2007

TR-2007015: Justification Logics and Conservative Extensions

Melvin Fitting

Justification Logics and Conservative Extensions

Melvin Fitting
Dept. Mathematics and Computer Science
Lehman College (CUNY), 250 Bedford Park Boulevard West
Bronx, NY 10468-1589
e-mail: melvin.fitting@lehman.cuny.edu
web page: comet.lehman.cuny.edu/fitting

July 2, 2007

Abstract

Several *justification logics* have evolved, starting with the logic LP, [2]. These can be thought of as explicit versions of modal logics, or logics of knowledge or belief in which the unanalyzed necessity operator has been replaced with a family of explicit justification terms. Modal logics come in various strengths. For their corresponding justification logics, differing strength is reflected in different vocabularies. What we show here is that for justification logics corresponding to modal logics extending T, extensions are actually conservative. Our method of proof is very simple, and general enough to also handle several justification logics not directly corresponding to modal logics. Our methods do not, however, allow us to prove comparable results for justification logics corresponding to modal logics that do not extend T. That is, we are able to handle explicit logics of knowledge, but not explicit logics of belief. This remains open.

1 Introduction

Let us begin with the obvious. In the sequence of modal logics T, S4, S5, each is stronger than the one before. They have the same vocabulary, so it does not make sense to ask if each is conservative over its predecessor. But each of these logics has an *explicit* counterpart. These are logics in which, instead of formulas of the form $\Box X$, we have formulas of the form $t{:}X$, read "t is an explicit justification, or reason, for X." These explicit justifications come equipped with certain machinery, and there is a small calculus involving this machinery. The first such logic was LP, an explicit counterpart of S4, introduced by Sergei Artemov in a series of papers culminating in [2]. The syntax for justification terms in LP allows a 'bang' operator, !. Dropping it produces a logic often called LP(T), an explicit counterpart of T. Adding an operator ? produces a logic often called LP(S5), an explicit counterpart of S5. (All this will be presented more formally below.) What it means to say these are explicit counterparts of the well-known modal logics is embodied in the *Realization Theorem*, a fundamental result first proved for LP by Artemov, see [2].

Theorem 1.1 (LP Realization Theorem) *Let X be a modal formula. A* realization *of X is a formula in the language of LP that results by replacing each occurrence of $\Box$ with some explicit justification, t. A realization is* normal *if negative occurrences of $\Box$ are replaced with distinct variables (which are always part of the language of explicit justification logics).*

If X is a theorem of S4 there is some normal realization of X that is a theorem of LP. Conversely (and much simpler), if some realization of X is a theorem of LP then X is a theorem of S4.

There are similar results connecting LP(T) with T, and LP(S5) with S5. Thus each of the standard modal logics T, S4, and S5 has an explicit counterpart in a very precise sense. Now, each of LP(T), LP = LP(S4), and LP(S5) is an extension of its predecessor, vocabularies are different, and in fact each is a *conservative* extension of its predecessor.

In this paper we will show the conservativity result just stated, as part of a broader family of similar results. This will be done using a very simple proof theoretic approach. Unfortunately, the approach has its limits, so there are open problems at the end.

2 Justification Logics

It is reasonable to assume a reader of this paper is familiar with the standard modal logics: K, T, K4, S4, S5. No other modal logics will be involved here. Justification logics are much less familiar, however, so we first introduce the language, then the axiomatic characterizations for several of them. We also introduce a non-standard system of designating them, which is particularly handy here. Of course we supply the names that are standard in the literature as well. Our non-standard nomenclature is not intended to be used outside this paper.

2.1 Language

We begin with the family of *justification terms*. (These were called *proof terms* in [2], for important reasons that are not part of our concern here.) Justification terms are built up from *variables*: x_1, $x_2, \ldots$; and *constant symbols*: c_1, $c_2, \ldots$. They are built up using the following *operation symbols*: $+$ and $\cdot$, both binary, and ! and ?, both unary. These are used as infix and prefix, respectively.

This is not the place for an elaborate discussion of the intended meaning of these operations. See [5] for something of a history of the subject. But here is a brief outline. $\cdot$ is an application operation. The intention is, if t is a justification of $X \supset Y$ and u is a justification of X then $t \cdot u$ is a justification of Y. $+$ combines justifications, $t + u$ justifies whatever t justifies and also whatever u justifies. ! is a kind of positive verifier, if t justifies X then $!t$ justifies the fact that t justifies X. And ? is a negative verifier, if t does not justify X then $?t$ justifies that fact.

Formulas are built up from *propositional letters*: P_1, $P_2, \ldots$, and a *falsehood constant*, $\bot$, using $\supset$, in the usual way, together with an additional rule of formation, $t{:}X$ is a formula provided t is a justification term and X is a formula.

We will be interested in sub-languages, and so the following notation will be used. If S is any subset of $\{+, \cdot, !, ?\}$ then $L(S)$ is that part of the language described above, all of whose justification operations come from the set S.

2.2 Axiomatics

Axiom systems for justification logics evolved from one for LP, either by removing or by adding machinery. To begin with, here is a list of axioms from which we will pick and choose; more properly these are axiom schemes.

Classical Axioms:	all tautologies
Truth Axioms:	$t{:}X \supset X$
	$(X \supset Y) \supset (t{:}X \supset Y)$
	$t{:}(X \supset Y) \supset (u{:}X \supset Y)$
+ Axioms:	$t{:}X \supset (t+u){:}X$
	$u{:}X \supset (t+u){:}X$
$\cdot$ Axiom:	$t{:}(X \supset Y) \supset (u{:}X \supset (t \cdot u){:}Y)$
! Axiom:	$t{:}X \supset !t{:}t{:}X$
? Axiom:	$\neg t{:}X \supset ?t{:}\neg t{:}X$

The Truth Axioms include two that are not standard. The last two of them are, in fact, easy consequences of the first Truth Axiom. Likewise as Classical Axioms we assume all tautologies, though a finite set of schemes would be sufficient. Both of these peculiarities arise for the same reason, and have to do with the role of constants. Further discussion is postponed until after their role in proofs has been introduced.

For rules, of course we have the standard one.

$$\textit{Modus Ponens}{:}\ \frac{X \qquad X \supset Y}{Y}$$

Finally there is a version of the modal necessitation rule, and here there is some non-uniformity. Constant symbols are intended to serve as justifications for truths that we cannot further analyize, but our ability to analyize is dependent on available machinery. Consequently, we have three different versions of the final rule; more will be said about this shortly.

Definition 2.1 The following are versions of a *Constant Necessitation* rule.

Axiom Necessitation If X is an axiom and c is a constant, then $c{:}X$ is a theorem.

Iterated Axiom Necessitation If X is an axiom and $c_1, c_2, \ldots, c_n$ are constants, then $c_1{:}c_2{:}\ldots c_n{:}X$ is theorem.

Theorem Necessitation If X is a theorem and $c_1, c_2, \ldots, c_n$ are constants, then $c_1{:}c_2{:}\ldots c_n{:}X$ is theorem.

The rules stated above are progressively stronger. Incidentally, the Theorem Necessitation rule could also have been given recursively: if X is a theorem and c is a constant, then $c{:}X$ is a theorem. An important feature of justification logics is *internalization*: if X is a theorem, then for some justification term t, $t{:}X$ is a theorem. Typically, the term t can be constructed from a proof of X, but the construction of t requires a certain minimal amount of machinery. Axioms themselves are never the result of elaborate proofs—we simply assume them. This is embodied in the *Axiom Necessitation* rule above, the weakest of the three versions. If we are working with a justification logic with ! available, and we have the ! axiom, then if X is an axiom, it has a constant justification, so we have $c{:}X$, this in turn has a justification, $!c{:}c{:}X$, this has its justification, $!!c{:}!c{:}c{:}X$, and so on. But if ! is not part of the machinery we cannot take this route, and so *Iterated Axiom Necessitation* is assumed instead. Finally, if we have a really weak justification logic, not containing $\cdot$, we lack machinery to analyze anything complex, and the *Theorem Necessitation* version will be assumed—everything provable has a justification, about which nothing very interesting can be said.

Clearly the role of constants has much to do with the choice of axioms. Replacing axioms with equivalent versions changes the use of constants in both the Axiom Necessitation and the

Interated Axiom Necessitation rules. It is not simple to say, then, what it means to have equivalent axiomatizations of a justification logic—it does not mean they have the same set of theorems. This will be addressed, very incompletely, in Section 4. For the time being, our peculiar choice of Truth Axioms is so that the behavior of constants is simple to describe when proofs are manipulated in the ways we will consider below.

Now we can properly specify the family of justification logics we will be considering.

Definition 2.2 Let S be a subset of $\{+, \cdot, !, ?\}$. We define two justification logics whose language is $L(S)$. They are denoted $K(S)$ (with K for knowledge) and $B(S)$ (with B for belief). These have axioms and rules specified as follows.

1. For axioms, both $K(S)$ and $B(S)$ have the Classical Axioms. $K(S)$ assumes the Truth Axioms, while $B(S)$ does not. Finally, both assume the $+$ axiom if $+$ is in S, and similarly for $\cdot$, $!$, and $?$.

2. For rules, both have *Modus Ponens*. If $!$ is in S, $K(S)$ and $B(S)$ have the *Axiom Necessitation* rule. If $\cdot$ is in S but $!$ is not, $K(S)$ and $B(S)$ have the *Iterated Axiom Necessitation* rule. Finally, if neither $!$ nor $\cdot$ is in S, $K(S)$ and $B(S)$ have the *Theorem Necessitation* rule.

The primary utility of the notation introduced here is that it makes it very easy to state our main results compactly below. Since our nomenclature is not standard, here are some correspondences with the literature. Besides these logics, there are others that have been considered in the literature, and there are also systems that can be characterized in present terms, for example $K(\{!\})$, that have not been considered in the literature. (It's probably not very interesting.)

Standard Name	Name Used Here	Origin	Modal Counterpart
LP(K)	$B(\{+, \cdot\})$	[3]	K
LP^-(K)	$B(\{\cdot\})$	[4]	
LP(T)	$K(\{+, \cdot\})$	[3]	T
LP^-(T)	$K(\{\cdot\})$	[4]	
LP(K4)	$B(\{+, \cdot, !\})$	[3]	K4
LP^-(K4)	$B(\cdot, !\})$	[4]	
LP	$K(\{+, \cdot, !\})$	[2]	S4
LP^-	$K(\{\cdot, !\})$	[1, 4]	
LP(S5)	$K(\{+, \cdot, !, ?\})$	[7, 9]	S5

All the logics considered here meet two fundamental properties common to many justification logics. Since these will be needed in Section 4, they are stated here for the record.

Proposition 2.3 (Substitution Closure) *For every $S \subseteq \{+, \cdot, !, ?\}$, both $K(S)$ and $B(S)$ are closed under term substitution. That is, if X is a theorem of one of these logics, and X' is the result of replacing all occurrences of a variable x with a justification term t, then X' is also a theorem.*

The proof for this is standard. It is true for axioms, since they are specified by axiom schemes. Then one shows it is true for each line of a proof by induction on proof length. It should be noted that the role of constants, what is usually called the *constant specification*, changes in moving from a provable formula to a provable term substitution instance. Suppose, for instance, that we are using the Axiom Necessitation rule. If, originally, $A(x)$ was an axiom, and we introduced $c{:}A(x)$ into the proof, then also $A(t)$ will be an axiom, and we can similarly introduce $c{:}A(t)$, but now

c has been associated with two different formulas, and so the constant specification is different than it was. This is a well-known phenomenon, but attention should be called to it at this point. Substitution closure holds for the usual axiomatization of LP, as found in [2], for instance, but with the same problems concerning constant specifications. It likewise holds for axiomatizations of several other justification logics in the literature.

Vladimir Krupski has pointed out (personal communication) that Substitution Closure is a kind of trick. It amounts to an implicit use of constants for schemes and not for individual formulas. There has been work on clarifying the role of substitution by bringing a substitution operation directly into the formalism of the logic, in [8, 10]. Here we are not engaged in a deep analysis, so we stay with the 'tricky' approach.

Proposition 2.4 (Internalization) *For every $S \subseteq \{+, \cdot, !, ?\}$, both $K(S)$ and $B(S)$ have the internalization property: if X is a theorem so is $t:X$ for some ground (that is, variable free) justification term t.*

If S contains $\cdot$, this proposition has a proof due to Artemov, [2]. If S does not contain $\cdot$, the proposition defaults to the Theorem Necessitation rule.

3 Results

Theorem 3.1 *Let $S_1, S_2 \subseteq \{+, \cdot, !, ?\}$ and suppose $S_1 \subsetneq S_2$. Then $K(S_2)$ is a conservative extension of $K(S_1)$.*

The argument for this Theorem shows how to convert proofs from logic extensions back into proofs in the logic being extended. It does this by eliminating operator symbols.

Definition 3.2 Let o be one of $+$, $\cdot$, $!$, or $?$. If X is a formula of $L(\{+, \cdot, !, ?\})$, by X^o we mean the result of eliminating all justification terms containing o. More precisely, we have the following recursive characterization. For propositional letters $P^o = P$, and also $\bot^o = \bot$. Of course $(X \supset Y)^o = (X^o \supset Y^o)$. And finally:

$$(t{:}X)^o = \begin{cases} X^o & \text{if } o \text{ occurs in } t \\ t{:}X^o & \text{if } o \text{ does not occur in } t \end{cases}$$

The central part of the proof of Theorem 3.1 is contained in the following Proposition. Note that its proof is constructive (and simple).

Proposition 3.3 (Operator Elimination) *Let $S \subseteq \{+, \cdot, !, ?\}$ and assume o is one of the operation symbols in S. If Z is one of the axioms of $K(S)$, then Z^o is an axiom of $K(S - \{o\})$.*

Proof There are several cases and subcases, depending on choice of axiom and choice of operation symbol. The argument in each case is straightforward. It might be simpler to construct your own argument rather than reading mine. Here are the cases.

Classical Axiom: If Z is a tautology, so is Z^o.

Truth Axiom: Z is either $t{:}X \supset X$ or $(X \supset Y) \supset (t{:}X \supset Y)$ There are two simple subcases

o does not occur in t. Then Z^o is again a Truth Axiom, of the same kind.

o occurs in t. Then Z^o is a Classical Axiom.

Truth Axiom: $Z = t{:}(X \supset Y) \supset (u{:}X \supset Y)$ Again there are simple subcases.

- o does not occur in t or in u. Then Z^o is again a Truth Axiom, of the same kind.
- o occurs in u but not in t. Then Z^o is $t{:}(X^o \supset Y^o) \supset (X^o \supset Y^o)$, a different kind of Truth Axiom.
- o occurs in t but not in u. Then Z^o is $(X^o \supset Y^o) \supset (u{:}X^o \supset Y^o)$, again a different kind of Truth Axiom.
- o occurs in both t and u. Then Z^o is $(X^o \supset Y^o) \supset (X^o \supset Y^o)$, a Classical Axiom.

$+$ Axiom: $Z = t{:}X \supset (t+u){:}X$ The other $+$ axiom is similar so only this one is considered.

- o occurs in t. Z^o is $X^o \supset X^o$, a Classical Axiom.
- o occurs in u but not in t. Z^o is $t{:}X^o \supset X^o$, a Truth Axiom.
- o occurs in neither t nor u, and o is not $+$. Z^o is $t{:}X^o \supset (t+u){:}X^o$, another $+$ axiom.
- o occurs in neither t nor u, and o is $+$. Z^o is $t{:}X^o \supset X^o$, a Truth Axiom.

$\cdot$ Axiom: $Z = (t{:}(X \supset Y) \supset (u{:}X \supset (t \cdot u){:}Y))$ The subcases are as follows.

- o occurs in both t and u. In this case Z^o is $(X^o \supset Y^o) \supset (X^o \supset Y^o)$, a Classical Axiom.
- o occurs in u but not in t. Then Z^o is $t{:}(X^o \supset Y^o) \supset (X^o \supset Y^o)$, an instance of the first Truth Axiom.
- o occurs in t but not in u. Then Z^o is $(X^o \supset Y^o) \supset (u{:}X^o \supset Y^o)$, an instance of the second Truth Axiom.
- o occurs in neither t nor u, and o is not $\cdot$. Then Z^o is $t{:}(X^o \supset Y^o) \supset (u{:}X^o \supset (t \cdot u){:}Y^o)$, an instance of the $\cdot$ Axiom.
- o occurs in neither t nor u, and o is $\cdot$. Then Z^o is $t{:}(X^o \supset Y^o) \supset (u{:}X^o \supset Y^o)$, an instance of the third Truth Axiom.

! Axiom: $Z = t{:}X \supset !t{:}t{:}X$ The cases are as follows.

- o occurs in t. Z^o is $X^o \supset X^o$, a Classical Axiom.
- o does not occur in t, and o is not !. Z^o is $t{:}X^o \supset !t{:}t{:}X^o$, a ! Axiom.
- o does not occur in t, and o is !. Z^o is $t{:}X^o \supset t{:}X^o$, a Classical Axiom.

? Axiom: $Z = \neg t{:}X \supset ?t{:}\neg t{:}X$ This case is similar to the ! case.

■

Now, finally, there is very little left to do.

Proof of Theorem 3.1 Suppose $S_1 \subsetneq S_2$, where both are subsets of $\{+, \cdot, !, ?\}$. Assume S_2 contains a single operation symbol o that is missing from S_1. (The case of multiple operation symbols is handled by iterating the single operator case.) Let X be a theorem of $K(S_2)$, where X does not contain any occurrence of o. We show X is a theorem of $K(S_1)$.

Consider a proof of X in $K(S_2)$. Replace each line, Z, of that proof with Z^o. Each axiom of $K(S_2)$ is replaced with an axiom of $K(S_1)$, by Proposition 3.3. Applications of *modus ponens* turn into other applications of *modus ponens*. Finally, applications of Constant Necessitation in $K(S_2)$ turn into applications of Constant Necessitation in $K(S_1)$, because $K(S_2)$ axioms turn into $K(S_1)$ axioms. Thus the entire proof converts to one in $K(S_1)$. Finally, since X did not contain o, it is still the last line of the proof, hence X is provable in $K(S_1)$. ■

4 Embedding and Equivalence

The role of constants in justification logics imposes certain peculiar complications. For instance consider LP, or $K(\{+,\cdot,!\})$ in the temporary terminology of this paper. There is much flexibility possible in its axiomatization. For example, we need an underpinning of classical logic, but that could be axiomatized in several ways—infinitely many different ways, in fact. But this affects applications of the *Constant Necessitation* rule. If, say, $X \supset X$ is an axiom, we can infer $c{:}(X \supset X)$ for a constant c. If instead we have an axiomatization of classical logic in which $X \supset X$ is not an axiom, nonetheless it will be a theorem, but then *Constant Necessitation* does not apply to it. We do, however, have the internalization feature to appeal to: for some justification term t, $t{:}(X \supset X)$ will be a theorem. In some sense these differences shouldn't matter very much. What is basic in one axiomatization (and so has a constant justification) is subject to proof in the other (and so has a more complex justification), but if we have some straightforward way of going back and forth between these two versions, that should be sufficient. In this section we formally address a very restricted version of the issue. We assume $\cdot$, $+$, and ! are present, so the version of Constant Necessitation used is Axiom Necessitation. We also assume a locality condition, defined below. Presumably some of this can be relaxed, but the technical details would become more complex. What is given here is enough to partially justify the presence of three Truth Axiom schema in Section 2.2 instead of the customary single one.

Recall that in this section we are assuming Axiom Necessitation is the version of Constant Necessitation we use. A *constant specification* $\mathcal{C}$ is an assignment of axioms to constants. A proof meets constant specification $\mathcal{C}$ provided that whenever $c{:}X$ is introduced using the Axiom Necessitation rule, then X is a formula that $\mathcal{C}$ assigns to constant c. A constant specification can be given ahead of time, or created during the course of a proof. A constant specification is *injective* if at most one formula is associated with each justification term.

Definition 4.1 We say one justification logic J_1 *locally embeds in another*, J_2, provided that for each theorem X of J_1, there is a mapping from constants of J_1 to justification terms of J_2 that converts X into a theorem of J_2.

We say two justification logics are *locally equivalent* if each embeds in the other.

A similar notion of uniform embedding, and of uniform equivalence, might be defined, in which the mapping on constants does not depend on the formula X. This is not investigated here. In the following theorem we consider two justification logics which differ only in axioms. They may, for instance, differ in how the underlying classical logic is axiomatized, or as in our case in Section 2.2, in what Truth Axioms are assumed.

Theorem 4.2 (Embedding) *Let J_1 and J_2 be two justification logics in the same language $L(S)$, where S is either $\{+,\cdot,!\}$ or $\{+,\cdot,!,?\}$. We assume the rules of infererence for J_1 and J_2 are* modus ponens *and Axiom Necessitation, as given in Section 2.2, but the choice of axioms may be entirely different. Suppose we have the following conditions.*

1. *J_1 is axiomatized so that if, in an axiom, all constant symbols are replaced with distinct terms, the result is again an axiom.*

2. *J_2 satisfies Substitution Closure: if X is a theorem, and X' is the result of replacing all occurrences of a variable x with a justification term t, then X' is also a theorem. (see Proposition 2.3).*

3. *J_2 satisfies Internalization: if X is a theorem so is $t{:}X$ for some ground justification term t (see Proposition 2.4).*

4. *Every axiom of J_1 is a theorem of J_2.*

Then J_1 locally embeds in J_2.

Note that the axiomatizations of Section 2.2 all meet condition 1. The same is true of the usual axiomatization of LP, as found in [2] for instance.

Proof Let X be a theorem of J_1, and let $\mathcal{P}$ be a proof of X in J_1. All this is fixed for what follows.

We must create a mapping from constants of $L(S)$ to terms. If c does not appear in $\mathcal{P}$, map c to itself. Now let c be a constant that occurs in $\mathcal{P}$. If c is not involved in an Axiom Necessitation rule in $\mathcal{P}$, we again map c to itself. Otherwise, say $c{:}A_1, \ldots, c{:}A_n$ are all the members of $\mathcal{P}$ that involve constant c and have been introduced using Axiom Necessitation. (Then all of A_1, $\ldots$, A_n are axioms of J_1. If the constant specification used in $\mathcal{P}$ is injective, $n = 1$ and the use of $+$ below is unnecessary.) Complications can arise due to the fact that some A_i may contain occurrences of c itself. If this happens c is said to be *self-referential*, and it was shown in [6] that such self-referentiality is essential. (Thanks to Sergei Artemov for suggestions on how to handle this.)

Let A_i be one of $A_1, \ldots, A_n$. Suppose $c_1, c_2, \ldots, c_k$ are all the constants occurring in A_i (in some standard order), where c may occur in the list. For clarity we write $A_i(c_1, c_2, \ldots, c_k)$ for A_i. Let $x_1, x_2, \ldots, x_k$ be variables not occurring in A_i (again in some standard order). Since A_i is an axiom of J_1, which meets condition 1, then $A_i(x_1, x_2, \ldots, x_k)$ will also be an axiom. Then by condition 4, $A_i(x_1, x_2, \ldots, x_k)$ is a theorem of J_2. Since J_2 meets condition 3, there is some ground justification term t_i such that $t_i{:}A_i(x_1, x_2, \ldots, x_k)$ is a theorem of J_2; if there is more than one such term, say we choose the first in some standard enumeration. In this way, for each $i = 1, \ldots, n$ we determine a ground term t_i. Now we map the constant c to the justification term $t_1 + \ldots + t_n$.

For each constant c occurring in $\mathcal{P}$, let $t(c)$ be the term that was assigned to c above. For each formula Z in $\mathcal{P}$, let $t(Z)$ be the formula that results when each constant c in Z is replaced by the justification term $t(c)$, and let $t(\mathcal{P})$ be the sequence that is like $\mathcal{P}$, but with each formula Z replaced with $t(Z)$. The sequence $t(\mathcal{P})$ is not, itself, a proof in J_2, but each item in it is a theorem of J_2. This has a straightforward proof by induction on position in $\mathcal{P}$.

If Z is an axiom of J_1 that occurs in $\mathcal{P}$, since J_1 meets condition 1, $t(Z)$ will also be an axiom, and hence a theorem of J_2 by condition 4.

Suppose Z occurs in $\mathcal{P}$, and follows from earlier items Y and $Y \supset Z$ in $\mathcal{P}$ by *modus ponens*, and the result is known for these items. Then $t(Z)$ similarly follows from $t(Y)$ and $t(Y \supset Z) = (t(Y) \supset t(Z))$ by *modus ponens*.

Finally we have the Axiom Necessitation case. Suppose Z is $c{:}A$ where A is a J_1 axiom. Say $A = A(c_1, c_2, \ldots, c_k)$, where all the constants of A are explicitly displayed. If $x_1, x_2, \ldots, x_k$ are variables not occurring in A, as seen above, there is a ground term t such that $t{:}A(x_1, x_2, \ldots, x_k)$ is a theorem of J_2, and hence so is $t(c){:}A(x_1, x_2, \ldots, x_k)$, on making use of the axioms for $+$. Since J_2 meets condition 2, $t(c){:}A(t(c_1), t(c_2), \ldots, t(c_k))$ is also a theorem, and this is $t(c{:}A) = t(Z)$. ■

Usually in the literature, the Truth Axiom is taken to be given by a single schema: $t{:}X \supset X$. We assumed two additional schemas: $(X \supset Y) \supset (t{:}X \supset Y)$ and $t{:}(X \supset Y) \supset (u{:}X \supset Y)$. It is an easy consequence of Theorem 4.2 that, for S either $\{+, \cdot, !\}$ or $\{+, \cdot, !, ?\}$, if we had axiomatized $K(S)$ with the usual single Truth schema instead of the way we did, the resulting logic would have been locally equivalent to the version we used, in the sense of Definition 4.1. Similarly we could have used “enough” tautologies instead of taking all of them as Classical Axioms, and that would have given locally equivalent logics as well. We made the choice we did because then the manipulations of Proposition 3.3 always turned axioms into axioms, and hence the behavior of Constant Necessitation was simple to describe.

5 Conclusion

The main thing left undone is quite obvious: there is no analog of Theorem 3.1 for logics of belief instead of knowledge—of the form $B(S)$ instead of $K(S)$. The methods of proof used here clearly do not extend to explicit logics of belief. Many of the cases involved in the proof of Proposition 3.3 yielded an instance of a Truth Axiom schema. Without the Truth Axioms, present methods cannot succeed. Nonetheless, either a belief analog of Theorem 3.1 holds, or it does not. A result either way would be of interest. The obvious conjecture is that it holds, but a proof is left to others.

One other item was left quite unfinished. Theorem 4.2 needed the presence of $+$, $\cdot$ and !. Producing a version not needing ! is probably straightforward, but a bit messy to state. Dropping $+$ complicates things, unless we start with an injective constant specification. A version without $\cdot$ probably has little intrinsic interest. Likewise, the status of local/global seems to depend on whether or not injective constant specifications are used, but this has not been investigated. A full examination of what ‘equivalence’ ought to mean for justification logics still awaits. The result proved here is just an easy first step.

References

[1] S. Artemov. Explicit provability: the intended semantics for intuitionistic and modal logic. Technical Report CFIS 98-10, Cornell University, 1998.

[2] S. Artemov. Explicit provability and constructive semantics. *The Bulletin for Symbolic Logic*, 7(1):1–36, 2001.

[3] V. Brezhnev. Explicit counterparts of modal logic (in Russian). Master’s thesis, Lomonosov Moscow State University, 1999.

[4] M. C. Fitting. The logic of proofs, semantically. *Annals of Pure and Applied Logic*, 132:1–25, 2005.

[5] M. C. Fitting. Reasoning with justifications. Forthcoming in *Studia Logica*, 2007.

[6] R. Kuznets. On self-referentiality in modal logic. *The Bulletin of Symbolic Logic*, 12(3):510, 2006.

[7] E. Pacuit. A note on some explicit modal logics. In C. Dimitracopoulos, editor, *Proceedings of the Fifth Panhellenic Logic Symposium*, pages 117–125, 2005.

[8] N. Rubtsova. Logic of proofs with substitution operation. To appear in Mat. Zametki (Russian).

[9] N. Rubtsova. Evidence reconstruction of epistemic modal logic S5. In D. Grigoriev, J. Harrison, and E. A. Hirsch, editors, *Computer Science — Theory and Applications*, Lecture Notes in Computer Science, vol 3967, pages 313–321. Springer-Verlag, 2006.

[10] N. Rubtsova and T. Yavorskaya. Operations on proofs and labels. To appear in Journal of Applied Nonclassical Logic.