City University of New York (CUNY)

CUNY Academic Works

Publications and Research

New York City College of Technology

2018

Character sums of Lee and Weintraub

Brad Isaacson

CUNY New York City College of Technology

[How does access to this work benefit you? Let us know!](#)

More information about this work at: https://academicworks.cuny.edu/ny_pubs/279

Discover additional works at: <https://academicworks.cuny.edu>

This work is made publicly available by the City University of New York (CUNY).

Contact: AcademicWorks@cuny.edu

Character sums of Lee and Weintraub

Brad Isaacson

*New York City College of Technology, Mathematics Department
300 Jay Street, Brooklyn, NY 11201, U.S.A.; bisaacson@citytech.cuny.edu*

Abstract

We prove two conjectures of Lee and Weintraub and one conjecture of Ibukiyama and Kaneko about character sums arising as fixed point contributions in the Atiyah-Singer holomorphic Lefschetz formula applied to finite group actions on the space of certain Siegel cusp forms.

Keywords: Character sums, Exponential sums

1. Introduction

Fix an odd prime p and let ψ denote the Legendre symbol mod p : $\psi(a) = \left(\frac{a}{p}\right)$. We put $\zeta = \exp(2\pi i/p)$. Let $P(x), Q(x), R(x)$ be polynomials with integer coefficients. Following Lee-Weintraub[1], we define the Lee-Weintraub sum $S[P, Q, R]$ by

$$S[P, Q, R] = - \sum'_{k, j(p)} \frac{\psi(k)}{(\zeta^{kP(j)} - 1)(\zeta^{kQ(j)} - 1)(\zeta^{kR(j)} - 1)},$$

where k and j each run over a complete residue system modulo p , and the prime on the summation sign means that the meaningless terms are to be excluded.

These sums are of interest for two reasons. Firstly, they are related to special values of L -functions which are then related to the theory of modular forms, so they are related to many fundamental objects of number theory and therefore deserve to be studied. Secondly, the polynomials P, Q, R may be of degree greater than 1, which makes these sums exotic. Accordingly, the treatment of these sums requires new techniques.

Lee-Weintraub encountered these sums while studying the action of the symplectic group $Sp(4, \mathbb{F}_p)$ on the space of Siegel cusp forms of degree 2 belonging to the principal congruence subgroup $\Gamma(p)$ by using the holomorphic Lefschetz fixed-point theorem of Atiyah-Singer, where traces of this action gave rise to certain Lee-Weintraub sums. Arakawa-Hashimoto on the other hand studied this action by using the Selberg trace formula, where certain traces of this action were expressed by special values of certain L -functions. In [2], we expressed the special values of these L -functions by finite sums of products of Bernoulli

polynomials. The shape of our result is the consequence of the Shintani method employed, first used by Arakawa in a special case [3]. On the other hand, 20 Ibukiyama-Saito[4] proved that these L -functions were up to elementary factors simply the Riemann zeta function, giving an alternative evaluation of the special values of principally simpler shape than those obtained by the method of Arakawa-Shintani. Comparing these two different calculations of the special values, we obtained formulas (called Arakawa Identities) expressing finite sums 25 of products of Bernoulli polynomials by generalized Bernoulli numbers. In this paper, we use the Arakawa Identities obtained in [2] (Theorem 3.1) to prove two conjectures of Lee and Weintraub and one conjecture of Ibukiyama and Kaneko regarding Lee-Weintraub sums.

We define the k th Bernoulli polynomial $B_k(x)$ by the generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

and the k th periodic Bernoulli polynomial $P_k(x)$ by

$$P_k(x) = \begin{cases} 0, & \text{if } k = 1, x \in \mathbb{Z}, \\ B_k(\{x\}), & \text{otherwise,} \end{cases}$$

where $\{x\}$ denotes the fractional part of x . We denote by $B_{k,\psi}$ the k th generalized Bernoulli number attached to ψ :

$$B_{k,\psi} = p^{k-1} \sum_{a(p)} P_k\left(\frac{a}{p}\right) \psi(a),$$

where the summation is over a complete residue system modulo p .

30 In [1], Lee-Weintraub made three conjectures expressing Lee-Weintraub sums as linear combinations of generalized Bernoulli numbers. One of these conjectures (Conjecture 3.5 in [1]) was proved by Ibukiyama-Saito (Corollary 1.3 in [4]) by deducing the Arakawa Identity given by (i) in Theorem 3.1. We refer to this identity as the Lee-Weintraub Identity.

Lee-Weintraub Identity. *For any odd prime number p , we get*

$$S[-x, x+1, x(x+1)] = \sqrt{\psi(-1)p} \left(\frac{p+1}{4} B_{1,\psi} + \frac{3}{2} B_{2,\psi} + \frac{1}{6} B_{3,\psi} \right).$$

35 **Remark.** This was both conjectured and proved in the case of $p \equiv 3(4)$.

In this paper, we prove the remaining two Lee-Weintraub conjectures by utilizing all of the Arakawa Identities given in Theorem 3.1. The following two theorems were conjectured by Lee and Weintraub in the case of $p \equiv 3(4)$ (see page 256 in [1]).

Theorem 5.1. *For any odd prime number p , we get*

$$S[-x, x+2, x(x+2)] = \sqrt{\psi(-1)p} \left(\frac{(p+1)(1+\psi(2))}{12} B_{1,\psi} + \frac{4+3\psi(2)}{4} B_{2,\psi} + \frac{4+\psi(2)}{18} B_{3,\psi} \right).$$

Theorem 5.2. *For any odd prime number p , we get*

$$S[-x, x+1, x(x+2)] = \sqrt{\psi(-1)p} \left(\frac{2p-1+3\psi(2)}{16} B_{1,\psi} + \frac{5+\psi(2)}{8} B_{2,\psi} + \frac{1}{16} B_{3,\psi} \right).$$

40 We next prove a dual version of the Lee-Weintraub Identity. We consider the sums $S[-x, x+1, x(x+1)]$, $S[1, x^2, (x+1)^2]$ dual to each other since the quadratic forms that naturally arise, namely $4bc - (a-b-c)^2$ and $ab+bc+ca$ respectively, are dual to each other. We refer to this identity as the Dual Lee-Weintraub Identity.

Theorem 5.3 (Dual Lee-Weintraub Identity). *For any odd prime number p , we get*

$$S[1, x^2, (x+1)^2] = \sqrt{\psi(-1)p} \left(-(p-2)B_{1,\psi} + \frac{3}{4}B_{2,\psi} - \frac{1}{6}B_{3,\psi} \right).$$

45 **Remark.** It turns out that there are many identities of the Lee-Weintraub type expressing $S[P, Q, R]$ as a linear combination of generalized Bernoulli numbers, some of which we plan to prove in a future paper.

As a corollary to the Dual Lee-Weintraub Identity, we prove the following conjecture of Ibukiyama and Kaneko in the case of $p \equiv 3(4)$ (see Example 8.19 and Remark 8.20 in [5]).

50 **Corollary 5.4.** *For any odd prime number p , we get*

$$\sum_{(a,b,c) \in T} abc \psi(abc) = \frac{p^2(p+1)}{2} B_{1,\psi} - \frac{3p^3}{4} B_{2,\psi} - \frac{p^2}{6} B_{3,\psi},$$

where $T = \{(a, b, c) \in \mathbb{Z}^3 \mid 1 \leq a, b, c \leq p-1, ab+bc+ca \equiv 0(p)\}$.

Lastly, we prove an innocent looking corollary which is not easy to prove using elementary means.

Corollary 5.5. *For any odd prime number p , we get*

$$\sum_{k,t(p)} P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{(k+t)^2}{p}\right) = \sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_1\left(\frac{k^2+kt}{p}\right) P_1\left(\frac{t^2+kt}{p}\right).$$

2. Preliminaries

55 In this section, we discuss all relevant background information needed to evaluate the Lee-Weintraub sums given in the introduction. We omit several proofs which can be found in the literature.

We fix the notation. Let p be an odd prime. We denote by ψ the Legendre symbol mod p : $\psi(a) = \left(\frac{a}{p}\right)$. We put $\zeta = \exp(2\pi i/p)$. Let $\tau(\psi)$ denote the Gaussian sum $\tau(\psi) = \sum_{n=1}^{p-1} \psi(n)\zeta^n$. Let $\delta_{x,y}$ be the Kronecker delta symbol ($\delta_{x,y} = 1$ if $x = y$, and 0 otherwise). We recall the periodic Bernoulli polynomials $P_k(x)$ given by

$$P_k(x) = \begin{cases} 0, & \text{if } k = 1, x \in \mathbb{Z}, \\ B_k(\{x\}), & \text{otherwise,} \end{cases}$$

where $B_k(x)$ is the k th Bernoulli polynomial (see Introduction) and $\{x\}$ denotes the fractional part of x . We remind the reader that the parity of P_k is the parity of k . We henceforth will apply this parity condition without mention.

We state the well-known multiplication formula for the periodic Bernoulli polynomials.

Lemma 2.1. *For any $k \in \mathbb{Z}$, $k \geq 0$, $n \in \mathbb{N}$, $x \in \mathbb{R}$, we have*

$$\frac{1}{n^{k-1}} P_k(nx) = \sum_{a(n)} P_k\left(x + \frac{a}{n}\right),$$

where the summation is over a complete residue system modulo n .

We next give two addition formulas for the periodic Bernoulli polynomials.

Lemma 2.2. *Let $x, y \in \mathbb{R}$. Then, we have*

$$\begin{aligned} & P_1(x)P_1(y) - P_1(x)P_1(x+y) - P_1(y)P_1(x+y) \\ &= -\frac{1}{2} \left(P_2(x) + P_2(y) + P_2(x+y) \right) + \frac{1}{4} \delta(x, y), \end{aligned}$$

where

$$\delta(x, y) = \begin{cases} 1, & \text{if } x, y \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This follows from Theorem 3.3 in [6] with lattice $L = \mathbb{Z}^2$ and vectors $\sigma_0 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} x \\ y \end{pmatrix}$. $\square$

Lemma 2.3. *Let $x, y \in \mathbb{R}$. Then, we have*

$$\begin{aligned} & P_1(x)P_2(x+y) + P_1(y)P_2(x+y) - P_1(x)P_2(y) - P_1(y)P_2(x) \\ &= \frac{1}{3} \left(P_3(x) + P_3(y) + 2P_3(x+y) \right). \end{aligned}$$

Proof. This follows from Theorem 3.3 in [6] with lattice $L = \mathbb{Z}^3$ and vectors $\sigma_0 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} x \\ y \\ x+y \end{pmatrix}$. $\square$

Remark. Addition formulas for the periodic Bernoulli polynomials such as Lemma 2.2, Lemma 2.3 have been studied in full generality by Gunnells and

Szzech[6].

Many of the sums in Section 5 will contain periodic Bernoulli polynomials whose arguments are homogeneous polynomials of degree 2. Thus we introduce the following proposition.

Proposition 2.4. *Let $n \in \mathbb{N}$, $\{k_j\}_{j=1}^n$ be a set of nonnegative integers with $\sum_{j=1}^n k_j \in 2\mathbb{Z} + 1$, and $\{q_j(k, t)\}_{j=1}^n$ be homogeneous polynomials of degree 2. Then for any $a \in \mathbb{Z}$, we have*

$$\begin{aligned} & \sum_{k, t(p)} P_{k_1}(aq_1(k, t)/p) P_{k_2}(aq_2(k, t)/p) \cdots P_{k_n}(aq_n(k, t)/p) \\ &= \psi(a) \sum_{k, t(p)} P_{k_1}(q_1(k, t)/p) P_{k_2}(q_2(k, t)/p) \cdots P_{k_n}(q_n(k, t)/p). \end{aligned}$$

Proof. In the case $p \equiv 1(4)$, there exists a c such that $c^2 \equiv -1(p)$. Replacing (k, t) by (ck, ct) , both sides of the proposition vanish for parity reasons. In the case $p \equiv 3(4)$, there exists a c such that $c^2 \equiv \psi(a)a^{-1}(p)$. Replacing (k, t) by (ck, ct) , we obtain the assertion of the proposition. $\square$

We next consider the sequence of functions $\{c_k(x)\}_{k=0}^\infty$ given in [7]. We set

$$c_k(x) = \sum'_{p \in \mathbb{Z}+x} \frac{1}{p^k} \quad (k \geq 1), \quad (2.1)$$

where x is a complex number and the prime on the summation sign indicates that meaningless terms are to be excluded. If $k = 1$, then the infinite sum on the right-hand side is understood to be

$$\lim_{t \rightarrow \infty} \left(\sum'_{\substack{p \in \mathbb{Z}+x \\ |p| < t}} \frac{1}{p} \right).$$

We also define a function $c_0(x)$ by $c_0(x) = -1$ if $x \in \mathbb{Z}$, and 0 otherwise. An alternative definition of $c_k(x)$ is obtained by applying Hecke summation to the Dirichlet series

$$c_k(x, s) = \sum'_{p \in \mathbb{Z}+x} \frac{1}{p^k |p|^s} \quad (\operatorname{Re}(s) > 1).$$

As a function of the complex variable s , $c_k(x, s)$ admits an analytic continuation to $s = 0$. Then, $c_k(x) = c_k(x, 0)$ holds true in all cases. We note that $c_1(x) = \pi \cot(\pi x)$ if $x \notin \mathbb{Z}$ and 0 otherwise. Taking derivatives term by term, we get the

following using facts that will be used in Section 4.

$$c_2(x) = \begin{cases} c_1^2(x) + \pi^2, & \text{if } x \notin \mathbb{Z}, \\ \pi^2/3, & \text{if } x \in \mathbb{Z}, \end{cases}$$

$$c_3(x) = c_1^3(x) + \pi^2 c_1(x) = c_1(x) c_2(x).$$

We now turn our attention to some of the various properties of the c_k 's. From the definition (2.1), we see that $c_k(x)$ is periodic with period 1, and that the parity of c_k is the parity of k . The c_k 's also possess the following multiplication formula which is analogous to Lemma 2.1. For any $k \in \mathbb{Z}$, $k \geq 0$, $n \in \mathbb{N}$, $x \in \mathbb{C}$, we have

$$n^k c_k(nx) = \sum_{a(n)} c_k\left(x + \frac{a}{n}\right). \quad (2.2)$$

The c_k 's also possess the following addition formula which is analogous to Lemma 2.2. Let $x, y \in \mathbb{R}$. Then, we have

$$\begin{aligned} c_1(x)c_1(y) - c_1(x)c_1(x+y) - c_1(y)c_1(x+y) \\ = c_0(x)c_2(y) + c_0(y)c_2(x+y) + c_0(x+y)c_2(x) + \pi^2. \end{aligned} \quad (2.3)$$

We now give the following finite Fourier expansions of the P_k 's and c_k 's. Let $k, n \in \mathbb{N}$, $x \in \mathbb{Z}$, and put $\zeta_n = \exp(2\pi i/n)$. Then, we have

$$\begin{cases} P_k\left(\frac{x}{n}\right) = -\frac{k!}{(2\pi i n)^k} \sum_{a(n)} c_k\left(\frac{a}{n}\right) \zeta_n^{ax}, \\ c_k\left(\frac{x}{n}\right) = -\frac{(-1)^k (2\pi i n)^k}{n k!} \sum_{a(n)} P_k\left(\frac{a}{n}\right) \zeta_n^{ax}. \end{cases} \quad (2.4)$$

3. Review of Arakawa Sums and Arakawa Identities

⁸⁰ In this section, we review the Arakawa sums and relevant Arakawa Identities obtained in [2]. As we shall see in Section 5, they play a vital role in evaluating Lee-Weintraub sums.

Fix an odd prime p . We keep the notation used previously. The Arakawa

sums $A_i(k_1, k_2, k_3)$ ($1 \leq i \leq 6$) are defined by

$$\begin{aligned}
 A_1(k_1, k_2, k_3) &= \sum_{k,t(p)} P_{k_1} \left(\frac{k^2 - 2kt}{p} \right) P_{k_2} \left(\frac{2kt}{p} \right) P_{k_3} \left(\frac{t^2 - k^2}{p} \right), \\
 A_2(k_1, k_2, k_3) &= \sum_{k,t(p)} P_{k_1} \left(\frac{k^2 - 2kt}{p} \right) P_{k_2} \left(\frac{kt}{p} \right) P_{k_3} \left(\frac{t^2 - k^2}{p} \right), \\
 A_3(k_1, k_2, k_3) &= \sum_{k,t(p)} P_{k_1} \left(\frac{k^2 - 2kt}{p} \right) P_{k_2} \left(\frac{4kt}{p} \right) P_{k_3} \left(\frac{t^2 - k^2}{p} \right), \\
 A_4(k_1, k_2, k_3) &= \sum_{k,t(p)} P_{k_1} \left(\frac{2(k^2 - 2kt)}{p} \right) P_{k_2} \left(\frac{2kt}{p} \right) P_{k_3} \left(\frac{t^2 - k^2}{p} \right), \\
 A_5(k_1, k_2, k_3) &= \sum_{k,t(p)} P_{k_1} \left(\frac{2(k^2 - 2kt)}{p} \right) P_{k_2} \left(\frac{kt}{p} \right) P_{k_3} \left(\frac{2(t^2 - k^2)}{p} \right), \\
 A_6(k_1, k_2, k_3) &= \sum_{k,t(p)} P_{k_1} \left(\frac{2(k^2 - 2kt)}{p} \right) P_{k_2} \left(\frac{kt}{p} \right) P_{k_3} \left(\frac{t^2 - k^2}{p} \right).
 \end{aligned} \tag{3.1}$$

For convenience, we write

$$\begin{cases} A_i = A_i(1, 1, 1) & (1 \leq i \leq 6), \\ a = A_1(1, 0, 2). \end{cases} \tag{3.2}$$

We now state the Arakawa Identities pertinent to the evaluation of Lee-Weintraub sums.

Theorem 3.1 (*Theorem 3.3 in [2]*). *We have*

$$\begin{aligned}
 (i) \quad A_1 - \frac{1}{3}a &= \frac{5p-1+2p\delta_{p,3}}{24p} B_{1,\psi} + \frac{11}{36p} B_{3,\psi}, \\
 (ii) \quad A_2 - \frac{1}{6}a &= \frac{4p+1-3\psi(2)-2p\delta_{p,3}}{24p} B_{1,\psi} + \frac{11}{72p} B_{3,\psi}, \\
 (iii) \quad A_3 - \frac{2}{3}a &= \frac{(9+\psi(2))p-2\psi(2)-4p\delta_{p,3}}{48p} B_{1,\psi} + \frac{11\psi(2)}{144p} B_{3,\psi}, \\
 (iv) \quad A_4 + A_5 - 2A_6 - \frac{\psi(2)}{12}a &= \frac{2(2\psi(2)-3)p-3+\psi(2)+8p\delta_{p,3}}{48p} B_{1,\psi} + \frac{11\psi(2)}{144p} B_{3,\psi}.
 \end{aligned}$$

Remark. The assertion (i) in Theorem 3.1 was established by Ibukiyama-Saito[4] from the work of Arakawa[3]. The remaining assertions are new.

The question arises whether a direct and elementary proof of these identities is possible. That seems to be a very difficult problem, similar in spirit to giving an elementary proof of Dirichlet's class number formula for $h(-p) = \#$ of reduced quadratic irrationalities in the complex upper half-plane of discriminant $-p$, $p \equiv 3(4)$, $p \geq 7$.

4. Auxiliary Sums

In this section, we obtain formulas for all of the auxiliary sums needed in Section 5. To fix our standpoint, we assume here that we are satisfied if an exponential sum or a character sum can be expressed in terms of generalized Bernoulli numbers. We note that all of the results in this section will be obtained using only elementary methods from algebra and number theory.

Fix an odd prime p . We keep the notation used previously. Let $a, b, c \in \mathbb{Z}$ with $a \not\equiv 0(p)$. We will frequently encounter two types of generalized quadratic Gaussian sums: $\sum_{k,t(p)} \zeta^{at^2+bkt+ck^2}$ and $\sum_{k,t(p)} \zeta^{k(at^2+bt+c)} \psi(k)$. Both are easily evaluated with the help of the following formulas.

$$\sum_{n(p)} \zeta^{an^2+bn} = \tau(\psi) \psi(a) \zeta^{-(4a)^{-1}b^2}, \quad (4.1)$$

$$\sum_{n(p)} \psi(an^2 + bn + c) = \begin{cases} -\psi(a), & \text{if } b^2 - 4ac \not\equiv 0(p), \\ (p-1)\psi(a), & \text{if } b^2 - 4ac \equiv 0(p). \end{cases} \quad (4.2)$$

We remind the reader of a frequently used formula.

$$\sum_{a(p)} P_k \left(\frac{a}{p} \right) \psi(a) = B_{k,\psi} / p^{k-1}. \quad (4.3)$$

This together with (2.4) yield the analogous result for the c_k 's.

$$\sum_{a(p)} c_k \left(\frac{a}{p} \right) \psi(a) = \frac{(-1)^{k+1} (2\pi i)^k \tau(\psi)}{k!} B_{k,\psi}. \quad (4.4)$$

We now prove a lemma which is used in Proposition 4.4, Proposition 4.5, Proposition 4.7, and Proposition 4.8.

Lemma 4.1. *We have*

$$\begin{aligned}
 (i) \quad & \sum_{a(p)} c_1 \left(\frac{a}{p} \right) c_2 \left(\frac{a}{p} \right) \psi(a) = \frac{(2\pi i)^3 \tau(\psi)}{6} B_{3,\psi}, \\
 (ii) \quad & \sum_{a(p)} c_1 \left(\frac{2a}{p} \right) c_2 \left(\frac{a}{p} \right) \psi(a) = \frac{(2\pi i)^3 \tau(\psi)}{12} (3(1 - \psi(2)) B_{1,\psi} + B_{3,\psi}), \\
 (iii) \quad & \sum_{a(p)} c_1 \left(\frac{a}{p} \right) c_2 \left(\frac{2a}{p} \right) \psi(a) = \frac{(2\pi i)^3 \tau(\psi)}{24} (3(\psi(2) - 1) B_{1,\psi} + B_{3,\psi}), \\
 (iv) \quad & \sum_{a(p)} c_1^2 \left(\frac{a}{p} \right) \psi(a) = 2\pi^2 \tau(\psi) B_{2,\psi}, \\
 (v) \quad & \sum_{a(p)} c_1 \left(\frac{2a}{p} \right) c_1 \left(\frac{a}{p} \right) \psi(a) = \pi^2 \tau(\psi) B_{2,\psi}, \\
 (vi) \quad & \sum_{a,b(p)} c_1 \left(\frac{a}{p} \right) c_1 \left(\frac{a+b}{p} \right) \psi(b) = 4\pi^2 \tau(\psi) B_{2,\psi}, \\
 (vii) \quad & \sum_{a,b(p)} c_1 \left(\frac{a}{p} \right) c_1 \left(\frac{2a+b}{p} \right) \psi(b) = (4 + \psi(2)) \pi^2 \tau(\psi) B_{2,\psi}, \\
 (viii) \quad & \sum_{a,b(p)} P_1 \left(\frac{a}{p} \right) P_1 \left(\frac{a+b}{p} \right) \psi(b) = B_{2,\psi}/2, \\
 (ix) \quad & \sum_{a,b(p)} P_1 \left(\frac{a}{p} \right) P_1 \left(\frac{2a+b}{p} \right) \psi(b) = B_{2,\psi}/4.
 \end{aligned}$$

Proof. The assertion (i) follows from $c_1(x)c_2(x) = c_3(x)$ and (4.4). The assertion (iv) follows from $c_1^2(x) = c_2(x) - \pi^2 (x \notin \mathbb{Z})$ and (4.4). We now prove the assertion (ii). We introduce the following trig function.

$$t_1(x) = \begin{cases} \pi \tan(\pi x), & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

From the trig identity $c_1(2x) = \frac{1}{2}(c_1(x) - t_1(x))$ together with $c_2(x) = c_1^2(x) + \pi^2$ for $x \notin \mathbb{Z}$ and $c_1(x)c_2(x) = c_3(x)$, we get

$$\begin{aligned}
 & \sum_{a(p)} c_1 \left(\frac{2a}{p} \right) c_2 \left(\frac{a}{p} \right) \psi(a) \\
 &= \frac{1}{2} \sum_{a(p)} c_3 \left(\frac{a}{p} \right) \psi(a) - \frac{\pi^2}{2} \sum_{a(p)} c_1 \left(\frac{a}{p} \right) \psi(a) - \frac{\pi^2}{2} \sum_{a(p)} t_1 \left(\frac{a}{p} \right) \psi(a).
 \end{aligned}$$

The first two sums can be evaluated by (4.4), so it remains to evaluate the third sum. Observe that $t_1(x) = -c_1(x + 1/2) = -2c_1(2x) + c_1(x)$ by the

multiplication formula (2.2). Hence, by (4.4), we obtain

$$\sum_{a(p)} t_1 \left(\frac{a}{p} \right) \psi(a) = (1 - 2\psi(2))\tau(\psi)2\pi i B_{1,\psi}.$$

Thus the assertion (ii) of the proposition follows. The assertions (iii) and (v) can be similarly verified, so we omit the proofs. We now prove the assertion (vi). Applying the addition formula (2.3) to the arguments $x = \frac{-a}{p}, y = \frac{a+b}{p}$, multiplying throughout by $\psi(b)$ and carefully summing over $a, b(p)$, we get

$$\sum_{a,b(p)} c_1 \left(\frac{a}{p} \right) c_1 \left(\frac{a+b}{p} \right) \psi(b) = 2 \sum_{a,b(p)} c_2 \left(\frac{b}{p} \right) \psi(b).$$

Thus the assertion (vi) follows by (4.4). We now prove the assertion (vii). Applying the addition formula (2.3) to the arguments $x = \frac{-a}{p}, y = \frac{2a+b}{p}$, multiplying throughout by $\psi(b)$ and carefully summing over $a, b(p)$, we get

$$\begin{aligned} & \sum_{a,b(p)} c_1 \left(\frac{a}{p} \right) c_1 \left(\frac{2a+b}{p} \right) \psi(b) \\ &= \frac{1}{2} \sum_{a,b(p)} c_1 \left(\frac{a}{p} \right) c_1 \left(\frac{a+b}{p} \right) \psi(b) + \frac{2 + \psi(2)}{2} \sum_{b(p)} c_2 \left(\frac{b}{p} \right) \psi(b). \end{aligned}$$

100 Thus the assertion (vii) follows from (vi) and (4.4). The assertions (viii), (ix) can be proved in a similar manner to (vi), (vii) where we apply the addition formula Lemma 2.2 rather than (2.3), so we omit the proofs. $\square$

We next prove a lemma which is used in Proposition 4.6.

Lemma 4.2. *For any odd prime number p , we get*

$$A_4(2, 0, 1) = -\frac{4 + \psi(2)}{3} a.$$

Proof. Applying the addition formula Lemma 2.3 with $x = \frac{k^2 - 2kt}{p} + \frac{1}{2}$, $y = \frac{t^2 - k^2}{p}$, then the multiplication formula Lemma 2.1 to all of the periodic

Bernoulli polynomials shifted by $1/2$ and summing over $k, t(p)$, we get

$$\begin{aligned}
 & \frac{1}{2} \sum_{k, t(p)} P_1 \left(\frac{2(k^2 - 2kt)}{p} \right) P_2 \left(\frac{2(t^2 - 2kt)}{p} \right) - \sum_{k, t(p)} P_1 \left(\frac{2(k^2 - 2kt)}{p} \right) P_2 \left(\frac{t^2 - 2kt}{p} \right) \\
 & - \frac{1}{2} \sum_{k, t(p)} P_1 \left(\frac{k^2 - 2kt}{p} \right) P_2 \left(\frac{2(t^2 - 2kt)}{p} \right) + \sum_{k, t(p)} P_1 \left(\frac{k^2 - 2kt}{p} \right) P_2 \left(\frac{t^2 - 2kt}{p} \right) \\
 & + \frac{1}{2} \sum_{k, t(p)} P_1 \left(\frac{t^2 - k^2}{p} \right) P_2 \left(\frac{2(t^2 - 2kt)}{p} \right) - \sum_{k, t(p)} P_1 \left(\frac{t^2 - k^2}{p} \right) P_2 \left(\frac{t^2 - 2kt}{p} \right) \\
 & - \sum_{k, t(p)} P_1 \left(\frac{2(k^2 - 2kt)}{p} \right) P_2 \left(\frac{t^2 - k^2}{p} \right) + \sum_{k, t(p)} P_1 \left(\frac{k^2 - 2kt}{p} \right) P_2 \left(\frac{t^2 - k^2}{p} \right) \\
 & - \frac{1}{2} \sum_{k, t(p)} P_1 \left(\frac{t^2 - k^2}{p} \right) P_2 \left(\frac{2(k^2 - 2kt)}{p} \right) + \sum_{k, t(p)} P_1 \left(\frac{t^2 - k^2}{p} \right) P_2 \left(\frac{k^2 - 2kt}{p} \right) \\
 & = \frac{1}{3} \sum_{k, t(p)} \left\{ \frac{1}{4} P_3 \left(\frac{2(k^2 + 2kt)}{p} \right) - P_3 \left(\frac{k^2 + 2kt}{p} \right) + P_3 \left(\frac{t^2 - k^2}{p} \right) \right. \\
 & \quad \left. + \frac{1}{2} P_3 \left(\frac{2(t^2 - 2kt)}{p} \right) - 2P_3 \left(\frac{t^2 - 2kt}{p} \right) \right\}.
 \end{aligned}$$

The right-hand side vanishes by (ii), (iv) in Proposition 4.3. We now express the ten sums on the left-hand side in terms of Arakawa sums. Replacing (k, t) by $(-k, t - k)$ in the first sum, we get $-A_5(1, 0, 2)$. Replacing (k, t) by $(-k, t - k)$ in the second sum, we get $-A_4(1, 0, 2)$. Replacing (k, t) by $(k - t, k)$ in the third sum, we get $A_4(2, 0, 1)$. Replacing (k, t) by $(-k, t - k)$ in the fourth sum, we get $-A_1(1, 0, 2)$. Replacing (k, t) by (t, k) in the fifth sum, we get $-A_4(2, 0, 1)$. Replacing (k, t) by $(-t, k - t)$ in the sixth sum, we get $A_1(1, 0, 2)$. The seventh sum is $A_4(1, 0, 2)$. The eighth sum is $A_1(1, 0, 2)$. The ninth sum is $A_4(2, 0, 1)$. Replacing (k, t) by $(k - t, -t)$ in the tenth sum, we get $-A_1(1, 0, 2)$. Hence, we have

$$-\frac{1}{2} A_5(1, 0, 2) - 2a - \frac{3}{2} A_4(2, 0, 1) = 0.$$

By Proposition 2.4, we have $A_5(1, 0, 2) = \psi(2)a$. Thus the assertion of the lemma follows. $\square$

Proposition 4.3. *Let $a, b, n, m \in \mathbb{Z}$ with $n, m \geq 0$, and let l be any integer*

prime to p . Then, we have

- (i) $\sum_{k(p)} P_n \left(\frac{lk}{p} \right) \psi(k) = \psi(l) B_{n,\psi} / p^{n-1},$
- (ii) $\sum_{k(p)} P_n \left(\frac{lk+a}{p} \right) = P_n(0) / p^{n-1},$
- (iii) $\sum_{k(p)} P_n \left(\frac{l(k+a)^2}{p} \right) = (\psi(l) B_{n,\psi} + P_n(0)) / p^{n-1},$
- (iv) if $n+m \in 2\mathbb{Z}+1$, then $\sum_{k,t(p)} P_n \left(\frac{a(t^2-k^2)}{p} \right) P_m \left(\frac{bkt}{p} \right) = 0,$
- (v) if $n+m \in 2\mathbb{Z}+1$, then $\sum_{k,t(p)} P_n \left(\frac{a(k^2+2kt)}{p} \right) P_m \left(\frac{b(t^2+kt)}{p} \right) = 0.$

Proof. The assertion (i) follows from (4.3) upon replacing k by $l^{-1}k$. The assertion (ii) follows from the multiplication formula Lemma 2.1. We next prove the assertion (iii). Replacing k by $k-a$, we have $\sum_{k(p)} P_n \left(\frac{lk^2}{p} \right) = \sum_{k(p)} P_n \left(\frac{lk}{p} \right) (\psi(k) + 1)$. Thus the assertion (iii) follows from (i) and (ii). The
110 assertion (iv) follows by parity reasons upon replacing (k, t) by $(-t, k)$. The assertion (v) follows from (iv) after replacing (k, t) by $(k-t, t)$. $\square$

Proposition 4.4. *We have*

$$\begin{aligned}
 (i) \quad & \sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_2\left(\frac{k^2+2kt}{p}\right) = \frac{1}{6p} B_{1,\psi} - \frac{1}{3p} B_{3,\psi}, \\
 (ii) \quad & \sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_2\left(\frac{2(k^2+2kt)}{p}\right) = \frac{4-3\psi(2)}{6p} B_{1,\psi} - \frac{\psi(2)}{6p} B_{3,\psi}, \\
 (iii) \quad & \sum_{k,t(p)} P_1\left(\frac{2t^2}{p}\right) P_2\left(\frac{k^2+2kt}{p}\right) = \frac{5\psi(2)-3}{12p} B_{1,\psi} - \frac{\psi(2)}{12p} B_{3,\psi}, \\
 (iv) \quad & \sum_{k,t(p)} P_1\left(\frac{2t^2}{p}\right) P_2\left(\frac{2(k^2+2kt)}{p}\right) = \frac{\psi(2)}{6p} B_{1,\psi} - \frac{\psi(2)}{3p} B_{3,\psi}, \\
 (v) \quad & \sum_{k,t(p)} P_1\left(\frac{k^2+2kt}{p}\right) P_2\left(\frac{t^2}{p}\right) = -\frac{1}{3p} B_{3,\psi}, \\
 (vi) \quad & \sum_{k,t(p)} P_1\left(\frac{k^2+2kt}{p}\right) P_2\left(\frac{2t^2}{p}\right) = \frac{1-\psi(2)}{2p} B_{1,\psi} - \frac{\psi(2)}{6p} B_{3,\psi}, \\
 (vii) \quad & \sum_{k,t(p)} P_1\left(\frac{2(k^2+2kt)}{p}\right) P_2\left(\frac{t^2}{p}\right) = \frac{\psi(2)-1}{4p} B_{1,\psi} - \frac{\psi(2)}{12p} B_{3,\psi}, \\
 (viii) \quad & \sum_{k,t(p)} P_1\left(\frac{2(k^2+2kt)}{p}\right) P_2\left(\frac{2t^2}{p}\right) = -\frac{\psi(2)}{3p} B_{3,\psi}, \\
 (ix) \quad & \sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_2\left(\frac{(k+t)^2+k^2}{p}\right) = \frac{3\psi(2)-1}{12p} B_{1,\psi} + \frac{1}{12p} B_{3,\psi}.
 \end{aligned}$$

Proof. We prove the assertions in the following way. We first apply the finite Fourier transform (2.4) to each of the periodic Bernoulli polynomials. We next evaluate the resulting generalized quadratic Gaussian sums. It then remains to
 115 evaluate sums involving c_k 's with linear arguments, which is accomplished by applying Lemma 4.1 and (4.4). We now prove the assertion (i) to illustrate.

If $p \equiv 1(4)$, there exists a c such that $c^2 \equiv -1(p)$. Replacing (k, t) by (ck, ct) , we see that $\sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_2\left(\frac{k^2+2kt}{p}\right) = 0$ for parity reasons. Since $B_{3,\psi}, B_{1,\psi}$ also vanish for parity reasons, the assertion is clear in the case $p \equiv 1(4)$. We now assume that $p \equiv 3(4)$. Applying the finite Fourier transform (2.4) to $P_1\left(\frac{t^2}{p}\right)$,

$P_2\left(\frac{k^2+2kt}{p}\right)$ together with (4.1), we find that

$$\begin{aligned}
 \sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_2\left(\frac{k^2+2kt}{p}\right) &= \frac{2}{(2\pi ip)^3} \sum_{a,b(p)} c_1\left(\frac{a}{p}\right) c_2\left(\frac{b}{p}\right) \sum_{k,t(p)} \zeta^{at^2+2bkt+bk^2} \\
 &= \frac{2}{(2\pi ip)^3} \left(\sum_{\substack{a,b(p) \\ a,b \equiv 0(p)}} + \sum_{\substack{a,b(p) \\ a \equiv 0(p) \\ b \not\equiv 0(p)}} + \sum_{\substack{a,b(p) \\ a \not\equiv 0(p) \\ b \equiv 0(p)}} + \sum_{\substack{a,b(p) \\ a,b \not\equiv 0(p) \\ a-b \equiv 0(p)}} + \sum_{\substack{a,b(p) \\ a,b \not\equiv 0(p) \\ a-b \not\equiv 0(p)}} \right) c_1\left(\frac{a}{p}\right) c_2\left(\frac{b}{p}\right) \sum_{k,t(p)} \zeta^{at^2+2bkt+bk^2} \\
 &= \frac{2}{(2\pi ip)^3} \left\{ p^2 c_1(0) c_2(0) + p c_1(0) \sum_{\substack{b(p) \\ b \not\equiv 0(p)}} c_2\left(\frac{b}{p}\right) + p \tau(\psi) c_2(0) \sum_{a(p)} c_1\left(\frac{a}{p}\right) \psi(a) \right. \\
 &\quad \left. + p \tau(\psi) \sum_{a(p)} c_1\left(\frac{a}{p}\right) c_2\left(\frac{a}{p}\right) \psi(a) + \tau^2(\psi) \sum_{a,b(p)} c_1\left(\frac{a}{p}\right) c_2\left(\frac{b}{p}\right) \psi(b) \psi(a-b) \right\}.
 \end{aligned}$$

Since c_1, ψ are odd and c_2 is even, the last sum vanishes for parity reasons. Thus the assertion (i) follows from $c_1(0) = 0, c_2(0) = \pi^2/3$, (4.4), and (i) in Lemma 4.1. The assertions (ii)-(ix) can be similarly verified, so we omit the proofs. $\square$

Proposition 4.5. *We have*

$$\begin{aligned}
 (i) \quad & \sum_{k,t(p)} P_1 \left(\frac{k^2 + 2kt}{p} \right) P_2 \left(\frac{2kt}{p} \right) = -\frac{1}{3p} B_{3,\psi}, \\
 (ii) \quad & \sum_{k,t(p)} P_1 \left(\frac{k^2 + 2kt}{p} \right) P_2 \left(\frac{4kt}{p} \right) = \frac{1 - \psi(2)}{2p} B_{1,\psi} - \frac{\psi(2)}{6p} B_{3,\psi}, \\
 (iii) \quad & \sum_{k,t(p)} P_1 \left(\frac{2(k^2 + 2kt)}{p} \right) P_2 \left(\frac{2kt}{p} \right) = \frac{\psi(2) - 1}{4p} B_{1,\psi} - \frac{\psi(2)}{12p} B_{3,\psi}, \\
 (iv) \quad & \sum_{k,t(p)} P_1 \left(\frac{2(k^2 + 2kt)}{p} \right) P_2 \left(\frac{4kt}{p} \right) = -\frac{\psi(2)}{3p} B_{3,\psi}, \\
 (v) \quad & \sum_{k,t(p)} P_1 \left(\frac{2kt}{p} \right) P_2 \left(\frac{k^2 + 2kt}{p} \right) = \frac{1}{3p} B_{3,\psi}, \\
 (vi) \quad & \sum_{k,t(p)} P_1 \left(\frac{2kt}{p} \right) P_2 \left(\frac{2(k^2 + 2kt)}{p} \right) = \frac{\psi(2) - 1}{2p} B_{1,\psi} + \frac{\psi(2)}{6p} B_{3,\psi}, \\
 (vii) \quad & \sum_{k,t(p)} P_1 \left(\frac{4kt}{p} \right) P_2 \left(\frac{k^2 + 2kt}{p} \right) = \frac{1 - \psi(2)}{4p} B_{1,\psi} + \frac{\psi(2)}{12p} B_{3,\psi}, \\
 (viii) \quad & \sum_{k,t(p)} P_1 \left(\frac{4kt}{p} \right) P_2 \left(\frac{2(k^2 + 2kt)}{p} \right) = \frac{\psi(2)}{3p} B_{3,\psi}, \\
 (ix) \quad & \sum_{k,t(p)} P_1 \left(\frac{k^2 + kt}{p} \right) P_2 \left(\frac{kt}{p} \right) = -\frac{1}{3p} B_{3,\psi}, \\
 (x) \quad & \sum_{k,t(p)} P_1 \left(\frac{k^2 + kt}{p} \right) P_2 \left(\frac{t^2 + kt}{p} \right) = -\frac{1}{3p} B_{3,\psi}, \\
 (xi) \quad & \sum_{k,t(p)} P_1 \left(\frac{k^2 + 2kt}{p} \right) P_2 \left(\frac{kt}{p} \right) = \frac{1 - \psi(2)}{4p} B_{1,\psi} - \frac{1}{12p} B_{3,\psi}, \\
 (xii) \quad & \sum_{k,t(p)} P_1 \left(\frac{kt}{p} \right) P_2 \left(\frac{k^2 + 2kt}{p} \right) = \frac{1 - \psi(2)}{2p} B_{1,\psi} + \frac{1}{6p} B_{3,\psi}.
 \end{aligned}$$

Since the assertions of this proposition can be proved in a manner similar to those in Proposition 4.4, we omit the proofs.

Proposition 4.6. *We have*

$$\begin{aligned}
 (i) \quad & \sum_{k,t(p)} P_1 \left(\frac{k^2 + 2kt}{p} \right) P_2 \left(\frac{t^2 + 2kt}{p} \right) = -a, \\
 (ii) \quad & \sum_{k,t(p)} P_1 \left(\frac{k^2 + 2kt}{p} \right) P_2 \left(\frac{2(t^2 + 2kt)}{p} \right) = -\frac{4 + \psi(2)}{3} a, \\
 (iii) \quad & \sum_{k,t(p)} P_1 \left(\frac{2(k^2 + 2kt)}{p} \right) P_2 \left(\frac{t^2 + 2kt}{p} \right) = -A_4(1, 0, 2), \\
 (iv) \quad & \sum_{k,t(p)} P_1 \left(\frac{2(k^2 + 2kt)}{p} \right) P_2 \left(\frac{2(t^2 + 2kt)}{p} \right) = -\psi(2)a.
 \end{aligned}$$

Proof. The assertions (i), (iii) follow by replacing (k, t) by $(k, t - k)$. The assertion (ii) follows by replacing (k, t) by $(t - k, k)$ to get $A_4(2, 0, 1)$, then applying Lemma 4.2. The assertion (iv) follows from Proposition 2.4 and (i). $\square$

Proposition 4.7. *We have*

$$\begin{aligned}
 (i) \quad & \sum_{a=1}^{p-1} a\psi(a) = pB_{1,\psi}, \\
 (ii) \quad & \sum_{a,b=1}^{p-1} ab\psi(a-b) = \frac{p^2}{2} B_{2,\psi}, \\
 (iii) \quad & \sum_{a,b=1}^{p-1} ab\psi(2a-b) = \frac{p^2}{4} (2(1-\psi(2))B_{1,\psi} + B_{2,\psi}), \\
 (iv) \quad & \sum_{a,b=1}^{p-1} ab\psi(a+b) = -\frac{p^2}{2} (2B_{1,\psi} + B_{2,\psi}).
 \end{aligned}$$

Proof. The assertion (i) is obvious. We now prove the assertion (ii). We have $\sum_{a,b=1}^{p-1} ab\psi(a-b) = \sum_{a,b(p)} [a][b]\psi(a-b)$, where $[n]$ denotes the least nonnegative residue of $n \bmod p$. Observe that $[x]/p = P_1(x/p) + 1/2$ if $x \notin \mathbb{Z}$ and $P_1(x/p)$ otherwise. Replacing b by $a+b$ and expressing everything in terms of periodic Bernoulli polynomials, with the help of (i), (ii) in Proposition 4.3, we get

$$\frac{\psi(-1)}{p^2} \sum_{a,b=1}^{p-1} ab\psi(a-b) = \sum_{a,b(p)} \frac{[a]}{p} \frac{[a+b]}{p} \psi(b) = \sum_{a,b(p)} P_1 \left(\frac{a}{p} \right) P_1 \left(\frac{a+b}{p} \right) \psi(b).$$

Thus the assertion (i) follows from (viii) in Lemma 4.1. The remaining assertions can be similarly verified, so we omit the proofs. $\square$

Remark. For an integer l , sums of the form $\sum_{a,b=1}^{p-1} ab\psi(la+b)$ have been studied in greater generality in [8].

Proposition 4.8. *We have*

$$\begin{aligned}
 (i) \quad & \sum_{k,t(p)} P_1 \left(\frac{2k}{p} \right) P_1 \left(\frac{2kt^2}{p} \right) \psi(k) = 0, \\
 (ii) \quad & \sum_{k,t(p)} P_1 \left(\frac{2k}{p} \right) P_1 \left(\frac{k(t+1)^2}{p} \right) \psi(k) = 0, \\
 (iii) \quad & \sum_{k,t(p)} P_1 \left(\frac{2kt^2}{p} \right) P_1 \left(\frac{k(t+1)^2}{p} \right) \psi(k) = -\frac{4+\psi(2)}{4p} B_{2,\psi}, \\
 (iv) \quad & \sum_{k(p)} P_1^2 \left(\frac{2k}{p} \right) \psi(k) = \frac{\psi(2)}{p} B_{2,\psi}, \\
 (v) \quad & \sum_{k(p)} P_1 \left(\frac{2k}{p} \right) P_1 \left(\frac{k}{p} \right) \psi(k) = \frac{4+\psi(2)}{4p} B_{2,\psi}, \\
 (vi) \quad & \sum_{k,t(p)} P_1 \left(\frac{k(t+1)^2+k}{p} \right) P_1 \left(\frac{kt^2}{p} \right) \psi(k) = \frac{p-4}{4p} B_{2,\psi}, \\
 (vii) \quad & \sum_{k,t(p)} P_1 \left(\frac{k(t+1)^2+k}{p} \right) P_1 \left(\frac{k}{p} \right) \psi(k) = \frac{1}{2} B_{2,\psi}, \\
 (viii) \quad & \sum_{k,t(p)} P_1 \left(\frac{kt^2}{p} \right) P_1 \left(\frac{k}{p} \right) \psi(k) = 0, \\
 (ix) \quad & \sum_{k,t(p)} P_1 \left(\frac{k(t+1)^2+k}{p} \right) \psi(k) = -B_{1,\psi}, \\
 (x) \quad & \sum_{k,t(p)} P_1 \left(\frac{kt(t+1)}{p} \right) P_1 \left(\frac{k(t+1)}{p} \right) \psi(k) = -\frac{1}{2} B_{2,\psi}, \\
 (xi) \quad & \sum_{k(p)} P_1 \left(\frac{kt(t+1)}{p} \right) P_1 \left(\frac{kt}{p} \right) \psi(k) = \frac{1}{2} B_{2,\psi}, \\
 (xii) \quad & \sum_{k(p)} P_1 \left(\frac{k(t+1)}{p} \right) P_1 \left(\frac{kt}{p} \right) \psi(k) = \frac{1}{2} B_{2,\psi}, \\
 (xiii) \quad & \sum_{k,t(p)} P_1 \left(\frac{kt(t+1)}{p} \right) \psi(k) = -B_{1,\psi}.
 \end{aligned}$$

130

Since the assertions of this proposition can be proved in a manner similar to those in Proposition 4.4, we omit the proofs.

5. Evaluation of Lee-Weintraub Sums

In this section, we evaluate the Lee-Weintraub sums given in the introduction. Since this work is new and there are no models, we will carefully explain

135 the results. We remind the reader of the notation.

Fix an odd prime p and let ψ denote the Legendre symbol mod p : $\psi(a) = \left(\frac{a}{p}\right)$. We put $\zeta = \exp(2\pi i/p)$. Let $\tau(\psi)$ denote the Gaussian sum $\tau(\psi) = \sum_{n=1}^{p-1} \psi(n)\zeta^n$. Let $\delta_{x,y}$ be the Kronecker delta symbol. We remind the reader that $A_i(k_1, k_2, k_3)$ are the Arakawa sums given in (3.1), and that A_i, a are the
140 Arakawa sums given in (3.2).

We will routinely make use of the multiplication formula Lemma 2.1 to express periodic Bernoulli polynomials shifted by $1/2$ as follows.

$$P_k(x + 1/2) = 1/2^{k-1} P_k(2x) - P_k(x) \quad (k \in \mathbb{Z}, k \geq 0, x \in \mathbb{R}). \quad (5.1)$$

We will also regularly call upon Proposition 2.4 to “factor out” a $\psi(2)$ from a sum of products of periodic Bernoulli polynomials.

We next introduce some new notation. Let $q_j(k, t)$ ($j = 1, 2, 3$) be polynomial expressions of k and t . We write $S(q_1(k, t), q_2(k, t), q_3(k, t))$ to denote the following sum of triple products of periodic Bernoulli polynomials.

$$S(q_1(k, t), q_2(k, t), q_3(k, t)) = \sum_{k, t(p)} P_1(q_1(k, t)/p) P_1(q_2(k, t)/p) P_1(q_3(k, t)/p).$$

In the theorems below, we frequently simplify sums of products of three periodic Bernoulli polynomials by applying the addition formula Lemma 2.2 to two of the terms, then multiplying throughout by the remaining term and carefully summing. Thus we introduce the following notation. We write $[a, b; c; (d) \text{ in } x.x, (e) \text{ in } y.y]$ to mean “applying Lemma 2.2 with $x = a/p, y = b/p$, multiplying throughout by $P_1(c/p)$ and summing over $k, t(p)$ with the help of (d) in Proposition x.x and (e) in Proposition y.y”. Through this process, we express our original sum of triple products as two new sums of triple products plus correction sums. Since all of the nontrivial correction sums have already been evaluated in Section 4, we will only explain manipulations of sums of triple products of periodic Bernoulli polynomials. Accordingly, we introduce the following notation. We write $S(a, b, c) \sim S(a, a + b, c) + S(b, a + b, c)$ to mean

$$S(a, b, c) = S(a, a + b, c) + S(b, a + b, c) + \text{“correction terms”},$$

where the correction terms are generalized Bernoulli numbers arising from the evaluation of correction sums. We will apply the same sequence of steps to
145 each new sum of triple products arising. In this way, we will obtain a tree of sums. In the end, we will condense the tree and express our original sum of triple products by Arakawa sums, from which we can evaluate by virtue of the Arakawa Identities given by Theorem 3.1. Occasionally some of the sums of triple products will vanish for parity reasons, perhaps not directly, but after a
150 suitable transformation (such as replacing k by $k - t$). We will not mention these sums at all.

We now prove the following theorem which was conjectured by Lee and Weintraub in the case of $p \equiv 3(4)$ (see page 256 in [1]).

Theorem 5.1. *For any odd prime number p , we get*

$$S[-x, x+2, x(x+2)] \\ = \sqrt{\psi(-1)p} \left(\frac{(p+1)(1+\psi(2))}{12} B_{1,\psi} + \frac{4+3\psi(2)}{4} B_{2,\psi} + \frac{4+\psi(2)}{18} B_{3,\psi} \right).$$

Proof. Let S denote the sum $S[-x, x+2, x(x+2)]$. Then, we have

$$S = - \sum'_{k,j(p)} \frac{\psi(k)}{(\zeta^{-kj} - 1)(\zeta^{k(j+2)} - 1)(\zeta^{kj(j+2)} - 1)}.$$

Since $\frac{1}{\zeta^n - 1} = \frac{1}{p} \sum_{a=1}^{p-1} a \zeta^{an}$ for any n with $(n, p) = 1$, together with (4.2) and (i) in Proposition 4.7, we get

$$\begin{aligned} S &= -\frac{1}{p^3} \sum_{a,b,c=1}^{p-1} abc \sum_{\substack{j(p) \\ j \not\equiv 0, -2(p)}} \sum_{\substack{k(p) \\ k \not\equiv 0(p)}} \zeta^{k\{cj^2 + (-a+b+2c)j+2b\}} \psi(k) \\ &= -\frac{\tau(\psi)}{p^3} \sum_{a,b,c=1}^{p-1} abc \left\{ \left(\sum_{j(p)} \psi(cj^2 + (-a+b+2c)j+2b) \right) - \psi(2b) - \psi(2a) \right\} \\ &= \tau(\psi) \left(\frac{(p-1)^2(2\psi(2)+1)}{4} B_{1,\psi} - \frac{1}{p^2} \sum_{\substack{a,b,c=1 \\ (-a+b+2c)^2 - 8bc \equiv 0(p)}}^{p-1} abc \psi(c) \right). \end{aligned}$$

We put $T = \{(a, b, c) \in \mathbb{Z}^3 \mid 1 \leq a, b, c \leq p-1, (-a+b+2c)^2 - 8bc \equiv 0(p)\}$, $U = \{(2k, 2kt^2, k(t+1)^2) \in (\mathbb{F}_p^\times)^3 \mid 1 \leq k, t \leq p-1\}$, and

$$A = \sum_{(a,b,c) \in T} abc \psi(c).$$

Then, we have

$$S = \sqrt{\psi(-1)p} \left(\frac{(p-1)^2(2\psi(2)+1)}{4} B_{1,\psi} - \frac{1}{p^2} A \right). \quad (5.2)$$

Next we evaluate A . To this end, we express everything in terms of periodic Bernoulli polynomials and take full advantage of their periodicity. By the 1-to-1 correspondence between the sets T and U , we have

$$A = \sum_{\substack{k,t(p) \\ kt(t+1) \not\equiv 0(p)}} [2k][2kt^2][k(t+1)^2] \psi(k(t+1)^2),$$

where $[n]$ denotes the least nonnegative residue of n mod p . Observe that

$[x]/p = P_1(x/p) + 1/2$ if $x \notin \mathbb{Z}$ and $P_1(x/p)$ otherwise. Then, with the help of (i), (ii) in Proposition 4.3 and (i)-(v) in Proposition 4.8, we get

$$\frac{1}{p^3}A = I + \frac{(p-2)(2\psi(2)+1)}{4}B_{1,\psi} - \frac{4+3\psi(2)}{4p}B_{2,\psi}, \quad (5.3)$$

where $I = \sum_{k,t(p)} P_1\left(\frac{2k}{p}\right) P_1\left(\frac{2kt^2}{p}\right) P_1\left(\frac{k(t+1)^2}{p}\right) \psi(k)$.

We now evaluate the sum I . Since $\sum_{k,t(p)} P_1\left(\frac{2k}{p}\right) P_1\left(\frac{2kt^2}{p}\right) P_1\left(\frac{k(t+1)^2}{p}\right) = 0$ for parity reasons, we have $I = \sum_{k,t(p)} P_1\left(\frac{2k}{p}\right) P_1\left(\frac{2kt^2}{p}\right) P_1\left(\frac{k(t+1)^2}{p}\right) (\psi(k) + 1) = \sum_{k,t(p)} P_1\left(\frac{2k^2}{p}\right) P_1\left(\frac{2k^2t^2}{p}\right) P_1\left(\frac{k^2(t+1)^2}{p}\right)$. Replacing (k, t) by $(k, k^{-1}t)$ whenever $k \not\equiv 0(p)$, we get

$$I = \sum_{k,t(p)} P_1\left(\frac{2k^2}{p}\right) P_1\left(\frac{2t^2}{p}\right) P_1\left(\frac{(k+t)^2}{p}\right).$$

We set

$$\begin{aligned} X_1 &= \sum_{k,t(p)} P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{(k+t)^2}{p}\right), \\ X_2 &= \sum_{k,t(p)} P_1\left(\frac{k^2}{p} + \frac{1}{2}\right) P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{(k+t)^2}{p}\right), \\ X_3 &= \sum_{k,t(p)} P_1\left(\frac{k^2}{p} + \frac{1}{2}\right) P_1\left(\frac{t^2}{p} + \frac{1}{2}\right) P_1\left(\frac{(k+t)^2}{p}\right). \end{aligned}$$

By the multiplication formula Lemma 2.1, we have

$$I = X_1 + 2X_2 + X_3. \quad (5.4)$$

155 We now express the X_j ($j = 1, 2, 3$) in terms of Arakawa sums by creating trees of sums of depth 4 using the notation introduced at the beginning of the section.

By $[-t^2, (k+t)^2; k^2; \text{(ii), (iii) in 4.3}]$, we get

$$X_1 \sim S(k^2, t^2, k^2 + 2kt) - S(k^2, (k+t)^2, k^2 + 2kt).$$

Replacing (k, t) by $(-k, k+t)$ in the second sum on the right, we obtain

$$X_1 \sim 2S(k^2, t^2, k^2 + 2kt). \quad (5.5)$$

By $[-k^2, k^2 + 2kt; t^2; \text{(ii), (iii) in 4.3, (i) in 4.4}]$, we get

$$S(k^2, t^2, k^2 + 2kt) \sim -S(t^2, 2kt, k^2 + 2kt). \quad (5.6)$$

By $[t^2, 2kt; k^2 + 2kt; (v) \text{ in } 4.4, (i) \text{ in } 4.5, (i) \text{ in } 4.6]$, we get

$$\begin{aligned} S(t^2, k^2 + 2kt, 2kt) \\ \sim S(t^2, t^2 + 2kt, k^2 + 2kt) + S(2kt, t^2 + 2kt, k^2 + 2kt) + \frac{1}{2}a. \end{aligned} \quad (5.7)$$

We now simplify the first sum on the right-hand side of (5.7). By $[t^2, t^2 + 2kt; k^2 + 2kt; (v) \text{ in } 4.3, (v) \text{ in } 4.4, (i) \text{ in } 4.6]$, we find that

$$S(t^2, t^2 + 2kt, k^2 + 2kt) \sim S(t^2 + 2kt, 2(t^2 + kt), k^2 + 2kt) + \frac{1}{2}a.$$

Replacing (k, t) by $(t - k, k)$ in the sum on the right, we get $-A_1$. Thus, we have

$$S(t^2, t^2 + 2kt, k^2 + 2kt) \sim -A_1 + \frac{1}{2}a. \quad (5.8)$$

We next simplify the second sum on the right-hand side of (5.7). By $[t^2 + 2kt, -(k^2 + 2kt); 2kt; (iv) \text{ in } 4.3, (v) \text{ in } 4.5]$, we get

$$S(2kt, t^2 + 2kt, k^2 + 2kt) \sim -S(2kt, t^2 + 2kt, t^2 - k^2) + S(2kt, k^2 + 2kt, t^2 - k^2).$$

Replacing (k, t) by $(t, -k)$ in the first sum and (k, t) by $(-k, t)$ in the second sum, we get A_1 and $-A_1$, respectively. Thus, we have

$$S(2kt, t^2 + 2kt, k^2 + 2kt) \sim -2A_1. \quad (5.9)$$

Then, from (5.5)-(5.9), we get

$$X_1 = 6 \left(A_1 - \frac{1}{3}a \right) - \frac{6p - 3 + 2p\delta_{p,3}}{4p} B_{1,\psi} - \frac{2}{p} B_{3,\psi}. \quad (5.10)$$

We next express X_2 in terms of Arakawa sums. By $[-t^2, (k + t)^2; k^2 + p/2; (ii), (iii) \text{ in } 4.3]$, we get

$$X_2 \sim S(k^2 + p/2, t^2, k^2 + 2kt) - S(k^2 + p/2, (k + t)^2, k^2 + 2kt).$$

Replacing (k, t) by $(k, t - k)$ in the second sum, we have

$$X_2 \sim 2S(k^2 + p/2, t^2, k^2 + 2kt). \quad (5.11)$$

By $[-k^2 + p/2, k^2 + 2kt; t^2; (ii), (iii) \text{ in } 4.3, (i) \text{ in } 4.4]$, we get

$$S(k^2 + p/2, t^2, k^2 + 2kt) \sim -S(t^2, 2kt + p/2, k^2 + 2kt). \quad (5.12)$$

By $[t^2, 2kt + p/2; k^2 + 2kt; (v)$ in 4.4, (i), (ii) in 4.5, (i), (ii) in 4.6], we get

$$\begin{aligned} & S(t^2, 2kt + p/2, k^2 + 2kt) \\ & \sim S(t^2, t^2 + 2kt + p/2, k^2 + 2kt) \\ & + S(2kt + p/2, t^2 + 2kt + p/2, k^2 + 2kt) \\ & + \frac{\psi(2) - 2}{12} a. \end{aligned} \quad (5.13)$$

We now simplify the first sum on the right-hand side of (5.13). By $[t^2, t^2 + 2kt + p/2; k^2 + 2kt; (v)$ in 4.3, (v) in 4.4, (i), (ii) in 4.6], we get

$$\begin{aligned} & S(t^2, t^2 + 2kt + p/2, k^2 + 2kt) \\ & \sim S(t^2 + 2kt + p/2, 2(t^2 + kt) + p/2, k^2 + 2kt) + \frac{\psi(2) - 2}{12} a. \end{aligned}$$

Replacing (k, t) by $(t - k, k)$ in the sum on the right and applying (5.1), we obtain

$$\begin{aligned} & S(t^2, t^2 + 2kt + p/2, k^2 + 2kt) \\ & \sim -S(2(k^2 - 2kt), 4kt, t^2 - k^2) - A_1 + A_3 + A_4 + \frac{\psi(2) - 2}{12} a. \end{aligned} \quad (5.14)$$

We next simplify the second sum on the right-hand side of (5.13). By $[t^2 + 2kt + p/2, -(k^2 + 2kt); 2kt + p/2; (iv)$ in 4.3, (v)-(viii) in 4.5], we get

$$\begin{aligned} & S(2kt + p/2, t^2 + 2kt + p/2, k^2 + 2kt) \\ & \sim -S(2kt + p/2, t^2 + 2kt + p/2, t^2 - k^2 + p/2) \\ & + S(2kt + p/2, k^2 + 2kt, t^2 - k^2 + p/2). \end{aligned}$$

Replacing (k, t) by $(t, -k)$ in the first sum and (k, t) by $(-k, t)$ in the second sum and applying (5.1) together with Proposition 2.4, we obtain

$$\begin{aligned} & S(2kt + p/2, t^2 + 2kt + p/2, k^2 + 2kt) \\ & \sim S(2(k^2 - 2kt), 4kt, t^2 - k^2) - \psi(2)A_1 + \psi(2)A_2 - A_4. \end{aligned} \quad (5.15)$$

Then, from (5.11)-(5.15), we get

$$\begin{aligned} X_2 &= 2(1 + \psi(2)) \left(A_1 - \frac{1}{3}a \right) - 2\psi(2) \left(A_2 - \frac{1}{6}a \right) - 2 \left(A_3 - \frac{2}{3}a \right) \\ &+ \frac{\psi(2) - 1}{4p} B_{1,\psi} - \frac{4 + \psi(2)}{6p} B_{3,\psi}. \end{aligned} \quad (5.16)$$

We now express X_3 in terms of Arakawa sums. By $[-t^2 + p/2, (k + t)^2; k^2 +$

$p/2$; (ii), (iii) in 4.3], we get

$$X_3 = S(k^2 + p/2, t^2 + p/2, k^2 + 2kt + p/2) - S(k^2 + p/2, (k+t)^2, k^2 + 2kt + p/2).$$

Replacing (k, t) by $(k, t - k)$ in the second sum on the right, we have

$$\begin{aligned} X_3 &= S(k^2 + p/2, t^2 + p/2, k^2 + 2kt + p/2) \\ &\quad + S(k^2 + p/2, t^2, k^2 + 2kt + p/2). \end{aligned} \quad (5.17)$$

We now simplify the first sum on the right-hand side of (5.17). By $[-k^2 + p/2, k^2 + 2kt + p/2; t^2 + p/2; (ii), (iii) in 4.3, (i)-(iv) in 4.4]$, we get

$$\begin{aligned} &S(k^2 + p/2, t^2 + p/2, k^2 + 2kt + p/2) \\ &\sim -S(t^2 + p/2, 2kt, k^2 + 2kt + p/2). \end{aligned} \quad (5.18)$$

By $[t^2 + p/2, 2kt; k^2 + 2kt + p/2; (v)-(viii) in 4.4, (i), (iii) in 4.5, (i)-(iv) in 4.6]$, we get

$$\begin{aligned} &S(t^2 + p/2, 2kt, k^2 + 2kt + p/2) \\ &\sim S(t^2 + p/2, t^2 + 2kt + p/2, k^2 + 2kt + p/2) \\ &\quad + S(2kt, t^2 + 2kt + p/2, k^2 + 2kt + p/2) \\ &\quad + \frac{\psi(2) + 1}{6}a - \frac{1}{2}A_4(1, 0, 2). \end{aligned} \quad (5.19)$$

We now simplify the first sum on the right-hand side of (5.19). By $[t^2 + p/2, t^2 + 2kt + p/2; k^2 + 2kt + p/2; (v) in 4.3, (v)-(viii) in 4.4, (i)-(iv) in 4.6]$, we find that

$$\begin{aligned} &S(t^2 + p/2, t^2 + 2kt + p/2, k^2 + 2kt + p/2) \\ &\sim S(t^2 + 2kt + p/2, 2(t^2 + kt), k^2 + 2kt + p/2) \\ &\quad + \frac{\psi(2) + 1}{6}a - \frac{1}{2}A_4(1, 0, 2). \end{aligned}$$

Replacing (k, t) by $(t - k, k)$ in the sum on the right and applying (5.1) together with Proposition 2.4, we obtain

$$\begin{aligned} &S(t^2 + p/2, t^2 + 2kt + p/2, k^2 + 2kt + p/2) \\ &\sim S(k^2 - 2kt, 2kt, 2(t^2 - k^2)) \\ &\quad - A_1 - \psi(2)A_2 + A_4 + \frac{\psi(2) + 1}{6}a - \frac{1}{2}A_4(1, 0, 2). \end{aligned} \quad (5.20)$$

We next simplify the second sum on the right-hand side of (5.19). By $[t^2 +$

$2kt + p/2, -(k^2 + 2kt) + p/2; 2kt$; (iv) in 4.3, (v), (vi) in 4.5], we get

$$\begin{aligned} & S(2kt, t^2 + 2kt + p/2, k^2 + 2kt + p/2) \\ & \sim -S(2kt, t^2 + 2kt + p/2, t^2 - k^2) + S(2kt, k^2 + 2kt + p/2, t^2 - k^2). \end{aligned}$$

Replacing (k, t) by $(t, -k)$ in the first sum and (k, t) by $(-k, t)$ in the second sum and applying (5.1), we get $A_4 - A_1$ and $-A_4 + A_1$, respectively. Thus, we have

$$S(2kt, t^2 + 2kt + p/2, k^2 + 2kt + p/2) \sim -2A_4 + 2A_1. \quad (5.21)$$

We next simplify the second sum on the right-hand side of (5.17). By $[-k^2 + p/2, k^2 + 2kt + p/2; t^2$; (ii), (iii) in 4.3, (i), (ii) in 4.4], we get

$$S(k^2 + p/2, t^2, k^2 + 2kt + p/2) \sim -S(t^2, 2kt, k^2 + 2kt + p/2). \quad (5.22)$$

By $[t^2, 2kt; k^2 + 2kt + p/2$; (v), (vii) in 4.4, (i), (iii) in 4.5, (i), (iii) in 4.6], we get

$$\begin{aligned} & S(t^2, 2kt, k^2 + 2kt + p/2) \\ & \sim S(t^2, t^2 + 2kt, k^2 + 2kt + p/2) + S(2kt, t^2 + 2kt, k^2 + 2kt + p/2) \\ & - \frac{1}{2}a + \frac{1}{2}A_4(1, 0, 2). \end{aligned} \quad (5.23)$$

We now simplify the first sum on the right-hand side of (5.23). By $[t^2, t^2 + 2kt; k^2 + 2kt + p/2$; (v) in 4.3, (v), (vii) in 4.4, (i), (iii) in 4.6], we find that

$$\begin{aligned} & S(t^2, t^2 + 2kt, k^2 + 2kt + p/2) \\ & \sim S(t^2 + 2kt, 2(t^2 + kt), k^2 + 2kt + p/2) - \frac{1}{2}a + \frac{1}{2}A_4(1, 0, 2). \end{aligned}$$

Replacing (k, t) by $(t - k, k)$ in the sum on the right and applying (5.1), we get

$$\begin{aligned} & S(t^2, t^2 + 2kt, k^2 + 2kt + p/2) \\ & \sim -S(k^2 + 2kt, 2kt, 2(t^2 - k^2)) + A_1 - \frac{1}{2}a + \frac{1}{2}A_4(1, 0, 2). \end{aligned} \quad (5.24)$$

We next simplify the second sum on the right-hand side of (5.23). By $[t^2 + 2kt, -(k^2 + 2kt) + p/2; 2kt$; (iv) in 4.3, (v), (vi) in 4.5], we get

$$\begin{aligned} & S(2kt, t^2 + 2kt, k^2 + 2kt + p/2) \\ & \sim -S(2kt, t^2 + 2kt, t^2 - k^2 + p/2) \\ & + S(2kt, k^2 + 2kt + p/2, t^2 - k^2 + p/2). \end{aligned} \quad (5.25)$$

Replacing (k, t) by $(t, -k)$ in the first sum and (k, t) by $(-k, t)$ in the second

sum and applying (5.1) together with Proposition 2.4, we obtain

$$S(2kt, t^2 + 2kt, k^2 + 2kt + p/2) \sim -\psi(2)A_2 + A_4. \quad (5.26)$$

Then, from (5.17)-(5.26), we get

$$\begin{aligned} X_3 = & -2 \left(A_1 - \frac{1}{3}a \right) + 2\psi(2) \left(A_2 - \frac{1}{6}a \right) + \frac{1-\psi(2)}{2} B_{1,\psi} \\ & + \frac{2-\psi(2)}{3p} B_{3,\psi}. \end{aligned} \quad (5.27)$$

Theorem 5.1 then follows immediately from (5.2)-(5.4), (5.10), (5.16), (5.27), and Theorem 3.1. $\square$

The following theorem was also conjectured by Lee and Weintraub in the case of $p \equiv 3(4)$ (see page 256 in [1]).

Theorem 5.2. *For any odd prime number p , we get*

$$S[-x, x+1, x(x+2)] = \sqrt{\psi(-1)p} \left(\frac{2p-1+3\psi(2)}{16} B_{1,\psi} + \frac{5+\psi(2)}{8} B_{2,\psi} + \frac{1}{16} B_{3,\psi} \right).$$

Proof. Let S denote the sum $S[-x, x+1, x(x+2)]$. Then, we have

$$S = - \sum'_{k,j(p)} \frac{\psi(k)}{(\zeta^{-kj} - 1)(\zeta^{k(j+1)} - 1)(\zeta^{kj(j+2)} - 1)}.$$

We proceed as in the proof of Theorem 5.1. By (4.2) and (i)-(iii) in Proposition 4.7, we get

$$S = \tau(\psi) \left(\frac{(p-1)(2p-1-\psi(2))}{4} B_{1,\psi} + \frac{3(p-1)}{8} B_{2,\psi} - \frac{1}{p^2} \sum_{\substack{a,b,c=1 \\ (-a+b+2c)^2 - 4bc \equiv 0(p)}}^{p-1} abc \psi(c) \right).$$

We put $T = \{(a, b, c) \in \mathbb{Z}^3 \mid 1 \leq a, b, c \leq p-1, (-a+b+2c)^2 - 4bc \equiv 0(p)\}$, $U = \{(k(t+1)^2 + k, kt^2, k) \in (\mathbb{F}_p^\times)^3 \mid 1 \leq k, t \leq p-1\}$, and

$$A = \sum_{(a,b,c) \in T} abc \psi(c).$$

Then, we have

$$S = \sqrt{\psi(-1)p} \left(\frac{(p-1)(2p-1-\psi(2))}{4} B_{1,\psi} + \frac{3(p-1)}{8} B_{2,\psi} - \frac{1}{p^2} A \right). \quad (5.28)$$

By the 1-to-1 correspondence between the sets T and U , we have

$$A = \sum_{\substack{k, t(p) \\ kt((t+1)^2+1) \not\equiv 0(p)}} [k(t+1)^2 + k][kt^2][k] \psi(k),$$

where $[n]$ denotes the least nonnegative residue of $n \bmod p$. With the help of (i), (ii) in Proposition 4.3, and (vi)-(ix) in Proposition 4.8, we get

$$\frac{1}{p^3} A = I + \frac{2p-3-\psi(2)}{4} B_{1,\psi} + \frac{3p-8-\psi(2)}{8p} B_{2,\psi}, \quad (5.29)$$

where $I = \sum_{k, t(p)} P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{(k+t)^2+k^2}{p}\right)$.

We now express the sum I in terms of Arakawa sums by creating a tree of sums of depth 1. By $[-k^2, (k+t)^2+k^2; t^2; \text{(iii) in 4.3, (ix) in 4.4}]$, we get

$$I = S(t^2, k^2, (k+t)^2) - S(t^2, (k+t)^2+k^2, (k+t)^2) + \frac{\psi(2)+1}{8p} B_{1,\psi} + \frac{1}{24p} B_{3,\psi}.$$

The first sum on the right is X_1 given in Theorem 5.1. Replacing (k, t) by $(k+t, -t)$ in the second sum, we get I . Thus, by (5.10), we obtain

$$I = 3 \left(A_1 - \frac{1}{3} a \right) - \frac{12p-7-\psi(2)+4p\delta_{p,3}}{16p} B_{1,\psi} - \frac{47}{48p} B_{3,\psi}. \quad (5.30)$$

Theorem 5.2 then follows immediately from (5.28)-(5.30) and Theorem 3.1. $\square$

We next prove an identity which is a dual version of the Lee-Weintraub Identity proved by Ibukiyama-Saito[4] involving the Lee-Weintraub sum $S[-x, x+1, x(x+1)]$ (see Introduction). We consider the sums $S[-x, x+1, x(x+1)]$, $S[1, x^2, (x+1)^2]$ dual to each other since the quadratic forms that naturally arise, namely $4bc - (a-b-c)^2$ and $ab+bc+ca$ respectively, are dual to each other. We refer to this identity as the Dual Lee-Weintraub Identity.

Theorem 5.3 (Dual Lee-Weintraub Identity). *For any odd prime number p , we get*

$$S[1, x^2, (x+1)^2] = \sqrt{\psi(-1)p} \left(-(p-2)B_{1,\psi} + \frac{3}{4}B_{2,\psi} - \frac{1}{6}B_{3,\psi} \right).$$

Proof. Let S denote the sum $S[1, x^2, (x+1)^2]$. Then, we have

$$S = - \sum'_{k, j(p)} \frac{\psi(k)}{(\zeta^k - 1)(\zeta^{kj^2} - 1)(\zeta^{k(j+1)^2} - 1)}.$$

We proceed as in the proofs of the previous two theorems. By (4.2) and (iv) in

Proposition 4.7, we get

$$S = \tau(\psi) \left(-\frac{3(p-1)}{2} B_{1,\psi} - \frac{3(p-1)}{4} B_{2,\psi} - \frac{1}{p^2} \sum_{\substack{a,b,c=1 \\ ab+bc+ca \equiv 0(p)}}^{p-1} abc \psi(b+c) \right).$$

We put $T = \{(a, b, c) \in \mathbb{Z}^3 \mid 1 \leq a, b, c \leq p-1, ab+bc+ca \equiv 0(p)\}$, $U = \{(kt(t+1), k(t+1), -kt) \in (\mathbb{F}_p^\times)^3 \mid 1 \leq k, t \leq p-1\}$, and

$$A = \sum_{(a,b,c) \in T} abc \psi(abc). \quad (5.31)$$

Since $\psi(b+c) = \psi(-1)\psi(abc)$ for every triple $(a, b, c) \in T$, we have

$$S = \sqrt{\psi(-1)p} \left(-\frac{3(p-1)}{2} B_{1,\psi} - \frac{3(p-1)}{4} B_{2,\psi} - \frac{\psi(-1)}{p^2} A \right). \quad (5.32)$$

By the 1-to-1 correspondence between the sets T and U , we have

$$A = \sum_{\substack{k,t(p) \\ kt(t+1) \not\equiv 0(p)}} [kt(t+1)][k(t+1)][-kt] \psi(-k^3 t^2(t+1)^2),$$

where $[n]$ denotes the least nonnegative residue of $n \bmod p$. With the help of (i), (ii) in Proposition 4.3, and (x)-(xiii) in Proposition 4.8, we get

$$\frac{\psi(-1)}{p^3} A = -I - \frac{3}{4} B_{1,\psi} - \frac{3}{4} B_{2,\psi}, \quad (5.33)$$

170 where $I = \sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_1\left(\frac{k^2+kt}{p}\right) P_1\left(\frac{t^2+kt}{p}\right)$.

We now express the sum I in terms of Arakawa sums by creating a tree of sums of depth 3. By $[kt, k^2+kt; t^2+kt; (v)$ in 4.3, (ix), (x) in 4.5], we get

$$I \sim S(kt, k^2+2kt, t^2+kt) + S(k^2+kt, k^2+2kt, t^2+kt).$$

Replacing (k, t) by $(-k, k+t)$ in the second sum on the right, we have

$$I \sim 2S(kt, k^2+2kt, t^2+kt). \quad (5.34)$$

By $[kt, t^2+kt; k^2+2kt; (v)$ in 4.3, (xi) in 4.5, (i) in 4.6], we get

$$\begin{aligned} & S(kt, k^2+2kt, t^2+kt) \\ & \sim S(kt, k^2+2kt, t^2+2kt) + S(t^2+kt, k^2+2kt, t^2+2kt) + \frac{1}{2}a. \end{aligned}$$

Replacing (k, t) by $(t-k, k)$ in the second sum on the right, we get $-A_2$. Thus,

we have

$$S(kt, k^2 + 2kt, t^2 + kt) \sim S(kt, k^2 + 2kt, t^2 + 2kt) - A_2 + \frac{1}{2}a. \quad (5.35)$$

By $[-(k^2 + 2kt), t^2 + 2kt; kt; \text{(iv) in 4.3, (xii) in 4.5}]$, we get

$$S(kt, k^2 + 2kt, t^2 + 2kt) \sim S(kt, k^2 + 2kt, t^2 - k^2) - S(kt, t^2 + 2kt, t^2 - k^2).$$

Replacing (k, t) by $(k, -t)$ in the first sum and (k, t) by $(t, -k)$ in the second sum, we get $-A_2$ and A_2 , respectively. Thus, we have

$$S(kt, k^2 + 2kt, t^2 + 2kt) \sim -2A_2. \quad (5.36)$$

Then, from (5.34)-(5.36), we get

$$I = -6 \left(A_2 - \frac{1}{6}a \right) + \frac{3p+3-3\psi(2)-2p\delta_{p,3}}{4p} B_{1,\psi} + \frac{3}{4p} B_{3,\psi}. \quad (5.37)$$

Theorem 5.3 then follows immediately from (5.32), (5.33), (5.37), and Theorem 3.1. $\square$

As a corollary to Theorem 5.3, we prove the following theorem which was conjectured by Ibukiyama and Kaneko in the case of $p \equiv 3(4)$ (see Example 8.19 and Remark 8.20 in [5]).

Corollary 5.4. *For any odd prime number p , we put $T = \{(a, b, c) \in \mathbb{Z}^3 \mid 1 \leq a, b, c \leq p-1, ab+bc+ca \equiv 0(p)\}$ and*

$$A = \sum_{(a,b,c) \in T} abc \psi(abc).$$

Then, we get

$$A = \frac{p^2(p+1)}{2} B_{1,\psi} - \frac{3p^3}{4} B_{2,\psi} - \frac{p^2}{6} B_{3,\psi}.$$

Proof. Observe that this is the same A given by (5.31) in Theorem 5.3. Thus the corollary follows immediately from (5.32) and Theorem 5.3. $\square$

Lastly, we prove a corollary which follows from the results above.

Corollary 5.5. *For any odd prime number p , we get*

$$\sum_{k,t(p)} P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{(k+t)^2}{p}\right) = \sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_1\left(\frac{k^2+kt}{p}\right) P_1\left(\frac{t^2+kt}{p}\right).$$

Proof. Observe that these are the sums X_1 in Theorem 5.1 and I in Theorem 5.3. Thus the corollary follows immediately from (5.10), (5.37), and Theorem 3.1, where we in fact see that both sums are equal to $-\frac{p-2}{4p} B_{1,\psi} - \frac{1}{6p} B_{3,\psi}$. $\square$

The sums in Corollary 5.5 are the essential parts of the dual Lee-Weintraub sums $S[-x, x+1, x(x+1)]$, $S[1, x^2, (x+1)^2]$, respectively. Letting $Q = \{(a, b, c) \in (\mathbb{F}_p^\times)^3 \mid 4bc - (a-b-c)^2 = 0(p)\}$ and $Q^* = \{(a, b, c) \in (\mathbb{F}_p^\times)^3 \mid ab+bc+ca =$

$0(p)\}$, we have the following restatement of Corollary 5.5.

$$\sum_{(a,b,c) \in Q} P_1\left(\frac{a}{p}\right) P_1\left(\frac{b}{p}\right) P_1\left(\frac{c}{p}\right) \psi(abc) = \sum_{(a,b,c) \in Q^*} P_1\left(\frac{a}{p}\right) P_1\left(\frac{b}{p}\right) P_1\left(\frac{c}{p}\right) \psi(abc).$$

The question arises whether Corollary 5.5 can be proved in an elementary way (avoiding the complicated method of Arakawa). This seems to be a very difficult problem.

Acknowledgments

I wish to give special thanks to Robert Sczech for his invaluable assistance with this manuscript. I also wish to thank Tomoyoshi Ibukiyama and Steven Weintraub for their insightful comments, and the referee for the constructive suggestions which greatly improved the presentation of this paper.

This paper is dedicated to the memory of Tsuneo Arakawa. His work in [3], a genuine tour de force, has been very inspirational to the author.

References

- [1] R. Lee, S. H. Weintraub, A generalization of a theorem of Hecke to the Siegel space of degree two, in: Algebraic Topology, Vol. 96 of Contemporary Math., Amer. Math. Soc., 1989, pp. 243–259. doi:10.1090/conm/096/1022685.
- [2] B. Isaacson, Special values of Ibukiyama-Saito L -functions, Kyushu J. Math. (in press).
- [3] T. Arakawa, Special values of L -functions associated with the space of quadratic forms and the representation of $Sp(2n, \mathbb{F}_p)$ in the space of Siegel cusp forms, in: Automorphic Forms and Arithmetic Varieties, Vol. 15 of Adv. Stud. Pure Math., Kinokuniya and Academic Press, 1989, pp. 99–169. doi:10.1016/B978-0-12-330580-0.50002-1.
- [4] T. Ibukiyama, H. Saito, On L -functions of ternary zero forms and exponential sums of Lee and Weintraub, J. Number Theory **48-2** (1994) 252–257. doi:10.1006/jnth.1994.1066.
- [5] T. Arakawa, T. Ibukiyama, and M. Kaneko, Bernoulli Numbers and Zeta Functions, Springer, 2014. doi:10.1007/978-4-431-54919-2.
- [6] P. Gunnells, R. Sczech, Evaluation of Dedekind sums, Eisenstein cocycles, and special values of L -functions, Duke Math. J. **118** (2003) 229–260. doi:10.1215/S0012-7094-03-11822-0.
- [7] A. Weil, Elliptic Functions According to Eisenstein and Kronecker, Springer, 1976. doi:10.1007/978-3-642-66209-6.
- [8] B. Isaacson, On a generalization of a theorem of Ibukiyama, Comment. Math. Univ. St. Pauli (in press).