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Abstract. In linear logic, a formula $!A$ with the modality ! means that A can be duplicated as many times as we like. We can paraphrase this as 'A holds with an empty resource'. Then it is natural to generalize formulas with modalities to the form $s : A$ with a term s, of which meaning is that A holds if there is a resource s. Terms are interpreted as elements of a well-ordered set; ! is interpreted as the least element. In this paper, we present an extended system of linear logic with such formulas $s : A$ as 'linear logic with explicit resources'. We show some basic properties of the logic: the soundness and completeness theorems with respect to a denotational semantics (Phase semantics), the cut-elimination theorem (normalization theorem). In addition, we consider the fragment containing no ! and prove the faithful embeddability of (ordinary) linear logic in the fragment.

§1 Introduction.

Linear logic, introduced in Girard [6], is a kind of resource-conscious logic. Some propositions which are valid in standard (classical or intuitionistic) logic become unvalid in linear logic. For example, it is not valid in linear logic that P implies 'P and P'; we cannot duplicate arbitrarily the resource P. Also, it is not valid that 'P and Q' imply Q; we cannot throw away the resource P. On the other hand, linear logic includes non resource-conscious fragment by equipping the modalities. We call this the modal fragment. A formula $!A$ with the modality ! means that A can be duplicated as many times as we like (including zero times). Then it is true that '$!P$ and Q' imply Q and $!P$ implies '$!P$ and $!P$'. Thus linear logic has both the resource-conscious fragment and the modal fragment.

Owing to the modal fragment, linear logic keeps the strength of intuitionistic logic. In fact, it is possible to embed intuitionistic logic into linear logic in a faithful way ([6]). So linear logic can be said as a refinement of intuitionistic logic with the explicit distinction of the two fragments hidden in it. (Cf. Girard [6], Okada [10].) In this paper, we shall extend the modal fragment and introduce an extended system of linear logic. A modal formula $!A$ can be interpreted to mean that A holds with an empty resource, which implies that A can be duplicated as many times as we like. So we can generalize modal formulas to the form $s : A$ of which meaning is that A holds if there is a resource s. The modality s is a term to be interpreted as an element of a well-ordered set. Then ! is an instance which means an empty resource and which should be interpreted as the least element of a well-ordered set. In this generalization of the modal fragment of linear logic, we are rightly said to be making explicit a hidden meaning of the original modality !, namely, 'with an empty resource' and making a generalization of modal formulas with terms. This is the reason why we call the system which we will be concerned with 'linear logic with explicit resources'. In addition, we introduce the operation $+$ on terms so that $s + t$ means the sum of the resources s and t. Then the examples of theorems in the logic are: $(s \otimes s : A) \multimap A, (s \leq t \otimes s : A) \multimap t : A, s : A \multimap (s + s) : (A \otimes A)$.

In recent years, the modal logic having the modality as terms has been studied a lot. For example, Hybrid Logic is a proof system internalizing Kripke-type possible world semantics. (Cf. [1], [5]). A modal formula $s : A$ there means that A holds at a possible world s. We can also raise Artemov's Logic of Proof ([2], [3]) as modal logic having the modality as terms. $s : A$ there means that s is a proof of A. This is a solution of the problem since Gödel: how to interpret the modal logic $S4$ in an arithmetical sense. It is also a formalization of what is called BHK-semantics for intuitionistic logic via the notion of proofs. In the Logic of Proofs, some elementary functions on terms played an important role. We borrow this idea to use the operation $+$ in our logic. Thus our extension of linear logic with explicit resources is along with such a recent trend of research of modal logic. [1]

In this paper, we show some basic results on linear logic with explicit resources, which we denote by LL^+. (By LL we mean the ordinary linear logic.) The content of the paper is as follows. In §2, we give a proof system of the logic and present some proofs in it. In §3, we give a denotational semantics for the logic, namely, Phase semantics. Then we prove the soundness theorem of the logic with respect to Phase semantics. In §4, we give a proof of the completeness theorem. As a corollary, we obtain cut-elimination theorem (normalization theorem). In §5, we focus on the subsystem not containing the modality !, which we will call the pure fragment. Then we show the embeddability of LL (containing only ! as modality) in the pure fragment. In §6, we conclude the paper.

§2 Proof system of LL^+

In this section we present the proof system of linear logic LL^+ with explicit resources. First the terms are defined as follows: constants $!, c, d, \dots$ are terms; when s and t are terms, so is $s + t$. Each term means some resource. In particular, ! means an empty resource. (In this paper we do not take ? into consideration for simplicity.) $+$ means the sum of resources. Formulas are usually defined as in the literature of linear logic with an additional formation rule: (1) when s, t are terms, $s \le t$ is a formula which is called a $\le$-formula. We assume that $\le$-formulas are not used to construct other formulas. [2] (2) when s is a term and A is a formula, $s : A$ is a formula. $s : A$ reads as 'if there is a resource s then A holds'. (We write $!A$ for $!{:}A$.) Then we consider a sequent of the form:

$t_1 \le u_1, \dots, t_m \le u_m \,; s_1, \dots, s_n \,; A_1, \dots, A_k \vdash B_1, \dots, B_h,$

where $(t_1 \le u_1, \dots, t_m \le u_m)$ and $(s_1, \dots, s_n)$ are multisets of $\le$-formulas and terms, respectively; $(A_1, \dots, A_k)$ and $(B_1, \dots, B_h)$ are multisets of formulas containing no $\le$-formulas.

This reads as 'if there are resources $s_1, \dots, s_n$ satisfying all $t_i \le u_i$ and $A_1 \otimes \cdots \otimes A_k$ holds, then $B_1 \parr \cdots \parr B_h$ holds'. Now the proof system of LL^+ is given as follows.

Axioms:

$\Sigma \,; \vec{s} \,; A \vdash A \qquad \Sigma \,; \vec{s} \,; \bot \vdash \qquad \Sigma \,; \vec{s} \,; \vdash 1$

$\Sigma \,; \vec{s} \,; \Gamma \vdash T, \Delta \qquad \Sigma \,; \vec{s} \,; 0, \Gamma \vdash \Delta$

Inference rules for additive connectives:

$$\frac{\Sigma \,; \vec{s} \,; A, \Gamma \vdash \Delta}{\Sigma \,; \vec{s} \,; A \,\&\, B, \Gamma \vdash \Delta} \,\&: l_1 \qquad \frac{\Sigma \,; \vec{s} \,; B, \Gamma \vdash \Delta}{\Sigma \,; \vec{s} \,; A \,\&\, B, \Gamma \vdash \Delta} \,\&: l_2$$

$$\frac{\Sigma \,; \vec{s} \,; \Gamma \vdash \Delta, A \quad \Sigma \,; \vec{s} \,; \Gamma \vdash \Delta, B}{\Sigma \,; \vec{s} \,; \Gamma \vdash \Delta, A \,\&\, B} \,\&: r$$

[1] This extended linear logic might be a solution to a research problem which Artemov presented on his webpage: http://web.cs.gc.cuny.edu/ sartemov/.

[2] We could permit terms themselves as formulas as in the case of Hybrid logic. In addition, we could permit terms and $\le$-formulas to construct formulas and then the proof system would become more complicated. We will investigate such systems elsewhere.

$$\frac{\Sigma\,;\vec{s}\,;A,\Gamma\vdash\Delta \quad \Sigma\,;\vec{s}\,;B,\Gamma\vdash\Delta}{\Sigma\,;\vec{s}\,;A\oplus B,\Gamma\vdash\Delta}\ \oplus:l$$

$$\frac{\Sigma\,;\vec{s}\,;\Gamma\vdash\Delta,A}{\Sigma\,;\vec{s}\,;\Gamma\vdash\Delta,A\oplus B}\ \oplus:r_1 \qquad \frac{\Sigma\,;\vec{s}\,;\Gamma\vdash\Delta,B}{\Sigma\,;\vec{s}\,;\Gamma\vdash\Delta,A\oplus B}\ \oplus:r_2$$

Inference rules for multiplicative connectives:

$$\frac{\Sigma\,;\vec{s}\,;A,B,\Gamma\vdash\Delta}{\Sigma\,;\vec{s}\,;A\otimes B,\Gamma\vdash\Delta}\ \otimes:l \qquad \frac{\Sigma_0\,;\vec{s}\,;\Gamma_0\vdash\Delta_0,A \quad \Sigma_1\,;\vec{t}\,;\Gamma_1\vdash\Delta_1,B}{\Sigma_0,\Sigma_1\,;\vec{s},\vec{t}\,;\Gamma_0,\Gamma_1\vdash\Delta_0,\Delta_1,A\otimes B}\ \otimes:r$$

$$\frac{\Sigma_0\,;\vec{s}\,;A,\Gamma_0\vdash\Delta_0 \quad \Sigma_1\,;\vec{t}\,;B,\Gamma_1\vdash\Delta_1}{\Sigma_0,\Sigma_1\,;\vec{s},\vec{t}\,;A\parr B,\Gamma_0,\Gamma_1\vdash\Delta_0,\Delta_1}\ \parr:l \qquad \frac{\Sigma\,;\vec{s}\,;\Gamma\vdash\Delta,A,B}{\Sigma\,;\vec{s}\,;\Gamma\vdash\Delta,A\parr B}\ \parr:r$$

$$\frac{\Sigma_0\,;\vec{s}\,;\Gamma_0\vdash\Delta_0,A \quad \Sigma_1\,;\vec{t}\,;B,\Gamma_1\vdash\Delta_1}{\Sigma_0,\Sigma_1\,;\vec{s},\vec{t}\,;A\multimap B,\Gamma_0,\Gamma_1\vdash\Delta_0,\Delta_1}\ \multimap:l \qquad \frac{\Sigma\,;\vec{s}\,;A,\Gamma\vdash\Delta,B}{\Sigma\,;\vec{s}\,;\Gamma\vdash\Delta,A\multimap B}\ \multimap:r$$

$$\frac{\Sigma\,;\vec{s}\,;A,\Gamma\vdash\Delta}{\Sigma\,;\vec{s}\,;\Gamma\vdash\Delta,A^{\perp}}\ \perp:r \qquad \frac{\Sigma\,;\vec{s}\,;\Gamma\vdash\Delta,A}{\Sigma\,;\vec{s}\,;A^{\perp},\Gamma\vdash\Delta}\ \perp:l$$

Inference rules for modalities:

$$\frac{\Sigma\,;\quad\vec{t}\,;\quad A,\Gamma\vdash\Delta}{\Sigma\,;\vec{s},\vec{t}\,;(+\vec{s}):A,\Gamma\vdash\Delta}\ \vec{s}:l \qquad \frac{\Sigma\,;\vec{s}\,;\vec{u}:\Gamma\vdash\quad B}{\Sigma\,;\quad\vec{u}:\Gamma\vdash(+\vec{s}):B}\ \vec{s}:r$$

$$\frac{\Sigma\,;\vec{s}\,;t:A,t:A,\Gamma\vdash\Delta}{\Sigma\,;\vec{s}\,;\quad t:A,\quad\Gamma\vdash\Delta}\ \mathit{contraction} \qquad \frac{\Sigma\,;\vec{s}\,;\quad\Gamma\vdash\Delta}{\Sigma\,;\vec{s}\,;t:A,\Gamma\vdash\Delta}\ \mathit{weakening}$$

$$\frac{\Sigma\,;!,\vec{s}\,;\Gamma\vdash\Delta}{\Sigma\,;\quad\vec{s}\,;\Gamma\vdash\Delta}\ !-\mathit{elimination} \qquad \frac{\Sigma\,;\quad\vec{s}\,;\Gamma\vdash\Delta}{\Sigma\,;t,\vec{s}\,;\Gamma\vdash\Delta}\ \mathit{weakening}$$

$$\frac{\Sigma\,;t,\vec{s}\,;\Gamma\vdash\Delta}{t\leq u,\Sigma\,;u,\vec{s}\,;\Gamma\vdash\Delta}\ \leq$$

Inference rules for constants and cut:

$$\frac{\Sigma\,;\vec{s}\,;\Gamma\vdash\Delta}{\Sigma\,;\vec{s}\,;\Gamma\vdash\Delta,\perp}\ \perp \qquad \frac{\vec{s}\,;\Sigma\,;\quad\Gamma\vdash\Delta}{\Sigma\,;\vec{s}\,;1,\Gamma\vdash\Delta}\ 1$$

$$\frac{\Sigma_0\,;\vec{s}\,;\Gamma_0\vdash\Delta_0,A \quad \Sigma_1\,;\vec{t}\,;A,\Gamma_1\vdash\Delta_1}{\Sigma_0,\Sigma_1\,;\vec{s},\vec{t}\,;\Gamma_0,\Gamma_1\vdash\Delta_0,\Delta_1}\ \mathit{cut}$$

Here $\vec{s}$ denotes a finite sequence of terms and $+\vec{s}$ denotes the sum of the terms regardless of the order of them; a capital greek letter $(\Gamma,\Delta,\ldots)$ denotes a multiset of formulas; $\vec{s}:\Gamma$ denotes a multiset of the form $(s_1:A_1,\ldots,s_n:A_n)$. Below, when $\vec{s}$ or Σ is empty in a sequent, we often omit the accompanying semicolons. We show some proofs of theorems of LL^{+}:

$$\cfrac{\cfrac{\cfrac{\cfrac{A\vdash A}{s\,;s:A\vdash A}\ s:l \quad \cfrac{B\vdash B}{t\,;t:B\vdash B}\ t:l}{s,t\,;s:A,t:B\vdash A\otimes B}\ \otimes:r}{s:A,t:B\vdash(s+t):(A\otimes B)}\ (s+t):r}{s:A\otimes t:B\vdash(s+t):(A\otimes B)}\ \otimes:l$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{s\,;s : A \vdash A}\,s:l}{s \le t\,;t\,;s : A \vdash A}\,\le}{s \le t\,;t\,;s : A, t : B \vdash A}\,weak. \qquad \dfrac{\dfrac{s \le t\,;B \vdash B}{s \le t\,;t\,;t : B \vdash B}\,t:l}{s \le t\,;t\,;s : A, t : B \vdash B}\,weak.}{s \le t\,;t\,;s : A, t : B \vdash A \,\&\, B}\,\&:r}{s \le t\,;s : A, t : B \vdash t : (A \,\&\, B)}\,t:r}{s \le t\,;s : A \otimes t : B \vdash t : (A \,\&\, B)}\,\otimes:l
$$

Other theorems of LL^+ concerning modalities include: $s : A \multimap t : s : A$, $t : s : A \multimap (s+t) : A$, $(s+t) : A \multimap (t+s) : A$, $s : A \multimap (s+s) : (A \otimes A)$. Note that the usual rules for modality ! are derivable ones:

$$
\dfrac{A, \Gamma \vdash B}{!A, \Gamma \vdash B}\ !:l \qquad \dfrac{!\Gamma \vdash B}{!\Gamma \vdash !B}\ !:r.
$$

§3 Phase semantics for LL^+

In this section, we give a denotational semantics, phase semantics, for LL^+. First we review phase semantics for LL. (Cf. [11], [12].) An *phase space* is a triple $(P, D, \perp)$ where P is a commutative monoid with the unit 1; D is a subset of the powerset of P with $\perp \in D$ and $P \in D$. For α, $\beta \in D$, we use the following notations.

- $\alpha \,\&\, \beta = \alpha \cap \beta$
- $\alpha \oplus \beta = cl(\alpha \cup \beta)$
- $\alpha \otimes \beta = cl(\alpha \cdot \beta)$
- $\alpha \multimap \beta = \{c \mid \alpha \cdot c \subseteq \beta\}$ [3]
- $\mathbf{1} = cl\{1\}$
- $\alpha^\perp = \alpha \multimap \perp$
- $\alpha \parr \beta = (\alpha^\perp \otimes \beta^\perp)^\perp$

Here $cl(\alpha)$ (*the closure of* α) is defined as $\bigcap\{\gamma | \alpha \subseteq \gamma, \gamma \in D\}$.

We impose the following conditions on D.

(P1) D is closed under arbitrary $\bigcap$, namely, if $\alpha_i \in D$ for any $i \in \Lambda$, then $\bigcap_{i\in\Lambda} \alpha_i \in D$, where Λ is any index set.

(P2) For any α, $\beta \in D$, $\alpha \multimap \beta \in D$.

Note that $cl(\alpha)$ is always an element of D by (P1).

A *topolinear space* is a quadruple $(P, D, \perp, \mathcal{O})$ where $(P, D, \perp)$ is a phase space; $\mathcal{O}$ is a subset of D satisfying the following conditions.

(O1) $\mathbf{1} \in \mathcal{O}$, and $\alpha \in \mathcal{O}$ implies $\alpha \subseteq \mathbf{1}$.

(O2) $\mathcal{O}$ is closed under arbitrary $\oplus$, namely, if $\alpha_i \in \mathcal{O}$ for any $i \in \Lambda$, then $\oplus_{i\in\Lambda}\alpha_i \in \mathcal{O}$, where Λ is any index set.

(O3) $\mathcal{O}$ is closed under finite $\otimes$, namely, for any α, $\beta \in \mathcal{O}$, $\alpha \otimes \beta \in \mathcal{O}$.

(O4) For any $\alpha \in \mathcal{O}$, $\alpha = \alpha \otimes \alpha$.

[3] In this paper, we use the notation $\alpha \cdot c$, $c \cdot \alpha$, $\alpha \cdot \beta$ to mean $\{a \cdot c \mid a \in \alpha\}$, $\{c \cdot a \mid a \in \alpha\}$, $\{a \cdot b \mid a \in \alpha, b \in \beta\}$, respectively.

We call an element of D *fact* and that of $\mathcal{O}$ *open fact*. (O1) says that $\mathbf{1}$ is the maximal open fact. Below, by $in(\alpha)$ we mean *the interior of* α, namely, the maximum open fact included in α.

Now we define an *extended topolinear space* $(P, D, \perp, \mathcal{O}, \mathcal{N})$ as a topolinear space with the set $\mathcal{N}$ of natural numbers, satisfying the following conditions (N1)-(N5). [4]

(N1) For any $\alpha \in \mathcal{O}$, α has a natural number as its *degree*, denoted by $\delta(\alpha)$.
(N2) For any $\alpha, \beta \in \mathcal{O}$, $\alpha \subseteq \beta$ implies $\delta(\alpha) = \delta(\beta)$.
(N3) For any $\alpha, \beta \in \mathcal{O}$, $\delta(\alpha \otimes \beta) = \delta(\alpha) + \delta(\beta)$.
Below, by $n : \alpha$ we mean an open fact which includes or is included in $in(\alpha)$.
(N4) For any $\alpha \in D$ and any $n \in \mathcal{N}$, $in(\alpha) = n : \alpha$ if and only if $\delta(in(\alpha)) = n$.
(N5) For any $\alpha \in D$ and any $n, m \in \mathcal{N}$, $n \leq m$ implies $n : \alpha \subseteq m : \alpha$.

Then we define the interpretations $*$ of terms and formulas of LL^+ on an extended topolinear space.

- $!^* = 0$.
- $s^* \in \mathcal{N}$, for a constant s.
- $(s + t)^* = s^* + t^*$.
- $(s \leq t)^* = s^* \leq t^*$; $p^* \in D$, for an atomic formula p.
- $(A \natural B)^* = A^* \natural B^*$, where $\natural$ is &, $\otimes$, $\oplus$, $\multimap$ or $\parr$.
- $(A^\perp)^* = (A^*)^\perp$.
- $(s : A)^* = s^* : A^*$.

Here $+$ in $s^* + t^*$ and $\leq$ in $s^* \leq t^*$ mean the addition and the inequality in the standard model of arithmetic, respectively. Below we consider only injective interpretations; if α and β are distinct constants or atomic formulas α^* and β^* are distinct.

We impose one condition on $*$: for any formulas A, B and any term t, $\delta(in(B^* \multimap C^*)) \leq t^*$ implies $\delta(in(C^*)) \leq t^*$.

Now by *a phase model* or, simply, *a model,* we understand an extended topolinear space together with an interpretation $*$ on it. Let $T \equiv \Sigma\ ;\ \vec{s}\ ;\ \Gamma \vdash \Delta$ be any sequent of LL^+. T is true in a model $\mathcal{M}$ with some assignment δ of degrees to open facts in it (in symbols, $\mathcal{M}, \delta \models T$) if and only if some arithmetical sentence in Σ^* is false, or the following conditions (C1) and (C2) are satisfied with some δ:

(C1) $\mathbf{1} \otimes (\otimes\Gamma)^* \subseteq (\parr\Delta)^*$
(C2) $\delta(\mathbf{1}) + \delta(in(\otimes\Gamma)^*) = \delta(in(\parr\Delta)^*) \leq (+\vec{s})^*$.

(C1) is the same as the truth condition in the case of phase semantics for LL. Note that the assignment functions assign degrees to occurrences of formulas in sequents, not to formulas themselves. Here we shall list some useful facts.

($\flat$1) $in(\alpha) \otimes in(\beta) \subseteq in(\alpha \otimes \beta)$. Because: by (O3), $in(\alpha) \otimes in(\beta) \in \mathcal{O}$. So $in(\alpha) \otimes in(\beta) = in(in(\alpha) \otimes in(\beta))$. On the other hand, as $in(\alpha) \subseteq \alpha$ and $in(\beta) \subseteq \beta$, $in(in(\alpha) \otimes in(\beta)) \subseteq in(\alpha \otimes \beta)$.

[4] We can put any well-ordered set in the place of the set of natural number in the definition. However, we consider the latter for intuitive readability.

($\flat$2) $\delta(in(\alpha \multimap \beta)) + \delta(in(\alpha)) = \delta(in(\beta))$. Because: by ($\flat$1), $in(\alpha \multimap \beta) \otimes in(\alpha) \subseteq in((\alpha \multimap \beta) \otimes \alpha)$. On the other hand, as $(\alpha \multimap \beta) \otimes \alpha \subseteq \beta$, $in((\alpha \multimap \beta) \otimes \alpha) \subseteq in(\beta)$. Therefore, $in(\alpha \multimap \beta) \otimes in(\alpha) \subseteq in(\beta)$. So by (N2), $\delta(in(\alpha \multimap \beta) \otimes in(\alpha)) = \delta(in(\beta))$ and by (N3), $\delta(in(\alpha \multimap \beta)) + \delta(in(\alpha)) = \delta(in(\beta))$.

Note that, by ($\flat$1), $\delta(in(\alpha)) + \delta(in(\beta)) = \delta(in(\alpha \otimes \beta))$. Below, by $\vdash_L^{(cf)} T$, we mean that a sequent T is provable in a proof system L (without using the cut rule). And we omit the description of $\leq$-formulas, when they are not essential in inferences.

The aim in this section is to show the following.

Theorem 3. 1 *(Soundness for LL^+) For any sequent T of LL^+ and any model $\mathcal{M}$,*
$\vdash_{LL^+} T$ *implies* $\mathcal{M}, \delta \models \mathcal{S}$ *with some* δ.

To prove this, we introduce the intuitionistic version ILL^+ of LL^+. The language of ILL^+ does not contain $⅋$. The sequents are defined in a similar way to the case of LL^+ except that the number of formulas on the right-hand side of sequents to one. The proof system of ILL^+ is obtained from LL^+ (i) by removing the inference rules for $⅋$ and (ii) by restricting the number of formulas on the right-hand side of sequents to one. We offer the proof system of ILL^+ in Appendix. For any sequent $T \equiv \Sigma\, ; \vec{s}\, ; \Gamma \vdash \Delta$ of LL^+, let T^i denote $\Sigma\, ; \vec{s}\, ; \Gamma \vdash (\otimes(\Delta^\perp))^\perp$. Then we have the following.

Proposition 3. 2 *Let T be any sequent of LL^+. Then,*
1. $\vdash_{LL^+} T$ *if and only if* $\vdash_{ILL^+} T^i$,
2. $\vdash^{cf}_{LL^+} T$ *if and only if* $\vdash^{cf}_{ILL^+} T^i$,
3. *for any assignment δ, we have:* $\mathcal{M}, \delta \models T$ *if and only if* $\mathcal{M}, \delta \models T^i$.

This Proposition 3.2 can be easily proved. So it is sufficient for Theorem 3.1 to prove the following.

Theorem 3. 3 *(Soundness for ILL^+) For any sequent S of ILL^+ and any model $\mathcal{M}$,*
$\vdash_{ILL^+} S$ *implies* $\mathcal{M}, \delta \models \mathcal{S}$ *with some* δ.

An assignment δ of degrees is said *suitable for* a sequent $\vec{s}\, ; \Gamma \vdash B$, when (1) δ satisfies the condition (C2) above and (2) for any $A \in \Gamma \cup \{\mathbf{1}\}$ and any natural number n, δ can be updated to another δ' so that (i) $\delta'(in(A^*)) = \delta(in(A^*)) + n$ and (ii) either $\delta'(in(\otimes\Gamma^-)^*) = \delta(in(\otimes\Gamma^-)^*) - n$ or $\delta'(in(B^*)) = \delta(in(B^*)) + n$, where Γ^- is $\Gamma \cup \{\mathbf{1}\}$ minus (one occurrence of) A.

We show the following lemma.

Lemma 3. 4 *For any sequent S, if $\vdash_{ILL^+} S$, then there is a suitable assignment δ for S.*

(Proof) Let P be a proof of $S \equiv \vec{s}\, ; A, \Gamma \vdash D$ in ILL^+ and δ be any assignment. We proceed by induction on the length of P.

(Base Case)

(i) P is an axiom of the form $\vec{s}\, ; A^0 \vdash A^1$. Here A^0 and A^1 are the same formulas but denote distinct occurrences. For all occurrences of atomic formulas p in A^0 and A^1, we can put $\delta(in(p)^*) = 0$ so that $\delta(\mathbf{1}) = \delta(in(A^0)^*) = \delta(in(A^1)^*) = 0$. Then, for a positive occurrence of an atomic p in A^0, we can increase $\delta(in(p^*))$ by n (any natural number). And for the corresponding occurrence of p in A^1, we can also increase $\delta(in(p^*))$ by n. Thus updated δ' satisfies $\delta'(in(A^0)^*) = \delta'(in(A^1)^*) = n$.

(ii) P is an axiom of the form $\vec{s}\,;\, \vdash 1$. We can put $\delta(\mathbf{1}) = \delta(\mathbf{1}) = 0$. We can update the assignment so that $\delta'(\mathbf{1}) = \delta'(\mathbf{1}) = n$.

The proofs for the other axioms are similar.

(Induction Step) We distinguish cases according to the last inference I in P. Let S_0 and S_1 be the left and right upper sequents of I, respectively.

(Case 1) I is $\multimap : l$.

$$\frac{\vec{s}\,; A, \Gamma \vdash D \quad \vec{t}\,; E, \Delta \vdash F}{\vec{s}, \vec{t}\,; A, D \multimap E, \Gamma, \Delta \vdash F} \multimap : l$$

We have three cases.

(i) If we increase the degree of $in(A^*)$ by n. We treat only the case when the formula A comes from S_0. The other case can be treated similarly.

By the induction hypotheses for S_0 and S_1, $\delta_0(\mathbf{1}) + \delta_0(in(A^*)) + \delta_0(in(\otimes\Gamma^*)) = \delta_0(in(D^*)) \leq (+\vec{s})^*$ and $\delta_1(\mathbf{1}) + \delta_1(in(E^*)) + \delta_1(in(\otimes\Delta^*)) = \delta_1(in(F^*)) \leq (+\vec{t})^*$. Let δ be the composition of δ_0 and δ_1 except that $\delta(\mathbf{1}) = \delta_0(\mathbf{1}) + \delta_1(\mathbf{1})$. Then

$\delta(\mathbf{1}) + \delta(in(A^*)) + \delta(in(\otimes\Gamma^*)) + \delta(in(E^*)) + \delta(in(\otimes\Delta^*)) = \delta(in(D^*)) + \delta(in(F^*))$.

So, by ($\flat$2),

$\delta(\mathbf{1}) + \delta(in(A^*)) + \delta(in(\otimes\Gamma^*)) + \delta(in(D \multimap E)^*) + \delta(in(\otimes\Delta^*)) = \delta(in(F^*)) \leq (+\vec{t})^*$.

Moreover, by the induction hypothesis for S_0, δ can be updated to δ':

$\delta'(\mathbf{1}) + \delta'(in(\otimes\Gamma^*)) = \delta(\mathbf{1}) + \delta(in(\otimes\Gamma^*)) - n$ or $\delta'(in(D^*)) = \delta(in(D^*)) + n$.

Then, in either case, $\delta'(\mathbf{1}) + \delta'(in(D \multimap E)^*) + \delta'(in(\otimes\Gamma^*)) + \delta'(in(\otimes\Delta^*)) = \delta(\mathbf{1}) + \delta(in(D \multimap E)^*) + \delta(in(\otimes\Gamma^*)) + \delta(in(\otimes\Delta^*)) - n$.

(ii) If we increase the degree of $in((D \multimap E)^*)$ by n, we can increase $\delta(in(E^*))$ by n. For simplicity, we may assume $A \cup \Gamma = \Gamma$. Then, in the same way as above, we obtain: $\delta(\mathbf{1}) + \delta(in(D \multimap E)^*) + \delta(in(\otimes\Gamma^*)) + \delta(in(\otimes\Delta^*)) = \delta(in(F^*)) \leq (+\vec{t})^*$.

Moreover, by the induction hypothesis for S_1, $\delta'(\mathbf{1}) + \delta'(in(\otimes\Delta^*)) = \delta(\mathbf{1}) + \delta(in(\otimes\Delta^*)) - n$ or $\delta'(in(F^*)) = \delta(in(F^*)) + n$.

In the former case, $\delta'(\mathbf{1}) + \delta'(in(\otimes\Gamma^*)) + \delta'(in(\otimes\Delta^*)) = \delta(\mathbf{1}) + \delta(in(\otimes\Gamma^*)) + \delta(in(\otimes\Delta^*)) - n$.

(iii) If we increase the degree of $\mathbf{1}$, the proof is similar.

(Case 2) I is weakening.

$$\frac{\vec{s}\,; \Gamma \vdash E}{\vec{s}\,; t : D, \Gamma \vdash E} \; weak.$$

We have three cases.

(i) If we increase the degree of $(t : D)^*$. By the induction hypothesis for S_0, $\delta_0(\mathbf{1}) + \delta_0(in(\otimes\Gamma^*)) = \delta_0(in(E^*)) \leq (+\vec{s})^*$, where δ_0 can be updated for the degree of $\mathbf{1}$ to be increased.

Let δ be the same as δ_0 except that $\delta(t : D)^* = \delta_0(\mathbf{1})$ and $\delta(\mathbf{1}) = 0$. Then $\delta(\mathbf{1}) + \delta(t : D)^* + \delta(in(\otimes\Gamma^*)) = \delta(in(E^*)) \leq (+\vec{s})^*$

Moreover, by using the induction hypothesis, we can update δ to δ' so that either

$\delta'(in(\otimes\Gamma^*)) = \delta(in(\otimes\Gamma^*)) - n$ or $\delta'(in(E^*)) = \delta(in(E^*)) + n$.

In the former case, we can add $\delta'(\mathbf{1})$ and $\delta(\mathbf{1})$ which are 0. Then

$\delta'(\mathbf{1}) + \delta'(in(\otimes\Gamma^*)) = \delta(\mathbf{1}) + \delta(in(\otimes\Gamma^*)) - n$.

(ii) If we increase the degree of $\mathbf{1}$ by n. The proof is made by modifying the above case in the definition of δ as follows: 'Let δ be the same as δ_0 except that $\delta(\mathbf{1}) = \delta_0(\mathbf{1})$ and $\delta(t : D)^* = 0$.'

(iii) If we increase the degree of an occurrence of formula in Γ, the proof is similar to the above cases.

(Case 3) I is contraction.

$$\frac{\vec{s}\,;t:D^0,t:D^1,\Gamma\vdash E}{\vec{s}\,;t:D,\Gamma\vdash E}\ cont.$$

We have three cases.

(i) If we increase the degree of $(t:D)^*$. By the induction hypothesis, $\delta_0(\mathbf{1})+\delta_0(t:D^0)^*+\delta_0(t:D^1)^*+\delta_0(in(\otimes\Gamma^*))=\delta_0(in(E^*))\leq(+\vec{s})^*$, where δ_0 can be updated for the degree of $(t:D^0)^*$ to be increased.

Let δ be the same as δ_0 except that $\delta(t:D)^*=\delta_0(t:D^0)^*+\delta_0(t:D^1)^*$. Then $\delta(\mathbf{1})+\delta(t:D)^*+\delta(in(\otimes\Gamma^*))=\delta(in(E^*))\leq(+\vec{s})^*$

Moreover, by the induction hypothesis for S_0, δ_0 can be updated so that we increase the degree of $(t:D^0)^*\otimes(t:D^1)^*$. Therefore, δ can be updated to δ':

$\delta'(\mathbf{1})+\delta'(in(\otimes\Gamma^*))=\delta(\mathbf{1})+\delta(in(\otimes\Gamma^*))-n$ or $\delta'(in(E^*))=\delta(in(E^*))+n$.

(ii) and (iii). If we increase by n the degree of $\mathbf{1}$ or of an occurrence of formula in Γ, the proof is similar to the above cases.

(Case 4) I is $(+\vec{s}):r$.

$$\frac{\vec{s}\,;\vec{u}:\Gamma\vdash B}{;\vec{u}:\Gamma\vdash(+\vec{s}):B}\ (+\vec{s}):r$$

By the induction hypothesis, we have $\delta(\mathbf{1})+\delta(\otimes(\vec{u}:\Gamma)^*)=\delta(in(B^*))$. By (N2) and (N5), $\delta((s:B)^*)=\delta(in(B^*))$. So, by taking $\delta((u:A)^*)=0$ for each $u:A\in\vec{u}:\Gamma$, $\delta(\mathbf{1})+\delta(\otimes(\vec{u}:\Gamma)^*)=\delta((s:B)^*)=0$. On the other hand, for any $u:A\in\vec{u}:\Gamma$, if we increase $\delta((u:A)^*)$ by n, $\delta((s:B)^*)$ is also increased by n.

Other cases are similar or easier. In particular, for the proofs of the case of inferences for additive connectives or modalities, we can easily make proofs by using the facts $\delta(in(A^*))=\delta(in(A\,\&\,B)^*)=\delta(in(A\oplus B)^*)$ and $\delta(in(A^*))=\delta((s:A)^*)$. These facts are immediately followed from (N2) and (N5). ■

Now we can prove Theorem 3.3.

(Proof of Theorem 3.3) Let S be any sequent and $\mathcal{M}$ be any model. If an $\leq$-formula in S is false in the interpretation of $\mathcal{M}$, S is trivially true in $\mathcal{M}$. So we assume otherwise.

Let P be a proof of S in ILL^+ with the last inference I. We proceed by induction on the length of P. Let $*$ be the interpretation in $\mathcal{M}$. The proof for the part (C2) was given in the previous lemma. The proof for the part (C1) is the same as the usual proof of soundness for ILL except for the cases concerning the inferences of modality. So we present only the proofs of (C1) for those cases.

(Case 1) I is $\vec{s}:l$.

$$\frac{\vec{t}\,;A,\Gamma\vdash B}{\vec{s},\vec{t}\,;(+\vec{s}):A,\Gamma\vdash B}\ \vec{s}:l$$

By the induction hypothesis, $A^*\otimes(\otimes\Gamma^*)\subseteq B^*$. To show $((+\vec{s}):A)^*\otimes(\otimes\Gamma^*)\subseteq B^*$, it suffices to show $((+\vec{s}):A)^*\subseteq in(A^*)$, under some assignment of degrees.

By Lemma 3.1, for some assignment δ suitable for the upper sequent, $\delta(in(A^*))+\delta(in(\otimes\Gamma^*))=\delta(in(B^*))\leq(+\vec{t})^*$. Then we can update δ to δ': $\delta'(in(A^*))=\delta(in(A^*))+(+\vec{s})^*$. Now δ' is suitable for S and $(+\vec{s})^*\leq\delta'(in(A^*))$. So $((+\vec{s}):A)^*\subseteq in(A^*)$.

(Case 2) I is $\vec{s}:r$.

$$\frac{\vec{s}\,;\vec{u}:\Gamma\vdash B}{;\vec{u}:\Gamma\vdash(+\vec{s}):B}\ (+\vec{s}):r$$

By the induction hypothesis, $\otimes(\vec{u}:\Gamma)^*\subseteq B^*$. To show $\otimes(\vec{u}:\Gamma)^*\subseteq((+\vec{s}):B)^*$, it suffices to show $\delta(in(B^*))\leq(+\vec{s})^*$, under some δ. But this is immediate by Lemma 3.4; for some δ suitable for the upper sequent, $\delta(\otimes(\vec{u}:\Gamma)^*)=\delta(in(B^*))\leq(+\vec{s})^*$.
■

§4 Completeness for LL^+

In this section, we establish the completeness theorem for LL^+ with respect to phase semantics. We make use of the method of Okada [8, 9], which is a refinement of Girard's original proof of completeness. It derives also the cut-elimination theorem as a corollary to the completeness theorem.

Theorem 4. 1 *(Strong completeness for LL^+) Let T be any sequent of LL^+. If $\mathcal{M}, \delta \models T$ for any model $\mathcal{M}$ with some δ, then $\vdash^{cf}_{LL^+} T$.*

By Proposition 3.1 in the previous section, it is sufficient for Theorem 4.1 to prove the following.

Theorem 4. 2 *(Strong completeness for ILL^+) Let S be any sequent of ILL^+. If $\mathcal{M}, \delta \models \mathcal{S}$ for any model $\mathcal{M}$ with some δ, then $\vdash^{cf}_{ILL^+} S$.*

Below we fix a sequent $S \equiv \Sigma\,; \vec{s}\,; \Gamma \vdash B$. We prove the theorem by constructing a specific phase model $\mathcal{M}$ such that the truth of S in $\mathcal{M}$ implies the cut-free provability of S in ILL^+.

We suppose that $\bigwedge \Sigma^*$ is true in $\mathcal{M}$. Let ST be the multiset of subterms of the terms occurring in $(+\vec{s}) : (\otimes\Gamma \multimap B)$. And let SF be the multiset of occurrences of subformulas of $(+\vec{s}) : (\otimes\Gamma \multimap B)$, where subformulas of a formula $(t+u) : A$ include $t : A$ and $u : A$. Note that ST and SF are both finite.

Now let P be a commutative monoid consisting of sub-multisets of SF. For Γ, $\Delta \in P$, $\Gamma \cdot \Delta$ is defined as the multiset (Γ, Δ) (the concatenation of Γ and Δ). We can take the empty set as the unit. For any A in SF and any $\vec{t}$ in ST, we define $[A]^{\vec{t}}$ and $[A]$ as follows.

$[A]^{\vec{t}} := \{\Gamma \subseteq SF |$ there is a cut-free proof of $\Sigma\,; \vec{t}\,; \Gamma \vdash A$ in ILL^+ $\}$.

$[A] := \{\Gamma | \Gamma \in [A]^{\vec{t}}$, for some $\vec{t}$ in ST $\}$.

We may assume that 'cut-free proof' in this definition has logical axioms only of the form $\Sigma\,; \vec{s}\,; p \vdash p$ (with p atomic). A fact is defined as a subset of P of the form $\bigcap_{i\in\lambda}[B_i]$, where λ is any index set. An open fact is a fact of the form $\bigcap_{i\in\lambda}[t : B_i]$ with a term t.

This definition surely gives us a topolinear space; the conditions (P1, 2) and (O1-4) are surely satisfied. Here we show the proof for (O3). (The proofs for the other conditions are proved in a similar way.) Let $\bigcap_{i\in\lambda_1}[s : B_i]$ and $\bigcap_{j\in\lambda_j}[t : C_j]$ be any open facts. It suffices to show that $\bigcap_{i\in\lambda_1}[s : B_i] \otimes \bigcap_{j\in\lambda_2}[t : C_j] \subseteq \bigcap_{i\in\lambda_1, j\in\lambda_2}[u : (s : B_i \otimes t : C_j)]$, for any term u in ST.

Suppose $\Gamma \in \bigcap_{i\in\lambda_1}[s : B_i]$ and $\Delta \in \bigcap_{j\in\lambda_2}[t : C_j]$. Fix any $i \in \lambda_1$ and $j \in \lambda_2$. We have the following cut-free proofs.

$$
\dfrac{\overset{(\alpha_0)}{\overline{s}\,; \vec{a}_0 : \Gamma_0 \vdash B_i}}{\vec{a}_0 : \Gamma_0 \vdash s : B_i}\; s : r \qquad \cdots \qquad \dfrac{\overset{(\alpha_k)}{\overline{s}\,; \vec{a}_k : \Gamma_k \vdash B_i}}{\vec{a}_k : \Gamma_k \vdash s : B_i}\; s : r
$$

$$
\begin{array}{c} \ddots \quad \vdots \quad \therefore \\ \vdots \\ \Gamma \vdash s : B_i \end{array}
$$

and

$$
\begin{array}{ccc}
\dfrac{\overset{(\beta_0)}{\overline{t}\,;\vec{b}_0 : \Delta_0 \vdash C_i}}{\vec{b}_0 : \Delta_0 \vdash t : C_j}\ t:r & \cdots & \dfrac{\overset{(\beta_h)}{\overline{t}\,;\vec{b}_h : \Delta_h \vdash C_i}}{\vec{b}_h : \Delta_h \vdash t : C_j}\ t:r
\end{array}
$$

$$
\vdots
$$

$$
\Delta \vdash t : C_j
$$

Here the indicated inferences $s : r$ are the ones introducing $s : B_i$ under consideration; the indicated $t : r$ inferences are similar; when $s \equiv u_1 + \cdots + u_n$, $\overline{s}$ denotes $(u_1 \ldots, u_n)$; $\overline{t}$ is similar. By using these proofs, we make a cut-free proof of $\Gamma, \Delta \vdash u : (s : B_i \otimes t : C_j)$ in the following way. First, combining the subproofs (α_0) and $(\beta_0) - (\beta_h)$, we can obtain cut-free proofs of

$$
\begin{gathered}
\vec{b}_0 : \Delta_0, \vec{a}_0 : \Gamma_0 \vdash u : (s : B_i \otimes t : C_j),\\
\vdots\\
\vec{b}_h : \Delta_h, \vec{a}_0 : \Gamma_0 \vdash u : (s : B_i \otimes t : C_j).
\end{gathered}
$$

Then, using the above second proof, we have a cut-free proof of

$$
\Delta, \vec{a}_0 : \Gamma_0 \vdash u : (s : B_i \otimes t : C_j).
$$

Likewise, we obtain cut-free proofs of

$$
\begin{gathered}
\Delta, \vec{a}_1 : \Gamma_k \vdash u : (s : B_i \otimes t : C_j),\\
\vdots\\
\Delta, \vec{a}_k : \Gamma_k \vdash u : (s : B_i \otimes t : C_j).
\end{gathered}
$$

Finally, using the above first proof, we have a cut-free proof of

$$
\Delta, \Gamma \vdash u : (s : B_i \otimes t : C_j).
$$

Now $\Gamma \cdot \Delta \in [u : (s : B_i \otimes t : C_j)]$. So, $\Gamma \cdot \Delta \in \bigcap_{i \in \lambda_1, j \in \lambda_2}[u : (s : B_i \otimes t : C_j)]$, and we have

$$
\bigcap_{i \in \lambda_1}[s : B_i] \cdot \bigcap_{j \in \lambda_2}[t : C_j] \subseteq \bigcap_{i \in \lambda_1, j \in \lambda_2}[u : (s : B_i \otimes t : C_j)].
$$

In general, when β is a fact, $\alpha \subseteq \beta$ implies $cl(\alpha) \subseteq \beta$. Therefore,

$$
\bigcap_{i \in \lambda_1}[s : B_i] \otimes \bigcap_{j \in \lambda_2}[t : C_j] \subseteq \bigcap_{i \in \lambda_1, j \in \lambda_2}[u : (s : B_i \otimes t : C_j)].
$$

Next we prove the converse inclusion. Let $[D]$ be any fact such that $\bigcap_{i \in \lambda_1}[s : B_i] \cdot \bigcap_{j \in \lambda_2}[t : C_j] \subseteq [D]$. Suppose $\Gamma \in \bigcap_{i \in \lambda_1, j \in \lambda_2}[u : (s : B_i \otimes t : C_j)]$. It suffices to show $\Gamma \in [D]$. Fix any $i \in \lambda_1$ and $j \in \lambda_2$. We have the following cut-free proof.

$$
\begin{array}{ccc}
\dfrac{\overline{u}\,;\vec{a}_0 : \Gamma_0 \vdash s : B_i \otimes t : C_j}{\vec{a}_0 : \Gamma_0 \vdash u : (s : B_i \otimes t : C_j)}\ u:r & \cdots & \dfrac{\overline{u}\,;\vec{a}_k : \Gamma_k \vdash s : B_i \otimes t : C_j}{\vec{a}_k : \Gamma_k \vdash u : (s : B_i \otimes t : C_j)}\ u:r
\end{array}
$$

$$
\vdots
$$

$$
\Gamma \vdash u : (s : B_i \otimes t : C_j)
$$

And, for any $0 \leq l \leq k$, we have the following cut free proof.

$$
\dfrac{\Delta_0^l \vdash s : B_i \quad \Sigma_0^l \vdash t : C_j}{\Delta_0^l, \Sigma_0^l \vdash s : B_i \otimes t : C_j} \otimes : r \qquad \cdots \qquad \dfrac{\Delta_m^l \vdash s : B_i \quad \Sigma_m^l \vdash t : C_j}{\Delta_m^l, \Sigma_m^l \vdash s : B_i \otimes t : C_j} \otimes : r
$$

$$
\vdots
$$

$$
\overline{u}\ ; \vec{a}_l : \Gamma_l \vdash s : B_i \otimes t : C_j
$$

For any $0 \leq n \leq m$, we consider the proof (say, P) above $\Delta_n^l \vdash s : B_i$. In P, there are some $s : r$ inferences introducing $s : B_i$ under consideration, and there is no term in the lower sequents of those $s : r$ inferences. If there are some inferences introducing terms ($\vec{s} : l$ or weakening) between those inferences and $\Delta_n^l \vdash s : B_i$, we can move them below $\otimes : r$ involved. Thus we may assume that Δ_n^l contains no term, as well as Σ_n^l.

Now $\Delta_n^l \in [s : B_i]$ and $\Sigma_n^l \in [t : C_j]$. The above proofs do not depend on i or j; if we take another i' from λ_1 and another j' from λ_2, we have the same proofs as above. So we have $\Delta_n^l \in \bigcap_{i \in \lambda_1}[s : B_i]$ and $\Sigma_n^l \in \bigcap_{j \in \lambda_2}[t : C_j]$. Hence

$$
(\Delta_n^l, \Sigma_n^l) \in \textstyle\bigcap_{i \in \lambda_1}[s : B_i] \cdot \bigcap_{j \in \lambda_2}[t : C_j].
$$

So, $(\Delta_n^l, \Sigma_n^l) \in [D]$, namely, $\Delta_n^l, \Sigma_n^l \vdash D$ is cut-free provable. Then, by using the above proofs, we obtain a cut-free proof of $\Gamma \vdash D$, namely, $\Gamma \in [D]$.

Thus the above definition surely gives a topolinear space. Moreover, we construct an extended topolinear space. Let $\mathcal{N}_0 = \{s^* | s \in ST\}$. Note that $\mathcal{N}_0$ is finite.

For any term s in ST, we set: $s^* : \bigcap_{i \in \lambda}[B_i] := \bigcap_{i \in \lambda}[s : B_i]$. Note that (i) $\delta(\bigcap_{i \in \lambda}[s : B_i]) = s^*$ and (ii) for any fact $\bigcap_{i \in \lambda}[C_i]$, its interior exists uniquely, and so does the degree of it, as $\mathcal{N}_0$ is finite.

Now we may assume that the conditions (N1-4) are satisfied. For (N5), it suffices to show that $s^* \leq t^*$ implies $\bigcap_{i \in \lambda}[s : B_i] \subseteq \bigcap_{i \in \lambda}[t : B_i]$. This is easily proved by the method of transforming cut-free proofs. (Note that $s \leq t \in \Sigma$.) Thus the specific model above defined is surely an extended topolinear space. Finally, for atomic p, we put $p^* = [p]$. This completes the definition of the specific model $\mathcal{M}$.

Lemma 4. 3 *Let A be any formula in SF and $\vec{t}$ be any terms in ST.*
Then $\delta(in(A^*)) \leq (+\vec{t})^*$ implies $A \in A^* \subseteq [A]^{\vec{t}}$.

By this lemma, we obtain the completeness theorem: Suppose that a sequent $S \equiv .\Sigma\ ; \vec{s}\ ; \Gamma \vdash B$ is true in any model such that $\bigwedge \Sigma^*$ is true, with an assignment of degrees. Then S is also true in the specific model $\mathcal{M}$ such that $\bigwedge \Sigma^*$ is true, with an assignment δ of degrees. Now $\delta(in(\otimes\Gamma \multimap B)^*) \leq (+\vec{s})^*$ and $1 \in (\otimes\Gamma \multimap B)^*$. By the lemma, $1 \in [\otimes\Gamma \multimap B]^{\vec{s}}$. So $\Sigma\ ; \vec{s}\ ; \Gamma \vdash B$ is cut-free provable in ILL^+.

(Proof of Lemma 4.2) Suppose $\delta(in(A^*)) \leq (+\vec{t})^*$. We proceed by induction on the length of A.

(Base Case) When A is an atomic formula p. By definition of p^*, it is clear that $p \in p^*$. Suppose that $\Gamma \in [p]$. Then we have a cut-free proof of $\vec{u}\ ; \Gamma \vdash p$, with some $\vec{u}$ in ST. If $(+\vec{t})^* \leq (+\vec{u})^*$, then we construct the following cut-free proof, with any $\vec{v}$ in ST,

$$
\dfrac{\dfrac{\vec{u}\ ; \Gamma \vdash p}{\vec{v}, \vec{u}\ ; \vec{v} : \Gamma \vdash p}\ \vec{v} : l}{\vec{v} : \Gamma \vdash ((+\vec{v}) + (+\vec{u})) : p}\ (\vec{v}, \vec{u}) : r
$$

and $[((+\vec{v}) + (+\vec{u})) : p]$ is not empty. This contradicts that $\delta(in(p^*)) \le (+\vec{t})^*$, as $\delta(in(p^*)) = \delta([((+\vec{v}) + (+\vec{u})) : p]) = (+\vec{u})^* + (+\vec{v})^* \ge (+\vec{t})^*$. Therefore, we have $(+\vec{u})^* \le (+\vec{t})^*$ and we may assume that Σ contains $(+\vec{u}) \le (+\vec{t})$.

(Induction Step) We distinguish cases according to the form of A.

(Case 1) A is of the form $B \multimap C$. First we show $B \multimap C \in B^* \multimap C^*$. Suppose $\Gamma \in B^*$. By the induction hypothesis, $B^* \subseteq [B]^{\vec{u}}$ with some $\vec{u}$. So $\Gamma \in [B]^{\vec{u}}$. On the other hand, put $C^* = \bigcap_i [D_i]$. By the induction hypothesis, $C \in C^*$ and so, $C \in \bigcap_i [D_i]$. For any i, we can construct the following cut free proof, with some $\vec{v}$ in ST.

$$\frac{\vec{u}\,; \Gamma \vdash B \quad \vec{v}\,; C \vdash D_i}{\vec{u}, \vec{v}\,; B \multimap C, \Gamma \vdash D_i} \multimap : l$$

Therefore, we obtain $B^* \cdot (B \multimap C) \subseteq C^*$, namely, $B \multimap C \in B^* \multimap C^*$. Next we show $B^* \multimap C^* \subseteq [B \multimap C]^{\vec{t}}$. As $\delta(in(B^* \multimap C^*)) \le (+\vec{t})^*$, $\delta(in(C^*)) \le (+\vec{t})^*$ by the definition of $*$. Suppose $\Gamma \in B^* \multimap C^*$. Then $B^* \cdot \Gamma \subseteq C^*$. By the induction hypothesis, $B \in B^*$ and $C^* \subseteq [C]^{\vec{t}}$. So we have cut-free proofs of $\vec{t}\,; B, \Gamma \vdash C$ and so, of $\vec{t}\,; \Gamma \vdash B \multimap C$. Hence, $\Gamma \in [B \multimap C]^{\vec{t}}$.

(Case 2) A is of the form $u : B$. First we show $u : B \in u^* : B^*$. Put $B^* = \bigcap_{i\in\lambda} [D_i]$. By the induction hypothesis, $B \in B^*$, and so, for any $i \in \lambda$, we have a cut-free proof of $\vec{v}\,; B \vdash D_i$ with some $\vec{v}$. Then we show that we can eliminate $\vec{v}$, by the induction on the length of B. When B is an atom, each term in $\vec{v}$ is present in logical axioms above or is introduced by weakening. In either case, we can eliminate $\vec{v}$ in the proof. When B is $b : C$, put $C^* = \bigcap_{i\in\lambda}[E_i]$. Then $B^* = b^* : C^* = \bigcap_{i\in\lambda}[b : E_i]$. By the induction hypothesis, we have cut-free proofs of $; C \vdash E_i$ and so, of $; b : C \vdash b : E_i$, which is a desirable sequent. Other cases are similar.

Thus we have cut-free proofs of $; B \vdash D_i$ and of $; u : B \vdash u : D_i$. So $u : B \in \bigcap_{i\in\lambda}[u : D_i] = u^* : \bigcap_{i\in\lambda}[D_i] = u^* : B^*$.

Next we show that $u^* : B^* \subseteq [u : B]^{\vec{t}}$. Suppose $\Gamma \in u^* : B^*$. Then we have a cut-free proof (say, P) of $\vec{v}\,; \Gamma \vdash u : D_i$. In P, there occur $u : r$ inferences introducing $u : D_i$.

$$\begin{array}{ccc} (\alpha_0) & & (\alpha_k) \\ \dfrac{u\,; \vec{a}_0 : \Gamma_0 \vdash D_i}{\vec{a}_0 : \Gamma_0 \vdash u : D_i}\, u : r & \cdots & \dfrac{u\,; \vec{a}_k : \Gamma_k \vdash D_i}{\vec{a}_k : \Gamma_k \vdash u : D_i}\, u : r \\ & \vdots & \\ & \vec{v}\,; \Gamma \vdash u : D_i & \end{array}$$

Now, for each $0 \le n \le k$, $\vec{a}_n : \Gamma_n \in \bigcap_{i\in\lambda}[D_i] = B^*$. So, by the induction hypothesis, $\vec{a}_n : \Gamma_n \in [B]^{\vec{t}}$. Therefore, we have cut-free proofs of $\vec{t}\,; \vec{a}_n : \Gamma_n \vdash B$ and so, of $\vec{t}\,; \vec{a}_n : \Gamma_n \vdash u : B$. Then replace each subproof α_n in P by the proof of $\vec{t}\,; \vec{a}_n : \Gamma_n \vdash u : B$ to obtain a cut-free proof of $\vec{t}, \vec{v}\,; \Gamma \vdash u : B$. If $\vec{v}$ is not empty (and not $\vec{!}$), then $[(+\vec{t}) + (+\vec{v}) : u : B]$ is not empty and neither is $[(+\vec{t}) + (+\vec{v}) + u : B]$, as $[(+\vec{t}) + (+\vec{v}) : u : B] \subseteq [(+\vec{t}) + (+\vec{v}) + u : B]$.

On the other hand, by the induction hypothesis, $B^* \subseteq [B]^{\vec{t}} \subseteq [B]$. So $\delta(in([B])) \le (+\vec{t})^*$, by (N2). Therefore, it is a contradiction that $[(+\vec{t}) + (+\vec{v}) + u : B]$ is not empty. Thus $\vec{v}$ is empty and we have $\Gamma \in [u : B]^{\vec{t}}$.

Other cases are proved in the similar way to the case of ILL. ∎

Now, by combining the soundness and the strong completeness theorems, we obtain the cut-elimination theorem for LL^+.

Corollary 4. 4 *(Cut-elimination for LL^+) For any sequent T,*
$\vdash_{LL^+} T$ *implies* $\vdash^{cf}_{LL^+} T$.

Consider a sequent $T^!$ containing only ! as modality. In a cut-free proof of $T^!$ in LL^+, if any, there occurs no terms other than ! in the proof, as $T^!$ contains only !. So the proof is also that in LL. Therefore we have: $\vdash^{cf}_{LL^+} T^!$ implies $\vdash^{cf}_{LL} T^!$. By combining this and Corollary 4.4, $\vdash_{LL^+} T^!$ implies $\vdash^{cf}_{LL} T^!$. Then we obtain the following two corollaries.

Corollary 4. 5 *(Conservativity of LL^+ over LL) For any sequent $T^!$ containing only ! as modality,*
$\vdash_{LL^+} T^!$ *implies* $\vdash_{LL} T^!$.

Corollary 4. 6 *(Cut-eliminataion for LL) For any sequent T of LL,*
$\vdash_{LL} T$ *implies* $\vdash^{cf}_{LL} T$.

Finally we remark that we can introduce an operation on terms $max(*,*)$ and eliminate $\leq$-formulas in the system. Here $max(s,t)$ means the larger one of s and t. The interpretation in phase model is defined as the maximum function in the standard model of arithmetic. Formally speaking, we add the following inference rule on the operation.

$$\frac{\Sigma\ ; t, \vec{s}\ ; \Gamma \vdash \Delta}{\Sigma\ ; max(t,u), \vec{s}\ ; \Gamma \vdash \Delta}\ max$$

Then let us consider a translation $\sharp$ among sequents of LL^+:

$$(\Sigma\ ; \vec{s}\ ; \Gamma \vdash \Delta)^\sharp := (\vec{s})^\sharp\ ; \Gamma^\sharp \vdash \Delta^\sharp$$

Here $(\vec{s})^\sharp$, $\Gamma^\sharp$ and $\Delta^\sharp$ are defined as the result of replacement: for any $t \leq u$ in Σ, if u occurs (i) in $\vec{s}$, (ii) negatively in Γ or (iii) positively in Δ, then replace the occurrence of u by $max(t,u)$. Then we have the following.

Theorem 4. 7 *For any sequent S of LL^+,*
$\vdash_{LL^+} S$ *if and only if* $\vdash_{LL^+} S^\sharp$.

It is easy to prove this and we do not make proof in detail. Instead we only remark that for the transformations between proofs of S and $S^\sharp$, the interchanging the inference rules $\leq$ and max is crucial. Other inferences are preserved through the translation $\sharp$. As an example, we take a proof of $s \leq t\ ; s : A \otimes t : B \vdash t : (A \,\&\, B)$ in §2. From it we can make a proof of $s : A \otimes t : B \vdash max(s,t) : (A \,\&\, B)$ as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{s\ ; s : A \vdash A}\ s:l}{max(s,t)\ ; s : A \vdash A}\ max}{max(s,t)\ ; s : A, t : B \vdash A}\ weak. \qquad \dfrac{\dfrac{\dfrac{B \vdash B}{t\ ; t : B \vdash B}\ t:l}{max(s,t)\ ; t : B \vdash B}\ max}{max(s,t)\ ; s : A, t : B \vdash B}\ weak.}{max(s,t)\ ; s : A, t : B \vdash A \,\&\, B}\ \&:r}{s : A, t : B \vdash max(s,t) : (A \,\&\, B)}\ t:r}{s : A \otimes t : B \vdash max(s,t) : (A \,\&\, B)}\ \otimes:l$$

We could define a variant of the system LL^+ by removing $\leq$-formulas and introducing the operation max. However, these are essentially the same systems, as Theorem 4.7 holds. There is a trade-off between the two systems. In such a variant, the proof system would be simpler owing to the absence of $\leq$-formulas than the original system LL^+. In turn, the original LL^+ would give us an intuitive readability by equipping $\leq$-formulas, as compared with such a variant.

§5 The pure fragment of LL^+

In this section, we focus on an interesting fragment of LL^+, namely, the fragment not containing the modality !. Formally, it is obtained from LL^+ just by removing ! from the definition of terms in §2. We shall call this fragment *the pure fragment* LL^p of LL^+. Some theorems in LL^+ raised in §2 can be viewed as those in LL^p:

$s : A \otimes t : B \vdash (s + t) : (A \otimes B)$,

$s \leq t\ ; s : A \otimes t : B \vdash t : (A \,\&\, B)$.

Moreover, these two theorems can be viewed as the results of labeling ! occurring in the following theorems with terms,

$!A \otimes !B \vdash !(A \otimes B)$,

$!A \otimes !B \vdash !(A \,\&\, B)$.

Now it is natural to ask, in general, how we can label the theorems in LL to obtain those in LL^p. In this section, we solve this question by presenting algorism for such a labeling. More specifically, we show that LL (the ordinary linear logic) can be faithfully embedded in LL^p by a proof-transformation method.

By a *labeling* l of a formula A of LL we mean the replacement of each occurrence of ! in A by a term of LL^p. By A^l we mean the result of the replacement l. For a sequent $\Gamma \vdash \Delta$ of LL, let S^l denote $\Sigma; \vec{s}; \Gamma^l \vdash \Delta^l$ with some Σ and $\vec{s}$. We show the faithful embeddability of LL in LL^p via this notion of labeling.

Theorem 5. 1 *For any sequent* $\Gamma \vdash \Delta$ *of* LL,
$\vdash_{LL} \Gamma \vdash \Delta$ *if and only if* $\vdash_{LL^p} \Sigma; \vec{s}; \Gamma^l \vdash \Delta^l$ *with some* l, Σ *and* $\vec{s}$.

This is an analogous result to Artemov's realization theorem of the modal logic $S4$ in LP (Logic of Proof) ([3], cf. [4]). To prove Theorem 5.1, we use the method by sequent calculi as in [3], [4].

The direction 'if' is easily proved by induction on the length of proof in LL. We treat only the direction 'only if'. A labeling l for a formula A is called *normal* when each negative occurrences of ! in A is replaced by a constant. (This notion of normality is also found in Artemov [3]). A labeling l of a sequent $\Gamma \vdash \Delta$ is *normal* when $(\otimes\Gamma \multimap \invamp \Delta)^l$ is a normal labeling.

(Proof of the direction 'only if' of Theorem 5.1) Let P be a proof of a sequent S in LL. Our aim is to construct a normal labeling l so that S^l is provable in LL^p.

We assume that P is cut-free and proceed by induction on the length of P. When P is an axiom itself, it is obvious. Let I be the last inference in P. We treat only crucial three cases.

(Case 1) I is $! : l$.

$$\frac{B, \Gamma \vdash \Delta}{!B, \Gamma \vdash \Delta}\ ! : l$$

By the induction hypothesis, we have a proof of $\Sigma\ ; \vec{s}\ ; B^l, \Gamma^l \vdash \Delta^l$ with some $\Sigma, \vec{s}, l$. Then, taking a fresh constant c, we construct the following proof.

$$\frac{\Sigma\ ; \vec{s}\ ; B^l, \Gamma^l \vdash \Delta^l}{\Sigma\ ; c, \vec{s}\ ; c : B^l, \Gamma^l \vdash \Delta^l}\ c : l$$

The updated labeling remains normal.

(Case 2) I is $\& : r$.

$$\frac{D, \Gamma \vdash \Delta, B \quad D, \Gamma \vdash \Delta, C}{D, \Gamma \vdash \Delta, B \,\&\, C}\ \& : r$$

By the induction hypotheses, we have proofs of:

$\Sigma_0 \,;\vec{s}\,; D^{l_0}, \Gamma^{l_0} \vdash \Delta^{l_0}, B^{l_0}$ and $\Sigma_1 \,;\vec{t}\,; D^{l_1}, \Gamma^{l_1} \vdash \Delta^{l_1}, C^{l_1}$.

Note that for the indicated formula D, D^{l_0} and D^{l_1} can be distinct. We show that l_0 and l_1 can be unified to l so that $D^l \equiv D^l$, by induction on the number of occurrences of ! in D. (For a formula E in Γ or Δ, we can unify the labeling in a similar way. So we suppose that $\Gamma^{l_0} \equiv \Gamma^{l_1}$ and $\Delta^{l_0} \equiv \Delta^{l_1}$.) Note that a positive (or nagative) occurrence of subformula of D is negative (or positive) in the sequent including D.

Suppose that an outmost positive occurrence of subformula of the form $!E$ in D is labeled as $c_0 : E^l$ and $c_1 : E^l$ in D^{l_0} and D^{l_1}, respectively. Then, taking a fresh constant c, unify the labeling of $!E$ as $c : E^l$ and replace all occurrences of c_0 and c_1 by c in the proof.

Next suppose that an outmost negative occurrence of subformula $!E$ in D (denoted by $D(!E)$) is labeled as $t_0 : E^l$ and $t_1 : E^l$ in D^{l_0} and D^{l_1}, respectively. It is easily checked that $D((t_0 + t_1) : E^l) \multimap D(t_i : E^l)$ is provable in LL^p $(i = 0, 1)$. Then we set $(!E)^l \equiv (t_0 + t_1) : E^l$ and we obtain the following proof. Note that we can add $\leq$-formulas freely as axioms above can take them.

$$\frac{D((t_0 + t_1) : E^l) \multimap D(t_0 : E^l) \quad \Sigma_0, \Sigma_1, \,;\vec{s}\,; D^{l_0}(t_0 : E^l), \Gamma^l \vdash \Delta^l, B^l}{\Sigma_0, \Sigma_1, \,;\vec{s}\,; D^l((t_0 + t_1) : E^l), \Gamma^l \vdash \Delta^l, B^l}\ cut$$

And we have a similar proof of $\Sigma_0, \Sigma_1, \,;\vec{t}\,; D^l((t_0 + t_1) : E^l), \Gamma^l \vdash \Delta^l, C^l$.

Furthermore, if (Σ_0, Σ_1) contains $(+\vec{s}) \leq (+\vec{t})$ or $(+\vec{t}) \leq (+\vec{s})$, we obtain

$\Sigma_0, \Sigma_1, \,;\vec{t}\,; D^l((t_0 + t_1) : E^l), \Gamma^l \vdash \Delta^l, B^l \,\&\, C^l$ or

$\Sigma_0, \Sigma_1, \,;\vec{s}\,; D^l((t_0 + t_1) : E^l), \Gamma^l \vdash \Delta^l, B^l \,\&\, C^l$.

Otherwise, we add $(+\vec{s}) \leq (+\vec{t})$ and obtain the former sequent. After all, the labeling l is normal.

(Case 3) I is $! : r$.

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ ! : l$$

By the induction hypothesis, we have a proof of $\Sigma \,;\vec{s}\,;\vec{c} : \Gamma^l \vdash A^l$ with some $\Sigma, \vec{s}, l$. Then we obtain the following proof, keeping the labeling normal.

$$\frac{\Sigma \,;\vec{s}\,;\vec{c} : \Gamma^l \vdash A^l}{\Sigma \,;\vec{c} : \Gamma^l \vdash (+\vec{s}) : A^l}\ (+\vec{s}) : r$$

■

§6 Conclusion Remark

In this paper, we presented an extended linear logic with explicit resources and proved some basic results for it: the soundness and completeness theorems with respect to a denotational semantics (the extended Phase semantics) and the cut-elimination theorem. Furthermore, we focus on the pure fragment containing no ! as the modality and show the embeddability of the ordinal linear logic (containing only ! as the modality) in the pure fragment.

Though the whole LL^+ is undecidable (as LL is undecidable [7]), we leave the decidability problem for the pure fragment of LL^+ open. It is expected that it might be decidable, when we recall that the axiom $!A \multimap A$ is an obstacle for decidability in the case of LL but in the pure fragment the corresponding axiom is $(s \otimes s : A) \multimap A$, not $s : A \multimap A$.

Appendix

The proof system of the intuitionistic linear logic ILL^+ with explicit resources is defined as follows.

Axioms:

$\Sigma\,;\vec{s}\,;A \vdash A \qquad \Sigma\,;\vec{s}\,;\bot \vdash \qquad \Sigma\,;\vec{s}\,;\vdash 1$

$\Sigma\,;\vec{s}\,;\Gamma \vdash T \qquad \Sigma\,;\vec{s}\,;0,\Gamma \vdash A$

Inference rules for additive connectives:

$$\frac{\Sigma\,;\vec{s}\,;A,\Gamma \vdash C}{\Sigma\,;\vec{s}\,;A \,\&\, B,\Gamma \vdash C}\ \&:l_1 \qquad \frac{\Sigma\,;\vec{s}\,;B,\Gamma \vdash C}{\Sigma\,;\vec{s}\,;A \,\&\, B,\Gamma \vdash C}\ \&:l_2$$

$$\frac{\Sigma\,;\vec{s}\,;\Gamma \vdash A \quad \Sigma\,;\vec{s}\,;\Gamma \vdash B}{\Sigma\,;\vec{s}\,;\Gamma \vdash A \,\&\, B}\ \&:r$$

$$\frac{\Sigma\,;\vec{s}\,;A,\Gamma \vdash C \quad \Sigma\,;\vec{s}\,;B,\Gamma \vdash C}{\Sigma\,;\vec{s}\,;A \oplus B,\Gamma \vdash C}\ \oplus:l$$

$$\frac{\Sigma\,;\vec{s}\,;\Gamma \vdash A}{\Sigma\,;\vec{s}\,;\Gamma \vdash A \oplus B}\ \oplus:r_1 \qquad \frac{\Sigma\,;\vec{s}\,;\Gamma \vdash B}{\Sigma\,;\vec{s}\,;\Gamma \vdash A \oplus B}\ \oplus:r_2$$

Inference rules for multiplicative connectives:

$$\frac{\Sigma\,;\vec{s}\,;A,B,\Gamma \vdash C}{\Sigma\,;\vec{s}\,;A \otimes B,\Gamma \vdash C}\ \otimes:l \qquad \frac{\Sigma_0\,;\vec{s}\,;\Gamma \vdash A \quad \Sigma_1\,;\vec{t}\,;\Delta \vdash B}{\Sigma_0,\Sigma_1\,;\vec{s},\vec{t}\,;\Gamma,\Delta \vdash A \otimes B}\ \otimes:r$$

$$\frac{\Sigma_0\,;\vec{s}\,;\Gamma \vdash A \quad \Sigma_1\,;\vec{t}\,;B,\Delta \vdash C}{\Sigma_0,\Sigma_1\,;\vec{s},\vec{t}\,;A \multimap B,\Gamma,\Delta \vdash C}\ \multimap:l \qquad \frac{\Sigma\,;\vec{s}\,;A,\Gamma \vdash B}{\Sigma\,;\vec{s}\,;\Gamma \vdash A \multimap B}\ \multimap:r$$

$$\frac{\Sigma\,;\vec{s}\,;A,\Gamma \vdash}{\Sigma\,;\vec{s}\,;\Gamma \vdash A^{\bot}}\ \bot:r \qquad \frac{\Sigma\,;\vec{s}\,;\Gamma \vdash A}{\Sigma\,;\vec{s}\,;A^{\bot},\Gamma \vdash}\ \bot:l$$

Inference rules for modalities:

$$\frac{\Sigma\,;\quad \vec{t}\,;\quad A,\Gamma \vdash B}{\Sigma\,;\vec{s},\vec{t}\,;(+\vec{s}):A,\Gamma \vdash B}\ \vec{s}:l \qquad \frac{\Sigma\,;\vec{s}\,;\vec{u}:\Gamma \vdash \quad B}{\Sigma\,;\quad \vec{u}:\Gamma \vdash (+\vec{s}):B}\ \vec{s}:r$$

$$\frac{\Sigma\,;\vec{s}\,;t:A,t:A,\Gamma \vdash B}{\Sigma\,;\vec{s}\,;\quad t:A,\quad \Gamma \vdash B}\ \mathit{contraction} \qquad \frac{\Sigma\,;\vec{s}\,;\quad \Gamma \vdash B}{\Sigma\,;\vec{s}\,;t:A,\Gamma \vdash B}\ \mathit{weakening}$$

$$\frac{\Sigma\,;!,\vec{s}\,;\Gamma \vdash B}{\Sigma\,;\quad \vec{s}\,;\Gamma \vdash B}\ ! - \mathit{elimination} \qquad \frac{\Sigma\,;\quad \vec{s}\,;\Gamma \vdash B}{\Sigma\,;t,\vec{s}\,;\Gamma \vdash B}\ \mathit{weakening}$$

$$\frac{\Sigma\,;t,\vec{s}\,;\Gamma \vdash B}{t \le u,\Sigma\,;u,\vec{s}\,;\Gamma \vdash B}\ \le$$

Inference rules for constants and cut:

$$\frac{\Sigma\,;\vec{s}\,;\Gamma \vdash}{\Sigma\,;\vec{s}\,;\Gamma \vdash \bot}\ \bot \qquad \frac{\vec{s}\,;\Sigma\,;\quad \Gamma \vdash B}{\Sigma\,;\vec{s}\,;1,\Gamma \vdash B}\ 1 \qquad \frac{\Sigma_0\,;\vec{s}\,;\Gamma \vdash A \quad \vec{t}\,;\Sigma_1\,;A,\Delta \vdash B}{\Sigma_0,\Sigma_1\,;\vec{s},\vec{t}\,;\Gamma,\Delta \vdash B}\ \mathit{cut}$$

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