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Abstract

The logic S4LP combines the modal logic S4 with the justification logic LP. This is the case both axiomatically and semantically, though of course the real story is in the details. We introduce a simple restriction on the behavior of constants in S4LP, a restriction that has no effect on the LP sublogic. With this assumed, some very powerful derived rules are established. Then these are used to show we have completeness relative to a semantics having what we call the *local realizability* property. This means that at each world of such a model, for each formula true at that world there is a realization also true at that world, where a realization is the result of replacing all modal operators with explicit justification terms. This is part of a attempt to understand the deeper aspects of Artemov's Realization Theorem, though it is not yet clear just how the results obtained here relate to that theorem.

1 Introduction

Logics of knowledge, Hintikka style, are long familiar tools, [10]. Recently a family of *justification logics* has been created. In these logics, instead of a single modal operator, *known*, there is an infinite structured family of explicit reasons according to which something is known. There are justification logic analogs of several standard single-knower Hintikka style logics, and work is proceeding on multiple-knower versions. Connections between the Hintikka versions and the explicit versions are quite close, via *Realization Theorems*. Essentially they say that any theorem of one of the standard Hintikka style logics of knowledge can be *realized*, its knowledge operators can be replaced with explicit justifications to produce a theorem of the corresponding explicit logic of knowledge. Thus the usual knowledge operators carry hidden explicit content.

Justification logics began with an analog of S4, due to Sergei Artemov, [1]. The original motivation was to supply a constructive, arithmetic semantics for propositional intuitionistic logic, thus completing a project begun by Gödel. Artemov succeeded in this. Consequently the justification logic created was called LP, standing for "logic of proofs," where explicit justifications were thought of as representing formal arithmetic proofs. It was soon realized that proofs were only one kind of justification, and LP was one of a family of similar logics. Nonetheless in order to keep the discussion relatively manageable here, I will frame the entire discussion in terms of LP, thinking of it as a representative member of the family but having historical precedence. What is developed here, in fact, applies to a range of modal and justification logics.

For LP, the Realization Theorem makes a fundamental connection with S4. It says that each theorem of S4 has a realization—there is a replacement of modal operators with explicit justification terms that produces a theorem of LP. (The converse is also true, and trivial.) Indeed, a realization can be chosen that is *normal*, negative occurrences of necessity can be replaced with distinct variables. Moreover a realization can be produced constructively, based on a proof of the modal version in S4. Given this fundamental relationship between S4 and LP, it is natural to consider a logic that combines features of LP and S4, so that both explicit and implicit notions of knowledge are present. This has been done, with the resulting logic known as S4LP, [4, 5]. Axiomatically, one simply provides the machinery of S4, the machinery of LP, and a connecting axiom saying that explicit knowledge implies implicit knowledge (there is a full formulation in Section 3.1). S4LP is a conservative extension of both S4 and LP. The Realization Theorem becomes a result about this single logic, rather than one that connects two different ones. Unfortunately, the only proofs known for the single-logic version of the Realization Theorem detour through the older proofs, via conservativity. (Unfortunately too, this paper does not shed any fresh light on this important issue.)

A Hintikka/Kripke semantics for S4 is standard and well-known. In [6, 8] a semantics for LP was presented, combining some justification logic machinery originating in [11] with the usual S4 semantics. This semantics has been adapted to S4LP in two quite distinct ways. First, one can use the LP semantics without change, since there is an underlying Kripke structure present for the interpretation of the modal operator. Axiomatic soudness was shown in [4, 5] and a completeness theorem is in [7]. The second semantics is the single agent version of an n-agent logic of knowledge with explicit common knowledge, [2, 3]. In this, separate accessibility relations are used for the modal knowledge operator and for the explicit justification terms. Again, soundness and completeness results have been shown.

We are not concerned here with the two-accessibility-relation version of S4LP semantics. That has applications to common knowledge. Instead we are concerned with the single accessibility relation version, as investigated in [4, 5, 7]. In this semantics justifications can be thought of as supplying an analysis of an individual's knowledge, and the connection between justifications and the modal/knowledge operator can be expected to be quite close.

Let us say an S4LP model meets the *local realizability condition* provided, at each possible world of the model, each formula that is true at that world has a realization that is also true at that world (normaliity of realizations is not required). The main result of this paper is that axiomatic S4LP is complete with respect to models meeting the local realizability condition, provided a certain condition is placed on the constant specifications allowed. What was called *strong* completeness for LP in [8] is an easy corollary, thus giving strong completeness two quite distinct proofs.

We first present the basics of LP and S4LP, axiomatically and semantically. Next, subject to a restriction on constant specifications, some proof-theoretic results are shown for S4LP. Finally, we use these results to show completeness of S4LP relative to a semantics in which models meet a local realizability condition.

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2 The Logic LP

Since we will be building directly on it, we begin with a brief sketch of the oldest of the justification logics, LP, from [1]. First, the language and an axiom system, and then a standard semantics.

2.1 LP Axiomatically

Justification terms or *proof terms* are built up from *variables*, x_1, x_2, ..., and *constant symbols*, c_1, c_2, ... They are built up using the following *operation symbols*: $+$ and $\cdot$, both binary, and !, unary. These are used as infix and prefix, respectively. The reader is referred to [1] and to [9] for a discussion of the intended meaning of these operation symbols.

Formulas are built up from *propositional letters*, P_1, P_2, ..., and a *falsehood constant*, $\bot$, using $\supset$ in the usual way, together with an additional rule of formation, $t:X$ is a formula provided t is a justification term and X is a formula. One reads it as "t is a justification for X." Other propositional connectives are introduced as abbreviations.

Axioms for LP are as follows; more properly these are axiom schemes.

Classical Axioms:	all tautologies
Truth Axioms:	$t{:}X \supset X$
+ Axioms:	$t{:}X \supset (t+u){:}X$
	$u{:}X \supset (t+u){:}X$
$\cdot$ Axioms:	$t{:}(X \supset Y) \supset (u{:}X \supset (t \cdot u){:}Y)$
! Axioms:	$t{:}X \supset !t{:}t{:}X$

We have two rules, one standard, one not.

$$\text{Modus Ponens: } \frac{X \qquad X \supset Y}{Y}$$

$$\text{Axiom Necessitation: If } X \text{ is an axiom and } c \text{ is a constant: } \frac{}{c{:}X}$$

A *constant specification* $\mathcal{C}$ is an assignment of axioms to constants. We can take it to be a set whose members are of the form $c{:}X$, where c is a constant and X is an axiom. A proof *meets constant specification* $\mathcal{C}$ provided that whenever $c{:}X$ is introduced using the Axiom Necessitation rule, then X is an axiom that $\mathcal{C}$ assigns to constant c. A constant specification can be given ahead of time, or can be created during the course of a proof. In this paper we will assume a constant specification has been fixed ahead of time. Various conditions can be imposed on constant specifications. A constant specification is *axiomatically appropriate* if all instances of axiom schemes have proof constants—in this paper *this will always be assumed*. Another common condition is being *injective*, which means that at most one formula is associated with each constant. We will need a condition, given in Section 4, that conflicts with injectivity, but which is nonetheless natural to consider.

2.2 LP Semantics

The semantics commonly used for LP comes from [8], and amounts to a blending of an earlier semantics from [11] with the usual Hintikka style semantics for logics of knowledge. A model is $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{A}, \mathcal{V} \rangle$, where $\langle \mathcal{G}, \mathcal{R} \rangle$ is a frame in the usual sense, with $\mathcal{R}$ a reflexive and transitive relation on $\mathcal{G}$. $\mathcal{V}$ maps propositional variables to subsets of $\mathcal{G}$. The item not standard in Kripke models is $\mathcal{A}$, which is an *admissible evidence function*. For each justification term t and formula X, $\mathcal{A}(t, X)$ is some subset of $\mathcal{G}$. The intuition is, $\mathcal{A}(t, X)$ is intended to be the set of worlds at which t is considered to be admissible evidence for X. Note that this does not mean certain evidence—just evidence that is relevant. Admissible evidence functions must meet certain conditions—for example, agreeing with a given constant specification. We give these conditions next. (In earlier work a mapping $\mathcal{E}$, called an evidence function, was used in place of $\mathcal{A}$. It assigned to each possible world Γ and to each justification t a set of formulas. The change in notation is essentially cosmetic, and is part of a move toward uniformity in the area of justification logics.)

Constant Specification Condition For the given constant specification $\mathcal{C}$, if $c{:}X \in \mathcal{C}$ then $\mathcal{A}(c, X) = \mathcal{G}$. If this condition is met, we say $\mathcal{A}$ meets constant specification $\mathcal{C}$.

$\cdot$ **Condition** $\mathcal{A}(s, X \supset Y) \cap \mathcal{A}(t, X) \subseteq \mathcal{A}(s \cdot t, Y)$.

$+$ **Condition** $\mathcal{A}(s, X) \cup \mathcal{A}(t, X) \subseteq \mathcal{A}(s + t, X)$.

$\mathcal{R}$ **Closure Condition** $\Gamma\mathcal{R}\Delta$ and $\Gamma \in \mathcal{A}(t, X)$ imply $\Delta \in \mathcal{A}(t, X)$.

! **Condition** $\mathcal{A}(t, X) \subseteq \mathcal{A}(!t, t{:}X)$.

Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{A}, \mathcal{V} \rangle$ be an LP model. The notation $\mathcal{M}, \Gamma \Vdash X$ is read: formula X is true at world $\Gamma \in \mathcal{G}$, of LP model $\mathcal{M}$. The conditions for it are as follows.

Atomic Condition For a propositional letter P, $\mathcal{M}, \Gamma \Vdash P$ if $\Gamma \in \mathcal{V}(P)$.

Classical Conditions $\mathcal{M}, \Gamma \Vdash X \supset Y$ iff $\mathcal{M}, \Gamma \not\Vdash X$ or $\mathcal{M}, \Gamma \Vdash Y$. Also $\mathcal{M}, \Gamma \not\Vdash \bot$.

Justification Condition $\mathcal{M}, \Gamma \Vdash t{:}X$ iff $\Gamma \in \mathcal{A}(t, X)$ and $\mathcal{M}, \Delta \Vdash X$ for all $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$.

We say X is *true at world* Γ if $\mathcal{M}, \Gamma \Vdash X$, and otherwise X is *false at* Γ. X is valid in a model $\mathcal{M}$ if X is true at every world of it.

The justification condition above says we have $t{:}X$ at Γ if X is knowable at Γ in the Hintikka sense, and t is admissible evidence for X at Γ. If we think of Hintikka semantics as capturing the idea of *true belief*, then what the present machinery captures is *justified* true belief.

The semantics as just described is sometimes called the *weak model semantics*. There is a stronger version. A model $\mathcal{M}$ is said to be *fully explanatory* provided, if $\mathcal{M}, \Delta \Vdash X$ for all $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$ then there is some justification t such that $\mathcal{M}, \Gamma \Vdash t{:}X$. More informally, $\mathcal{M}$ is fully explanatory provided knowability of X at Γ (in the Hintikka semse) implies there is a justification for X at Γ. The fully explanatory condition will be examined in a bit more detail in Section 6.

In [8] soundness and completeness was shown, in the following sense. If $\mathcal{C}$ is an axiomatically appropriate constant specification, then X has an axiomatic proof using $\mathcal{C}$ if and only if X is valid in every weak LP model meeting $\mathcal{C}$ if and only if X is valid in every strong LP model meeting $\mathcal{C}$.

While strong models are interesting, all applications of the semantics have so far only needed the weak model semantics. This is a puzzling circumstance, which the results of this paper will only make more puzzling.

3 The Logic S4LP

The logics LP and S4 are connected intimately via the Realization Theorem, as noted in the Introduction. Consequently it is natural to consider a logic combining the two, and this is S4LP, originating in [4, 5]. This section presents the usual formulation, axiomatically and via one of the two standard semantics.

3.1 S4LP Axioms

First, the language of LP is extended with the formation rule: if X is a formula, so is $\Box X$. Next, the axiomatization of LP as given in Section 2.1 is extended with S4 machinery, and a connecting axiom.

$\Box$ Axioms:	$\Box X \supset X$
	$\Box(X \supset Y) \supset (\Box X \supset \Box Y)$
	$\Box X \supset \Box\Box X$
Connecting Axiom:	$t{:}X \supset \Box X$

The usual necessitation rule is added.

$$\Box \text{ Necessitation: } \frac{X}{\Box X}$$

Finally, it is assumed that the LP Axiom Necessitation rule also applies to the new axioms just added, and that constant specifications also take these new axioms into account.

We should note that $\Box$ Necessitation can be shown to be a redundant rule, but doing so involves proving an Internalization Theorem, whose statement and proof we skip here.

3.2 An S4LP Semantics

As was noted in the introduction, there are two quite different kinds of semantics for S4LP, with different motivations. One, presented in [2, 3], allows not just one but multiple agents, each with its own knowledge operator, K_i, but with justifications being meaningful to all and playing the role of justified common knowledge. If there is a single agent, the logic reduces to S4LP. In the corresponding semantics, one has multiple accessibility relations, one for each agent, and one for justification terms. In the S4LP case there is, in effect, a single agent and so there are two relations, one used Hintikka style to supply an interpretation for $\Box$, the other, combined with an admissible evidence function, is used as in Section 2.2. For this semantics, both weak and strong completeness theorems are provable. However, this is not the semantics that will concern us here, and it will not be mentioned further.

The semantics that interests us in this paper understands knowledge as having an explicit (justification term) aspect and an implicit (modal) aspect. Justification terms provide an analysis of our knowledge, rather than being the items of knowledge we share with other agents. This approach originated in [4, 5, 7]. In this version a *single* accessibility relation is used for both implicit and explicit knowledge. We now specify this semantics a bit more properly.

First, LP models are fundamental, and these are exactly as in Section 2.2. This provides the semantics for justification terms. But since we are using an extended language now, we can also adopt the following, familiar from modal logic.

Necessitation Condition $\mathcal{M}, \Gamma \Vdash \Box X$ iff $\mathcal{M}, \Delta \Vdash X$ for all $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$.

In other words, justification terms are interpreted using the single accessibility relation and the admissible evidence function, while the modal operator uses the accessibility relation but does not take the admissible evidence function into account. Note that the Justification Condition now can be given a somewhat simpler expression: $\mathcal{M}, \Gamma \Vdash t{:}X$ iff $\Gamma \in \mathcal{A}(t, X)$ and $\mathcal{M}, \Gamma \Vdash \Box X$.

As with LP itself, we can still introduce notions of weak and strong models, but now the Fully Explanatory condition is simpler to state: for each possible world Γ and for each formula X, if $\mathcal{M}, \Gamma \Vdash \Box X$ then there is some justification term t such that $\mathcal{M}, \Gamma \Vdash t{:}X$. Equivalently, if $\mathcal{M}, \Gamma \Vdash \Box X$ then there is some justification term t such that $\Gamma \in \mathcal{A}(t, X)$.

Axiomatic soundness comes from [4, 5]. Completeness of S4LP with respect to the weak model semantics for S4LP was shown in [7]. Completeness with respect to the strong model semantics, with models satisfying the fully explanatory condition, is an open problem in full generality. Here it will be a special case of a more general result, but the general result will be proved subject to a special condition on constant specifications.

4 Some Derived S4LP Rules

The rules presented here will form part of the basis for our S4LP completeness proof in the next section. We must, however, impose a restriction on constant specifications, one that is at odds with injectivity. In order to state the restriction simply, we begin with the notion of replacing terms containing a variable with a $\Box$ operator.

Definition 4.1 Let Z be a formula of S4LP, and let x be a variable. by $Z(x/\Box)$ we mean the result of replacing every justification term in Z that contains x with $\Box$. More specifically:

$$\begin{aligned} P(x/\Box) &= P \text{ for } P \text{ atomic} \\ [X \supset Y](x/\Box) &= [X(x/\Box) \supset Y(x/\Box)] \\ [\Box X](x/\Box) &= \Box[X(x/\Box)] \\ [t{:}X](x/\Box) &= \begin{cases} t{:}[X(x/\Box)] & \text{if } x \text{ does not occur in } t \\ \Box[X(x/\Box)] & \text{if } x \text{ occurs in } t \end{cases} \end{aligned}$$

The following Lemma essentially says why we are interested in this notion. It exemplifies the symmetry inherent in S4LP.

Lemma 4.2 *If Z is an axiom of S4LP, so is $Z(x/\Box)$ for every variable x.*

Proof We do one case as an example, and leave the others to you. Consider the axiom $t{:}X \supset (t+u){:}X$. For it, we have the following three subcases.

1. x does not occur in either t or u. Then $[t{:}X \supset (t+u){:}X](x/\Box) = [t{:}X(x/\Box) \supset (t+u){:}X(x/\Box)]$, which is also a + axiom.
2. x occurs in u but not in t. Then $[t{:}X \supset (t+u){:}X](x/\Box) = [t{:}X(x/\Box) \supset \Box X(x/\Box)]$, which is a connecting axiom.
3. x occurs in t. Then $[t{:}X \supset (t+u){:}X](x/\Box) = [\Box X(x/\Box) \supset \Box X(x/\Box)]$, a classical axiom.

■

Now we give our constant specification restriction.

Definition 4.3 ($\Box$ Closed) Let $\mathcal{C}$ be a constant specification for S4LP. We say $\mathcal{C}$ is $\Box$ *closed* provided that whenever $c{:}Z \in \mathcal{C}$ then also $c{:}Z(x/\Box) \in \mathcal{C}$, for each variable x.

Note that a constant specification that is $\Box$ closed cannot be injective (though it may be when restricted to LP formulas, not containing $\Box$). Here is the main result concerning $\Box$ closed constant specifications.

Theorem 4.4 *Suppose the constant specification $\mathcal{C}$ is $\Box$ closed. If Z is provable in* S4LP *using constant specification $\mathcal{C}$, so is $Z(x/\Box)$, for every variable x.*

Proof By induction on axiomatic proof length. Lemma 4.2 takes care of axioms. Modus ponens and $\Box$ necessitation are straightforward. Axiom necessitation is covered by the assumption that the constant specification is $\Box$ closed. ■

Note that if we do not assume the constant specification is $\Box$ closed, it is still the case that if Z is provable in S4LP, so is $Z(x/\Box)$, but using a different constant specification.

Corollary 4.5 *Assuming a constant specification that is $\Box$ closed, the following is a derived rule of* S4LP*. If x does not occur in A or B:*

$$\frac{x{:}A \supset B}{\Box A \supset B}$$

This corollary points out a similarity in behavior between the $\Box$ operator in S4LP and the existential quantifier in first-order logic. This similarity has been an important motivating factor in the development of justification logics.

5 Completeness

We begin with a proper definition of local realizability, and a statement of the completeness theorem whose proof will occupy the rest of this section. It should be noted that the completeness argument has similarities with one often used for first-order modal logic with the Barcan formula.

Definition 5.1 For a formula X of S4LP, a *realization* is a formula X' whose structure is like that of X, but in which every occurrence of $\Box$ has been replaced with a justification term. More precisely we have the following, in which the shorter term "realizes" has been used for readability.

$$\begin{aligned} P &\text{ realizes } P \text{ if } P \text{ is atomic} \\ X' \supset Y' &\text{ realizes } X \supset Y \text{ if } X' \text{ realizes } X \text{ and } Y' \text{ realizes } Y \\ t{:}X' &\text{ realizes } t{:}X \text{ if } X' \text{ realizes } X \\ t{:}X' &\text{ realizes } \Box X \text{ if } t \text{ is a term and } X' \text{ realizes } X \end{aligned}$$

Note that this definition extends the usual notion of realization. Ordinarily it is S4 formulas that are realized, while the definition given above includes a case covering justification terms which are not part of the language of S4. Also note, we are not considering *normal* realizations.

Definition 5.2 Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{A}, \mathcal{V} \rangle$ be an S4LP model. We say it *meets the local realizability condition* provided: for every world $\Gamma \in \mathcal{G}$ and for every S4LP formula X, if $\mathcal{M}, \Gamma \Vdash X$ then there is some realization X' of X such that $\mathcal{M}, \Gamma \Vdash X'$.

Here is the completeness result that will be proved in the remainder of the section.

Theorem 5.3 *Let $\mathcal{C}$ be a constant specification that is $\Box$ closed (Definition 4.3). If Z_0 is not provable using this constant specification, then Z_0 is false at some world of an* S4LP *model meeting this constant specification and* meeting the local realizability condition.

We begin with some standard items. We will not need to restrict things to a $\Box$ closed constant specification for a while, so for the time being just assume all proofs use a constant specification $\mathcal{C}$ that may or may not meet this condition.

For a set S of formulas and a single formula X, we write $S \vdash X$ if there is some finite subset $\{Y_1, \ldots, Y_n\}$ of S such that $(Y_1 \wedge \ldots \wedge Y_n) \supset X$ is a theorem of S4LP. Note that with this definition, the deduction theorem is immediately true. We say S is *inconsistent* if $S \vdash \bot$, and *consistent* if it is not inconsistent. As with the deduction theorem, compactness is also immediate. *Maximal* consistency has its usual meaning.

Definition 5.4 Let X be a formula of S4LP, and assume an occurrence of $\Box$ in X has been designated. Let t be a justification term. By $X(t)$ we mean the result of replacing the designated occurrence of $\Box$ in X with t.

For instance, say X is $\Box(P \supset x{:}\dot{\Box}Q) \supset \Box R$ and the designated occurrence of $\Box$ is marked with a dot. Then $X(t) = \Box(P \supset x{:}t{:}Q) \supset \Box R$. The notation $X(t)$ is really incomplete, in the sense that which occurrence of $\Box$ in X is designated is understood, and is not represented in the notation itself.

Definition 5.5 We say a set S of S4LP formulas has the $\Box$ *instantiation property* provided, for every formula X, if $S \cup \{X\}$ is consistent, then for each designated occurrence of $\Box$ in X there is some term t such that $S \cup \{X(t)\}$ is also consistent.

Trivially an inconsistent set has the $\Box$ instantiation property, though this vacuous fact will have no use for us. More importantly, if S is not just consistent but *maximally* so, then being consistent with S is equivalent to being a member of S. So for maximally consistent sets the $\Box$ instantiation property becomes: if $X \in S$ then for each designated occurrence of $\Box$ in X there is some t such that $X(t) \in S$. This, finally, is what we will need, but the more general version plays an essential role along the way.

Definition 5.6 Let S be a set of S4LP formulas. $S^\sharp$ is defined to be $\{X \mid \Box X \in S\}$.

It is common in constructing canonical modal models to take as possible worlds all maximally consistent sets, and for two such sets, to say Δ is accessible from Γ if $\Gamma^\sharp \subseteq \Delta$. Here, in addition to maximal consistency, we will require sets serving as possible worlds to have the $\Box$ instantiation property. We begin with a result connecting that notion with the operation just defined.

Proposition 5.7 *Suppose S is a maximally consistent set of* S4LP *formulas that has the $\Box$ instantiation property. Then $S^\sharp$ also has the $\Box$ instantiation property.*

Proof Assume the hypothesis. Let X be an S4LP formula in which an occurrence of $\Box$ has been designated, and suppose $S^\sharp \cup \{X\}$ is consistent. We will show that for some t, $S^\sharp \cup \{X(t)\}$ is consistent. In doing this it will be convenient to use the modal operator $\Diamond$, which has its usual definition, $\Diamond Z = \neg\Box\neg Z$; indeed negation itself is not primitive, $\neg Z = (Z \supset \bot)$.

Since $S^\sharp \cup \{X\}$ is consistent, so is $S \cup \{\Diamond X\}$ by the following argument. If it were not consistent, $S, \Diamond X \vdash \bot$, and so $S \vdash (\Diamond X \supset \bot)$, or equivalently, $S \vdash \Box\neg X$. Since S is maximally consistent, $\Box\neg X \in S$, hence $\neg X \in S^\sharp$, so $S^\sharp \cup \{X\}$ would not have been consistent.

Since S has the $\Box$ instantiation property, for some justification term t, $S \cup \{\Diamond X(t)\}$ is consistent.

Finally, since $S \cup \{\Diamond X(t)\}$ is consistent, it follows that so is $S^\sharp \cup \{X(t)\}$, which finishes the argument. Again the verification is a proof by contradiction. Suppose $S^\sharp \cup \{X(t)\}$ is not consistent. Then $S^\sharp, X(t) \vdash \bot$, and so $S^\sharp \vdash \neg X(t)$. Then for some $Y_1, \ldots, Y_n \in S^\sharp$, the formula $(Y_1 \wedge \ldots \wedge Y_n) \supset$

$\neg X(t)$ is provable. It follows using the Rule of Necessitation and standard modal theorems, that $(\Box Y_1 \wedge \ldots \wedge \Box Y_n) \supset \Box\neg X(t)$ is provable. Since $Y_1, \ldots, Y_n \in S^\sharp$, we must have $\Box Y_1, \ldots, \Box Y_n \in S$, and since S is maximally consistent, we must also have $\Box\neg X(t) \in S$, contradicting the fact that $S \cup \{\neg\Box\neg X(t)\}$ is consistent. ■

Next we address the problem of extending a set that has the $\Box$ instantiation property to a maximally consistent set that still has this property. First a Lemma, and then the main item.

Lemma 5.8 *Suppose S is a set of* S4LP *formulas that has the $\Box$ instantiation property, and F is a finite set of formulas. If $S \cup F$ is consistent then $S \cup F$ also has the $\Box$ instantiation property.*

Proof Let X be an S4LP formula in which an occurrence of $\Box$ has been designated. Assume that $S \cup F \cup \{X\}$ is consistent. We will show that for some t, $S \cup F \cup \{X(t)\}$ is also consistent. The argument is very simple.

Say $F = \{Y_1, \ldots, Y_n\}$. Then $S \cup \{Y_1 \wedge \ldots \wedge Y_n \wedge X\}$ is consistent. Since S has the $\Box$ instantiation property, for some t we have that $S \cup \{Y_1 \wedge \ldots \wedge Y_n \wedge X(t)\}$ is consistent, using the original designated occurrence of $\Box$ in X. But this implies that $S \cup F \cup \{X(t)\}$ is consistent. ■

Proposition 5.9 *If S is a consistent set of* S4LP *formulas that has the $\Box$ instantiation property, then S can be extended to a set that is maximally consistent and has the $\Box$ instantiation property.*

Proof Assume S is consistent and has the $\Box$ instantiation property. We extend S using a modified version of the usual Lindenbaum construction. Enumerate the formulas of S4LP, say X_0, X_1, ... Then define a sequence of sets of formulas, S_0, S_1, ..., in which each set extends its predecessor, is consistent, and has the $\Box$ instantiation property. The definition is as follows.

To start, $S_0 = S$.

Suppose S_n has been defined, is consistent, and has the $\Box$ instantiation property. If $S_n \cup \{X_n\}$ is not consistent, set $S_{n+1} = S_n$. Otherwise, proceed as follows. X_n has a finite number of $\Box$ occurrences, say k of them. Choose one of them as designated. $S_n \cup \{X_n\}$ is consistent and, by Lemma 5.8, it has the $\Box$ instantiation property. Then, using the designated occurrence of $\Box$, there must be some justification term t such that $S_n \cup \{X_n, X_n^1\}$ is consistent, where $X_n^1 = X_n(t)$. By Lemma 5.8 this set too has the $\Box$ instantiation property. Now repeat this with a different designated occurrence of $\Box$ in X_n, getting a set $S_n \cup \{X_n, X_n^1, X_n^2\}$, consistent and with the $\Box$ instantiation property. And so on for each of the k occurrences of $\Box$ in X_n. Let $S_{n+1} = S_n \cup \{X_n, X_n^1, X_n^2, \ldots, X_n^k\}$.

Let $S = S_0 \cup S_1 \cup S_2 \cup \ldots$. By a standard argument, S is maximally consistent. But also it has the $\Box$ instantiation property by the following argument. Suppose X is a formula with a designated occurrence of $\Box$, and suppose $S \cup \{X\}$ is consistent. Say $X = X_n$. Then $S \cup \{X_n\}$ is consistent and so at stage n of the construction above, not only is $X_n = X$ in S_{n+1}, but also $X(t)$ is in S_{n+1} for some justification term t, and hence $X(t)$ is in S, so trivially $S \cup \{X(t)\}$ is consistent. ■

All results to this point have had the form: some set has the $\Box$ instantiation property provided some other set does. We do not yet know there are any such sets at all, except for inconsistent ones. This is taken care of by the following Lemma and Proposition, but now the $\Box$ closure of the constant specification comes into play.

Lemma 5.10 *Suppose we have a constant specification that is $\Box$ closed. Let F be a consistent finite set of* S4LP *formulas, and let X be a single formula with a designated occurrence of $\Box$. Then $F \cup \{X \supset X(x)\}$ is consistent, where x is a variable that does not occur in F or in X.*

Proof Assume the hypothesis, and also assume $F \cup \{X \supset X(x)\}$ is not consistent, where x does not occur in F or in X. Then $[\bigwedge F \wedge (X \supset X(x))] \supset \bot$ is provable. By Theorem 4.4 we also have provability of $\{[\bigwedge F \wedge (X \supset X(x))] \supset \bot\}(x/\Box)$, but this is just $[\bigwedge F \wedge (X \supset X)] \supset \bot$, and it follows that F is not consistent. ■

Proposition 5.11 *Assume we have a constant specification that is $\Box$ closed, Definition 4.3. If F is a finite, consistent set of* S4LP *formulas then F can be extended to a consistent set that has the $\Box$ instantiation property.*

Proof Enumerate the formulas of S4LP, X_0, X_1, X_2, ... We define a chain F_0, F_1, F_2, ... of consistent finite sets, as follows.

$F_0 = F$.

Assume F_n has been defined. Consider formula X_n. Choose an occurrence of $\Box$ in X_n and take it to be designated (if there are none, this step is vacuous). Let x be a variable that does not occur in the finite set F_n or in X_n. Then $F_n \cup \{X_n \supset X_n(x)\}$ is consistent, by Lemma 5.10. Further extend this set by consistently adding an instantiation implication for a different designated occurrence of $\Box$ in X_n, and so on until each designated occurrence has had a corresponding implication added. Call the resulting set F_{n+1}. Clearly it is finite and consistent.

Finally, set F^* to be $\cup_n F_n$. Then F^* is consistent, and it is easy to see it will have the $\Box$ instantiation property. ■

Completeness for LP was proved in [8], and that proof was extended to S4LP, as usually formulated, in [7]. Now we modify that proof to establish our main result, which we restate here for convenience.

Theorem 5.3 Let $\mathcal{C}$ be a constant specification that is $\Box$ closed (Definition 4.3). If Z_0 is not provable using this constant specification, then Z_0 is false at some world of an S4LP model meeting this constant specification and *meeting the local realizability condition*.

Proof Since Z_0 is not provable, $\{\neg Z_0\}$ is consistent. Using Proposition 5.11, this set extends to a set that has the $\Box$ instantiation property, and by Proposition 5.9, this further extends to a set that is maximally consistent and has the $\Box$ instantiation property. Call this set Γ_0.

Construct a model as follows. Let $\mathcal{G}$ be the set of all maximally consistent sets of formulas that have the $\Box$ instantiation property. (Note that $\Gamma_0 \in \mathcal{G}$.) If $\Gamma \in \mathcal{G}$, let $\Gamma^\sharp = \{X \mid \Box X \in \Gamma\}$, and set $\Gamma \mathcal{R} \Delta$ if $\Gamma^\sharp \subseteq \Delta$. This gives us a frame, $\langle \mathcal{G}, \mathcal{R} \rangle$. It is easily shown to be reflexive and transitive. Define a mapping $\mathcal{A}$ by setting $\mathcal{A}(t, X) = \{\Gamma \in \mathcal{G} \mid t{:}X \in \Gamma\}$. Finally, define a mapping $\mathcal{V}$ by specifying that for an atomic formula P, $\Gamma \in \mathcal{V}(P)$ if and only if $P \in \Gamma$. This gives us a structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{A}, \mathcal{V} \rangle$. We begin by showing that $\mathcal{M}$ is an S4LP model.

First we verify that our candidate for an admissible evidence function, $\mathcal{A}$, meets the $\cdot$ Condition. Suppose we have $\Gamma \in [\mathcal{A}(s, X \supset Y) \cap \mathcal{A}(t, X)]$. By the definition of $\mathcal{A}$, we must have $t{:}X \in \Gamma$ and $s{:}(X \supset Y) \in \Gamma$. Since $s{:}(X \supset Y) \supset (t{:}X \supset (s \cdot t){:}Y)$ is an S4LP axiom, and Γ is maximally consistent, it follows that $(s \cdot t){:}Y \in \Gamma$, and hence $\Gamma \in \mathcal{A}(s \cdot t, Y)$.

Next we verify the $\mathcal{R}$ Closure Condition. Suppose $\Gamma, \Delta \in \mathcal{G}$ and $\Gamma \mathcal{R} \Delta$. Also assume $\Gamma \in \mathcal{A}(t, X)$. By definition of $\mathcal{A}$, we have $t{:}X \in \Gamma$. But $t{:}X \supset !t{:}t{:}X$ is an S4LP axiom, and so is $!t{:}t{:}X \supset \Box t{:}X$, so we have $\Box t{:}X \in \Gamma$, and hence $t{:}X \in \Gamma^\sharp \subseteq \Delta$. Then $\Delta \in \mathcal{A}(t, X)$.

Verifying that $\mathcal{A}$ meets the + and ! Conditions is similar, and is omitted. Likewise it is straightforward to check that $\mathcal{M}$ meets constant specification $\mathcal{C}$.

We have now verified that $\mathcal{M}$ is an S4LP model.

Next, a Truth Lemma can be shown: for each formula X and each $\Gamma \in \mathcal{G}$

$$X \in \Gamma \Longleftrightarrow \mathcal{M}, \Gamma \Vdash X \tag{1}$$

Many of the cases are familiar from standard S4 completeness proofs. I'll verify only one case. Suppose (1) is known for X, and we are considering the formula $t{:}X$.

Suppose first that $t{:}X \in \Gamma$. Then, using the Connecting Axiom, $t{:}X \supset \Box X$, $\Box X \in \Gamma$, and so $X \in \Gamma^\sharp$. Then if Δ is an arbitrary member of $\mathcal{G}$ with $\Gamma \mathcal{R} \Delta$ we have $\Gamma^\sharp \subseteq \Delta$ and hence $X \in \Delta$. By the induction hypothesis, $\mathcal{M}, \Delta \Vdash X$. Also since $t{:}X \in \Gamma$, we have $\Gamma \in \mathcal{A}(t, X)$. It follows that $\mathcal{M}, \Gamma \Vdash t{:}X$.

Next, suppose $\mathcal{M}, \Gamma \Vdash t{:}X$. This case is trivial. As part of the definition of $\Vdash$ we must have $\Gamma \in \mathcal{A}(t, X)$, and by definition of $\mathcal{A}$ for $\mathcal{M}$, we must then have $t{:}X \in \Gamma$.

Thus we have the Truth Lemma. It follows immediately that $\mathcal{M}$ meets the local realizability condition. Here is the argument. Suppose $\mathcal{M}, \Gamma \Vdash X$. Then by the Truth Lemma, $X \in \Gamma$. Designate an occurrence of $\Box$ in X. Since members of $\mathcal{G}$ have the $\Box$ instantiation property, for some t, $\Gamma \cup \{X(t)\}$ is consistent, hence $X(t) \in \Gamma$ since Γ is maximal. If there are occurrences of $\Box$ in $X(t)$ repeat this step, eliminating a second occurrence. And so on. When all occurrences of $\Box$ are gone, we have a realization X' of X, with $X' \in \Gamma$. But then $\mathcal{M}, \Gamma \Vdash X'$, by the Truth Lemma again.

Finally, since $\neg Z_0 \in \Gamma_0$, and $\Gamma_0 \in \mathcal{G}$, we have $\mathcal{M}, \Gamma_0 \not\Vdash Z_0$, completing the proof. ■

It should be noted that we have *not* proved a semantic version of compactness. Essentially our failure to do so comes down to the restriction that the set F be finite in Proposition 5.11.

6 Fully Explanatory

The LP semantics given in Section 2.2 was introduced in [6, 8] where two versions were given, weak and strong. The weak version is basic in this paper. For the strong version, an additional condition was placed on models. An LP model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{A}, \mathcal{V} \rangle$ was called *fully explanatory* provided, for each world $\Gamma \in \mathcal{G}$, if $\mathcal{M}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, then $\mathcal{M}, \Gamma \Vdash t{:}X$ for some justification term t. Strong models were defined to be weak models meeting the fully explanatory condition. Completeness of axiomatic LP with respect to strong models was demonstrated, using a proof that closely mirrored a central part of the usual proof of completeness for modal logics, using canonical models. In the conventional modal case one shows that if $\{\Box X_1, \ldots, \Box X_n, \neg\Box Y\}$ is consistent, then so is $\{X_1, \ldots, X_n, \neg Y\}$. In the LP case one shows that, for fixed choice of t_1, $\ldots$, t_n, if $\{t_1{:}X_1, \ldots, t_n{:}X_n, \neg u{:}Y\}$ is consistent for every choice of u, then $\{X_1, \ldots, X_n, \neg Y\}$ is also consistent.

As it happens, strong completeness and the fully explanatory condition have found no applications. In all applications of the LP semantics so far, the weak version has sufficed. Nonetheless, it seems reasonable to suppose that an aspect of justification logics that is closely connected to such a fundamental modal argument must play a significant role at some point. Perhaps this is wishful thinking, but some wishful thinking has a degree of plausibility to it.

The reason the fully explanatory condition for LP comes up here is: the work of this paper provides a second proof of strong completeness for LP, along completely different lines from that referred to above. The argument follows.

Suppose X is a formula of LP, and X is not provable using a constant specification $\mathcal{C}$ that is axiomatically appropriate (for LP axioms only, of course). It is simple to extend $\mathcal{C}$ to a constant specification that is axiomatically appropriate for all of S4LP and is $\Box$ closed; we leave it to you to check this. Call the extension $\mathcal{C}^*$. Then X is not provable in S4LP using $\mathcal{C}^*$ either, since an LP counter model for X easily extends to an S4LP countermodel for X, using the weak notion of S4LP model from [7]. Since X is not provable in S4LP using $\mathcal{C}^*$, by the completeness proof of the present paper, X is falsified at some world of an S4LP model meeting the local realizability condition. If we ignore $\Box$ in such a model we have an LP model, and local realizability immediately gives us the fully explanatory condition.

It is curious that the fully explanatory condition can be approached from such seemingly different directions—via a generalization of a standard modal argument, and via a generalization of a Henkin completeness argument. It is also curious that no use has been found for the condition. This report ends on a note of genuine puzzlement.

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