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Stable Commutator Length in Amalgamated Free Products

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Abstract

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We show that stable commutator length is rational on free products of free Abelian groups amalgamated over $\mathbb{Z}^k$, a class of groups containing the fundamental groups of all torus knot complements. We consider a geometric model for these groups and parameterize all surfaces with specified boundary mapping to this space. Using this work we provide a topological algorithm to compute stable commutator length in these groups. We then use the combinatorics of this algorithm to prove that for a word w in the (p, q)–torus knot complement, $\mathrm{scl}(w)$ is quasirational in p and q. Finally, we analyze central extensions, and prove that under certain conditions the projection map preserves stable commutator length.

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Chapter 1

Introduction

The motivating question for this thesis concerns the description of maps of surfaces to a space whose fundamental groups split as an amalgamated free product. We develop combinatorial tools, arising from equations in the homology of the space, for simplifying a map of a surface into the space. These tools are used to provide a finite dimensional parameterization of surfaces with boundary mapping to a space whose fundamental group is an amalgamated free product of free Abelian groups and compute stable commutator length in these groups.

1.1 Overview of the Main Results

Given a group G, and an element g of its commutator subgroup, it is natural to ask what the smallest number of commutators is whose product is g. This question of *commutator length* was studied by Culler in [Cul81] using topological methods. It is shown there that, even in the free group, this quantity behaves in unexpected ways. Bavard introduced the notion of *stable commutator length* in [Bav91], which is the linear growth rate of the commutator length of g^n, see Chapter 2 for a more precise definition.

It remains difficult, still, to compute stable commutator length in particular examples and the first major breakthrough in this field was by Calegari in [Cal09b], where it was shown that stable commutator length is rational for elements of free groups and can be computed algorithmically.

Calegari also showed, in [Cal11], that it is possible to compute stable commutator length in free products of Abelian groups and that the answer is again rational. The main results presented below provide new examples of groups where stable commutator length can be computed, extending Calegari's results to include some amalgamated free products.

Theorem 6.2.1 *Let $G = A *_{\mathbb{Z}^k} B$, where A and B are free abelian groups or rank at least k. Then stable commutator length is rational on G and there exists an algorithm to compute stable commutator length for any rational chain in the real one-boundaries of G.*

In particular, the class of groups in Theorem 6.2.1 with $A = B = \mathbb{Z}$ contains the class of fundamental groups of all torus knot complements.

A generalization of this result can be achieved allowing more than two groups in the amalgamated free product.

Theorem 6.2.2 *Let $G = *_{\mathbb{Z}^k} A_i$, where $\{A_i\}$ is a collection of free abelian groups of rank at least k, and all of the A_i share a common $\mathbb{Z}^k$ subgroup. Then stable commutator length is rational on G and there exists an algorithm to compute stable commutator length for any rational chain in the real one-boundaries of G.*

To prove Theorems 6.2.1 and 6.2.2 we study maps of surfaces with boundary to a space X, whose fundamental group is G. By cutting the space along a natural codimension one subspace, we can cut the surface S into simple pieces along properly embedded arcs and simple closed curves, which will be called *loop components*. Of independent interest are the following two propositions, describing a simplifying procedure for surfaces with boundary.

Proposition 4.3.2 *Let S be a surface and $f : S \to X$ a continuous map. Then, up to replacing S by a surface with higher Euler characteristic, all cuts along loop components disconnect the surface. Further, if there are l loop components, then the chain $f_*(\partial S)$ is split into at least $(l+1)$ subchains by the loops, each belonging to span$\{a_1^{r_1}, \ldots, a_k^{r_k}\} \subseteq H_1(G;\mathbb{R})$.*

To compute scl we will need to consider combinatorial covers of surfaces.

A combinatorial m-fold cover of a surface S is a surface which is cut along the same arc types as S, but with each combinatorial class of arc appearing m-times. The boundary of this cover is an m-fold cover of ∂S. The proposition below says, informally, that taking combinatorial covers does not result in more cuts along loops.

Proposition 6.1.2 *Suppose that S contains k loop components. Then there is a surface S', which is a combinatorial double cover of S and contains at most k loop components.*

The proofs of Theorems 6.2.1 and 6.2.2 provide a combinatorial way to describe surfaces with boundary mapping to X. This is particularly interesting in the torus knot complement case, since surfaces in three-manifolds are a subject of recent and deep results in topology and geometric group theory.

We also study the groups G as central extensions of Abelian groups with boundary. Central extensions are closely linked to the second cohomology of a group — as in [Bro82] — using the Euler class. Futher, Gersten proved in [Ger92] that when the Euler class of an extension is bounded, the extension is quasi-isometric to a direct product. Here, we provide a sufficient condition to conclude that the Euler class of an extension is torsion.

Proposition 7.1.1 *Let $1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$ be a central extension with*

A a free Abelian group so that $i(A) \cap [E, E]$ is empty. Then the induced map $H^2(G; \mathbb{R}) \to H^2(E; \mathbb{R})$ is injective. Thus, the Euler class of the extension is a torsion class in $H^2(G; A)$.

We use the injectivity on real second cohomology to prove the following Theorem, relating stable commutator length indirectly to the geometry of the central extension. In particular, scl also can not distinguish between the group E and the group $G \times A$.

Theorem 7.2.2 *Let*

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

be a central extension so that the induced map $H^2(G; \mathbb{R}) \to H^2(E; \mathbb{R})$ is injective. Then the projection map $E \xrightarrow{\pi} G$ preserves scl.

Using Theorem 7.2.2 and the methods of Theorem 6.2.1, we can gain explicit control of the effect of torsion on scl. In particular, we prove the following, largely answering a question of Walker in [Wal13].

Corollary 7.3.2 *Let $G_{p_1,p_2,\ldots,p_k} = \prod_{i=1}^{k} \mathbb{Z}\big/_{p_i\mathbb{Z}}$. Given a word $w \in F_k$, the free group on k generators, let $\mathrm{scl}_{p_1,p_2,\ldots,p_k}(w)$ be the* scl *of the projection of w to $G_{p_1,p_2,\ldots,p_k}$, then for p_i sufficiently large, $\mathrm{scl}_{p_1,p_2,\ldots,p_k}(w)$ is a ratio of two quasipolynomial functions in p_i.*

1.2 Historical Context: Maps of Surfaces

Maps of surfaces and surfaces with boundary have been of interest to topologists and geometric group theorists for many years. Recently, there has been a huge jump in understanding for closed surfaces in three-manifolds. Agol and Wise's methodology that lead to the proof of the Virtual Haken conjecture in [Wis12, Ago] showed the power of using combinatorial methods in analyzing surfaces.

Many questions about closed surfaces can be naturally generalized to questions about surfaces with a prescribed boundary. For example, in three-manifolds stable commutator length is analogous to the Thurston norm on $H_2(M;\mathbb{R})$. In fact, Calegari conjectured in [Cal09b] that many of the properties of the Thurston norm carry over to scl.

Other open questions remain about surfaces in three-manifolds, such as Gabai's Simple Loop Conjecture [Gab85], asking whether every non-injective map of a close surface can be compressed along a simple loop. Calegari proved in [Cal13] that stable commutator length can be related to geometric incompressibility of closed surfaces. One of the main motivations of the results here is to provide an understanding of scl and develop more machinery to understand maps of surfaces with boundary in other spaces and apply these results to deeper topological questions.

On the group theoretic side, there is also interest in understanding when groups contain surface subgroups. Gromov famously asked whether every hyperbolic group contains a surface subgroup. Using combinatorial methods developed for the computation of stable commutator length, Calegari and Walker showed in [CWb, CWa] that groups formed by random constructions almost surely contain surface subgroups.

Other methods have also been used to prove the existence of surface subgroups in different types of group. In right-angled Artin groups, this was initiated by Servatius, Droms and Servatius in [SDS89] and furthered by Crisp, Sageev and Sapir in [CSS08]. Kim studied the inverse question in [Kim10], showing which right-angled Artin groups do not contain such subgroups.

Independent interest in commutator length, and the topology of surfaces with boundary, began with Culler in [Cul81], among others. After Bavard defined stable commutator, and proved its relationship to quasimorphisms and bounded cohomology in [Bav91] (see Section 2.4 below) many people became interested in computation. In particular, computing stable commutator length using a topological viewpoint is evident in [Cal09b, Cal11, Wal13, CFL], and will be discussed in Chapter 3.

Theorem 6.2.1, in particular, sheds light on both the topology of surfaces with boundary in certain three-manifolds, and the effect of amalgamation on stable commutator length. While still open, the following conjecture — a weak version of a conjecture of Calegari — asks if the same work can be carried out on precisely those groups which are both fundamental groups of three-manifolds and right-angled Artin groups. By work of Hermiller and Meier in [HM99] those groups can be realized as the fundamental group of a graph of groups where the underlying graph is a tree, the vertex groups are $\mathbb{Z}^2$ and the edge groups are either $\mathbb{Z}$ or trivial. These are also the fundamental groups of flip manifolds, which were studied extensively in [BN08].

Conjecture 1.2.1. *Let Γ be a finite tree and $\mathcal{A}(\Gamma)$ the right-angled Artin group with generating graph Γ. Stable commutator length is a piece-wise rational linear norm on the one-boundaries of $\mathcal{A}(\Gamma)$.*

A more combinatorial and algebraic viewpoint, obtaining lower bounds of stable commutator length using quasimorphisms began with work of Matsumoto and Morita in [MM85], who first proved that groups which are uniformly perfect have no nontrivial quasimorphisms. This work was carried out in many different kinds of groups: specifically for free groups [Bro81], hyperbolic groups [Fuj98], groups acting on hyperbolic spaces with the WPD condition (*e.g.*, mapping class groups) [BF02, BF09], free products with amalgamation [Fuj00], lattices in higher rank Lie groups [BM99], right-angled Artin groups [BC12], and many others. Lower bounds and spectral gaps in the values of stable commutator length have been extensively studied and are of continued interest, as they often arise from aspects of negative curvature in groups.

1.3 Outline of Subsequent Chapters

In chapters 2 and 3 we will provide the reader with definitions and a summary of previous results regarding stable commutator length. In chapter 3 we will give specific context for the main results by outlining previous techniques of Calegari and Walker from [Cal09b, Cal11, Wal13] that have led to precise computations of stable commutator length.

The next three chapters are devoted to the proof of Theorems 6.2.1 and 6.2.2. In chapter 4 we will form the necessary parameter space for maps of surfaces, formed by analyzing pieces formed by cutting along a one-submanifold. We show that this analysis is natural by offering a new, simple proof of a classical theorem of Waldhausen. We then delve into the topology of these maps by introducing a system of gluing equations for the pieces. The equations are used to prove the main technical lemma involved in the proof of Theorem 6.2.1, giving conditions under which a surface can be simplified.

Chapter 5 is devoted to the construction of a function to estimate the Euler characteristic of a surface. We study the combinatorics of our parameter space and a covering procedure. The function we need counts disc pieces to estimate the Euler characteristic and is a piecewise rational linear function.

We then show that given a vector in our parameter space, it is possible to form a surface with boundary for which the gluing equations can be solved. We prove that, despite possibly making the surface more complicated, that we can still use the function from Chapter 5 to estimate Euler characteristic. From here we proof of Theorem 6.2.1 will follow by using linear programming. As an example, we compute the stable commutator length of certain simple commutators using the methods described in Chapters 4, 5 and 6.

In Section 6.4 we will look at the dynamics of scl for a single word in a free group as we change groups. We will show that scl of a word varies essentially like a rational function in p and q for (p, q) torus knots, giving a partial answer to a question of Walker from [Wal13].

In Chapter 7 we analyze central extensions, showing that a certain natural class of extensions have torsion Euler class. The groups in Theorems 6.2.1 and 6.2.2 naturally arise as these sorts of extensions, and we prove that the projection map in the extension preserves stable commutator length.

Chapter 2

Background

In this chapter we will provide the necessary background on stable commutator length. Much of this material can be found in [Cal09a], and proofs are included where they are enlightening and relevant to the proof of Theorem 6.2.1. In the first section we will define stable commutator length, and include some results that are immediate from the definition. In the second section we will provide an equivalent definition of scl using surfaces with boundary, and we will prove the equivalence. In the final two sections we will present the third equivalent definition of scl using quasimorphisms and detail prior results using this definition.

2.1 Stable Commutator Length

Let G be a group, and $[G, G]$ be its commutator subgroup. By definition $[G, G]$ is generated by all elements of the form $[a, b] = aba^{-1}b^{-1}$, where $a, b \in$

G. Given $g \in [G, G]$, we define its commutator length, denoted $\mathrm{cl}_G(g)$, as the minimal number of commutators whose product is g, or equivalently ts word length in $[G, G]$.

Even for simple examples of groups $\mathrm{cl}_G(g)$ is difficult to compute. Culler showed in [Cul81] that when $G = F_2 = \langle a, b\rangle$ the following holds:

$$\mathrm{cl}_{F_2}([a,b]^n) = \begin{cases} \frac{n}{2} + 1, & \text{if } n \text{ is even} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}. \tag{2.1}$$

Thus $\mathrm{cl}([a,b]^3) = 2$, and so $\mathrm{cl}(g^n)$ is not homogenous, even for simple examples, although it is naturally subadditive in n, and so we can homogenize the function.

Definition 2.1.1. *Given $g \in [G, G]$, the* stable commutator length *of g is defined as:*

$$\mathrm{scl}_G(g) = \lim_{n\to\infty} \frac{\mathrm{cl}_G(g^n)}{n}.$$

Where no confusion is possible, we will drop the subscript.

Since $n \mapsto \mathrm{cl}(g^n)$ is subadditive, this limit always exists and can be thought of as the linear growth rate of $\mathrm{cl}(g^n)$. The following proposition is immediate from the definition of scl.

Proposition 2.1.2. *Let G, H be groups. Then:*

1. *If* $f\colon G \to H$ *is a homomorphism, then* $\mathrm{scl}_H(f(g)) \leq \mathrm{scl}_G(g)$*;*

2. $\mathrm{scl}_{G\times H}((g,h)) = \max\{\mathrm{scl}_G(g), \mathrm{scl}_H(h)\}$.

2.2 Surfaces with Boundary

Let S be a surface with genus g and one boundary component. Then it is well known that $\pi_1(S) = F_{2g}$, and further that any representative of the free homotopy class of ∂S is a product of g commutators. In fact, for any representative γ in the free homotopy class of ∂S:

$$[\gamma] = [a_1, b_1]\,[a_2, b_2] \cdots [a_g, b_g]\,,$$

where $a_1, b_1, \ldots a_g, b_g$ are some standard set of generators for $\pi_1(S)$ and $[\gamma]$ denotes the homotopy class of γ.

Let X be a topological space with $\pi_1(X) = G$. Since a surface with boundary is a $K(\pi, 1)$, given any $h \in [G, G]$ and representation of h as a product of g commutators there is a continuous map $f\colon S \to X$, where S is a surface of genus g with one boundary component and $f(\partial S)$ is (freely) homotopic to a representative of the class h.

Definition 2.2.1. *A map* $f\colon S \to X$ *of a surface* S *is called* admissible *for*

$h \in [G,G]$ if for some representative $\gamma : S^1 \to X$ of h, we have:

$$\begin{array}{ccc} \partial S & \xrightarrow{i} & S \\ \Big\downarrow{\scriptstyle \partial f} & & \Big\downarrow{\scriptstyle f} \\ S^1 & \xrightarrow{\gamma} & X \end{array}$$

such that $(\partial f)_*\big([\partial S]\big) = n(S)[S^1]$ and the diagram commutes up to homotopy.

Given a (not necessarily connected) surface S, let $\chi^-(S)$ be the Euler characteristic of S, taken after discarding any components with non-negative Euler characteristic. The following theorem allows us to use χ^- to compute scl of any element.

Theorem 2.2.2. *[Cal09a, Proposition 2.10] Let X be a topological space with $\pi_1(X) = G$ and let $g \in [G,G]$. Then:*

$$\mathrm{scl}(g) = \inf \left\{ \frac{-\chi^-(S)}{2n(S)} \right\},$$

where the infimum is taken over all admissible maps $f \colon S \to X$ for g, where S is a surface with one boundary component.

Proof. First we show that

$$\mathrm{scl}(g) \le \inf \left\{ \frac{-\chi^-(S)}{2n(S)} \right\}.$$

Let S be any surface with genus h with a continuous map $f \colon S \to X$, admissible for g with $n(S) = n$. Then g^n is writable as a product of h commutators, and

$$\frac{-\chi^-(S)}{2n} = \frac{h}{n} - \frac{1}{2n}.$$

Let $M > 0$ and take an M-fold regular, in fact cyclic, cover of S, which must have M boundary components. The genus of this cover is $1-M(1-h)$. Glue on an $M+1$ holed sphere where one boundary component is mapped to $f(\partial S)^M$ and the remainder are mapped to $f(\partial S)$. Call the resulting surface S_M, notice that $n(S_M) = Mn$. Then the genus of S_M is $2 - 2M + Mh$ and $\chi^-(S_M) = -M\chi^-(S) + 1 - M$. So

$$\frac{\mathrm{cl}(g^{Mn})}{Mn} \leq \frac{2}{Mn} - \frac{2}{n} + \frac{h}{n}, \tag{2.2}$$

and

$$\frac{-\chi^-(S_M)}{2Mn} = \frac{h}{n} - \frac{1}{n} + \frac{1}{2Mn}. \tag{2.3}$$

Taking a limit as $M \to \infty$, and comparing equations 2.2 and 2.3 yields the desired inequality.

Now, for the reverse inequality let $\epsilon > 0$ and let N be such that $\frac{\mathrm{cl}(g^N)}{N} \leq \mathrm{scl}(g) + \epsilon$. Now let $h = \mathrm{cl}(g^N)$, then there exists a surface of genus h and a continuous map $f \colon S \to X$ of a surface with one boundary component admissible for g with $n(S) = N$. Then:

$$\frac{-\chi^-(S)}{2N} = \frac{h}{N} - \frac{1}{2N} \leq \frac{h}{N} \leq \mathrm{scl}(g) + \epsilon.$$

Thus, for every $\epsilon > 0$ there exists such a with:

$$\frac{-\chi^-(S)}{2n(S)} - \epsilon \leq \mathrm{scl}(g).$$

So $\inf\left\{\frac{-\chi^-(S)}{2n(S)}\right\} \leq \mathrm{scl}(g)$, as desired. □

We can also define stable commutator length for formal sums of elements of G which are zero in $H_1(G;\mathbb{Z}) = {}^{G}\big/_{[G,G]}$ similarly to Definition 2.1.1. These are the *one-boundaries* of G, which we will denote $B_1(G;\mathbb{Z})$ (note that there is also a way to define $B_1(G;\mathbb{Z})$ using the bar complex for group homology). We can, however, use Theorem 2.2.2 to define this in an equivalent way, by allowing surfaces with multiple boundary components.

Definition 2.2.3. *Let $g_1 + g_2 + \cdots + g_k \in B_1(G;\mathbb{Z})$. Then*

$$\mathrm{scl}(g_1 + \cdots + g_k) = \inf\left\{\frac{-\chi^-(S)}{2n(S)}\right\},$$

where the infimum is taken over all continuous maps $f\colon S \to X$, where S is a surface with k boundary components which is admissible for the chain. Further, any surface realizing the infiumum above is called an extremal *surface.*

Here admissible means that there is a commutative diagram similar to Definition 2.2.1, replacing S^1 with $\bigsqcup_{i=1}^{k} S^1_i$ and requiring the $f_*(\partial S) = n(S)\left(\sum [S^1_i]\right)$. In other words, the covering degree on every S^1 is $n(S)$.

With this definition, scl forms a pseudo-norm on $B_1(G;\mathbb{R})$. However, for every $g, h \in G$, $\mathrm{scl}(ng - g^n) = 0$ and $\mathrm{scl}(hgh^{-1} - g) = 0$. Thus, scl descends to a pseudo-norm on

$$B_1^H(G;\mathbb{R}) = {}^{B_1(G;\mathbb{R})}\Big/_{\langle g^n - ng, hgh^{-1} - g\rangle}.$$

Consequently, a homomorphism which preserves this pseudo-norm is called an *isometry* for scl.

2.3 Quasimorphisms

There is a description of scl dual to Definition 2.2.3 using bounded cohomology and quasimorphisms. Gromov defined bounded (co)homology in [Gro82] using the l_1-completion of the standard chain complex for homology, and its dual – bounded functions on the set of n-simplices – for cohomology. The cochains in this complex are often called *bounded cochains*. The homology of the bounded cochain complex is denoted $H_b^*(X;\mathbb{R})$.

When G is a group, we can define $H_b^n(G;\mathbb{R})$ using the l_1-completion of the bar complex. Further, Gromov showed that when X is a $K(G,1)$, $H_b^n(X;\mathbb{R}) = H_b^n(G;\mathbb{R})$.

The inclusion of chain complexes leads to a comparison map $H_b^n(G;\mathbb{R}) \to H^n(G;\mathbb{R})$. When $n = 2$ the kernel of this map is the space of *real quasiho-*

momorphisms of G.

Definition 2.3.1. *A function* $\phi\colon G \to \mathbb{R}$ *is called a* quasihomomorphism *(or* quasimorphism*) if there exists a least non-negative real number* $D(\phi)$ *such that for any* $g, h \in G$*:*

$$|\phi(gh) - \phi(g) - \phi(h)| \leq D(\phi).$$

The number $D(\phi)$ is called the *defect* of ϕ, and the space of quasimorphisms of a group G is denoted $QH(G)$. Any bounded function is trivially a quasimorphism, and further a function is a homomorphism if and only if its defect is zero. A quasimorphism is called *homogeneous* if $\phi(g^n) = n\phi(g)$, and the space of homogeneous quasimorphisms is denoted $HQH(G)$.

There is a natural map $QH(G) \to HQH(G)$ which takes a quasimorphism ϕ to its homogenization. In particular,

$$\phi \mapsto \overline{\phi}(g) = \lim_{n\to\infty} \frac{\phi(g^n)}{n}.$$

Further, for a given ϕ, $|\phi(g) - \overline{\phi}(g)|$ is bounded, independently of g. Letting $\mathrm{Bdd}(G;\mathbb{R})$ be the set of bounded functions from G to $\mathbb{R}$, this implies that the kernel of the comparison map above is precisely:

$$QH(G)\Big/ H^1(G;\mathbb{R}) \oplus \mathrm{Bdd}(G;\mathbb{R}) \cong HQH(G)\Big/ H^1(G;\mathbb{R}),$$

since the coboundary of any unbounded quasimorphism which is not a homomorphism is a bounded function on G^2, and so is a 2-cocycle, but the quasimorphism is not a bounded 1-chain. We will frequently refer to this space as the space of quasimorphisms.

Many groups have either large spaces of quasimorphisms or none at all. For example, Abelian and amenable groups have no quasimorphisms, nor do lattices in higher rank Lie groups [BM99]. On the other hand, Brooks showed in [Bro81] that free groups of rank at least 2 have an infinite dimensional space of quasimorphisms. More precisely, the Brooks counting quasimorphisms span an infinite dimensional subspace of quasimorphisms.

This work was expanded by Fujiwara, and Bestvina-Fujiwara to show that hyperbolic groups have an infinite dimensional space of quasimorphisms [Fuj98] as well as groups acting on hyperbolic (or CAT(0)) spaces with the weak proper discontinuity property [BF02, BF09]. Mapping class groups have this property and Behrstock and Charney proved in [BC12] that non-abelian right-angled Artin groups do as well.

2.4 The Bavard Duality Theorem

Quasimorphisms provide a method that is precisely dual to that of Theorem 2.2.2 for computing scl. The connection is suggested by the following proposition.

Proposition 2.4.1. *Let ϕ be a homogenous quasimorphism on G. Then for any $a, b \in G$, $|\phi([a,b])| \leq D(\phi)$.*

Proof. First we show that ϕ is constant on conjugacy classes. Let $a, b \in G$, then for any n we have that

$$\left|\phi(ab^n a^{-1}) - \phi(a) - \phi(b^n) - \phi(a^{-1})\right| \leq D(\phi).$$

But $ab^n a^{-1} = (aba^{-1})^n$, so since ϕ is homogenous this reduces to $|n\phi(aba^{-1}) - n\phi(b)| \leq D(\phi)$ for all n. Thus $|\phi(aba^{-1}) - \phi(b)| = 0$.

Now, consider $[a,b] = aba^{-1} \cdot b$. We have that

$$\left|\phi([a,b]) - \phi(aba^{-1}) - \phi(b^{-1})\right| \leq D(\phi).$$

Since ϕ is constant on conjugacy classes we have that $|\phi([a,b])| \leq D(\phi)$, as desired. □

In fact, if $\mathrm{cl}(g) = m$ then $|\phi(g)| \leq (2m-1)D(\phi)$ for homogenous quasimorphisms. Thus scl and quasimorphisms are closely related. By extending quasimorphisms linearly to the one-chains of G, we can formalize the relationship in the following theorem, known as the Bavard Duality Theorem.

Theorem 2.4.2. *Let G be a group and $\eta \in B_1^H(G;\mathbb{R})$. Then*

$$\mathrm{scl}(\eta) = \sup_{\phi \in HQH(G)/H^1} \frac{\phi(\eta)}{2D(\phi)}.$$

This supremum is always achieved, and any ϕ with that property is called an *extremal* quasimorphism for g.

Calegari and Fujiwara used the Bavard Duality theorem in [CF10] to show that there are lower bounds for scl of most elements of hyperbolic groups and amalgamated free products. These results are frequently referred to as "spectral gaps." More precise estimates for well-aligned elements of groups acting on trees were given by Clay, Forrester and Louwsma in [CFL].

Hull and Osin showed in [HO] that quasimorphisms on hyperbolically embedded subgroups extend to quasimorphisms on the ambient group. By controlling the effect on defect, the showed that the inclusion map of hyperbolically embedded subgroups is bilipschitz on scl.

Chapter 3

PQL Groups

In this chapter we will describe previous results computing stable commutator lengths in families of groups. First, we will define the notion of a PQL group, and present a conjecture which motivates our main results. We will then briefly describe algorithms for computing scl in free groups and free products of cyclic groups, foreshadowing the methods of proof and strategy involved in the proof of Theorem 6.2.1.

3.1 Definition

In general, the problem of computing scl in any group is hard. Other than cases where it is known to be identically zero, it was previously known only for:

1. free groups [Cal09b];

2. free products of Abelian groups [Cal11];

3. free products of cyclic groups (with an efficient algorithm) [Wal13];

4. the universal central extension of subgroups of $Homeo(S^1)$ [Cal09a, Zhu08];

5. free products of free Abelian groups amalgamated over $\mathbb{Z}^k$ subgroups [Theorems 6.2.1, 6.2.2];

6. certain elements of Baumslag-Solitar groups [CFL].

It should be noted than in all but the fourth example in the list, scl takes on only rational values. In fact, scl is a piecewise rational linear (pseudo) norm on the space of one-boundaries.

Definition 3.1.1. *A group G is called PQL - pronounced 'pickle' - if* scl *is a piecewise rational linear (pseudo) norm on $B_1^H(G;\mathbb{R})$.*

Piecewise rational linear norms appear naturally in other contexts, in particular the Thurston norm on the second homology of a three-manifold. There is a natural analogy between scl in groups and the Thurston norm in three-manifolds. Consequently, Calegari has conjectured the following:

Conjecture 3.1.2. *Let G be the fundamental group of a compact three-manifold. Then G is PQL.*

Theorem 6.2.1 gives a partial answer to Calegari's conjecture in the case of torus knot complements.

3.2 Scallop and Scylla

Calegari and Walker provided algorithms in [Cal09b, Wal13] (called Scallop and Scylla, respectively) that, given a chain in a free group or free product of cyclic groups, produce an extremal surface which bounds the chain. The basis of the algorithm is that maps of surfaces with boundary to a wedge of two circles can be modeled by "fatgraphs", originally shown by Culler in [Cul81].

A fatgraph is a graph with a cyclic order on the link of every vertex. When one "fattens" the graph each edge becomes a rectangle and each vertex a polygon. The cyclic order on the links give instructions for what sides of each polygon should be adjacent.

To compute scl one forms a vector space whose basis vectors represent possible rectangles and polygons (or triangles in scylla). There is then a system of linear equations which dictates when a sum of rectangles and polygons is admissible (*i.e.* glues up and has the correct boundary). Now, Euler characteristic can be measured as a linear function on this polyhedron by using an orbifold Euler characteristic on triangles and rectangles. Then, using linear programming one can compute scl.

Walker produced a similar algorithm which accounts for relations in cyclic groups, by allowing more polygons which can contain sides on the boundary. These algorithms run in polynomial time and scylla has been proven to be efficient.

The combinatorics of the algorithm and fatgraphs were used in [CWb, CWa] to prove the existence of surface subgroups in large families of groups.

3.3 SSS

The algorithms described in the previous section are efficient and useful, but have defied extension to other groups. The algorithm that will be described in the remaining sections of this dissertation bears more semblance to Calegari's algorithm for computing scl in free products of Abelian groups [Cal11].

First, let G be a free product of free Abelian groups. For convenience, say $G = A * B$. We form tori T_A and T_B and take their wedge product to obtain a space X with $\pi_1(X) = G$. Further, denoted the wedge point p. Let $f\colon S \to X$ be a continuous map of a surface with boundary. We can cut S along $f^{-1}(p)$, which is a properly embedded one-submanifold. The result is a collection of planar (after some adjustments) surfaces mapping to T_A and T_B whose boundary components are made of two types of alternating arcs:

τ-edges from ∂S and σ-edges from $f^{-1}(p)$.

The following summarizes the main results of [Cal11]:

Theorem 3.3.1. *Given $\eta \in B_1^H(G;\mathbb{R})$, there exists a rational polyhedron V inside the vector space spanned by the σ-edges parameteriizing all surface with boundary mapping to η. Further, for any $v \in V$ there is a piecewise rational linear function $\chi_V(v)$ so that for any $\epsilon > 0$ there is an n and a surface S parameterized by nv so that*

$$\left|\chi_V(v) - \frac{\chi(S)}{n}\right| < \epsilon.$$

As noted in the previous section, this immediately implies that G is PQL, since one can use linear programming to compute scl. It is worth noting that, unlike the free group case, not every chain has an extremal surface: the algorithm constructs a family of surfaces which approaches the infimum in Theorem 2.2.2. In fact, there exist chains for which no extremal surfaces exist.

Calegari and Walker used the combinatorics of this algorithm to show in [CW13] that scl takes on quasirational (*i.e*, rational with coefficients depending only on the modulus of some parameter with respect to a fixed integer) values in what they call "surgery families" of elements of free groups. In F_2 these families are of the form $a^{\alpha_1(p)}b^{\beta_1(p)}\cdots a^{\alpha_k(p)}b^{\beta_k(p)}$, where the α_i and β_j

are linear functions of p. Key to the proof of their theorem is that the words in the family are structurally similar in the free groups, although their word lengths tend to infinity. The steps in their proof will be discussed in Section 6.4 and extended to include the groups covered by Theorem 6.2.1.

Implementations of the algorithms based on the theorem above, and to compute slices of the scl-norm ball in $B_1^H(F_r;\mathbb{R})$ exist, though they are thought to be infeasible.

Chapter 4

Maps of Surfaces

In this and the following three chapters we will consider groups $G = A *_{\mathbb{Z}^k} B$, where A and B are free Abelian groups of rank at least k. In particular, let $A = \langle a_1, \dots a_n \rangle$ and $B = \langle b_1, \dots b_m \rangle$ be free Abelian groups. Then, up to isomorphism, we can assume that

$$G = \langle a_1, \dots, a_n, b_1, \dots, b_m \mid a_1^{r_1} = b_1^{s_1}, \dots, a_k^{r_k} = b_k^{s_k}, [a_i, a_j] = [b_p, b_q] = 1 \rangle .$$

We will analyze scl in these groups using Definition 2.2.3. Let T_A be an n-torus, whose fundamental group we identify with A and T_B be an m-torus whose fundamental group is identified with B. Let X be the space formed by connecting T_A and T_B by a cylinder $T^k \times [0,1]$ where $T \times \{0\} \subset A$ represents the free Abelian subgroup generated by $a_1^{r_1}, \dots a_k^{r_k}$ and similarly for $T^k \times \{1\} \subset B$. By construction $\pi_1(X) = G$.

The goal of this chapter is to analyze maps of surfaces with (possibly) empty boundary $f\colon S \to X$. The main result is Proposition 4.3.2, describing the structure of surface with boundary mapping to X and a simplification procedure for surfaces which increases Euler characteristic. We conclude the chapter by giving an effective parameterization of surfaces with boundary which are admissible for a fixed chain η and develop a scheme for analyzing the Euler characteristic of such surfaces similar to [Cal11], which will be used in Chapter 5.

4.1 Cutting Up Surfaces

Let $\eta \in B_1^H(G)$ be any rational chain and $f\colon S \to X$ a surface map which is admissible for η. We say that η is in a normal form if the sum of the exponents for each a_i is zero and every cyclically reduced summand of η conjugate into the amalgamating subgroup is written only with generators coming from A (or B).

Lemma 4.1.1. *Given any integral chain $\eta \in B_1^H(G)$, it is possible to put η in a normal form.*

Proof. Let η be a chain in $B_1^H(G)$. First, rewrite every summand conjugate into the amalgamating subgroup using only generators from A. For the remainder of the proof, those summands will be left alone. Let k_i be the sum of all exponents of a_i and l_i be the sum of all exponents of b_i. Since $H_1(G) = \langle a_i, b_i \mid r_i a_i - s_i b_i = 0 \rangle$, for each i there must be an $n_i \in \mathbb{Z}$ so that

$$k_i a_i + l_i b_i = n_i \left(r_i a_i - s_i b_i \right).$$

Without loss of generality, assume that $n_i > 0$. By replacing an occurrence of a_i with $b_i^{s_i} a_i^{1-r_i}$, we obtain a chain η' where $k_i' = k_i - r_i$ and $l_i' = l_i + s_i$. Thus,

$$k_i a_i + l_i b_i = (n_i - 1)(r_i a_i - s_i b_i)$$

and $\eta' = \eta$ in G. Repeating n_i many times, for each i, yields the result. □

Remark. *It is possible to collapse these summands into a single summand in $B_1^H(G)$, though this will not be necessary.*

Using the conventions above, let $C = T^k \times \{\frac{1}{2}\}$. Then $X \setminus C$ is homotopic to the disjoint union of T_A and T_B. Given an admissible map $f \colon S \to X$ for η (which may be 0, in which case we mean that S is closed) we can cut S along $f^{-1}(C)$. We would like to study these pieces, and to do so we require a nice form for $f(\partial S)$.

Definition 4.1.2. *We say that a loop γ representing any $g \in G$ is* tight *if $\gamma \cap T^k \times (0,1)$ is an arc of the form $\{p\} \times (0,1)$ for some fixed $p \in T^k$.*

Given $f\colon S \to X$, we can homotope f so that $f(\partial S)$ is a union of tight loops. Further, we can homotope f so that $f(S)$ intersects C transversely (*i.e.*, crossing any arc in $f^{-1}(C) \subset S$ results in changing from a component mapping to T_A to a component mapping to T_B, or vice-versa). We will only consider maps of surfaces with these two properties.

Note that $f^{-1}(C)$ is necessarily a properly embedded codimension-1 submanifold. That is, it is a disjoint union of properly embedded arcs and simple closed curves on S. To understand S, we will analyze components of $S \setminus f^{-1}(C)$.

First, assume that some component of $S \setminus f^{-1}(C)$ is not planar: then it is a surface with boundary mapping to either T_A or T_B (which has fundamental group either A or B, both free Abelian — without loss of generality, assume it is A). This map cannot be π_1-injective. If S has boundary, then the sum of the boundary components must be trivial in A, so replace the component with a vanKampen diagram for the sum — a planar surface. This procedure increases Euler characteristic, thus we may assume that each component of $S \setminus f^{-1}(C)$ is planar.

Secondly, it is possible that $f^{-1}(C)$ contains isotopic copies of a curve. In this case, depending on whether the number of copies is odd or even the number of copies can be reduced to either one or zero by cutting out all of the annuli formed by the isotopies and gluing together the new boundary components. Either the two sides of the remaining curve map to opposite components of $X \setminus C$ or the same. In the first case, we keep the remaining curve. In the second, since the map is no longer transverse to C, homotope the map to eliminate the loop.

Now, boundary components of the closures of a component of $S \setminus f^{-1}(C)$ come in two types, which alternate. Using the notation of [Cal11], components of $f^{-1}(C)$ will be called *σ-edges*, which alternate with *τ-edges*, which come from ∂S. For convenience we will distinguish between components of $f^{-1}(C)$ depending other where they are properly embedded arcs or simple closed curves. We will call the latter *σ-loops*. Further, boundary components of S have two forms. Either:

1. the image is contained entirely in the torus T_A or T_B, which we will call *Abelian* loops;

2. the image alternates between tight loops in T_A and tight loops in T_B.

In order to consider the Abelian loops, pretend that there is a σ-edge on each Abelian loop called a *dummy* edge; any other is called *genuine*.

Consider each component of $S \setminus f^{-1}(C)$. The division between σ- and τ-edges gives a polygonal structure on each planar surface. From a collection of pieces, it must be possible to determine the Euler characteristic of S. To accommodate the gluings, it is necessary to use an Euler characteristic with corners, which is simply a version of an orbifold Euler characteristic. The following formula comes from thinking about each σ-edge intersecting ∂S at a right angle.

Definition 4.1.3. *Given a surface with a polygonal structure on its boundary, let $c(S)$ be the number of corners. Its orbifold Euler characteristic is given by:*

$$\chi_o(S) = \chi(S) - \frac{c(S)}{4}.$$

In the next section we will see that these definitions are natural, and show precisely what happens in the case where S is a closed surface.

4.2 Waldhausen's Theorem on Incompressible Surfaces

Let $G = \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$. Then $G = \langle a, b \mid a^p = b^q \rangle$, and when p and q are coprime, then $G = \pi_1(S^3 \backslash K)$, where K is a torus knot complement. For an alternative to the explanation given below, see [Hat02]. Since $S^3 \setminus K$ is a $K(\pi, 1)$, and so is X, they are homotopy equivalent. We can see this by by replacing T_A and T_B with the interior of genus 1 handlebodies in a Heegaard splitting of S^3 and the adjoining cylinder with $A = T^2 \setminus K$, which is an annulus.

Using the cutting procedure above, we can decompose maps of closed surfaces to the knot complement and analyze them. Recall that a map of a surface $S \to M$ is *incompressible* if it is π_1-injective. We can now provide an elementary proof of a classical theorem of Waldhausen for torus knot complements.

Theorem 4.2.1 (Waldhausen [Wal67])**.** *Let $M = S^3 \setminus K$, where K is a torus knot. If S is a closed, embedded, incompressible surface, then S is a boundary parallel torus.*

Proof. Let $f : S \to M$ be the embedding of the surface S and let T be the torus boundary of the genus 1 Heegaard splitting of S^3 with $K \subset T$. Isotope f so that it is transverse to $A = T \setminus K$ and $f(S) \cap A$ has a minimal number of components. Equivalently, no component of $S \setminus f^{-1}(A)$ is a disc. We cut S along $f^{-1}(A)$. As noted above, since S is incompressible each component must be planar. If any component of $S \setminus f^{-1}(A)$ has more than two boundary components, since each component of $M \setminus A$ is the interior of a genus 1 handle body we obtain a homomorphism from a free group on at least two generators to $\mathbb{Z}$, which cannot be injective. Thus, each component of $S \setminus f^{-1}(A)$ is an annulus and S is a torus.

Further, $f(S) \cap A$ is an embedded collection of curves in A parallel to K

and the surface connects these curves from above and below. Let C be some component of $S \setminus f^{-1}(A)$. Since the number of components of $f(S) \cap A$ is minimal, the boundary components of C split T into two pieces, exactly one of which is enclosed by C and contains K. Locally, C "jumps" over strands of K. If C jumps more than one strand, some arc in ∂C would be enclosed by C, but this is impossible if C is embedded. See Figure 4.1 for a diagram illustrating this. Thus, S is boundary parallel.

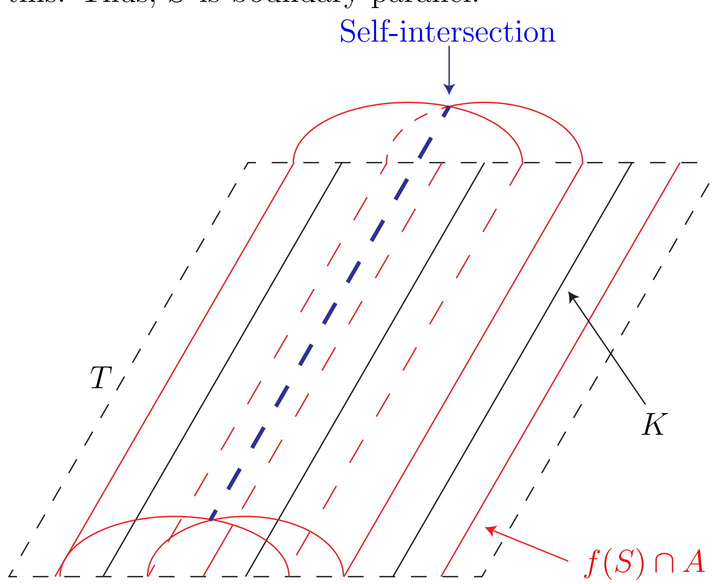


Figure 4.1: A component jumping over more than one strand of a torus knot.

□

4.3 Gluing Equations

On a component of $S \setminus f^{-1}(C)$, the sum of all the τ-edges, need not be zero in homology; it must, however, be an element of $\langle a_1^{r_1}, \ldots, a_k^{r_k} \rangle \leq H_1(A; \mathbb{Z})$. This homology deficit must be made up in the genuine σ-edges, since f describes how the image of the boundary bounds a surface in X, and thus sum of the images of all the σ- and τ- edges in a component must be zero.

Since $f(\partial S)$ is tight, each vertex maps to the same point, thus there is an element of $\mathbb{Z}^k$ attached to each genuine σ-edge and σ-loop describing how it wraps around C. We think of this element is a "*generalized winding number*" for the edge.

Encoding this information, label the σ-edges $1, \ldots n$ and let

$l_i = (l_{i,1}, \ldots, l_{i,k})$ be the element of $\mathbb{Z}^k$ attached to the σ-edge i. Further, let l_{C_j} be the element attached to the loop components of $f^{-1}(C)$. Each component of $S \setminus f^{-1}(C)$ describes an equation in homology. In particular, the sum of all the σ-edges and σ-loops in a component must cancel out the sum of all the τ-edges.

There are two systems of linear equations, one for components mapping to the torus T_A and one for components mapping to the torus T_B. Each row in the system corresponds to a component of $S \setminus f^{-1}(C)$ and since each

σ-edge is used only once it appears in exactly one equation in each system. Since gluings are orientation reversing, it appears with the coefficient 1 in the system for T_A and -1 in T_B. Thus, two surfaces can be glued along their σ-edges if and only if there is an integral solution to the system.

Lemma 4.3.1. *Consider a linear system of equations as follows*

$$\begin{bmatrix} M_A \\ \\ M_B \end{bmatrix} v = \begin{bmatrix} N_1 \\ \\ N_2 \end{bmatrix},$$

such that all entries of M_A and M_B are either 0 or 1; each column in M_A and each column in M_B has a 1 in exactly one row; N_1 and N_2 are integral vectors and the system is consistent. Letting A_j be a row from M_A and B_i a row from M_B, it is possible to row reduce the matrix so that each fully reduced row is of the form

$$\sum \epsilon_i B_i - \sum \delta_j A_j,$$

where $\epsilon_i, \delta_j \in \{0, 1\}$. Consequently, the system has an integral solution.

Proof. Put M_A is in row-echelon form by switching columns in the matrix. Let m be the number of rows of M_A, arrange that the left-most $m \times m$ block of A is an $m \times m$ identity matrix. By switching the rows of M_B, put M_B in row-echelon form (although it will not necessarily begin with an identity matrix).

Denote the (original) rows of M_A as A_j and the rows of M_B as B_i. No reductions need to be done on the matrix A, since it is already in row-echelon form. To reduce a row of M_B, first subtract off rows of M_A that share 1's with it in the first m-columns. Note that since there is only one 1 in each column of M_B and the first $m \times m$ block of M_A is an $m \times m$ identity matrix, if B_k and B_l are reduced by the same A_j, then $k = l$.

Now, the first m columns of M_B are eliminated and the reduced matrix M'_B has entries 1, 0 and -1. Further, each column of M'_B has at most one 1 and at most one -1. Continue to reduce M'_B by adding together rows (after possibly making some row switches). Let n be the number of rows in M_B. B'_1 requires no reduction, as its first possible non-zero entry is in the $(m+1)$-st column, after being reduced by the rows of M_A.

If B'_2 needs to be reduced, this is done by adding the row $B'_1 = B''_1$. Call each fully reduced row B''_i. We claim that B'_i does not get reduced by B''_k if B''_k was used to reduce a previous row. Let's say that B'_i and B'_l are both reduced by B''_k. Then B'_i and B'_l must both have an entry of 1 or -1 in the left-most entry of B''_k, a contradiction to each column having at most one 1 and one -1. Thus, each reduced row is achieved as a sum

$$B''_k = \sum_{i=1}^{n} \epsilon_i B_i - \sum_{j=1}^{m} \delta_j A_j,$$

where $\epsilon_i, \delta_j \in \{0, 1\}$, since each B'_i is the row B_i minus a sum of rows from M_A. In this way, the reduced matrix has entries 1, 0, -1. Since the original system is consistent and the image vector has integral entries, this is also true of the reduced system, and so there is an integral solution. □

Recall that every component of $S \setminus f^{-1}(C)$ produces an equation – thus each row of the matrices M_A and M_B corresponds to some component. From the first statement of Lemma 4.3.1, we can interpret taking a sum of rows in M_B and M_A as taking a union of the components. In particular, if a row in M_B and a row in M_A have a column where they are both non-zero, then they share a σ-edge, and in their union we glue along the shared edge. Thus, the lemma says that to reduce the matrix, it is only necessary to *use each edge once.* Further, every reduced row represents some set of glued components (along σ-edges) and the remaining 1's and -1's represent σ-edges in the boundary of that. By considering this combinatorial model for gluings, the next proposition about the structure of σ-loops in admissable maps follows.

Proposition 4.3.2. *Let S be a surface and $f : S \to X$ a continuous map. Then, up to replacing S by a surface with higher Euler characteristic, all loop components of $f^{-1}(C)$ are separating. Further, if there are l loop components, then the chain $f_*(\partial S)$ is split into at least $(l+1)$ subchains by the loops, each* *belonging to span* $\{a_1^{r_1}, \ldots, a_k^{r_k}\} \subseteq H_1(G; \mathbb{R})$.

Proof. The lemma in the case $k = 1$ is equivalent to saying that no row reduction in the proof of Lemma 4.3.1 will produce an equation $l_C = n$, for $n \neq 0$, or else it is possible to set $l_C = 0$ and still get a solution to the system. Since row reductions are geometric, by the comments above, this is the same as saying that there is some union of components mapping to T_B and T_A that results in a surface whose only remaining σ-edge is our loop. Thus, the loop is separating, and the chain represented by the boundary components from S in this surface plus $a_1^{r_1 n}$ must be zero. Thus the sum of those boundary components is in the span of $a_1^{r_1}$ in $H_1(G; \mathbb{R})$.

To prove the general case, repeat the row reduction operations k-times, one for each component of the solution vectors (which are in $\mathbb{Z}^k$).

Up to this point, it was implicitly assumed S is connected. If not, repeat the above procedure on each component of S separately to complete the proof. □

4.4 Parameterizing Pieces

In this section we will parameterize all surface maps $S \to X$ which are admissible for a chain in some finite dimensional subspace of $B_1^H(G)$. To do this, we analyze the components of $S \setminus f^{-1}(C)$ mapping to T_A and T_B.

Definition 4.4.1. *Let $T(A)$ be the set of all τ-edges coming from A. Similarly, let $T_2(A)$ be the set of ordered pairs of elements of $T(A)$, except whenever i is an abelian loop the only ordered pair including i is (i,i).*

Note that elements of $T(A)$ correspond to maximal subwords of components of a chain η (or more generally a basis element of the finite dimensional subspace) coming from A. Further, $T_2(A)$ corresponds to all possible σ-edges and the condition on abelian loops corresponds to the addition of dummy edges. We think of an element $(v,w) \in T_2(A)$ as a σ-edge that begins at the τ-edge v and ends at the τ-edge w.

Let $C(A)$ be the real vector space spanned by $T(A)$ and $C_2(A)$ the real vector space spanned by $T_2(A) \cup \{a_1^{r_1}, \dots, a_k^{r_k}\}$. Abusing notation, we denote a vector $v \in C_2(A)$ by

$$v = v' + \sum l_i a_i^{r_i},$$

where $v' \in \text{span } T_2(A)$. Given a surface S and a continuous map $f : S \to X$, let $v(S)$ be the sum (in $C_2(A)$) of all of the σ-edges in S, plus the sum of their winding numbers. We say that $v(S)$ *parameterizes* S.

There are two natural maps on $C_2(A)$:

$$\partial : C_2(A) \to C(A)$$

which is defined by $\partial(a,b) = a - b$ and $\partial(a_i^{r_i}) = 0$; and

$$h : C_2(A) \to A \otimes \mathbb{R}$$

defined by $h(a,b) = \frac{1}{2}(a+b)$ and $h(a_i^{r_i}) = -r_i a_i$.

In order for a boundary component to close, it must be that $\partial(v) = 0$. In particular, every τ-edge appears as a beginning of exactly one σ-edge and also the end of exactly one, but the two need not necessarily be distinct. Further, any surface S must have $h(v) = 0$, since the sum of σ-edges and τ-edges must be zero in homology.

Let V_A be the set of all vectors in $C_2(A)$ such that $h(v) = 0$ and $\partial(v) = 0$, with all components non-negative. Since it is defined by finitely many equations and inequalities, V_A is the cone on a finite sided rational polyhedron.

To understand the structure of the cone V_A, we form a graph Γ with vertices corresponding to elements of $T(A)$ and edges corresponding to elements of $T_2(A)$ so that the edge identified with (a,b) is a directed edge from the vertex a to the vertex b. A vector $v \in V_A$ gives weights on the edges of the graph. Let $|\Gamma(v)|$ be the number of connected components of the graph weighted by v, after we throw out edges with zero weight. The following lemma connects this graph with surfaces, and is the key to understanding how to compute Euler characteristic.

Lemma 4.4.2. *For any integral vector $v \in V_A$ there is a planar surface S and a map $f : S \to T_A$ so that the vector in $C_2(A)$ coming from S is v and the number of boundary components of S is $|\Gamma(v)|$. Further, any surface giving rise to the vector v has at least $|\Gamma(v)|$ boundary components.*

Lemma 4.4.2 is a restatement of [Cal11, Lemma 3.4], and the proof follows immediately, using winding numbers to cancel out homology deficits from τ-edges.

Let $|v|$ be the sum of the components of v coming from genuine σ-edges. There are exactly $2|v|$ corners on a surface, S, parameterized by v, thus

$$\chi_o(S) = \chi(S) - \frac{|v|}{2} = 2 - (\#\text{ of boundary components of } S) - \frac{|v|}{2}.$$

Noticing that $|\Gamma(nv)| = |\Gamma(v)|$, for any planar component S_1 of S, parameterized by v_1 with negative Euler characteristic and any $\epsilon > 0$ there is an integer n so that there is a surface parameterized by nv_1 (which we call nS_1) with $-\chi(nS_1)/n < \epsilon$. Thus, components of S with negative Euler characteristic are projectively negligible and contribute only corners to χ_o. To compute scl we need to study those components with positive Euler characteristic, *i.e.*, discs.

Chapter 5

Estimating Euler Characteristic

In Lemma 4.4.2 we showed that the minimal number of boundary components required for a surface to be parameterized by a vector $v \in V_A$ is determined by the number of connected components of the weighted graph $\Gamma(v)$. Further, it is clear that for any $v \in V_A$ this number is the same for nv as for v. Thus, to estimate Euler characteristic it is only necessary to count the maximal number of discs in any surface parameterized by v. In this chapter we analyze the function that counts the maximal number of discs in such a surface and also show that the estimate obtained for the Euler characteristic is "projectively optimal".

This chapter culminates in Lemma 5.3.2 and Corollary 5.3.3 which describe the particular form of the estimate.

5.1 An Example

Let $G = \mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}^2$ where the first $\mathbb{Z}^2$ is generated by a and b, and the last is generated by c and d and we amalgamate along the shared subgroup $b = c$. Let $w = [b, d]$, which is clearly the identity. This word is not in normal form, but we can still find the disc that a tight representative of w bounds.

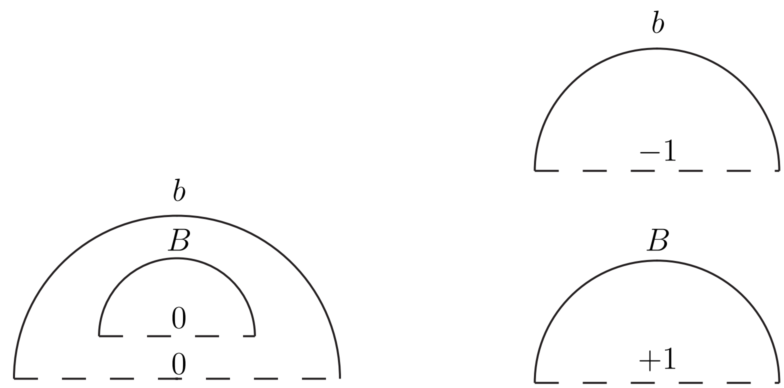
Using the notation for the vector spaces above, we have 4 τ-edges, b, B, d and D, using the convention $B = b^{-1}$. Consider the components mapping to the torus representing the first $\mathbb{Z}^2$. $T(A)$ has dimension 2 and $T_2(A)$ is generated by the σ-edges (b, b), (b, B), (B, b) and (B, B), as well as the additional basis vector b. Label these σ-edges e_1, e_2, e_3, e_4 and b, and the coordinates of each vector by $(v_1, v_2, v_3, v_4, v_b)$. We can easily see that

$$\ker(\partial) = \operatorname{span}\{(1,0,0,0,0),(0,1,1,0,0),(0,0,0,1,0),(0,0,0,0,1)\},$$

$$\ker(h) = \operatorname{span}\{(1,0,0,0,1),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,-1)\}.$$

Consider the vector $(1, 0, 0, 1, 0) \in V_A$. Note that since our word is in normal form and $v_b = 0$, which is a vector which can be obtained by cutting an admissible surface along σ-edges.

Both an annulus and two discs are parameterized by v, as shown in Figure 5.1. In each diagram, we must attach integers to each σ-edge, as noted in the lead up to Lemma 4.3.1: in Figure 5.1a we label each σ-edge 0, while in



(a) Annulus (b) Discs

Figure 5.1: Representations of $(1, 0, 0, 1, 0)$ as Planar Surfaces

Figure 5.1b we label one σ-edge 1 and the other -1. We see this by realizing $(1, 0, 0, 1, 0) = (1, 0, 0, 0, -1) + (0, 0, 0, 1, 1)$.

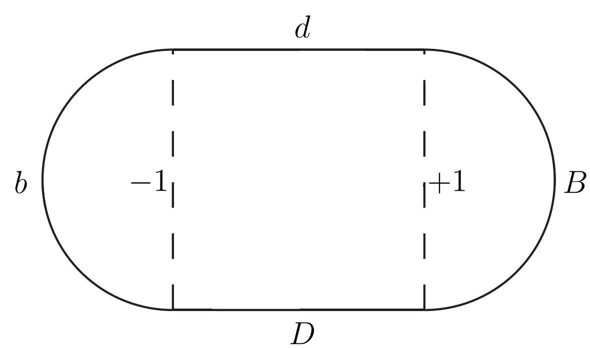


Figure 5.2: The disc bounding the word $bdBD$

We can now create the disc, as promised, by using the vector in V_B given by $(d, D) + (D, d)$, as shown in Figure 5.2. Each σ-edge is drawn in the figure with dashed lines and is labelled with an integer indicating its class in $H_1(C)$, while each τ-edge is labeled by the maximal subword it represents. This disc confirms that $[b, d] = 1$. If, however, the annulus representation in Figure 5.1a is used, the result is a genus 1 surface with one boundary component. Thus, it is necessary to consider multiple surfaces parameterized by the same vector to compute scl.

5.2 Disc Vectors and the Klein Functions

We call any integral vector v which is realized by a map of a disc into T_A a *disc vector*. From Lemma 4.4.2, we conclude that these are exactly the vectors v for which $|\Gamma(v)| = 1$. In order to estimate the Euler characteristic, we must estimate the number of disc vectors in a surface parameterized by some vector.

Let $\mathcal{D}_A$ be the set of disc vectors in V_A. As seen in the previous section, many surfaces are parameterized by one vector. Thus, we look for representations of $v \in V_A$ by the most disc vectors *i.e.*, representations of the form $v = \sum t_i v_i + v'$ such that $v_i \in \mathcal{D}_A$, $t_i > 0$ and $v' \in \overline{V_A}$. We call these *acceptable representations*. Following [Cal11] we make the following definition.

Definition 5.2.1. *The* Klein function *on V_A is defined by $\kappa(v) = \max \sum t_i$, where the maximum is taken over all acceptable representations $v = \sum t_i v_i + v'$.*

As in Calegari's algorithm, we use κ to approximate the Euler characteristic of a vector. However, κ will always consider any dummy σ-edge corresponding to an abelian loop representing an element in the shared $\mathbb{Z}^k$ subgroup as a disc. In light of this, we define

$$\chi_o(v) = \kappa(v) - \frac{|v|}{2} - v_{ab},$$

as the orbifold Euler characteristic of a *vector*, where v_{ab} is the l_1-norm of the component of v coming from dummy σ-edges corresponding to abelian loops labelled with an element of the amalgamating subgroup.

The following Lemma shows that $\chi_o(v)$ is, in fact, an estimate of the orbifold Euler characteristic of a surface parameterized by v. Further, this estimate is projectively optimal, as suggested at the end of Chapter 4. The proof of Lemma 5.2.2 describes a covering procedure which we will continue to use in Chapter 6. While the statement is similar to that of [Cal11, Lemma 3.9], the proof is made significantly more complicated by the presence of an amalgamation.

Lemma 5.2.2. *Let S_A be a surface with $v(S_A) = v$, then $\chi_o(v) \geq \chi_o(S_A)$. Further, for any rational vector $v \in V_A$ and $\epsilon > 0$ there exists a surface S_A with $v(S_A) = nv$ and $\chi_o(v) \leq \chi_o(S_A)/n + \epsilon$.*

Proof. This first statement follows from the fact that every disc component has Euler characteristic 1. Thus, disc components of S_A can add at most

$\kappa(v) - v_{ab}$ to the Euler characteristic, since κ will always count those dummy σ-edges as loops. All other components contribute zero or a negative number to the Euler characteristic along with their corners, so $\chi(S_A)$ is at most $\kappa(v) - v_{ab} - \frac{|v|}{2}$.

For the second, let $v = \sum t_i v_i + v'$ be an acceptable representation of v that realizes $\kappa(v)$. By perturbing the t_i slightly, we can assume each is rational, and their sum is within $\frac{\epsilon}{2}$ of $\kappa(v)$. Choose n so that: $nv = n_i v_i + v''$, where $n_i \in \mathbb{Z}$ and v'' is an integral vector. Now, form a surface using n_i discs parameterized by v_i, when v_i is not a dummy σ-edge representing an element of the amalgamating subgroup, and additional components (which are not discs) corresponding to v''. If v'' is non-zero, then we add all abelian loops labelled by elements of $\langle a_1^{r_1}, \ldots a_k^{r_k} \rangle$ to one of the components in the surface parameterized by v''. If $v'' = 0$, then we are forced to add the abelian loops into a disc.

Now take covers so that the Euler characteristic divided by the covering degree of the union of all non-disc components is negligible – as described in Chapter 4, simply take a connected cover of each of the boundary components. In particular if S'_A is the union of those components, find m so that $\chi(S'_A)/m > -\frac{\epsilon}{2}$. If $v'' = 0$ but $v_{ab} \neq 0$, then there is some disc vector w in our surface, so that the corresponding disc is not realized (since the component will contain the abelian loop). In this case, create m discs corresponding to w in mv, one of which contains the abelian loop (and so is not realized as a disc). As a result, we obtain a surface S_A whose Euler characteristic (divided by its covering degree) is within $\epsilon + \frac{1}{m}$ of $\kappa(v) - v_{ab}$. □

5.3 Sails and Piecewise Linearity

To compute $\kappa(v)$ we need to find admissible representations by disc vectors and thus need to know the structure of the set $\mathcal{D}_A$. Recall from Lemma 4.4.2 that the disc vectors are precisely the vectors for which the weighted graph $\Gamma(v)$ is connected. Thus, since this corresponds to allowing certain wights to be zero while others are strictly positive, $\mathcal{D}_A$ is the set of integral points in certain open faces of V_A. The following lemmas describe the geometry of this set and the function $\kappa(v)$.

Lemma 5.3.1. *The sets conv($\mathcal{D}_A$) and conv($\mathcal{D}_A + V_A$) are finite sided, convex, rational polytopes, whose vertices are elements of $\mathcal{D}_A$.*

Proof. By [Cal11, Lemma 4.8] the faces of $\ker(\partial)$ are in one-to-one correspondence with recurrent subgraphs of Γ, so that the disc vectors in V_A are contained in the open faces corresponding to recurrent connected subgraphs of Γ. If F is one of them, we know that $\text{conv}(F \cap \mathcal{D}_A)$ is a finite sided polytope, since the integer points in F form a finitely generated module over the semigroup of integer points in $\overline{F}$ (see [Mou00, CLO91] as a reference). Now, there are only finitely many connected recurrent subgraphs of Γ, so only finitely many of these faces to consider. Thus the convex hull is also a finite sided polytope. Similarly, a slight generalization of Dickson's Lemma implies that $\text{conv}(\mathcal{D}_A + V_A)$ is covered by only finitely many translates of V_A, implying the result. □

The structure of the set $\mathcal{D}_A$ is familiar to topologists, as it is used to study normal surfaces in three-manfolds (see [Gor] for a full discussion of normal surfaces). The finite generation of the module is equivalent to the existence of finitely many fundamental normal surfaces, out of which all others can be formed by taking a Haken sum. See [JO84] for an application to determining if a given manifold is Haken, and [Kan03] for an application showing that the Figure-eight knot complement is virtually fibered.

Lemma 5.3.2. *$\kappa(v)$ is a non-negative, concave function on V_A that is linear on rays and $\kappa = 1$ exactly on the boundary of conv$(\mathcal{D}_A + V_A)$ inside V_A.*

Proof. Clearly, $\kappa(v)$ is non-negative. Now, given two vectors v and w and admissible representations $v = \sum t_i v_i + v'$ and $w = \sum s_i w_i + w'$, we have that $v + w = (\sum t_i v_i + \sum s_i w_i) + (v' + w')$. Thus $\kappa(v + w) \geq \kappa(v) + \kappa(w)$. Now consider the vector rv. Clearly $rv = \sum r t_i v_i + rv'$ is an admissible representation, so $\kappa(rv) \geq r\kappa(v)$. Suppose that $rv = \sum t_i' v_i + v''$ is an admissible representation. Then $v = \sum \frac{t_i'}{r} v_i + \frac{1}{r} v''$ is an admissible representation for v, so $\kappa(rv) = \kappa(v)$.

Finally, let v be a vector on the boundary of $\text{conv}(\mathcal{D}_A + V_A)$. Since the vertices are all elements of $\mathcal{D}_A$, according to Lemma 5.3.1, we know that $v = \sum t_i v_i$, with $t_i \geq 0$, $\sum t_i = 1$ and v_i elements of $\mathcal{D}_A$. Thus $\kappa(v) \geq 1$. Suppose that $\kappa(v) > 1$ and let $v = \sum t_i' v_i' + v''$ be an admissable representation with $\sum t_i' > 1$. Then, $v / \sum t_i'$ has an admissable representation, with the sum of the coefficients of the disc vectors equal to 1. Thus, $v / \sum t_i' \in \text{conv}(\mathcal{D}_A) + V_A = \text{conv}(\mathcal{D}_A + V_A)$ by Lemma 5.3.1. □

The boundary of $\text{conv}(\mathcal{D}_A + V_A)$ is frequently called a *sail*, which was studied by Arnold in [Arn98]. Analysis of integer hulls and the sail are key to the computation described in the next chapter and play a role in Calegari-Walker's results on surgery families in [CW13].

From the previous two lemmas, we can compute the function $\kappa(v)$.

Corollary 5.3.3. *The functions $\kappa(v)$ and $\chi_o(v)$ are each equal to the minimum of the finite set of rational linear functions.*

Proof. Note that any function that is linear on rays and exactly 1 on the boundary of $\text{conv}(\mathcal{D}_A + V_A)$ must be equal to $\kappa(v)$. As noted in [CLO91], there is a primitive inward pointing normal vector v_j for each codimension 1 face F_j of $\text{conv}(\mathcal{D}_A + V_A)$, so that $\langle v, v_j \rangle = a_j$ for all $v \in F_j$, and for all other vectors in the polytope, $\langle v, v_j \rangle > a_j$. Set $\kappa'(v) = \min_j \left\{ \left\langle v, \frac{v_j}{a_j} \right\rangle \right\}$. This function clearly meets the criteria of Lemma 5.3.2, and so it must be equal to $\kappa(v)$. Further, each of the functions in the minimum is linear and there are only finitely many, since by Lemma 5.3.1 $\text{conv}(\mathcal{D}_A + V_A)$ has only finitely many sides. □

Chapter 6

Computing SCL

From Corollary 5.3.2 we know that, given a vector v, it is possible to estimate the optimal Euler characteristic of a surface parameterized by v. Given vectors $v \in V_A$ and $w \in V_B$ which have compatible σ-edges, it is still necessary to show that it is possible to solve the gluing equations.

In the first section of this chapter we will show that it is always possible to solve the gluing equations for compatible vectors, though it may be necessary to add some σ-loops to do so. We then show in Proposition 6.1.3 that it is still possible to estimate the Euler characteristic of the glued surface using the piecewise rational linear function χ_o^A on V_A and χ_o^B on V_B.

In the second section we prove Theorem 6.2.1, and show how to adapt the methods described to prove Theorem 6.2.2. In Section 6.3 we provide an example of the algorithm, giving a formula for scl of certain words.

Finally, In section 6.4 describe the behavior of scl of a fixed word or chain as we vary the group. This applies previous work of Calegari and Walker on surgery families in [CW13] and we show that scl depends quasirationally on parameters associated to the group.

6.1 Gluings and Loops

Recall that A and B are two free Abelian groups of rank at least k and $G = A *_{\mathbb{Z}^k} B$. Fix an $\eta \in B_1^H(G)$. The methods of Chapters 4 and 5 give us two polytopes V_A and V_B, as well as two piecewise rational linear functions χ_o^A and χ_o^B, respectively. We now consider the polytope $V_A \times V_B$.

We say two σ-edges $v = (\tau_1, \tau_2)$ and $w = (\tau_1', \tau_2')$ are *compatible* if τ_1 is followed by τ_2' and τ_1' is followed by τ_2 in η, considering each word in η cyclically. Two vectors $v \in V_A$ and $w \in V_B$ are *compatible* if their components from compatible σ-edges are equal. Let $V \subset V_A \times V_B$ be the set of pairs (v, w) such that v and w are compatible as vectors. V is defined by finitely many linear equations with rational coefficients and so V is also a finite-sided rational polytope.

To complete the proof of Theorem 6.2.1 we first need to prove that given compatible vectors it is possible to construct a surface admissible for η. We will prove in Lemma 6.1.1 that we can accomplish this by adding σ-loops. We then simplify the surface, using Proposition 4.3.2, and show in Lemma

6.1.2 and Lemma 6.1.3 that we can control the added σ-loops in a strong sense.

Lemma 6.1.1. *Given two compatible vectors $v \in V_A$ and $w \in V_B$ and surfaces parameterized by them, there is a solution to the gluing equations for those surfaces, formed by adding σ-loops.*

Proof. Since each σ-edge carries a weight, and weights attached to glued σ-edges must be opposite, we need to guarantee a solution to the gluing equations from section 2. We will do that by adding σ-loops to components coming from T_A and T_B and gluing those accordingly.

For each component coming from T_A, if the sum of its τ-edges is $\sum m_i a_i^{r_i}$, add $\sum |m_i|$ loops to the component each labelled with $\pm a_i^{r_i}$ and similarly for T_B. We can glue loops labelled with $\pm a_i^{r_i}$ to loops labelled $\mp b_i^{s_i}$.

If T_A has more loops labelled with $\pm a_i^{r_i}$ than T_B then glue as many loops as possible. Since w is in normal form, the remaining loops contain the same number of $a_i^{r_i}$ labels as $-a_i^{r_i}$ labels. Choose a component of S coming from T_B and add as many loops as there are remaining in T_A. Label half with $b_i^{s_i}$ and half with $-b_i^{s_i}$ and complete the gluing. See Figure 6.1 for an example.

By labeling all of the remaining σ-edges 0, we obtain the required gluing. $\square$

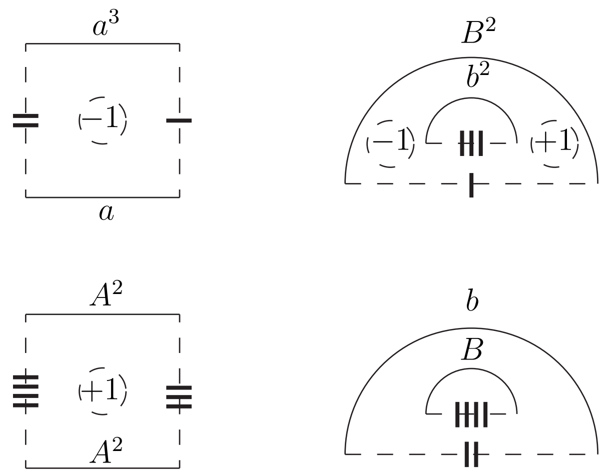


Figure 6.1: A surface bounding the chain $a^3B^2ab+A^2BA^2b^2$ in $\langle a, b \mid a^4 = b^3\rangle$. All σ-edges are labelled zero and we glue each $+1$ loop to a -1 loop.

After adding σ-loops in the proof of the proposition we have altered the Euler characteristic of the resulting surface. We show in the next proposition that it is possible to alter the winding numbers on σ-edges and σ-loopsand simplify the surface as in Proposition 4.3.2 We then show that the resulting Euler characteristic remains in line with the estimate obtained in Proposition 5.2.2.

First, if η is either a word or does not contain a subchain in the span of $\{a_1^{r_1}, \dots a_k^{r_k}\}$, Proposition 4.3.2 guarantees an integral solution to the gluing equations with all loops eliminated, since there are no subchains with sum in the span of $\{a_1^{r_1}, \dots, a_k^{r_k}\}$. Further, Lemma 5.2.2 guarantees that $\chi_o^A + \chi_o^B$ is an estimate for the Euler characteristic of this surface.

If η contains a subchain in the span of the amalgamating words, then

Proposition 4.3.2 implies that the gluing equations have a solution, leaving a limited number of separating loops. The following proposition shows us that we can strongly control the propagation of loops when taking covers in the sense of Lemma 5.2.2.

Proposition 6.1.2. *Let $v \in V_A$ and $w \in V_B$ be compatible integral vectors. Let S be a surface parameterized by the pair with the least number of loop components. Suppose that S contains k loop components. Then there is a surface S' parameterized by $2v$ and $2w$ which contains at most k loop components.*

Proof. Take v, w and S as above. According to Proposition 4.3.2, since there are k loop components ∂S is divided into $(k+1)$ subcollections $\partial_1, \dots, \partial_{k+1}$ so that each subcollection is mapped to a chain over G which is in the span of the amalgamating words. Further, no component of $S \setminus f^{-1}(C)$ can contain τ-edges from two of the subcollections and no boundary component of ∂_i can be joined to one in ∂_j by an arc in S which does not cross a loop component of $f^{-1}(C)$.

Form S' as in Lemma 5.2.2, so that $\partial S'$ is a double cover of ∂S. The surface S' may have more boundary components than S, meaning that some boundary components may be doubled in S'. There are two possible cases.

If some component with a τ-edge from ∂_i does not correspond to a disc vector, then in S' each lift of a boundary component in ∂_i can be connected to the other lift of that boundary component by an arc which remains in that component. Thus, in S' the lifts of ∂_i cannot be separated into two subcollections.

If every component with τ-edges in ∂_i is in $\mathcal{D}_A \cup \mathcal{D}_B$, consider the most naïve gluing of the σ-edges, where we duplicate the gluing pattern from S. The two lifts of each boundary component of ∂_i give two boundary components of S'. There are necessarily two subcollections, which we will call ∂_i and ∂_i'.

From the naive gluing since each σ-edge appears twice, we can switch the pairing. Since ν is in normal form, not all components of from $\mathcal{D}_A$ or $\mathcal{D}_B$ that appear are bigons. Thus, be obtain a new gluing in this way with ∂_i joined to ∂_i' by an arc which does not cross a loop. Thus ∂_i and ∂_i' are not disconnected in this surface and can therefore not be separated into two subcollections.

Thus, in S' we can arrange for there to be $k+1$ subcollections of boundary components which can be pairwise separated by loops. Thus only at most k loops are needed to solve the gluing equations, as desired. $\square$

Alternatively, as in [CFL], we can describe the proof of Proposition 6.1.2 in terms of the graph with a vertex for every piece from V_A or V_B, and edges in one-to-one correspondence with the σ-edges. The proof of theorem is then saying that in collections of discs arising in the cover, which are represented in the graph as disjoint subgraphs with no edge path connecting them, can be connected by switching the terminal vertices of two edges that come from the same type of σ-edge. Since ν is in normal form, the graph obtained by switching terminal vertices is not isomorphic to the original graph.

Proposition 6.1.3. *Let v and w be compatible rational vectors, then for any $\epsilon > 0$ there exists a positive integer N and a surface S parameterized by Nv and Nw so that*

$$\chi_o^A(v) + \chi_o^B(w) - \frac{\chi(S)}{N} < \epsilon.$$

Proof. Since by Lemma 5.2.2 the only obstruction in the proof of this proposition is the propagation of σ-loops, the statement follows from Proposition 6.1.2. □

6.2 SCL is Rational

Theorem 6.2.1. *Let $G = A *_{\mathbb{Z}^k} B$, where A and B are free abelian groups or rank at least k. G is PQL and there exists an algorithm to compute stable commutator length for any rational chain in $B_1^H(G)$.*

Proof. From Proposition 6.1.3, the Euler characteristic of a surface an be approximated by the piecewise rational linear function $\chi_o^A + \chi_o^B$, even accounting for the loops needed to solve the gluing equations. Further, the condition that S is admissible for η is rational linear, so $\chi_o^A + \chi_o^B$ can be maximized on the intersection of this affine subspace with V, a finite-sided polyhedron, using linear programming. Since all constraints are rational linear and the objective function is piecewise rational linear, the result is rational. □

As mentioned before, the methods above naturally extend to amalgamations of several free Abelian groups over a single $\mathbb{Z}^k$. Let X be the graph of spaces formed from, tori T_{A_i}, with T_{A_i} connected to $T_{A_{i+1}}$ by a cylinder. Using this space, we obtain the following, more general theorem.

Theorem 6.2.2. *Let $G = *_{\mathbb{Z}^k} A_i$, where $\{A_i\}$ is a collection of free abelian groups of rank at least k, and all of the A_i share a common $\mathbb{Z}^k$ subgroup. Then G is PQL and there exists an algorithm to compute stable commutator length for any rational chain in $B_1^H(G)$.*

Proof. Since each torus has only one set of gluing equations coming from the amalgamation, we write out the gluing system as a block matrix. Call the blocks M_i and use row switches to put each one in row-echelon form. Note that, by construction, the block M_i has non-zero entries in columns where only either M_{i-1} and M_{i+1} do as well. Call the overlap $M_{i,i-1}$ and $M_{i,i+1}$, respectively. Note that the $M_{i,i+1}$ blocks are in “block row-echelon form”, as shown in Figure 6.2.

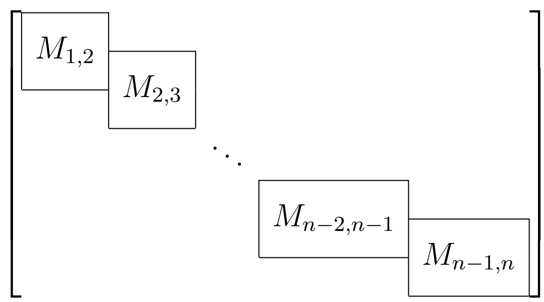


Figure 6.2: The form of the gluing matrix if there are more than two tori.

We begin row reducing the $M_{1,2}$ block, as in Lemma 4.3.1. At the end of this procedure, either the matrix is fully row reduced, and we can eliminate all loops not gluing a component from T_{A_1} to one from T_{A_2} and we are finished, or else there are reduced rows in M_2 with zeroes in every column of $M_{1,2}$.

In the first case, the σ-loops arise only when there are collections of τ-edges from T_{A_1} and T_{A_2} which cannot be connected by paths that do not cross into a component from T_{A_3}. This is a weak form of Proposition 4.3.2.

In the second we row reduce the next block using these zero rows. These rows correspond to unions of components from T_{A_1} and T_{A_2} with only σ-edges between T_{A_2} and T_{A_3} remaining. Thus, two such reduced rows must correspond to non-overlapping sums. Thus, we can continue to apply the method of Lemma 4.3.1 and show that loops arise only when there are τ-edges from T_{A_i} which cannot be connected by a path which stays in components from T_{A_i} and $T_{A_{i-1}}$ or T_{A_i} and $T_{A_{i+1}}$ or crosses σ-loops. From this weak form of Proposition 4.3.2, we can still use the covering procedure of Proposition 6.1.2 and obtain the result. □

6.3 Computations

While the algorithm described in the previous section is unwieldy, even in simple situations, and suspected to run in double exponential time [Cal11], it is possible to do some calculations without appealing to all of the machinery described.

Proposition 6.3.1. *Let $G = \langle a, b \mid a^p = b^q \rangle$, then*

$$\mathrm{scl}([a^m, b^n]) = \max\left\{\min\left\{\frac{1}{2} - \frac{m}{lcm(m,p)}, \frac{1}{2} - \frac{n}{lcm(n,q)}\right\}, 0\right\}$$

.

Remark. *If $[a^m, b^n] = 1$ in the group G, the stable commutator length is 0. However, the algorithm will give the answer $-\frac{1}{2}$ and produce a disc bounding the word. Thus, we must include the maximum in the formula above.*

Proof. Form the vector space $C_2(A)$ with basis $e_1 = (a^m, a^m)$, $e_2 = (a^m, a^{-m})$, $e_3 = (a^{-m}, a^m)$, $e_4 = (a^{-m}, a^{-m})$ and $e_5 = a^p$. The boundary and homology maps tell us that for an admissable vector $\sum v_i e_i$, $v_2 = v_3$ and $mv_1 - mv_4 - pv_5 = 0$. It remains to determine the Klein function. Clearly, the vector $e_2 + e_3$ forms a disc, and less obviously $\frac{\mathrm{lcm}(m,p)}{m} e_1 + \frac{\mathrm{lcm}(m,p)}{p} a^p$ makes a disc, and similarly for e_4 (see Figure 6.3 for an example when $m = 4$, $p = 6$). Thus

$$\kappa(v) = \frac{m}{\mathrm{lcm}(m,p)} v_1 + \frac{1}{2} v_2 + \frac{1}{2} v_3 + \frac{m}{\mathrm{lcm}(m,p)} v_4.$$

Similar formulas hold for $C_2(B)$, spanned by $f_1 = (b^n, b^n)$, $f_2 = (b^n, b^{-n})$ and so on, where a vector is given by $w = \sum w_i f_i$. In particular:

$$\kappa(w) = \frac{n}{\mathrm{lcm}(n,q)} w_1 + \frac{1}{2} w_2 + \frac{1}{2} w_3 + \frac{n}{\mathrm{lcm}(n,q)} w_4.$$

Further, the compatibility and admissability equations impose the constraints $v_1 = w_3$, $v_2 = w_1$, $v_3 = w_4$ and $v_4 = w_2$ and $|v| = |w| = 2$. Thus:

$$\begin{aligned} \frac{-\chi(v,w)}{2} = & \left(\frac{1}{4} - \frac{m}{2\mathrm{lcm}(m,p)}\right) v_1 + \left(\frac{1}{4} - \frac{m}{2\mathrm{lcm}(m,p)}\right) v_4 \\ & + \left(\frac{1}{4} - \frac{n}{2\mathrm{lcm}(n,q)}\right) w_1 + \left(\frac{1}{4} - \frac{n}{2\mathrm{lcm}(n,q)}\right) w_4. \end{aligned}$$

Since the word is in normal form we can look only in the subspace where $v_5 = w_5 = 0$. Thus, $v_1 = v_4$ and $w_1 = w_4$ and the above simplifies to:

$$\frac{-\chi(v,w)}{2} = \left(\frac{1}{2} - \frac{m}{\mathrm{lcm}(m,p)}\right) v_1 + \left(\frac{1}{2} - \frac{n}{\mathrm{lcm}(n,q)}\right) w_1.$$

It is clear the the minimum is either achieved when $v_1 = 1$ or when $w_1 = 1$, which are mutually exclusive possibilities. □

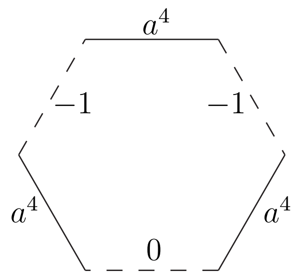


Figure 6.3: A disc for the vector $3(a^4, a^4)$ in the case $p = 6$.

This formula is similar to one that appears in [Wal13, Corollary 5.3], and we will show in the next chapter that it is, in fact, a generalization. It is notable that the formula in the proposition depends only on residue of $m \pmod p$ and $n \pmod q$. Further, it is quasilinear in $\frac{1}{p}$ and $\frac{1}{q}$, which is in line with [Wal13, Conjecture 5.4].

The formula also allows for choices of m, n, p, q where scl is zero, but the word is not trivial. The simplest example is when $m = n = 1$ and $p = 2$, $q = 3$. The group is then the fundamental group of the trefoil knot complement, and the extremal surface for the word $[a, b]$ is an annulus, shown in Figure 6.4. From the figure, it is clear that $[a, b] = [A, b]$ and is thus conjugate to its inverse. Thus, we see what is frequently referred to as a *mirror* in the group.

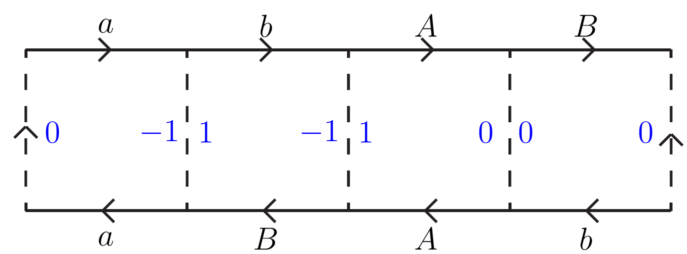


Figure 6.4: A mirror in the trefoil knot complement

6.4 Quasirationality

Calegari-Walker proved in [CW13] that stable commutator length in surgery families is quasirational in the parameter p, see Section 3.3 for a definition. To do this, they showed in subspaces of $B_1^H(F_r)$ spanned by k surgery families $\{w_i(p)\}_{i=1}^k$ that the scl unit norm ball quasiconverges in those subspaces, and that the vertices depend quasirationally on p, modulo identification.

Central to their argument is the following Theorem, proving essentially certain families of integer hulls have quasilinear vertices.

Theorem 6.4.1. *[CW13, Theorem 3.5] For* $1 \leq i \leq k$ *let* $v_i(n)$ *be a vector in* $\mathbb{R}^d$ *whose coordinates are rational functions of* n *of size* $O(n)$ *and let* $V_n = \{v_i(n)\}_{i=1}^k$*. Let* S_n *be the convex hull of the integer points in the convex hull of* V_n*. Then for* $n >> 0$*, there exists some* $\pi \in \mathbb{Z}$ *so that the vertices of* $S_{\pi n+i}$ *are the columns of a matrix whose entries are integer polynomials in the variable* n*.*

They call this condition *QIQ*. It is important to be careful here because for different values of i, it is possible that there are different numbers of vertices, so that the matrices for each i are not the same size. The process of looking at $\pi n + i$ instead of n is called *passing to a cycle*. It is also clear that, since these integer points are in the convex hull of V_n, their coordinates are also of size at most $O(n)$. Thus we obtain the following corollary.

Corollary 6.4.2. *Given the assumptions of Theorem* 6.4.1*, after passing to a cycle, the vertices of the integer hull are linear in* n*.*

Further, if V_n is instead a collection of integral linear extremal vectors of a polyhedral cone, the analogous result is true.

Corollary 6.4.3. *[CW13, Corollary 3.7] Let* V_n *be a family of cones with integral linear extremal vectors. Then the integer hull (open or closed) of* $V_n - 0$ *is QIQ and each polynomial is linear.*

Consider some word $w \in F_2 = \langle a, b \rangle$. By $w_{p,q}$ we mean the image of that word in the group $G_{p,q} = \langle a, b \mid a^p = b^q \rangle$. It's obvious that $\mathrm{scl}(w_{p,q}) \leq \mathrm{scl}_{F_2}(w)$. Additionally, Walker showed in [Wal13] that $\mathrm{scl}(w_{p,q}) \geq \mathrm{scl}_{F_2}(w) - \frac{C_1}{p} - \frac{C_2}{q}$, where C_1 and C_2 depend only on the word w.

For every choice of p, q we obtain polyhedra $V_A(p)$ and $V_B(q)$ and corresponding Klein functions $\kappa_{A,p}$ and $\kappa_{B,q}$ corresponding to data about the integer hulls of the disc faces of $V_A(p)$ and $V_B(q)$. Since the $w_{p,q}$ are combinatorially identical, we can prove the following lemma, similar to a statement in [Cal11, Section 4.4], which states that the polyhedron $V_A(p)$ satisfies the criteria of Corollary 6.4.3.

Lemma 6.4.4. *There are extremal integral vectors for $V_A(p)$ whose coordinates are quasilinear in p.*

Proof. This follows from the description of extremal rays given in [Cal11, Lemma 4.11]. Recall from Section 4.4 that Γ is graph whose vertices are in one-to-one corresponce with $T(A)$ and edges in one-to-one correspondence with $T_2(A)$. For an edge-path ϕ in Γ, we will let $h(\phi)$ be the sum of the values of h on the elements of $T_2(A)$ corresponding to the edges in ϕ. Then the extremal rays of V_A correspond to either:

1. embedded oriented cycles ϕ in Γ with $h(\phi) = 0$;

2. $h(\phi')\phi - h(\phi)\phi'$, where ϕ, ϕ' are distinct oriented embedded cycles in Γ with $h(\phi') > 0$ and $h(\phi) < 0$;

3. a pair ϕ and an integer v_ϕ so that $h(\phi + v_\phi) = 0$.

Note that only the third option is affected by the choice of p. In fact, in the case of the groups $G_{p,q}$ it is possible to ignore the second option all together, since for a high enough power of any oriented embedded cycle, $h(\phi)$ will have a multiple divisible by pa. The necessary multiple, and v_ϕ are determined quasilinearly in p, with period $h(\phi)$. □

Now, the combinatorics of $V_A(p)$ are identical for every p, the disc faces of $V_A(p)$ are the same and, by the previous lemma, satisfy the criteria of Corollary 6.4.3. Thus, we can conclude that that vertices of $\mathcal{D}_A(p)$ are QIQ in p, with all polynomials linear.

Definition 6.4.5. *For a given $p, q \in \mathbb{Z}$, let $\mathrm{scl}_{p,q}(w) = \mathrm{scl}(w_{p,q})$ in the group $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} = \langle a, b \mid a^p = b^q \rangle$. Further, denote by $\|\cdot\|_{p,q}$ the induced semi norm on $B_1^H(F_2;\mathbb{R})$.*

To analyze the dynamics of scl in these groups for a fixed word, we need to analyze the unit balls in the norms described in Definition 6.4.5 for every p, q. It follows from [Wal13] and Chapter 7, below, that as $p, q \to \infty$ the norms $\|\cdot\|_{p,q}$ converge to the standard scl norm on $B_1^H(F_2;\mathbb{R})$. In particular, the unit balls in the norms converge to the scl norm ball in the free group.

To understand this convergence we need to study the function $\kappa_{A,p}(v)$, which is the Klein function on $\mathrm{conv}(\mathcal{D}_A(p) + V_A(p))$. Recall from Lemma 5.3.1 that the set of vertices of this polyhedron is a subset of the vertices of $\mathrm{conv}(\mathcal{D}_A(p))$ and thus by Corollary 6.4.3 have coordinates which are quasilinear polynomials in p.

From here, we can determine precisely how the functions $\kappa_p(v)$ depend on p. For the next Theorem, it is necessary that we have a chain w in normal form, as defined in Chapter 4. Then every surface admissible for w comes from a fixed subset of V_A. Let v_a be the component of v corresponding the winding numbers from the amalgamation. We define:

$$V_A^{(0)} = V_A \cap \{v_a = 0\}.$$

Note that that when varying p and q, the polyhedral cones $V_A^{(0)}(p)$ and $V_B^{(0)}(p)$ are completely independent of p and q. In what follows, we will drop the dependence on p and q.

Theorem 6.4.6. *Let $w_1, \ldots, w_k$ be a collection of rational chains in $B_1^H(F_2;\mathbb{R})$ and let $B_{p,q}$ be the unit ball in the norm $\|\cdot\|_{p.q}$. Then the vertices of $B_{p,q}$ have coordinates which are eventually quasirational in p and q.*

Proof. Corollary 5.3.3 implies that the function $\kappa_p(v)$ on $V_A^{(0)}$ varies like a quotient of two linear polynomials in p. The result then follows immediately from [CW13, Theorem 4.6] by considering the linear map from compatible vectors in $V_A^{(0)} \times V_B^{(0)}$ to $B_1^H(F_2;\mathbb{R})$, whose image, by construction, is precisely the subspace spanned by $w_1, \ldots, w_k$. The image of the set of vectors with $\chi(v) \leq 1$ is then the unit norm ball in this subspace. □

Corollary 6.4.7. *For p and q sufficiently large $\mathrm{scl}_{p,q}(w)$ is a quasirational function in p and q for any fixed chain $w \in B_1^H(F_2;\mathbb{R})$.*

Proof. This follows from Theorem 6.4.6 by considering the case $k = 1$. □

Notably, [Wal13] shows that the $\|\cdot\|_{p,q}$ norm converges to the standard scl norm for F_2 at least as fast as $O(\frac{1}{p}+\frac{1}{q})$. Corollary 6.4.7 says that this is the slowest rate possible, though it is still an open question whether it is possible to obtain faster convergence.

A direct generalization of Corollary 6.4.7 is also true for for groups of the form $\mathbb{Z}^m *_{\mathbb{Z}^k} \mathbb{Z}^n$ and chains $w \in B_1^H(F_{m+n};\mathbb{R})$ using a proof identical to the one above. Even more, the same is true for the groups covered in Theorem 6.2.2, *i.e.* groups of the form $*_{\mathbb{Z}^k}\mathbb{Z}^{m_i}$ and chains $w \in B_1^H(F_{\sum m_i};\mathbb{R})$. We will use this extension in the next chapter.

Chapter 7

Applications

The groups G studied in the previous chapters arise naturally as central extensions of free products of Abelian groups with torsion. In this chapter we will study the Euler class of the these central extensions, which by the Bavard duality theorem has implications for stable commutator length. In particular, we will prove in Theorem 7.2.2 that the projection map of the central extension is an scl preserving map (*i.e.*, an isometry).

7.1 Central Extensions

Recall that a central extension of a group G by an Abelian group A is a group E and an exact sequence

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1,$$

with $i(A) \subseteq Z(E)$. Associated to any central extension is a cocycle $[e] \in H^2(B, A)$, which measures the failure of this extension to split, see [Bro82, Chapter IV] or [MS74] for a topological interpretation. Further, associated to such an extension is the Hochschild-Serre spectral sequence:

$$1 \to H^1(G; M) \to H^1(E; M) \to H^1(A; M) \to H^2(G; M) \to H^2(E; M)$$

where M is a trivial G-module. When $M = A$, it is clear from computation that the map $H^2(G; A) \to H^2(E; A)$ must have a kernel that contains the Euler class of the central extension. For more information, the interested reader can consult [McC85].

When $G = \mathbb{Z}^n *_{\mathbb{Z}^k} \mathbb{Z}^m$, then G arises naturally as a central extension of the form:

$$1 \to \mathbb{Z}^k \xrightarrow{i} G \xrightarrow{\pi} \left(\mathbb{Z}^{n-k} \times \prod_{i=1}^{k} \mathbb{Z}\big/ r_i\mathbb{Z}\right) * \left(\mathbb{Z}^{m-k} \times \prod_{i=1}^{k} \mathbb{Z}\big/ s_i\mathbb{Z}\right).$$

This central extension is of a very particular form, since $i(\mathbb{Z}^k) \cap [G, G] = \emptyset$.

Proposition 7.1.1. *Let $1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$ be a central extension with A a free Abelian group so that $i(A) \cap [E, E]$ is empty. Then the induced map $H^2(G; \mathbb{R}) \to H^2(E; \mathbb{R})$ is injective. Thus, the Euler class of the extension is a torsion class in $H^2(G; A)$.*

Proof. We consider the spectral sequence

$$1 \to H^1(G;\mathbb{R}) \to H^1(E;\mathbb{R}) \to H^1(A;\mathbb{R}) \to H^2(G;\mathbb{R}) \to H^2(E;\mathbb{R})$$

Since the real first cohomology of any group is simply the space of homomorphisms to $\mathbb{R}$, we need to show that the restriction map $\mathrm{Hom}(E;\mathbb{R}) \to \mathrm{Hom}(A;\mathbb{R})$ is surjective (*i.e*, that every homomorphism $A \to \mathbb{R}$ extends to a homomorphism $E \to \mathbb{R}$).

Let $\phi\colon A \to \mathbb{R}$ be a homomorphism. We will extend this to a homomorphism $\phi\colon E \to \mathbb{R}$. To do this, it suffices to define $\overline{\phi}$ on $E/[E,E]$.

Since $i(A) \cap [E,E] = \emptyset$, the induced map $A \xrightarrow{i} E \to E/[E,E]$ is injective, and A injects into the free part of the abelianization of E. We will denote elements of the abelianization by $\overline{g}$. Let $A = \langle a_1, \ldots, a_k \rangle$, then there exist generators $\overline{g_1}, \ldots, \overline{g_k}, \ldots, \overline{g_n}$ of $E/[E,E]$ so that $\overline{i(a_i)} = r_i \overline{g_i}$. Define

$$\overline{\phi}(\overline{g_i}) = \begin{cases} \dfrac{\phi(a_i)}{r_i}, & \text{if } 1 \leq i \leq k \\ 0, & \text{otherwise} \end{cases}.$$

Now, $\overline{\phi} : E/[E,E] \to \mathbb{R}$ lifts to a homomorphism $\phi : E \to \mathbb{R}$. Thus the map $H^1(E;\mathbb{R}) \to H^1(A;\mathbb{R})$ is surjective.

By the exactness of the sequence, this implies that

$$\ker\left(H^2(G;\mathbb{R}) \to H^2(E;\mathbb{R})\right) = 0.$$

The second statement of the Proposition follows immediately from the above and the Universal Coefficient Theorem [Hat02, Chapter 3.A], since $H^*(G; A) = H^*(G; \mathbb{Z}) \otimes A$ whenever A is a torsion free. □

Remark. *Gersten proved in [Ger92] that if $A = \mathbb{Z}$, and the image of the Euler class in $H^2(G; \mathbb{R})$ is bounded, then it is a bounded integral class. In the same paper, he proved that extensions with bounded Euler class are quasi-isometric to $G \times \mathbb{Z}$. Thus, since the central extensions above have Euler class which is 0 in $H^2(G; \mathbb{R})$, we have that E is quasi-isometric to a direct product, though not necessarily commensurable with $G \times \mathbb{Z}$.*

7.2 Projection is an Isometry

Section 2.4 discussed the relationship between quasimorphism and scl. Specifically, the Bavard Duality Theorem (Theorem 2.4.2) shows that one can compute scl from the values of quasimorphisms.

Recall that quasimorphisms arise as the kernel of the comparison map $H_b^2(G; \mathbb{R}) \to H^2(G; \mathbb{R})$. Therefore, given a homomorphism $\phi\colon G \to H$ information about the induced map on both bounded and regular homology can provide information about the effects on scl.

The following Lemma complements Proposition 7.1.1, providing information about bounded comology in central extensions.

Lemma 7.2.1. *Given a central extension,*

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1,$$

the induced map $\pi^*\colon H_b^2(G;\mathbb{R}) \to H_b^2(E;\mathbb{R})$ *is an isomorphism*

Proof. This is a well known consequence of Bouarich's Theorem on left exactness of bounded cohomology in dimension 2 [Cal09a, Theorem 2.49] and the fact that whenever A is amenable $H_b^n(A;\mathbb{R}) = 0$. □

Using Proposition 7.1.1 and Lemma 7.2.1, the following is an extension of [Cal09a, Proposition 4.30] and the proof an adaptation of the one found there.

Theorem 7.2.2. *Let*

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

be a central extension so that the induced map $H^2(G;\mathbb{R}) \to H^2(E;\mathbb{R})$ *is injective. Then the projection map* $E \xrightarrow{\pi} G$ *is an isometry for* scl.

Proof. From the comparison map and Lemma 7.2.1, there is the following commutative diagram, with exact rows and columns.

$$\begin{array}{ccccccc}
H^1(G;\mathbb{R}) & \longrightarrow & Q(G) & \xrightarrow{\delta} & H^2_b(G;\mathbb{R}) & \longrightarrow & H^2(G;\mathbb{R}) \\
\downarrow & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\
H^1(E) & \longrightarrow & Q(E) & \xrightarrow{\delta} & H^2_b(E;\mathbb{R}) & \longrightarrow & H^2(E;\mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1(A) & \longrightarrow & Q(A) & \longrightarrow & 0 & &
\end{array}$$

We need to show that given $\beta \in Q(E)$ there exists $\alpha \in Q(G)$ with $\pi^*\alpha - \beta \in H^1(E;\mathbb{R})$. First, consider $\delta\beta$. Since $\pi^*\colon H^2_b(G;\mathbb{R}) \to H^2(E;\mathbb{R})$ is an isomorphism, there exists some $\nu \in H^2_b(G;\mathbb{R})$ with $\pi^*\nu = \delta\beta$. By exactness, the image of $\delta\beta$ in $H^2(E;\mathbb{R})$ is zero. So, since $\pi^*\colon H^2(G;\mathbb{R}) \to H^2(E;\mathbb{R})$ is injective the image of ν in $H^2(G;\mathbb{R})$ is also zero.

Thus, by exactness, there exists $\alpha \in Q(H)$ with $\delta\alpha = \nu$, and by commutativity and exactness we must have that $\pi^*\alpha - \beta \in H^1(E)$. Thus $Q(H) \to Q(G) \to {Q(G)}/{H^1(G)}$ is surjective. Further, it is a general fact that $D(\pi^*\alpha) = D(\alpha)$, since π is a surjection.

Thus, by the Bavard Duality theorem, it follows that $\mathrm{scl}_G(\pi(\eta)) = \mathrm{scl}_E(\eta)$ for any chain $\eta \in B_1^H(E;\mathbb{R})$ and so π is an isometry. □

7.3 Consequences

Recall from above that the groups covered by Theorems 6.2.1 and 6.2.2 arose as central extensions of free products of Abelian groups with torsion. The following corollary is immedaite from Theorem 7.2.2

Corollary 7.3.1. *The projection map* $*_{\mathbb{Z}^k} A_i \xrightarrow{\pi} * \left(\mathbb{Z}^{n_i} \times \prod_{j=1}^{k} \mathbb{Z} \big/ r_{i,j}\mathbb{Z} \right)$ *is an isometry for* scl*.*

A particular example of the corollary is the case where $G = \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$, in which case the projection is a free product of cyclic groups. In fact, it is possible to project to any free product of cyclic groups using a group $*_{\mathbb{Z}} A_i$, where each $A_i = \mathbb{Z}$. Walker proved that there is an efficient algorithm to compute scl in these groups in [Wal13], and asked about quasilinearity of scl in the reciprocals of the orders of the cyclic groups. The following corollary is a partial answer to this question.

Corollary 7.3.2. *Let* $G_{p_1,p_2,\ldots p_k} = \prod_{i=1}^{k} \mathbb{Z} \big/ p_i\mathbb{Z}$*. Given a word* $w \in F_k$*, the free group on* k *generators, let* $\mathrm{scl}_{p_1,p_2,\ldots,p_k}(w)$ *be the* scl *of the projection of* w *to* $G_{p_1,p_2,\ldots,p_k}$*, then for* p_i *sufficiently large* $\mathrm{scl}_{p_1,p_2,\ldots,p_k}(w)$ *is a ratio of two quasipolynomial functions in the* p_i*'s.*

Proof. This follow immediately from Corollary 6.4.7 and Theorem 7.2.2. □

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