to Z$ of $\mathbf{C}$, we set

$$\langle f,\ g\rangle = (f \times g)\Delta_X : X \to Y \times Z.$$

iii) Let $\mathrm{tr}_\times$ be the endofunctor on $\mathbf{C} \times \mathbf{C}$ given by $(X, Y) \mapsto (Y, X)$. There is an isomorphism

$$\mathrm{tr}_\times : \times \to \times \circ \mathrm{tr}_\times$$

of functors $\mathbf{C}^2 \to \mathbf{C}^2$ whose component $\mathrm{tr}_{\times X,Y}$ is given by

$$\langle p_{1X,Y},\ p_{0X,Y}\rangle : X \times Y \to Y \times X.$$

The isomorphisms $\mathrm{ass}_\times$ and $\mathrm{tr}_\times$ are natural in each of their component variables.

One might say that the projections of a P-category are "half natural"–while $p_0(\varphi \times 1_X) = \varphi p_0$ holds, the companion equation $p_1(\varphi \times 1_X) = 1_X p_1$ fails in general (the right side is always total, the left may not be). In general, the projections $p_{i,X,Y}$ of a P-category $\mathbf{C}$ are natural in both variables if and only if for any morphisms $\varphi_i : Y \to X_i$ $(i = 0, 1)$ in $\mathbf{C}$, $p_i\langle\varphi_0, \varphi_1\rangle = \varphi_i$. Cockett and Lack observe that P-categories are "symmetric monoidal categories in which each object has a a monoidal natural cocommutative coassociative comultiplication (and possibly an unnatural counit)" [CL02].

A category is *isotypical* if any two objects are isomorphic. A *system of zero morphisms* is a collection of morphisms $0_{X,Y} : X \to Y$ for each pair of objects X and Y of $\mathbf{C}$ such that for objects W, Z and morphisms $f : W \to X$ and $g : Y \to Z$ of $\mathbf{C}$, one has $g0_{X,Y}f = 0_{W,Z}$. A system of zero morphisms is unique if it exists. A P-category is *prodominical* if it contains a system of zero morphisms and, for any $\phi : A \to B$, $\phi \times 0_{C,D} = 0_{A\times C,B\times D}$. We write 0_X for $0_{X,X}$ [Hel90]. We will always assume that our P-categories are prodominical.

2.1. **Domains and total morphisms.** We recall the definition of the domain of a morphism in a P-category [DPH87, Ros86].

The domain of a morphism $\varphi : X \to Y$ in a P-category is a morphism $\partial(\varphi) : X \to X$, defined by

$$\partial(\varphi) = p_0(1_X \times \varphi)\Delta_X = p_1(\varphi \times 1_X)\Delta_X. \tag{1}$$

The morphism φ is *total* if $\partial(\varphi) = 1_X$. We will use the following properties of domains:

$$\varphi = \varphi\partial(\varphi) \tag{2}$$

$$\partial(\psi\varphi) = \partial(\partial(\psi)\varphi) \tag{3}$$

$$\partial(\psi(\partial\varphi)) = \partial\psi\partial\varphi \tag{4}$$

For any object X of $\mathbf{C}$, the collection of domains $\partial\varphi$ for some $\varphi : X \to Y$ is denoted by $\mathrm{Dom}(X)$.[1] The set $\mathrm{Dom}(X)$ forms a commutative meet-semilattice under composition, with least element $0_{X,X}$ and greatest element 1_X.

The collection $\mathbf{C}_T$ of total morphisms of $\mathbf{C}$ forms a subcategory [Ros86]; the near product of a P-category becomes a categorical product on this subcategory.

In general, a P-category is equivalent to one obtained from a category with products, following a procedure analogous to that by which the category **Pfn** of sets and partial functions can be obtained from the category **Set** of sets [Ros88, Theorem 1.2].

2.2. **Turing morphisms and P-recursion categories.** A *Turing morphism* in a prodominical isotype is a morphism $\tau : W \times X \to Y$ such that for any $\phi : V \times X \to Y$ there exists a total $g : V \to W$, such that the following diagram commutes.

$$\begin{tikzcd} V \times X \arrow[r, "g \times 1_X"] \arrow[dr, "\phi"'] & W \times X \arrow[d, "\tau"] \\ & Y \end{tikzcd}$$

A *P-recursion category* (or, simply, a *recursion category*) is a prodominical isotype which contains a Turing morphism.[2] Examples of Turing morphisms and recursion categories are given in [DPH87, Mon89, Hel90, DPM91, Ste93, Len04].

A morphism $e : X \to X$ is *idempotent* if $e^2 = e$. A morphism $t : X \times X \to X$ in a prodominical isotype $\mathbf{C}$ is an *(idempotent) weak Turing morphism* if for every morphism $f : X \to X$, there exists a total (idempotent) morphism $e : X \to X$ such that the following diagram commutes.

$$\begin{tikzcd} X \times X \arrow[r, "e \times 1_X"] \arrow[d, "p_{1X}"] & X \times X \arrow[d, "t"] \\ X \arrow[r, "f"] & X \end{tikzcd} \tag{5}$$

We call the morphism e in (5) a *t-index of f*. More generally, a *t-index* is a total morphism e such that for some morphism f, the diagram (5) commutes. If t is a Turing morphism, then in the situation of diagram (5), t is also a weak Turing morphism, and we say that e is a t-index of f.

Every Turing morphism in a P-recursion category is a weak Turing morphism; conversely, in a prodominical isotype, a Turing morphism can be obtained from a weak Turing morphism [Hel90, Lemma 9.1].

Lemma 2.0.1. *If a prodominical isotype has a weak Turing morphism, then it has a Turing morphism.*

3. Index composers

A total morphism $s : X \times X \to X$ in a P-recursion category $\mathbf{C}$ is a *composer relative to a morphism* $t : X \times X \to X$ if for each pair of total morphisms f, g :

[1] Since our categories are assumed to be small, $\mathrm{Dom}(X)$ will be a set.

[2] Previously, the term *recursion category* was used for dominical P-recursion categories.

$X \to X$, the following identity holds [Ste93].

$$t(s(f \times g)\Delta_X \times 1_X)\Delta_X = t(f \times 1_X)\Delta_X t(g \times 1_X)\Delta_X. \tag{6}$$

The apparent motivation, from classical recursion theory, of this definition is that there exists a recursive function s of two variables such that for all natural numbers f and g,

$$\varphi_{s(f,g)} = \varphi_f \varphi_g.$$

This assertion is translated in P-categories by equation (6) if one assumes that index morphisms behave somewhat like constants; however, constant morphisms need not exist in a P-category.

Let $t : X \times X \to X$ be a Turing morphism in a P-category $\mathbf{C}$. Recall that t is also a weak Turing morphism. In a P-recursion category $\mathbf{C}$ with Turing morphism $t : X \times X \to X$, a *t-index composer* is a total morphism $s : X \times X \to X$ such that for each weak t-index $f : X \to X$, and for each weak t-index $g : X \to X$, the identity (6) holds.

In the notation of the preceding, let $\varphi, \psi : X \to X$ be morphisms in the P-recursion category $\mathbf{C}$, and suppose that s is a t-index composer. If f is a t-index of φ and if g is a t-index of ψ, then equation (6) immediately yields

$$t(s(f \times g)\Delta_X \times 1_X)\Delta_X = \varphi\psi. \tag{7}$$

This shows that composition in the monoid $\mathbf{C}(X, X)$ is definable uniformly in terms of the left hand expression in (7) and the collection of t-indices.

Theorem 3.1. *Every P-recursion category $\mathbf{C}$ contains an index composer.*

This fact follows from the coassociativity of the near-product of a P-category; for completeness, we include a statement and proof of coassociativity.

Proposition 3.1.1. *In a P-category, the following diagram commutes.*

$$\begin{tikzcd}
 & X \times X \arrow[r, "\Delta_X \times 1_X"] & (X \times X) \times X \arrow[d, "\mathrm{ass}_{\times X,X,X}"] \\
X \arrow[ur, "\Delta_X"] \arrow[dr, "\Delta_X"'] & & \\
 & X \times X \arrow[r, "1_X \times \Delta_X"] & X \times (X \times X)
\end{tikzcd}$$

Proof.

$$\begin{aligned}
&\mathrm{ass}_{\times X,X,X}(\Delta_X \times 1_X)\Delta_X \\
=&\langle p_{0X,X}p_{0X\times X,X}, \langle p_{1X,X}p_{0X\times X,X}, p_{1X\times X,X}\rangle\rangle(\Delta_X \times 1_X)\Delta_X \\
=&\langle p_{0X,X}p_{0X\times X,X}(\Delta_X \times 1_X), \langle p_{1X,X}p_{0X\times X,X}, p_{1X\times X,X}\rangle(\Delta_X \times 1_X)\rangle\Delta_X \\
=&\langle p_{0X,X}\Delta_X p_{0X,X}, \langle p_{1X,X}p_{0X\times X,X}(\Delta_X \times 1_X), p_{1X\times X,X}(\Delta_X \times 1_X)\rangle\rangle\Delta_X \\
=&\langle p_{0X,X}, \langle p_{0X,X}, p_{1X,X}\rangle\rangle\Delta_X = (p_{0X,X} \times 1_{X\times X})\Delta_{X\times X}\Delta_X \\
=&(p_{0X,X} \times 1_{X\times X})(\Delta_X \times \Delta_X)\Delta_X = (p_{0X,X}\Delta_X \times \Delta_X)\Delta_X \\
=&(1_X \times \Delta_X)\Delta_X
\end{aligned}$$

□

We let $t : X \times X \to X$ be a Turing morphism in $\mathbf{C}$. A t-index composer will be exhibited. We make the following observation.

Proposition 3.1.2. *If $e : X \to X$ is a t'-index of a morphism $\psi : X \to X$, where $t' : X \times X \to X$ is a weak Turing morphism, then for any morphism $\varphi : X \to X$, $t'(e \times \varphi)\Delta_X = \psi\varphi = t(e \times 1_X)\Delta_X\varphi$.*

Proof. The first identity follows from the diagram below.

$$\begin{tikzcd}
X \arrow[r, "\Delta_X"] \arrow[dr, "1_X"'] & X \times X \arrow[r, "1_X \times \varphi"] \arrow[d, "p_{1X}"] & X \times X \arrow[r, "e \times 1_X"] \arrow[d, "p_{1X}"] & X \times X \arrow[d, "t'"] \\
& X \arrow[r, "\varphi"] & X \arrow[r, "\psi"] & X
\end{tikzcd}$$

The second identity follows from the diagram below.

$$\begin{tikzcd}
X \arrow[r, "\Delta_X"] \arrow[dr, "1_X"'] & X \times X \arrow[r, "e \times 1_X"] \arrow[d, "p_{1X}"] & X \times X \arrow[d, "t'"] \\
& X \arrow[r, "\psi"] & X
\end{tikzcd}$$

□

3.1. **Proof of Theorem 3.1.**

Proof. Since t is a Turing morphism, there exists a total morphism $s : X \times X \to X$ such that

$$\begin{tikzcd}
(X \times X) \times X \arrow[r, "s \times 1_X"] \arrow[d, "\mathrm{ass}_\times"'] & X \times X \arrow[d, "t"] \\
X \times (X \times X) \arrow[r, "t(1_X \times t)"'] & X
\end{tikzcd}$$

Here $\mathrm{ass}_\times$ is the natural associativity isomorphism.

Compute as follows; the third and fourth lines use coassociativity, and the last line uses Proposition 3.1.2 with $e = f$ and $\varphi = t(g \times 1_X)\Delta_X$.

$$\begin{aligned}
t(s(f \times g)\Delta_X \times 1_X)\Delta_X &= t(s \times 1_X)((f \times g) \times 1_X)(\Delta_X \times 1_X)\Delta_X \\
&= t(1_X \times t)\mathrm{ass}_\times((f \times g) \times 1_X)(\Delta_X \times 1_X)\Delta_X \\
&= t(1_X \times t)(f \times (g \times 1_X))\mathrm{ass}_\times(\Delta_X \times 1_X)\Delta_X \\
&= t(f \times t(g \times 1_X))(1_X \times \Delta_X)\Delta_X \\
&= t(f \times t(g \times 1_X)\Delta_X)\Delta_X \\
&= t(f \times 1_X)\Delta_X t(g \times 1_X)\Delta_X.
\end{aligned}$$

□

Note that the property that g is a weak t-index was not used.

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References

[CL02] J. R. B. Cockett and Stephen Lack, *Restriction categories. I. Categories of partial maps*, Theoret. Comput. Sci. **270** (2002), no. 1-2, 223–259.

[DPH87] Robert A. Di Paola and Alex Heller, *Dominical categories: recursion theory without elements*, J. Symbolic Logic **52** (1987), no. 3, 594–635.

[DPM91] Robert A. Di Paola and Franco Montagna, *Some properties of the syntactic p-recursion categories generated by consistent, recursively enumerable extensions of Peano arithmetic*, J. Symbolic Logic **56** (1991), no. 2, 643–660.

[Hel90] Alex Heller, *An existence theorem for recursion categories*, J. Symbolic Logic **55** (1990), no. 3, 1252–1268.

[Len04] Florian Lengyel, *More existence theorems for recursion categories*, Ann. Pure Appl. Logic **125** (2004), no. 1-3, 1–41.

[Mon89] Franco Montagna, *"Pathologies" in two syntactic categories of partial maps*, Notre Dame J. Formal Logic **30** (1989), no. 1, 105–116.

[Ros86] G. Rosolini, *Continuity and effectiveness in topoi*, D.phil. thesis, University of Oxford, 1986.

[Ros88] ———, *Representation theorems for special p-categories*, Categorical Algebra and its Applications (Louvain-la-Neuve) (F. Borceux, ed.), Lectures Notes in Mathematics, vol. 1348, Springer-Verlag, 1988, pp. 307–315.

[Ste93] Stefano Stefani, *A relativization mechanism in recursion categories*, J. Symbolic Logic **58** (1993), no. 4, 1251–1267. MR 95c:03110

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