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Abstract

This paper studies a particular interpretation of propositional modal logic into propositional modal logic. Under an epistemic reading of the modality this interpretation can be understood as taking knowledge to be true belief. All normal modal logics of belief that under this definition of knowledge give rise to S5 as the logic of knowledge are determined. And all the normal modal logics of belief that give rise to S4w5 as the logic of knowledge are determined. Among the latter KD45 shows up as a maximal such logic.

1 Introduction

In an epistemic setting the meaning of a propositional modal formula $\Box A$ can be understood as either 'A is known' or 'A is believed'. If we understand *knowledge* as *true belief*[1] and if we have a formula F and wonder whether it holds when we read the $\Box$'s in F as 'knowing' then we might consider the following reduction. First replace each sub-formula of the form $\Box H$ in F by $\Box H \wedge H$. Then ask whether the resulting formula holds when the $\Box$'s are read as 'believing'. For this to work it is important that the notions of belief and knowledge match up in this way, by which I mean the following. Formally speaking the above translation, lets call it $*$, is an interpretation of the language of knowledge into the language of belief. Let L_k be a logic of knowledge and L_b a logic of belief. That these logics match up under this interpretation means that $*$ is a faithful interpretation of L_k into L_b. In this paper we determine all logics of belief for which $*$ is a faithful interpretation of S5 and of S4w5 Among the logics that faithfully interpret S4w5, we will see that the well-known KD45 is a maximal such logic.

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[1] There are all kinds of philosophical problems with such a definition but to handle those one needs more machinery, see [AN05b, AN05a]. In any case I don't claim that knowledge is true belief, but try to approach such an assumption in a formal way.

The reader is referred to [BdRV01] for the necessary background on modal logic, especially the notions of canonical formula, canonical models and their use in Kripke completeness proofs, filtrations and bounded morphisms. A short summary is given in Subsection 2.1.

The rest of this paper is organized as follows. In Section 2 the necessary modal logic machinery is presented. In Section 3 the general formal setting of the current research is set out and two specific questions are formally stated. These two questions are solved, using the machinery from Section 2, in Sections 4 and 5 respectively.

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2 Modal Logic Preliminaries

In this section we define the modal logics we will be concerned with in the rest of this paper. In subsection 2.1 some notational conventions used in the rest of this paper will be introduced and in subsection 2.2 we show a general result that will be useful on two occasions later on.

To begin with, the following common schemes will pop-up frequently.

$$\Box(A \to B) \to \Box A \to \Box B \qquad \mathsf{K}$$
$$\Box A \to A \qquad \mathsf{T}$$
$$\Box A \to \Box\Box A \qquad \mathsf{4}$$
$$\Diamond A \to \Box\Diamond A \qquad \mathsf{5}$$
$$A \to \Box\Diamond A \qquad \mathsf{B}$$
$$\Box A \to \Diamond A \qquad \mathsf{D}$$

And in addition the following three less common schemes will show up.

$$\Box A \wedge A \to \Box\Box A \qquad \mathsf{w4}$$
$$A \to \Box(\neg A \to \Diamond A) \qquad \mathsf{wB}$$
$$\Diamond A \wedge \neg A \to \Box\Diamond A \qquad \mathsf{w5}$$
$$\Diamond A \wedge \neg A \to \Box(\neg A \to \Diamond A) \qquad \mathsf{ww5}$$

The schemes w4, w5, ww5 and wB pop up automatically in the next section, however Kw4 is known to be the logic of all topological spaces if the $\Diamond$ is interpreted as the derivative operation [Esa04].[2] The author does not have any knowledge of literature on the schemes w5, ww5 and wB though as the following lemma shows wB is equally strong as B.

Lemma 2.1. KwB = KB

[2]In [Esa04] Kw4 is written as wK4.

Proof. It is enough to show, and this is what we will do, that for any formula A there exists a theorem X of KwB such that $\mathsf{K} \vdash X \to (A \to \Box\Diamond A)$.

Let $M = \langle W, R, \Vdash\rangle$ be a Kripke model and $w \in W$ and assume that the following formulas are forced at w.

$$A \to \Box(\neg A \to \Diamond A) \ , \tag{1}$$

$$A \wedge \Diamond A \to \Box(\neg A \vee \Box\neg A \to \Diamond(A \wedge \Diamond A)) \ . \tag{2}$$

Suppose for a contradiction that $w \Vdash A \wedge \Diamond\Box\neg A$. Then there exists some x with

$$wRx \Vdash \Box\neg A \ . \tag{3}$$

Since $w \Vdash A$, wRx and $w \Vdash (1)$ and (3) we get $x \Vdash A$. Now using (2) we get $x \Vdash \Diamond A$, contradicting (3).

By the completeness of K we thus have

$$\mathsf{K} \vdash (1) \wedge (2) \to (A \to \Box\Diamond A)$$

and we are done. □

Definition 2.2 (Weak-transitivity)**.** A binary relation is *weakly-transitive* when $xRyRz$ and $x \neq z$ implies xRz

Theorem 2.3. $\mathsf{w4}$ *is canonical for weak-transitivity*

Proof. We first show that $\mathsf{w4}$ is valid on all frames $\langle W, R\rangle$ where R is weakly-transitive. Fix some forcing relation $\Vdash$, let $x \in W$ and suppose $x \Vdash \Box A \wedge A$. Let $y, z \in W$ such that $xRyRz$. If $z = x$ then $z \Vdash A$ and if $z \neq x$ then by weak-transitivity we get xRz and thus $z \Vdash A$.

Now we show that if L is a normal modal logic that contains all instances of $\mathsf{w4}$ then the canonical model for L is weakly transitive. Let us first show that

$$\mathsf{Kw4} \vdash \Box A \wedge B \to \Box\Box(\neg A \to B) \ . \tag{4}$$

Reason in $\mathsf{Kw4}$ and assume $\Box A$, B and $\Diamond\Diamond(\neg A \wedge \neg B)$. Then by $\mathsf{w4}$, $\Diamond(\neg A \wedge \neg B) \vee (\neg A \wedge \neg B)$. Thus since B we have $\Diamond(\neg A \wedge \neg B)$ and thus $\Diamond\neg A$, contradicting $\Box A$ and thus (4) is shown.

Now let $M = \langle W, R, \Vdash\rangle$ be the canonical model for L and suppose $\Gamma R \Delta R \Theta$ where $\Theta \neq \Gamma$. There exists some B such that $B \in \Gamma$ and $B \notin \Theta$ (or $B \notin \Gamma$ and $B \in \Theta$, but since Γ and Δ are maximal consistent this amounts to the same thing). Now let $\Box A \in \Gamma$. By (4) we have $\Box\Box(\neg A \to B) \in \Gamma$ and thus $\neg A \to B \in \Theta$. Since $B \notin \Theta$ we have $\neg A \notin \Theta$ and thus $A \in \Theta$. □

Definition 2.4 (Weak-weak-euclidean)**.** A binary relation is *weak-weak-euclidean* when $wRx\ wRy$, $w \neq x$ and $x \neq y$ implies yRx

Theorem 2.5. $\mathsf{ww5}$ *is canonical for weak-weak-euclidean-ness*

Proof. Let $F = \langle W, R \rangle$ be a weak-weak-euclidean frame and let $\Vdash$ be any forcing relation on W. Let $w, y \in W$ and suppose $w \Vdash \Diamond F \wedge \neg F$, wRy and $y \Vdash \neg F$. We have to show that $y \Vdash \Diamond F$. As $w \Vdash \Diamond F$ there exist x such that wRx and $x \Vdash F$. Since $y \Vdash \neg F$ and $w \Vdash \neg F$ we have $x \neq y$ and $x \neq w$ and thus by weak-weak-euclidean-ness we get yRx, which gives $y \Vdash \Diamond F$, and we are done.

Now let L be any normal modal logic that contains every instance of $\mathsf{ww5}$ and let $M = \langle W, R, \Vdash \rangle$ be the canonical model for L. Suppose $\Gamma R \Delta$, $\Gamma R \Theta$, $\Gamma \neq \Delta$ and $\Delta \neq \Theta$. We have to show that $\Theta R \Delta$. First fix formulas B and C such that $B, C \in \Delta$, $\neg B \in \Gamma$ and $\neg C \in \Theta$ and let $\Box A \in \Theta$. Suppose for a contradiction that $A \notin \Delta$. Then $\Diamond(\neg A \wedge B \wedge C) \in \Gamma$ and, as $\neg B \in \Gamma$ also $\neg(\neg A \wedge B \wedge C) \in \Gamma$. Thus by $\mathsf{ww5}$ we get $\Box(\neg(\neg A \wedge B \wedge C) \to \Diamond(\neg A \wedge B \wedge C)) \in \Gamma$ and thus $\neg(\neg A \wedge B \wedge C) \to \Diamond(\neg A \wedge B \wedge C) \in \Theta$. Since $\neg C \in \Theta$ this implies $\Diamond(\neg A \wedge B \wedge C) \in \Theta$ and thus $\Diamond \neg A \in \Theta$, a contradiction. □

2.1 Notations and Conventions

Bounded morphisms With a bounded morphism we will either mean a bounded morphism between frames or a bounded morphism between models. The context will make clear which is meant. If P is a finite set of propositional variables then a *P-bounded morphism* is a bounded morphism where the base case is restricted to the variables in P.

Some transformations on frames and models If L is a normal modal logic then with $\mathcal{F}_L^{\mathrm{pg}}$ we denote its class of finite point-generated frames. Let $F = \langle W, R \rangle$ be a frame and let $\circ, \bullet \notin W$. We define the frames $F^* = \langle W^*, R^* \rangle$, $F^\circ = \langle W^\circ, R^\circ \rangle$ and $F^\bullet = \langle W^\bullet, R^\bullet \rangle$ as follows.

$$\begin{aligned} W^* &= W & R^* &= R \cup \{\langle x, x \rangle \mid x \in W\} \ , \\ W^\circ &= W \cup \{\circ\} & R^\circ &= R \cup \{\langle \circ, x \rangle \mid x \in W\} \ , \\ W^\bullet &= W \cup \{\bullet\} & R^\bullet &= R \cup \{\langle \bullet, x \rangle \mid x \in W\} \cup \{\langle \bullet, \bullet \rangle\} \ . \end{aligned}$$

Thus F^* is the reflexive closure of F, F° is obtained by adding a non-reflexive root to F and $F^\bullet$ is obtained by adding a reflexive root. We sometimes use $\star$ as a name for either $\circ$ and $\bullet$.

The operations $*$, $\circ$ and $\bullet$ are lifted to classes of frames $\mathcal{F}$ in the obvious way:

$$\begin{aligned} \mathcal{F}^* &= \{F^* \mid F \in \mathcal{F}\} \ , \\ \mathcal{F}^\circ &= \{F^\circ \mid F \in \mathcal{F}\} \ , \\ \mathcal{F}^\bullet &= \{F^\bullet \mid F \in \mathcal{F}\} \ . \end{aligned}$$

I trust that the reader can guess how these operations lift to models and to classes of models.

Induced subframes and submodels Let $F = \langle W, R \vdash\rangle$ be a frame. If $W' \subseteq W$ then the *subframe of F induced by W'* is the frame $\langle W', R'\rangle$ where R' is the restriction of R to W'. If $w \in W$ then with $w{\uparrow}^+$ we denote the subframe of F induced by $\{x \in W \mid wRx,\ w \neq x\}$. Again these things lift to models in the obvious way.

Some properties of frames and models A model M is *distinguishing* if for every world w there exists a formula ϕ_w such that $M, u \Vdash \phi_w$ iff $u = w$. A model $N = \langle W, R, \Vdash'\rangle$ is a *variant* of a model $M = \langle W, R, \Vdash\rangle$. That is, N is a variant of M when both M and N are based on the same frame. A model $N = \langle W, R, \Vdash\rangle$ is a *definable variant* of a model $M = \langle W, R, \Vdash\rangle$ if for every propositional variable p there exists a formula ϕ_p such that $N, x \Vdash p$ iff $M, x \Vdash \phi_p$.

The crucial consequences of these properties are as follows. For a finite distinguishing model, every variant is definable. And if a model for which every variant is definable validates a logic L then so does its underlying frame [BdRV01].

2.2 Logics extending $\mathsf{Kw4ww5}$

In this subsection we will show that all normal modal logics extending $\mathsf{Kw4ww5}$ have the finite frame property. The proof follows the idea of Fine's proof [Fin71, BdRV01] of Bull's Theorem but, thanks to the simple model theory of $\mathsf{Kw4ww5}$ is much simpler.

For $n \geq 1$ let

$$\mathcal{G}_n = \{F \mid \forall x, y(x \neq y \to xRy) \text{ and } |F| = n\} \ ,$$

Some easy observations: $\mathcal{G}^*_n \subseteq \mathcal{G}_n$, $\mathcal{F}^{\mathrm{pg}}_{\mathsf{Kw4B}} = \bigcup_{n\geq 1} \mathcal{G}_n$ and $\mathcal{F}^{\mathrm{pg}}_{\mathsf{S5}} = \bigcup_{n\geq 1} \mathcal{G}^*_n$.

Lemma 2.6. *We have the following.*

1. $\mathcal{F}^{\mathrm{pg}}_{\mathsf{KD45}} = \bigcup_{n\geq 1} \mathcal{G}^{*\circ}_n \cup \mathcal{G}^*_n$,
2. $\mathcal{F}^{\mathrm{pg}}_{\mathsf{S4w5}} = \bigcup_{n\geq 1} \mathcal{G}^{*\bullet}_n \cup \mathcal{G}^*_n$,
3. $\mathcal{F}^{\mathrm{pg}}_{\mathsf{Kw4ww5}} = \bigcup_{n\geq 1} \mathcal{G}^{\circ}_n \cup \mathcal{G}^{\bullet}_n \cup \mathcal{G}_n$.

Proof. All $\supseteq$'s are easy to check so let us just do the $\subseteq$'s.

Suppose $F \in \mathcal{F}^{\mathrm{pg}}_{\mathsf{KD45}}$. Let $|F| = n$ and suppose $F \notin \mathcal{G}^*_n$. Let w generate F. As $\mathsf{KD45} \vdash \Box(\Box p \to p)$ we have $w{\uparrow}^+ \in \mathcal{G}^*_{n-1}$. In case wRw then by $\mathsf{5}$ we have that for all x, xRw and thus $F \in \mathcal{G}^*_n$ which is not so. If xRw for some $x \neq w$ then as also wRx we get by $\mathsf{w4}$ that wRw, which we already excluded. Thus $F \in {\mathcal{G}^*}^{\circ}_{n-1}$.

Suppose $F \in \mathcal{F}^{\mathrm{pg}}_{\mathsf{S4w5}}$. Let $|F| = n$ and suppose $F \notin \mathcal{G}^*_n$. Let w generate F. By reflexiveness and weak-euclidean-ness of F we have $w{\uparrow}^+ \in \mathcal{G}^*_{n-1}$. In case for some $x \neq w$ we have xRw then for all $x \neq w$, xRw and $F \in \mathcal{G}^*_n$ which is not so. Thus $F \in {\mathcal{G}^*}^{\bullet}_{n-1}$.

Suppose $F \in \mathcal{F}^{\mathrm{pg}}_{\mathsf{Kw4ww5}}$. Let $|F| = n$ and suppose $F \notin \mathcal{G}_n$. Let w generate F. By weak-weak-euclidean-ness we have for all $x, y \in w\uparrow^+$, if $x \neq y$ then xRy. Thus $w\uparrow^+ \in \mathcal{G}_{n-1}$. Suppose there exist $x \neq w$ such that xRw. Then by w4 for all $x \neq w$, xRw and thus $F \notin \mathcal{G}_n$ which is not so. Thus $F \notin \mathcal{G}^{\circ}_{n-1} \cup \mathcal{G}^{\bullet}_{n-1}$. □

Theorem 2.7. *Let $F = \langle W, R\rangle$ be a p.g.* Kw4ww5 *frame and let P be a finite set of P.V.'s If $M = \langle F, \Vdash\rangle$ is a model based on F then there exists*

1. *a finite and distinguishing model N and*
2. *a P-bounded morphism from M onto N .*

Proof. Let F be generated by w_0. For $x, y \in M$ put

$$x \sim y \qquad \text{iff} \qquad \forall p{\in}P\ M, x \Vdash p \Leftrightarrow M, y \Vdash p$$

and

$$|x| = \{x' \mid x' \sim x\} \ .$$

First let $N' = \langle W', R', \Vdash'\rangle$ where

- $W' = \{|x| \mid x \neq w_0\}$,
- $|x|R'|y|$ iff $\exists x' \in |x| \exists y' \in |y| x'Ry'$,
- $N', |x| \Vdash p$ iff $M, x \Vdash p$ and $p \in P$.

Notice that we still could have $|w_0| \in W'$. We will first show that

$$\forall x \neq w_0 \forall z \neq w_0 (|x|R'|z| \rightarrow \exists z' \in |z| xRz') \ . \tag{5}$$

Let $x \neq w_0$ and $z \neq w_0$. In case $x \neq z$ we have xRz. In case $x = z$ then $|x|R'|z|$ only if there exits $z' \neq z$ and $z' \in |z|$ or xRx. Thus we have show (5).

Suppose that there exists $x_0 \neq w_0$ such that $x_0 \sim w_0$ and $|x_0|R'|x_0|$. Notice that

$$\forall x\ |x_0|R'|x| \ . \tag{6}$$

Define $g : M \longrightarrow N'$ by

$$g(x) = \begin{cases} |x| & \text{if } x \neq w_0 \ , \\ |x_0| & \text{otherwise.} \end{cases}$$

We will now show that

$$\forall z\ (g(w_0)R'g(z) \rightarrow \exists z'\ (g(z') = g(z) \wedge w_0Rz')) \ . \tag{7}$$

In case $z = w_0$ let $z' = x_0$ and in case $z \neq w_0$ take $z' = z$.

Since N' is clearly distinguishing the lemma will be shown in this case if g is a P-bounded morphism Clearly $M, x \Vdash p$ iff $N', g(x) \Vdash p$ for all $p \in P$. Assume xRy. In case both $x, y \neq w_0$ then also $g(x) = |x|R'|y| = g(y)$. In case $x = w_0$ then $g(x) = |x_0|$ and thus by (6) we get $g(x) = |x_0|R'g(y)$. Assume $g(x)R'g(y)$.

Since $g(x_0) = g(w_0)$ we can assume w.l.o.g. that $y \neq w_0$. In case $x = w_0$ then we find some y' with $g(y') = g(y)$ and xRy' by (7). In case $x \neq w_0$ we find such a y' using (5).

Now let us assume that no such x_0 exist. Put

$$N = \begin{cases} (N')^\bullet & \text{if } w_0 R w_0 \ , \\ (N')^\circ & \text{otherwise.} \end{cases}$$

Let us write $\star$ for the root of N ($\star = \bullet$ in case $w_0 R w_0$ and $\star = \circ$ otherwise). Define $f : M \longrightarrow N$ by

$$f(x) = \begin{cases} |x| & \text{if } x \neq w_0 \ , \\ \star & \text{otherwise.} \end{cases}$$

That f is a P-bounded morphism can now be shown just using (5) and thus we are done if we can show that N is distinguishing. For each x let

$$\pi_x = \bigwedge\{p \mid x \Vdash p,\ p \in P\} \wedge \bigwedge\{\neg p \mid x \nVdash p,\ p \in P\} \ .$$

If for some $x \neq w_0$, $x \sim w_0$. Then we do not have $|x|R'|x|$ and thus $|x| \Vdash \Box\neg\pi_x$ and $\star \Vdash \Diamond\pi_x$. Thus if we put

$$\phi_{|x|} = \begin{cases} \pi_x & x \not\sim w_0 \ , \\ \pi_x \wedge \Box\neg\pi_x & x \sim w_0 \end{cases}$$

and

$$\phi_\star = \pi_x \wedge \Diamond\pi_x$$

then for all $n, m \in N$, $m \Vdash \phi_n$ iff $n = m$. In case no such x exist then we can simply put $\phi_{|x|} = \pi_x$ and $\phi_\star = \pi_{w_0}$. □

Corollary 2.8. Any extension L of Kw4ww5 has the finite frame property

Proof. Let L be any extension of Kw4ww5 and suppose $L \nVdash A$. Let M be the sub-model of the canonical model of L generated by a MCS Γ with $\neg A \in \Gamma$. Apply Theorem 2.7 with $P = PV(A)$ to obtain a finite distinguishing model N that is a bounded morphic image of a definable variant of M that falsifies A at some world v. As M validates L so does N. And as N is finite and distinguishing its underlying frame is a frame for L. □

3 Interpreting Knowledge into Belief

In this section we set out the general idea for which two special cases are worked out in Section 4 and Section 5 below.

The most prominent logics of knowledge in the propositional modal setting found in the literature are S4 and S5 [FHMV95]. Logics of belief are usually obtained from these logics by dropping the T scheme and adding D, thus obtaining $\mathsf{KD4}$ and $\mathsf{KD45}$. As such belief is thus taken to be knowledge without the assurance that what we know is true, though our believes should be consistent. Or conversely, knowledge should be obtainable from belief as soon as we find out that what we belief is true. In line with this we will consider the following translation $*$ from modal formulas A to modal formulas A^*.

Definition 3.1 (Interpretation of Knowledge in Belief)**.** For any formula A define a formula A^* with induction on A as follows.

- ⋆ $p^* = p$,
- ⋆ $\bot^* = \bot$,
- ⋆ $(A \to B)^* = A^* \to B^*$,
- ⋆ $(\Box A)^* = \Box A^* \wedge A^*$.

The translation $*$ is a well-studied object in the field of provability logics. Using this translation a formal provability interpretation can be given to the $\Box$ that, unlike GL, preserves $\mathsf{S4}$ [Boo93]. No further reference to these uses of the translation $*$ are made in this paper so let us go back to the problem at hand.

Let L_k be a logic of knowledge and let L_b be a logic of belief (not necessarily any of the above, the reader can take his favorite system). If A is a modal formula then the validity of A when interpreting the $\Box$ as 'knowing' is thus formally expressed by $L_k \vdash A$. Similarly the validity of A when interpreting the $\Box$ as 'believing' is expressed by $L_b \vdash A$. If we now agree that knowledge is defined to be true belief then the validity of A in the knowledge interpretation can also be formally expressed as $L_b \vdash A^*$, and if we agree that knowledge is true belief then it is desirable, if not mandatory that these two match up. We thus ask ourselves the following two questions. Take $L_k = \mathsf{S5}$, find all the normal modal logics L_b such that

$$\mathsf{S5} \vdash A \qquad \text{iff} \qquad L_b \vdash A^* \ . \tag{8}$$

Similarly if we take $L_b = \mathsf{KD45}$, find the unique normal modal logic L_k and all the normal modal logics L'_b such that

$$L_k \vdash A \qquad \text{iff} \qquad \mathsf{KD45} \vdash A^* \qquad \text{iff} \qquad L'_b \vdash A^* \ . \tag{9}$$

In more fancy terms we thus look for L_b that faithfully interprets $\mathsf{S5}$ respectively look for L_k such that $\mathsf{KD45}$ faithfully interprets L_k.

There are of course more choices for L_k and L_b to start of with than just $\mathsf{S5}$ and $\mathsf{KD45}$. Among the standard schemes considered in epistemic logic it is the $\mathsf{5}$ scheme that, as far as the line of the research of this paper goes is the least trivial. We therefore restrict ourselves to logics that contain this scheme.

We finish with some basic facts that come in handy in the rest of the paper.

Lemma 3.2. $\mathsf{K} \vdash A^* \leftrightarrow A^{**}$

Proof. Induction on A. The base case and propositional connectives are trivial and so is the $\Box$ case since if $A \equiv \Box B$ then $A^* \equiv \Box B^* \wedge B^*$ and $A^{**} \equiv \Box B^{**} \wedge B^{**} \wedge B^{**}$. $\Box$

Lemma 3.3. *Let $M = \langle W, R, \Vdash \rangle$ be a model Then for all $w \in W$ and all formulas A we have $M^*, w \Vdash A$ iff $M, w \Vdash A^*$*

Proof. In what follows we write $x \Vdash$ for $M, x \Vdash$ and $x, \Vdash^*$ for $M^*, x \Vdash$. We show the lemma with induction on A. The base case and the cases for the propositional connectives are trivial. So assume $A \equiv \Box B$. ($\Leftarrow$) If $w \nVdash^* \Box B$ then there exists v with $wR^*v \Vdash^* \neg B$. By (IH) we get $v \Vdash \neg B^*$ and since wR^*v implies $w = v$ or wRv we get $w \Vdash \neg B^* \vee \neg \Box B^*$ and thus $w \nVdash B^* \wedge \Box B^*$. ($\Rightarrow$) If wRv then wR^*v thus $v \Vdash^* B$ and thus again by (IH) $v \Vdash B^*$. This implies $w \Vdash \Box B^*$. Moreover by reflexivity of R^* we get $w \Vdash^* B$, thus by (IH) $w \Vdash B^*$ and thus $w \Vdash (\Box B)^*$. $\Box$

Notice that this lemma can be used in two ways, we either start with M and construct M^* by adding all pairs $\langle x, x \rangle$ for $x \in W$ to R, or we start with a reflexive model M^* and remove some, but not necessarily all, of those pairs from R^* to obtain R.

Corollary 3.4. $\mathsf{T} \vdash A \leftrightarrow A^*$

Proof. By Lemma 3.3 $A \leftrightarrow A^*$ is valid on all reflexive frames and thus the corollary follows from the completeness of T for such frames. $\Box$

4 Interpreting S5 Knowledge into Belief

In this section we find all solutions for (8). We will do this as follows. We will first find two solutions to (8) (Theorem 4.1 and 4.2) and then show that any other extends one of these (Theorem 4.3) and is contained in the other (Theorem 4.5).

Theorem 4.1. $\mathsf{S5} \vdash A$ *iff* $\mathsf{S5} \vdash A^*$

Proof. By Corollary 3.4 above. $\Box$

Theorem 4.2. $\mathsf{S5} \vdash A$ *iff* $\mathsf{Kw4B} \vdash A^*$

Proof. It is easy to see that $\mathsf{S5} \vdash A$ implies $\mathsf{Kw4B} \vdash A^*$. Suppose that $\mathsf{S5} \nvdash A$. Then there exists a model $M = \langle W, R, \Vdash \rangle$ such that R is an equivalence on W and $w \in W$ with $w \Vdash \neg A$. In particular R is weakly transitive and symmetric and thus M is also a $\mathsf{Kw4B}$ model. Moreover since R is reflexive by Lemma 3.3 we get $w \Vdash \neg A^*$ and thus $\mathsf{Kw4B} \nvdash A^*$. $\Box$

Theorem 4.3. *If for all A,* $\mathsf{S5} \vdash A$ *iff* $\mathsf{L} \vdash A^*$ *then* $\mathsf{Kw4B} \subseteq L$

Proof. Let L be any normal modal logic satisfying the conditions of the lemma. That $L \vdash \Box A \wedge A \to \Box\Box A$ is clear. To show that $L \vdash A \to \Box\Diamond A$ note that

$$(\Diamond B \to \Diamond\Box B)^* = \Diamond B^* \vee B^* \to \Box(\Diamond B^* \vee B^*) \wedge (\Diamond B^* \vee B^*) \ .$$

Thus if $(\Diamond B \to \Diamond\Box B)^*$ is provable for an arbitrary formula B then so is

$$B^* \to \Box(\neg B^* \to \Diamond B^*) \ .$$

Thus $\mathsf{KwB} \subseteq L$ and by Lemma 2.1 $L \vdash A \to \Box\Diamond A$. □

Lemma 4.4. *Let $m \geq 1$ and $F \in \mathcal{G}^*_m$. Then for any $n \geq 2m$ and any $F' \in \mathcal{G}_n$ there exists a bounded morphism from F' onto F*

Proof. Any function $f : F' \longrightarrow F$ with $|f(x)^{-1}| \geq 2$ is a bounded morphism □

Theorem 4.5. *If for all A, $\mathsf{S5} \vdash A$ iff $L \vdash A^*$. Then $L \subseteq \mathsf{S5}$*

Proof. Assume the hypothesis of the theorem and suppose for a contradiction that $L \not\subseteq \mathsf{S5}$. By Theorem 4.3 we have $\mathsf{Kw4B} \subseteq L$. As $\bigcup_{m\geq 1} \mathcal{G}^*_m$ are just the finite $\mathsf{S5}$ frames there exists some n such that for all $m \geq n$, $\mathcal{G}^*_m \cap \mathcal{F}_L = \emptyset$. But then by Lemma 4.4 we thus have that for all $m \geq 2n$, $\mathcal{G}_m \cap \mathcal{F}_L = \emptyset$. Choose $k \geq 1$ such that $2^k \geq 2n$. Let

$$\Pi = \{\overline{p}_1 \wedge \cdots \wedge \overline{p}_k \mid \overline{p}_i \in \{p_i, \neg p_i\}\} \ .$$

Let $\pi_1, \ldots, \pi_{2^k}$ be an enumeration of Π. We have

$$\mathcal{F}_L \models (\Box\neg\pi_1 \wedge \neg\pi_1) \vee \cdots \vee (\Box\neg\pi_{2^k} \wedge \neg\pi_{2^k}) \ .$$

And as $\mathsf{Kw4B}$ extends $\mathsf{Kw4ww5}$ we thus have by Corollary 2.8 that

$$L \vdash (\Box\neg\pi_1 \wedge \neg\pi_1) \vee \cdots \vee (\Box\neg\pi_{2^k} \wedge \neg\pi_{2^k})$$

which by the assumption of the theorem implies

$$\mathsf{S4w5} \vdash \Box\neg\pi_1 \vee \cdots \vee \Box\neg\pi_{2^k} \ .$$

Which is clearly not so. □

5 Interpreting S4w5 Knowledge into Belief

In this section we find all solutions to (9). This will be done as follows. First we will find three solutions to (2) (Theorem 5.3, 5.4 and 5.5). Then we will show that any other solution extends one of these (Theorem 5.6) and is contained in either of the other two (Theorem 5.9).

We start with two lemmata that will be helpful in finding a solution to (2).

Lemma 5.1. *Let $\langle W, R\rangle$ be an $\mathsf{S4w5}$ frame and let R' be defined as $xR'y$ iff xRy and if $x = y$ then there exists $v \neq x$ such that vRx. Then $\langle W, R'\rangle$ is an $\mathsf{K45}$ frame*

Proof. Suppose $xR'yR'z$. We have $xRyRz$ thus xRz and thus in case $x \neq z$ we have $xR'z$. Now assume $x = z$. In case $y = x$ we are done at once and in case $y \neq x$ we have $xR'z$.

Suppose wRx, y. Then wRx, y and thus if $x \neq w$ we have yRx. Additionally if $y \neq x$ we get $yR'x$ immediately and if $y = x$ then we get $yR'x$ since wRx and $x \neq w$. Now suppose $x = w$. We thus want to show that $yR'w$ and we thus can assume that $y \neq w$. Since wRw there exists some $u \neq w$ such that uRw. By transitivity of R we get uRy since if $u = y$ we would be done we can assume that $u \neq y$ and thus by w5 we get yRw. Since $w \neq y$ we conclude $yR'w$. □

Lemma 5.2. *For any formula A,* $\mathsf{K45} \vdash A^*$ *iff* $\mathsf{KD45} \vdash A^*$

Proof. Suppose $\mathsf{K45} \nvdash A^*$. Let $\langle W, R, \Vdash\rangle$ be a K45 model, $w \in W$ and $w \Vdash \neg A^*$. In case $w \Vdash \Diamond\top$ then $\mathsf{K45} \nvdash \Diamond\top \to A^*$ and thus as $\mathsf{K45} \vdash \Diamond\top \to \Box\Diamond\top$ we have $\mathsf{KD45} \nvdash A^*$. In case $w \Vdash \Box\bot$ then we can assume that $W = \{w\}$. Let $R' = R \cup \{\langle w, w\rangle\}$ and $M' = \langle W, R' \Vdash'\rangle$. By Lemma 3.2 $w \Vdash \neg A^{**}$ and we thus have by Lemma 3.3 that $w \Vdash' \neg A^*$. Clearly M' is an KD45 model and thus $\mathsf{KD45} \nvdash A^*$. □

Theorem 5.3. $\mathsf{S4w5} \vdash A$ *iff* $\mathsf{KD45} \vdash A^*$

Proof. For the left to right direction we will only show

$$\mathsf{KD45} \vdash (\Diamond A \wedge \neg A \to \Box\Diamond A)^* \ . \tag{10}$$

We have

$$(\Diamond A \wedge \neg A \to \Box\Diamond A)^* =$$
$$(\Diamond A^* \vee \ A^*) \wedge \neg A^* \to \Box(\Diamond A^* \vee A^*) \wedge (\Diamond A^* \vee \ A^*) \ .$$

Thus (10) follows since $\mathsf{KD45} \vdash \Diamond A^* \to \Box(\Diamond A^* \vee A^*)$.

For the right to left direction suppose that $\mathsf{S4w5} \nvdash A$. As S4w5 is complete we find a S4w5 Kripke model $M^* = \langle W, R^*, \Vdash^*\rangle$ and $w \in W$ such that $w \nVdash^* A$. Applying Lemma 5.1 we find a K45 model $M = \langle W, R, \Vdash\rangle$ such that $R^* = R + \{\langle x, x\rangle \mid x \in W\}$. Thus by Lemma 3.3 we have $w \nVdash A^*$ thus $\mathsf{K45} \nvdash A^*$ and thus by Lemma 5.2 $\mathsf{KD45} \nvdash A^*$. □

Theorem 5.4. $\mathsf{S4w5} \vdash A$ *iff* $\mathsf{S4w5} \vdash A^*$

Proof. By Corollary 3.4. □

Theorem 5.5. $\mathsf{S4w5} \vdash A$ *iff* $\mathsf{Kw4ww5} \vdash A^*$

Proof. The right to left direction follows from $\mathsf{Kw4ww5} \subseteq \mathsf{S4w5}$ and Corollary 3.4. For the left to right direction it is enough to show that for any formula A

$$\mathsf{Kw4ww5} \vdash (\Box A \to \Box\Box A)^* \tag{11}$$

and

$$\mathsf{Kw4ww5} \vdash (\Diamond A \wedge \neg A \to \Box\Diamond A)^* \ . \tag{12}$$

(11) is clear. To show (12) note that

$$(\Diamond A \wedge \neg A \to \Box\Diamond A)^* =$$
$$(\Diamond A^* \vee A^*) \wedge \neg A^* \to \Box(\Diamond A^* \vee A^*) \wedge (\Diamond A^* \vee A^*) \ .$$

From which (12) follows. □

Theorem 5.6. *If for all A,* S4w5 $\vdash A$ *iff* $L \vdash A^*$ *then* Kw4ww5 $\subseteq L$

Proof. Assume the hypothesis of the theorem. We clearly have Kw4 $\subseteq L$. We have

$$(\Diamond p \wedge \neg p \to \Box\Diamond p)^* = (\Diamond p \vee p) \wedge \neg p \to \Box(\Diamond p \vee p) \wedge (\Diamond p \vee p)) \ .$$

Thus as $L \vdash (\Diamond p \wedge \neg p \to \Box\Diamond p)^*$ we also have

$$L \vdash \Diamond p \wedge \neg p \to \Box(\Diamond p \vee p)$$

and thus also Kww5 $\subseteq L$. □

Lemma 5.7. *Let* $2k \leq m$*,*

1. *if* $F \in \mathcal{G}^{\circ}_m$ *and* $F' \in \mathcal{G}^{*\circ}_k$ *then there exists a bounded morphism from* F *onto* F'*,*
2. *if* $F \in \mathcal{G}^{\bullet}_m$ *and* $F' \in \mathcal{G}^{*\bullet}_k$ *then there exists a bounded morphism from* F *onto* F'*.*

Proof. In both cases any function $f : F \longrightarrow F'$ with $|f(x)^{-1}| \geq 2$ is a bounded morphism □

Corollary 5.8. If Kw4ww5 $\subseteq L$, $L \not\subseteq$ KD45 and $L \not\subseteq$ S4w5 then there exists $k \geq 1$ such that $F \in \mathcal{F}^{\mathrm{pg}}_L$ and $|F| = m \geq k$ then $F \in \mathcal{G}_m$.

Proof. If $L \not\subseteq$ KD45 then, as by Corollary 2.8 KD45 has the finite frame property, there exists some u and a frame $F \in \mathcal{G}^{*\circ}_u \cup \mathcal{G}^{*}_u$ such that $F \notin \mathcal{F}^{\mathrm{pg}}_L$. By possibly adjoining a root to F we can assume that $F \in \mathcal{G}^{*\circ}_u$. Similarly since $L \not\subseteq$ S4w5 there exists some v and $F_2 \in \mathcal{G}^{*\bullet}_v$ such that $F_2 \notin \mathcal{F}^{\mathrm{pg}}_L$.

Let $k = 2 \cdot \max(u, v)$. Now as Kw4ww5 $\subseteq L$ we have that for all $F \in \mathcal{F}^{\mathrm{pg}}_L$ if $|F| = m$ then $F \in \mathcal{G}^{\circ}_m \cup \mathcal{G}^{\bullet}_m \cup \mathcal{G}_m$. By choice of k and Lemma 5.7 we thus get that if $k \leq m$ then $F \in \mathcal{G}_m$. □

Theorem 5.9. *If for all A,* S4w5 $\vdash A$ *iff* $L \vdash A^*$ *then* $L \subseteq$ KD45 *or* $L \subseteq$ S4w5

Proof. Assume the hypothesis of the theorem and assume for a contradiction that $L \not\subseteq$ KD45 and $L \not\subseteq$ S4w5. By Theorem 5.6 we get Kw4ww5 $\subseteq L$. Let k be as given by Corollary 5.8. Let n be such that $2^n \geq k + 1$. Let

$$\Pi = \{\overline{p}_1 \wedge \cdots \wedge \overline{p}_n \mid \overline{p}_i \in \{p_i, \neg p_i\}\} \ .$$

Let $\pi_1, \ldots, p_{2^n}$ be some enumeration of Π. By the conclusion of Corollary 5.8 we thus have

$$\mathcal{F}_L^{\mathrm{pg}} \models (\Diamond\pi_1 \vee \pi_1) \wedge \cdots \wedge (\Diamond\pi_{2^n} \vee \pi_{2^n}) \to (r \to \Box\Diamond r) \ .$$

By Corollary 2.8 we thus have

$$L \vdash (\Diamond\pi_1 \vee \pi_1) \wedge \cdots \wedge (\Diamond\pi_{2^n} \vee \pi_{2^n}) \to (r \to \Box\Diamond r)$$

and thus also

$$L \vdash (\Diamond\pi_1 \vee \pi_1) \wedge \cdots \wedge (\Diamond\pi_{2^n} \vee \pi_{2^n}) \to (r \to \Box(\Diamond r \vee r) \wedge (\Diamond r \vee r)) \ .$$

But then we must have

$$\mathsf{S4w5} \vdash \Diamond\pi_1 \wedge \cdots \wedge \Diamond\pi_{2^n} \to (r \to \Box\Diamond r)$$

which is clearly not so. □

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