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Abstract

Justification Logic is the study of a family of logics used to reason about *justified true belief*. Dynamic Epistemic Logic is the study of a family of logics obtained by adding various kinds of communication to the language of multi-modal logic, yielding languages for reasoning about communication and *true belief*. This paper is a first-step in merging these two areas, in that it brings the most basic kind of communication studied in Dynamic Epistemic Logic—the public announcement—over to Justification Logic. This gives us a language for reasoning about public announcements and justified true belief.

After giving an overview of Justification Logic, the paper introduces a notion of bisimulation for Justification Logic. Bisimulation allows us to study the affect on language expressivity when we add various kinds of communication to the language. Among a number of expressivity results, we show that adding public announcements to the language of Justification Logic strictly increases language expressivity. This stands in contrast to the Plaza-Gerbrandy Theorem, which states that adding public announcements to multi-modal logic does not increase language expressivity. This leads us to extend the language of Justification Logic in order to provide a Plaza-Gerbrandy analog of multi-modal logic that we can use to reason about justified true belief.

1 Introduction

The predominant approach in formal epistemology defines knowledge using Kripke's possible world semantics [11]. This *Kripke-knowledge* is defined as having true (correct) belief. Here *belief* is just Hintikka's notion: I *Hintikka-believe* something exactly when that something is true in all of those states of affairs that look the same to me as the actual state of affairs [16]. And to say that the Hintikka-belief of p is *true* (as in *correct*) means that p is in fact true in the actual state of affairs. So I Kripke-know something exactly when I Hintikka-believe that something and I am correct in this belief.

Now if a formal language has a possible worlds semantics that makes true Hintikka-belief expressible in the language, then this language can be used to reason about Kripke-knowledge. The language of modal logic is an example: if K is a modal and φ is a formula, then the modal formula $K\varphi$—read it as "φ is known"—expresses the true Hintikka-belief of φ when we interpret this language via Kripke's semantics.

Since Kripke's semantics for modal logic gives us a formal meaning for *true belief*, we are quite close to a formalization of Plato's definition of knowledge: knowledge is *justified true belief*. But while our interpretation of modal logic allows us to formalize the last two components of Plato's three-part definition, it falls short when we wish to formalize the first component, *justification*. Let us see why.

Consider a formula of the form $K\varphi \supset K\psi$. Such a formula is a statement of conditional knowledge that says my knowledge of ψ follows from my knowledge of φ. But notice that while such a formula describes a connection between my knowledge of one thing and my knowledge of another, the formula fails to provide a *reason* as to why this connection holds, something we certainly want of our logical language if we are to say that this language incorporates a notion of *justification*. It is thus more accurate for us to read the formula $K\varphi$ as "φ is known for some reason" because this formula merely asserts the existence of knowledge—it does not say *why* we have this knowledge.

Justification Logic has recently been suggested as a means of remedying this shortcoming [5, 6, 4, 12]. The basic language of Justification Logic extends the language of propositional logic by introducing formula-labeling terms, allowing us to take a term t and a formula φ and form the new formula $t{:}\varphi$. Terms can be nested, so in the formula $t{:}\varphi$, the formula φ may itself contain terms. But the most important feature of terms is the fact that they have a certain derivation-compatible structure: for each derivation $\mathcal{D}$ of a theorem φ (in a later-defined system), we can construct a term t whose structure mimics that of $\mathcal{D}$ in such a way that $t{:}\varphi$ is also a theorem. This allows us to think of the term t as a particular reason that explains why it is that φ is true. Justification Logic thus has a built-in notion of *justification* that, when combined with a possible worlds semantics [2, 13, 3], again allows us to capture true Hintikka-belief. Accordingly, we read $t{:}\varphi$ as "φ is known for reason t." We then have a formalization of *true belief* in a logic with in-language *justification*, thereby capturing all three components of Plato's definition.

So far Justification Logic has only been used to model static situations of knowledge (justified true belief). In this paper, we introduce *public announcements* into the language of Justification Logic. A public announcement is a kind of truthful public communication whose purpose is to create common knowledge among the hearers of the announcement. Public announcements are a basic concept in Dynamic Epistemic Logic, an area that studies communication and knowledge (true belief) by introducing various kinds of communication into the language of modal logic [23]. Our paper is thus a first-step in merging the areas of

Justification Logic and Dynamic Epistemic Logic.

Our task in this paper is to extend the language of Justification Logic so as to reason about public announcements alongside knowledge (justified true belief). After we introduce the syntax and semantics of of Justification Logic, we will define a notion of bisimulation for this language. Bisimulation allows us to study how language expressivity is affected when we introduce additional syntax to reason in the language about a given kind of communication such as public announcements. We will use our notion of bisimulation to show that adding public announcements to the language of Justification Logic strictly increases language expressivity, in contrast to the Plaza-Gerbrandy Theorem, which shows that adding public announcements to the language of modal logic does not increase language expressivity [18, 15]. We will conclude by defining a natural extension for the language of Justification Logic. This extension has the property that adding public announcements does not increase language expressivity, and so this extension may be considered a Plaza-Gerbrandy analog of modal logic that we can use to reason about justified true belief.

2 About Justification Logic

Justification Logic refers to a family of logics that have a certain close relationship with LP, Artemov's Logic of Proofs [7]. In this section we will define the syntax and semantics of LP and its multi-modal extensions.

2.1 Syntax of Justification Logic

Definition 2.1 (LP^P)**.** Let P be a set of propositional letters and let $\bot$ be the propositional constant for falsity. Then the *language of* LP^P is given by the following grammar.

$$\mathsf{LP}^P \text{ is } \begin{cases} \varphi & ::= & p \mid \bot \mid \varphi_1 \supset \varphi_2 \mid t{:}\varphi & \text{for } p \in P \\ t & ::= & c_i \mid x_j \mid t_1 + t_2 \mid t_1 \cdot t_2 \mid\, !t & \text{for } i, j \in \mathbb{N} \end{cases}$$

Some abbreviations for formulas: let $\neg\varphi$ be $\varphi \supset \bot$, let $\varphi \wedge \psi$ be $\neg(\varphi \supset \neg\psi)$, and let $\varphi \equiv \psi$ be $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$. For each $i, j \in \mathbb{N}$, we call c_i a *constant* and x_j a *variable*. The constants and variables make up the *atomic terms*, and (general) *terms* are formed according to the grammar for t. The intended reading of $t{:}\varphi$ is "φ is known for reason t." We will omit the superscript P in LP^P, instead writing LP, when doing so ought not cause confusion.

Adding a finite number of modals to the language of LP gives us the language of Justification Logic, JL.

Definition 2.2 ($\mathsf{JL}^{P,n}$)**.** Let P be a set of propositional letters, let n be a positive integer, let $\bar{n} := \{i \in \mathbb{N} : i \leq n\}$, and let $\bot$ be the propositional constant for falsity. Then the *language of (n-agent) Justification Logic (over P)*, written $\mathsf{JL}^{P,n}$, is given by the following grammar.

$$\mathsf{JL}^{P,n} \text{ is } \begin{cases} \varphi & ::= & p \mid \bot \mid \varphi_1 \supset \varphi_2 \mid t{:}\varphi \mid K_i\varphi & \text{for } p \in P \text{ and } i \in \bar{n} \\ t & ::= & c_i \mid x_j \mid t_1 + t_2 \mid t_1 \cdot t_2 \mid\, !t & \text{for } i, j \in \mathbb{N} \end{cases}$$

The meanings of $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \equiv \psi$, *constant*, *variable*, *atomic term*, and *term* are as in Definition 2.1. The intended reading of $K_i\varphi$ is "φ is known to i for some reason." We will omit one or both of the P or n when writing $\mathsf{JL}^{P,n}$ when doing so ought not cause confusion.

2.2 Theory for LP

Definition 2.3. The *theory of* LP consists of the following axiom schemes and rules of inference.

- *Classical propositional logic*

  A. A finite collection of axiom schemes for classical propositional logic

  RA. Modus Ponens: infer ψ from $\varphi \supset \psi$ and φ

- *Evidence management*

  LP1. $t\!:\!(\varphi \supset \psi) \supset (s\!:\!\varphi \supset (t \cdot s)\!:\!\psi)$

  LP2. $t\!:\!\varphi \supset\, !t\!:\!(t\!:\!\varphi)$

  LP3. $t\!:\!\varphi \supset (t+s)\!:\!\varphi$
  $s\!:\!\varphi \supset (t+s)\!:\!\varphi$

  LP4. $t\!:\!\varphi \supset \varphi$

  RLP. Constant Necessitation: infer $c\!:\!A$ whenever c is a constant and A is an axiom

Let us now give informal readings for the axiom schemes for evidence management. **LP1** says that if we have a reason t for an implication and a reason s for this implication's antecedent, then $t \cdot s$ is a reason for the consequent. So reasons are closed under Modus Ponens using the '$\cdot$' operation.

LP2 says that if t is a reason for φ, then $!t$ checks that t is indeed a reason for φ. So we use the '!' operation to verify our reasons.

LP3 says that if t is a reason for φ, then $t+s$ and $s+t$ are each reasons for φ. So the operation '+' allows us to combine reasons monotonically: $t+s$ is a reason for those things for which one or both of t or s is a reason.

LP4 says that having a reason for φ is sufficient for φ to be true. This tells us that our notion of evidence is quite strong, in that we cannot have a reason for something that is false.

Finally, the rule of Constant Necessitation (**RLP**) says that constants are reasons for our most basic facts, the axioms. Note that in the rule of Constant Necessitation, we infer $c\!:\!A$ whenever c is a constant c and A is an axiom, where A is any instance of the axiom schemes **A** of classical propositional logic or of the axiom schemes **LP1** through **LP4** of evidence management.

Remark 2.4. The reader whose conscience was shocked by one or more of **LP2** and **LP4** may rest assured we can safely drop one or both of these schemes, yielding various weaker notions of evidence [14].[1]

2.3 Theories for JL^n

We study a number of theories for JL^n, with the individual theory given by the choice for each positive integer $i \leq n$ of a modal logic Λ_i that is to govern the behavior of the modal K_i. For present purposes, we will choose $\Lambda_i \in \{\mathsf{K}, \mathsf{T}, \mathsf{S4}, \mathsf{S5}\}$ for each positive integer $i \leq n$. In naming these theories, we will include "LP" in the name because each theory will be defined as a certain conservative extension of the theory of LP. So for some example theories: the theory $\mathsf{LP}(\otimes_{i=1}^n \mathsf{S4})$ is the theory where each modal K_i is governed by $\mathsf{S4}$; the theory $\mathsf{LP}(\otimes_{i=1}^n \mathsf{S5})$ is like $\mathsf{LP}(\otimes_{i=1}^n \mathsf{S4})$, except that each K_i is $\mathsf{S5}$; the theory $\mathsf{LP}(\mathsf{S4} \otimes \mathsf{S5})$ is the theory where $n = 2$, the modal K_1 is $\mathsf{S4}$, and the modal K_2 is $\mathsf{S5}$. Let us now give a precise definition of such theories.

Definition 2.5 (L^{P,K_i})**.** Let P be a set of propositional letters and let i be a positive integer. The language L^{P,K_i} is given by the following grammar.

$$L^{P,K_i} \text{ is } \quad \varphi \quad ::= \quad p \mid \bot \mid \varphi_1 \supset \varphi_2 \mid K_i\varphi \quad \text{for } p \in P$$

Definition 2.6 (Theory $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)^P$)**.** Let P be a set of propositional letters and let n be a positive integer. For each positive integer $i \leq n$, let $\Lambda_i \in \{\mathsf{K}, \mathsf{T}, \mathsf{S4}, \mathsf{S5}\}$ be in the language L^{P,K_i}. Then the theory $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)^P$ is given by the following axiom schemes and rules of inference.

- *The theory of* LP^P: the axiom schemes and rules of inference for LP^P
- *Fusion of* $\{\Lambda_i\}_{i=1}^n$*:* for each positive integer $i \leq n$, the axiom schemes and rules of inference for Λ_i
- *The Connection Principle*: for each positive integer $i \leq n$, the axiom scheme $t{:}\varphi \supset K_i\varphi$

We may omit the superscript P when writing $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)^P$ when doing so ought not cause confusion.

Informally, the Connection Principle says that if there is justified true belief for something, then each agent i has true belief of that something. Artemov and Antonakos have each studied how this connection between reasons and knowledge gives us a notion of evidence-based common knowledge [1, 3]. For present purposes, we may understand the Connection Principle as a stipulation that the agents share a common source of trustworthy evidence.

Notice that in the theory $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, the rule of Constant Necessitation (**RLP**) has us infer $c{:}A$ from whenever c is a constant and A is an axiom, where A is an instance

[1]Note that if we drop **LP2**, then it is desirable to replace **RLP** with something stronger such as the following schematic description of axioms: whenever A is an axiom and c is a constant, then $c{:}A$ is also an axiom. See [14, 4] for details.

of an LP axiom scheme (whether a scheme **A** of classical propositional logic or a scheme **LP1** through **LP4** of evidence management), an instance of an axiom scheme of Λ_i for any positive integer $i \leq n$, or an instance of the Connection Principle.

2.4 The Internalization Property

One of the most important properties of Justification Logic is the *Internalization Property*, which is due to Artemov [7, 3].

Theorem 2.7 (Internalization [7, 3])**.** If φ is a theorem of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, then there is a term t such that $t{:}\varphi$ is also a theorem of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. The same also holds of the theory LP.

Proof (from [7, 3]). By induction on the length of the derivation of φ in $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. In case φ is an axiom, then $c{:}\varphi$ is a theorem by the rule of Constant Necessitation. Otherwise, if φ is not an axiom, then the last step in the derivation of φ is Modus Ponens, Constant Necessitation, or Necessitation of the modal K_i. We consider each case separately.

Suppose φ was derived by Modus Ponens from $\psi \supset \varphi$ and ψ. It then follows from the induction hypothesis that there are terms t and s such that both $t{:}(\psi \supset \varphi)$ and $s{:}\psi$ are also theorems. Applying **LP1**, we have that $(t \cdot s){:}\varphi$ is a theorem.

Suppose φ was derived by Constant Necessitation, so that φ is of the form $c{:}A$ for an axiom A. Then it follows from **LP2** that $!c{:}(c{:}A)$ is a theorem.

Suppose φ was derived by Necessitation of the modal K_i, so that φ is of the form $K_i\psi$ for a theorem ψ. It then follows from the induction hypothesis that there is a term t such that $t{:}\psi$ is also a theorem, and thus $!t{:}(t{:}\psi)$ is a theorem by **LP2**. Since $t{:}\psi \supset K_i\psi$ is an instance of the Connection Principle, $c{:}(t{:}\psi \supset K_i\psi)$ is a theorem by Constant Necessitation with constant c. Since $c{:}(t{:}\psi \supset K_i\psi)$ and $!t{:}(t{:}\psi)$ are theorems, it follows from **LP1** that $(c \cdot !t){:}(K_i\psi)$ is a theorem, which completes the proof. □

It is in the sense of the Internalization Theorem (Theorem 2.7) that we say Justification Logic *internalizes its proofs*. Internalization makes precise our intuitive understanding of terms as reasons (or evidence) for the formulas they label.

2.5 Possible Worlds Semantics

The possible world semantics for LP is due to Fitting [13]. Artemov extended this semantics to $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ [2, 5, 6, 3].

Definition 2.8. A *constant specification* is a function $\mathcal{C}$ that maps each constant c to a possibly empty set $\mathcal{C}(c)$ of formulas (in some fixed extension of the language of LP).

Definition 2.9. Let $N = (W, \{R_i\}_{i=1}^n, V)$ be a Kripke model[2] and let $\mathcal{C}$ be a constant

[2] So W is a nonempty set whose elements we call *worlds*, R_i is a binary relation on W for each positive integer $i \leq n$, and V is a *valuation on* $(W, \{R_i\}_{i=1}^n)$, which is a function that assigns to each world Γ a possibly empty set $V(\Gamma)$ of propositional letters. To say that Γ *is a world in* the Kripke model $(W, \{R_i\}_{i=1}^n, V)$ means that $\Gamma \in W$.

specification. An *evidence function (based on N and $\mathcal{C}$)* is a function $\mathcal{E}$ that assigns to each world $\Gamma \in W$ and each term t a possibly empty set $\mathcal{E}(\Gamma, t)$ of formulas (in some fixed extension of the language of LP) subject to the following restrictions.

- *Evidence Closure*
  - *Application.* If $\varphi \supset \psi \in \mathcal{E}(\Gamma, t)$ and $\varphi \in \mathcal{E}(\Gamma, s)$, then $\psi \in \mathcal{E}(\Gamma, t \cdot s)$.
  - *Verification.* If $\varphi \in \mathcal{E}(\Gamma, t)$, then $t{:}\varphi \in \mathcal{E}(\Gamma, !t)$.
  - *Sum.* $\mathcal{E}(\Gamma, t) \cup \mathcal{E}(\Gamma, s) \subseteq \mathcal{E}(\Gamma, t + s)$.
  - *Constant Specification.* $\mathcal{C}(c) \subseteq \mathcal{E}(\Gamma, c)$ for each constant c.
- *Evidence Monotonicity.* If $\varphi \in \mathcal{E}(\Gamma, t)$ and $\Gamma R_e \Delta$, then $\varphi \in \mathcal{E}(\Delta, t)$.

Informally, $\mathcal{E}(\Gamma, t)$ is understood as the set of formulas for which t is admissible as evidence at world Γ.[3]

Definition 2.10. Let $N = (W, \{R_i\}_{i=1}^n, V)$ be a Kripke model and let $\mathcal{C}$ be a constant specification. An *(n-agent) Fitting model (based on $\mathcal{C}$)* is a tuple

$$M = (W, \{R_i\}_{i=1}^n, V, \mathcal{E}, R_e) \ ,$$

where $\mathcal{E}$ is an evidence function (based on N and $\mathcal{C}$) and R_e is a reflexive and transitive binary relation on W such that $\bigcup_{i=1}^n R_i \subseteq R_e$ (that is, $\Gamma R_i \Delta$ implies $\Gamma R_e \Delta$ for each positive integer $i \leq n$). N is called the Kripke model *underlying* M. To say that Γ *is a world in* a Fitting model M means that Γ is a world in the Kripke model underlying M. A *pointed (n-agent) Fitting model* is a pair (M, Γ), where Γ is a world in the Fitting model M; the world Γ is called the *point* of (M, Γ). For a pointed Fitting model (M, Γ) and a formula $\varphi \in \mathsf{JL}^n$, we will write $M, \Gamma \models \varphi$ to mean that the formula φ is *true at* (M, Γ). The negation of $M, \Gamma \models \varphi$ is written $M, \Gamma \not\models \varphi$. Truth at a pointed Fitting model is defined by induction on the construction of φ as follows.

- $M, \Gamma \models p$ means that $p \in V(\Gamma)$, where p is a propositional letter.
- $M, \Gamma \not\models \bot$.
- $M, \Gamma \models \psi \supset \chi$ means that $M, \Gamma \models \psi$ implies $M, \Gamma \models \chi$.
- $M, \Gamma \models K_i\psi$ means that $M, \Delta \models \psi$ for each world Δ in M satisfying $\Gamma R_i \Delta$.

[3] To say that t is *admissible (as evidence)* for φ means that t is possible evidence for φ. Possible evidence is not the same as actual evidence. If t is *possible* evidence for φ, then t may be taken into account when considering the truth of φ. However, it need not be the case that t is itself sufficient to guarantee the truth of φ, something we require of *actual* evidence. Thus the viewpoint of this paper is that (actual) evidence is a rather strong notion because it is *conclusive*. This is not to say that weaker notions of evidence are not of interest—it is just that weaker notions have only recently been addressed (see [14]) because Justification Logic originated from proof-theoretic considerations (and a proof is quite a strong notion of evidence). See Remark 2.4.

- $M, \Gamma \models t{:}\varphi$ means that both $\varphi \in \mathcal{E}(\Gamma, t)$ and $M, \Delta \models \varphi$ for each world Δ in M satisfying $\Gamma R_e \Delta$.

A Fitting model M is said to have a property $\mathfrak{P}$ of binary relations (example: reflexivity) iff each binary relation in the Kripke model underlying M has property $\mathfrak{P}$; a pointed Fitting model (M, Γ) is said to have a property $\mathfrak{P}$ of binary relations iff M has property $\mathfrak{P}$.

Validity for JL-formulas is defined relative to a class of pointed Fitting models (M, Γ) such that the Kripke model underlying M has a structure appropriate for a fixed theory of Justification Logic.

Definition 2.11. Let T be an extension of the theory $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. The Kripke model $(W, \{R_i\}_{i=1}^n, V)$ is said to be *for the theory* T exactly (W, R_i) is a frame for the logic Λ_i for each positive integer $i \leq n$. (To say that (W, R_i) *is a frame for the logic* Λ_i means that each Λ_i-theorem is valid on the frame (W, R_i).) A Fitting model M based on the constant specification $\mathcal{C}$ is said to be *for the theory* T exactly when we have both that the Kripke model underlying M is for T and that $A \in \mathcal{C}(c)$ for each axiom A of T and for each constant c. A pointed Fitting model (M, Γ) is said to be *for the theory* T iff M is for the theory T.

Examples for Definition 2.11: An n-agent Fitting model is for the theory $\mathsf{LP}(\otimes_{i=1}^n \mathsf{S4})$ iff the model is reflexive and transitive and the constant specification $\mathcal{C}$ on which the model's evidence function is based includes each $\mathsf{LP}(\otimes_{i=1}^n \mathsf{S4})$-axiom in $\mathcal{C}(c)$ for each constant c; an n-agent Fitting model is for the theory $\mathsf{LP}(\otimes_{i=1}^n \mathsf{S5})$ iff the model is reflexive, transitive, and Euclidean and the constant specification $\mathcal{C}$ on which the model's evidence function is based includes each $\mathsf{LP}(\otimes_{i=1}^n \mathsf{S5})$-axiom in $\mathcal{C}(c)$ for each constant c; a two-agent Fitting model is for the theory $\mathsf{LP}(\mathsf{S4} \otimes \mathsf{S5})$ iff R_1 is reflexive and transitive, R_2 is an equivalence relation, and the constant specification $\mathcal{C}$ on which the model's evidence function is based includes each $\mathsf{LP}(\mathsf{S4} \otimes \mathsf{S5})$-axiom in $\mathcal{C}(c)$ for each constant c.

Definition 2.12. Let T be an extension of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ and let φ be a JL^n-formula. To say that φ is *valid (for T)* means that for each pointed Fitting model (M, Γ) for the theory T, we have that $M, \Gamma \models \varphi$.

Theorem 2.13 (Fitting-Artemov [13, 2, 5, 6, 3])**.** For each $\Lambda \in \{\mathsf{K}, \mathsf{T}, \mathsf{S4}, \mathsf{S5}\}$, a formula $\varphi \in \mathsf{JL}^{P,n}$ is provable in $\mathsf{LP}(\otimes_{i=1}^n \Lambda)$ if and only if φ is valid for $\mathsf{LP}(\otimes_{i=1}^n \Lambda)$. Likewise, a formula ψ in the language of LP is provable in LP if and only if ψ is valid for LP.

Corollary 2.14 ([13, 2, 5, 6, 3])**.** For each $\Lambda \in \{\mathsf{K}, \mathsf{T}, \mathsf{S4}, \mathsf{S5}\}$, the theory $\mathsf{LP}(\otimes_{i=1}^n \Lambda)$ is a conservative extension both of LP and of $\otimes_{i=1}^n \Lambda$.

2.6 Bisimulation for Justification Logic

We now define a notion of bisimulation for JL. Bisimulation allows us to study how language expressivity changes when we introduce additional syntax to reason in the language about a given kind of communication. We will use our notion of bisimulation later when we show that public announcements add expressive power to JL.

Definition 2.15. Given the Fitting model $M = (W, \{R_i\}_{i=1}^n, V, \mathcal{E}, R_e)$, a world $\Gamma \in W$, and a formula φ, to say that φ is *knowable at* Γ means that $M, \Delta \models \varphi$ for each $\Delta \in W$ with $\Gamma R_e \Delta$.

Definition 2.16. Given Fitting models

$$M_1 = (W_1, \{R_i\}_{i=1}^n, V_1, \mathcal{E}_1, R_e) \text{ and } M_2 = (W_2, \{S_i\}_{i=1}^n, V_2, \mathcal{E}_2, S_e),$$

a nonempty binary relation $B \subseteq (W_1 \times W_2)$ is a *bisimulation* between M_1 and M_2 exactly when each of the following conditions hold.

- The *frame bisimulation* conditions:

  For each binary relation R of M_1 and S of M_2 both sharing the same subscript:

  1. If $\Gamma_1 R \Delta_1$ and $\Gamma_1 B \Gamma_2$, then there is a $\Delta_2 \in W_2$ such that $\Gamma_2 S \Delta_2$ and $\Delta_1 B \Delta_2$;
  2. If $\Gamma_2 S \Delta_2$ and $\Gamma_1 B \Gamma_2$, then there is a $\Delta_1 \in W_1$ such that $\Gamma_1 R \Delta_1$ and $\Delta_1 B \Delta_2$.

- *Agreement of propositional valuation:* if $\Gamma_1 B \Gamma_2$, then $V_1(\Gamma_1) = V_2(\Gamma_2)$.

- *Agreement of evidence for knowable formulas:* if $\Gamma_1 B \Gamma_2$ and φ is knowable at Γ_1 or at Γ_2, then $\varphi \in \mathcal{E}_1(\Gamma_1, t)$ iff $\varphi \in \mathcal{E}_2(\Gamma_2, t)$ for each term t.

Two models are said to be *bisimilar* iff there exists a bisimulation between them. For pointed Fitting models (M_1, Γ_1) and (M_2, Γ_2), the points Γ_1 and Γ_2 are said to be *bisimilar* iff there is a bisimulation B between M_1 and M_2 satisfying $\Gamma_1 B \Gamma_2$, in which case we write $(M_1, \Gamma_1) \simeq_B (M_2, \Gamma_2)$. We may omit the subscript B when doing so ought not cause confusion.

Remark 2.17. By restricting to the case $n = 0$ in Definition 2.16, we obtain bisimulation for formulas in the the language of LP itself.

What is new about Definition 2.16 is the condition for evidence agreement on knowable formulas. While we could have defined bisimulation so that there is evidence agreement for *all* formulas, this turns out to be too strong of a requirement because it obscures the expressivity results we are able to obtain with this weaker notion of bisimulation. Regardless, our notion of bisimulation is correct, as the following theorem shows.

Theorem 2.18. Let M_1 and M_2 be as in Definition 2.16. If $(M_1, \Gamma_1) \simeq_B (M_2, \Gamma_2)$ and $\varphi \in \mathsf{JL}^n$, then $M_1, \Gamma_1 \models \varphi$ iff $M_2, \Gamma_2 \models \varphi$.

Proof. By induction on the construction of φ. All cases are routine except the inductive case for formulas of the form $t{:}\varphi$. We check this remaining case in detail, as follows. $M_1, \Gamma_1 \models t{:}\varphi$ means φ is knowable at Γ_1 and $\varphi \in \mathcal{E}_1(\Gamma_1, t)$. By the induction hypothesis, $\Gamma_1 B \Gamma_2$ implies φ is knowable at Γ_2 and, by the definition of bisimulation, we have $\varphi \in \mathcal{E}_2(\Gamma_2, t)$. Hence $M_2, \Gamma_2 \models t{:}\varphi$. Interchanging the models M_1 and M_2 in this argument gives the converse. □

We now give two examples of bisimilar models; each model which will be important for later results in the paper. The reader may wish to skip over the examples until they are later referenced.

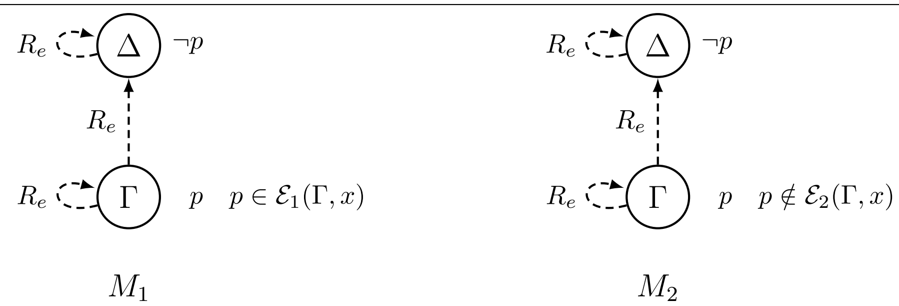

Figure 1. Diagrammatic representation of Fitting models M_1 and M_2 from Example 2.19.

Example 2.19. Let p be a propositional letter, let x be a variable, let $W := \{\Gamma, \Delta\}$, let $R_e := W \times W$, let R_i be a binary relation on W for each positive integer $i \leq n$, let $V(\Gamma) := \{p\}$, and let $V(\Delta) := \emptyset$. We will define evidence functions $\mathcal{E}_1$ and $\mathcal{E}_2$ so that the Fitting models

$$M_1 = (W, \{R_i\}_{i=1}^n, V, \mathcal{E}_1, R_e) \text{ and } M_2 = (W, \{R_i\}_{i=1}^n, V, \mathcal{E}_2, R_e)$$

are bisimilar, $p \in \mathcal{E}_1(\Gamma, x)$, and $p \notin \mathcal{E}_2(\Gamma, x)$. We first specify $\mathcal{E}_1$ and $\mathcal{E}_2$ and then show that M_1 and M_2 are bisimilar. See Figure 1 for a diagrammatic representation of these models.

Let $\mathcal{E}_1$ be the (unique) evidence function with the smallest graph such that $p \in \mathcal{E}_1(\Gamma, x)$. It then follows that $\varphi \notin \mathcal{E}_1(\Gamma, x)$ for any formula $\varphi \neq p$. We also have that $\mathcal{E}_1(\Delta, t) = \mathcal{E}_1(\Gamma, t)$ for all terms t.

For each $w \in W$ and for each term t, $\mathcal{E}_2$ is defined as follows:

$$\mathcal{E}_2(w, t) = \begin{cases} \mathcal{E}_1(w, t) & \text{if } t \neq x, \\ \emptyset & \text{if } t = x. \end{cases}$$

It is not difficult to show that $\mathcal{E}_2$ is also an evidence function; most of the evidence function properties follow immediately or else from the fact that $\mathcal{E}_1$ is itself an evidence function. It is also clear that $p \notin \mathcal{E}_2(\Gamma, x)$.

M_1 and M_2 are clearly frame bisimulations and agree on their propositional valuations. What remains is to show that they also satisfy the condition on agreement of evidence for knowable formulas. So suppose that φ is knowable at Γ in M_1. Certainly it cannot be the case that φ is p, for p is not knowable at Γ in M_1. Thus we have $\varphi \notin \mathcal{E}_1(\Gamma, x)$ because $\varphi \neq p$, and we also have $\varphi \notin \mathcal{E}_2(\Gamma, x)$ because $\mathcal{E}_2(\Gamma, x) = \emptyset$. So, in the case $t = x$, the evidence functions agree on the knowable formula φ. In case $t \neq x$, then we have $\mathcal{E}_1(\Gamma, t) = \mathcal{E}_2(\Gamma, t)$ by definition, so the evidence functions clearly agree on the knowable formula φ. The case where φ is knowable at Γ in M_2 is shown in the same way. A similar argument also applies at the world Δ. Hence M_1 and M_2 are bisimilar.

Example 2.20. Let $W := \{\Gamma\}$, $R_e = \{(\Gamma, \Gamma)\}$, let R_i be a binary relation on W for each positive integer $i \leq n$, and let $V(\Gamma) := \emptyset$. Let x be a variable. Then there are Fitting models

$$M_1 = (W, \{R_i\}_{i=1}^n, V, \mathcal{E}_1, R_e) \text{ and } M_2 = (W, \{R_i\}_{i=1}^n, V, \mathcal{E}_2, R_e)$$

such that M_1 and M_2 are bisimilar, $\bot \in \mathcal{E}_1(\Gamma, x)$, and $\bot \notin \mathcal{E}_2(\Gamma, x)$; here $\bot$ is the propositional constant for falsity. The construction of the evidence functions $\mathcal{E}_1$ and $\mathcal{E}_2$ is analogous to that given in Example 2.19, as is the verification that M_1 and M_2 are bisimilar.

3 Expressivity

Expressivity is the comparative study of the propositions expressible in two languages sharing a common semantics. The intuitive question this study attempts to answer is the following: can one language say everything that the other language can say?

3.1 Expressivity Defined

Definition 3.1. Suppose $\mathcal{I}$ is a class of interpretations for languages L_1 and L_2, where we write $I \models \varphi$ for an interpretation $I \in \mathcal{I}$ and a formula $\varphi \in (L_1 \cup L_2)$ to mean that φ is true under interpretation I. A *translation function (from L_1 to L_2 over $\mathcal{I}$)* is a function $u : L_1 \to L_2$ mapping each formula $\varphi \in L_1$ to a formula $\varphi^u \in L_2$ such that for each formula $\psi \in L_1$ and each $I \in \mathcal{I}$, we have $I \models \psi$ iff $I \models \psi^u$.

Remark 3.2. Using the notation of Definition 3.1, a translation function $u : L_1 \to L_2$ is typically defined by an induction on the depth of L_1-formulas, where the notion of L_1-formula depth is defined so as to ensure that $u : L_1 \to L_2$ is indeed a translation function [23]. But notice that we do not require our translation functions to be so well-behaved. As an example: a translation function could be non-computable.[4]

Definition 3.3 ($L_1 \hookrightarrow_{\mathcal{I}} L_2$)**.** Suppose $\mathcal{I}$ is a class of interpretations for languages L_1 and L_2 (as in Definition 3.1). Then we write $L_1 \hookrightarrow_{\mathcal{I}} L_2$ to mean that there exists a translation function from L_1 to L_2. The negation of $L_1 \hookrightarrow_{\mathcal{I}} L_2$ is written $L_1 \not\hookrightarrow_{\mathcal{I}} L_2$. When writing $L_1 \hookrightarrow_{\mathcal{I}} L_2$ or its negation, we will omit the subscript $\mathcal{I}$ when doing so ought not cause confusion.

Remark 3.4. We may view $\hookrightarrow_{\mathcal{I}}$ as a binary relation over the set $\mathcal{L}(\mathcal{I})$ of all languages L such that for each L-formula φ and for each $I \in \mathcal{I}$, we have exactly one of $I \models \varphi$ and $I \not\models \varphi$. It is easy to see that this relation is transitive and reflexive.

$L_1 \hookrightarrow L_2$ means that L_2 can express every proposition expressible by L_1, which is our formal understanding of the intuitive statement "L_2 can say at least as much as L_1 can say." Likewise, $L_1 \not\hookrightarrow L_2$ means that L_2 cannot express every proposition expressible by L_1, which is our formal understanding of the intuitive statement "L_2 cannot say everything that L_1 can say."

[4]Rohit Parikh and Evan Goris both suggested the following well-known example of a necessarily non-computable translation function. Take L_1 to be the language of arithmetic and take $L_2 := \{\bot, \top\}$, where $\bot$ is the propositional constant for falsity and $\top$ is the propositional constant for truth. Then let $\mathcal{I} := \{\mathbb{N}\}$, where $\mathbb{N}$ is the standard model of arithmetic. Since the validity problem for arithmetic is non-computable, a translation function $u : L_1 \to L_2$ is necessarily non-computable.

Definition 3.5 (Relative Expressivity)**.** Let $\mathcal{I}$ be a class of interpretations for the languages L_1 and L_2 (as in Definition 3.1).

- L_1 is *more expressive (for $\mathcal{I}$)* than L_2 iff both $L_1 \not\hookrightarrow_{\mathcal{I}} L_2$ and $L_2 \hookrightarrow_{\mathcal{I}} L_1$.
- L_1 and L_2 are *equally expressive (for $\mathcal{I}$)* iff both $L_1 \hookrightarrow_{\mathcal{I}} L_2$ and $L_2 \hookrightarrow_{\mathcal{I}} L_1$.
- L_1 and L_2 are *expressively incomparable (for $\mathcal{I}$)* iff both $L_1 \not\hookrightarrow_{\mathcal{I}} L_2$ and $L_2 \not\hookrightarrow_{\mathcal{I}} L_1$.

Example 3.6. Let P be a nonempty set of propositional letters, let $\perp$ be the propositional constant for falsity, and let $\mathcal{I}$ be the set of all truth assignments over P. We then define three propositional languages according to the following grammars.

$$
\begin{array}{llllll}
L^P_{\perp\supset\wedge\vee\neg} & \text{is} & \varphi & ::= & p \mid \perp \mid \varphi_1 \supset \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi & \text{for } p \in P \\
L^P_{\perp\supset} & \text{is} & \psi & ::= & q \mid \perp \mid \psi_1 \supset \psi_2 & \text{for } q \in P \\
L^P_{\wedge} & \text{is} & \chi & ::= & r \mid \chi_1 \wedge \chi_2 & \text{for } r \in P
\end{array}
$$

Then we have $L^P_{\perp\supset\wedge\vee\neg} \hookrightarrow L^P_{\perp\supset}$ and $L^P_{\perp\supset} \hookrightarrow L^P_{\perp\supset\wedge\vee\neg}$, which means that $L^P_{\perp\supset\wedge\vee\neg}$ and $L^P_{\perp\supset}$ are equally expressive. This is another way of saying that implication and falsity are sufficient to express all formulas in the full language $L_{\perp\supset\wedge\vee\neg}$ of propositional logic (over P).

We also have that $L^P_{\perp\supset\wedge\vee\neg} \not\hookrightarrow L^P_{\wedge}$, and $L^P_{\wedge} \hookrightarrow L^P_{\perp\supset\wedge\vee\neg}$, which means that $L^p_{\perp\supset\wedge\vee\neg}$ is more expressive than $L^P_{\wedge}$. This is another way of saying that conjunction is not enough to express the full language $L_{\perp\supset\wedge\vee\neg}$ of propositional logic (over P).

3.2 Proving Expressivity Results

To prove $L_1 \hookrightarrow L_2$, it is typical to define a translation function u from L_1 to L_2 (Definition 3.1) by induction on the depth of L_1-formulas, where the notion of L_1-formula depth is defined so as to ensure that $u : L_1 \to L_2$ is indeed a translation function [18, 15, 9, 23, 17]. This, for example, is essentially how the results in Example 3.6 are typically proven in a logic textbook [20]. Proving that $L_1 \not\hookrightarrow L_2$ is usually a bit trickier.

Definition 3.7 ($I_1 \equiv_L I_2$)**.** Suppose I_1 and I_2 are interpretations for language L. We write $I_1 \equiv_L I_2$ to mean that for each formula $\varphi \in L$, we have $I_1 \models \varphi$ exactly when $I_2 \models \varphi$.

Definition 3.8. If L is a language, then a function $d : L \to \mathbb{N}$ is called a *depth function (for L)*. If $d : L \to \mathbb{N}$ is a depth function for L, then for each $k \in \mathbb{N}$, we define

$$L^{d<k} := \{\varphi \in L : d(\varphi) < k\} \ .$$

We will omit the d in $L^{d<k}$, instead writing $L^{<k}$, when doing so ought not cause confusion.

The following lemma describes two methods for showing that $L_1 \not\hookrightarrow L_2$ [9, 23, 17].

Lemma 3.9. Suppose that $\mathcal{I}$ is a class of interpretations for languages L_1 and L_2 (as in Definition 3.1). Then each of the following statements implies $L_1 \not\hookrightarrow_{\mathcal{I}} L_2$.

1. There is a formula $\psi \in L_1$ and there are $I, I' \in \mathcal{I}$ such that $I \equiv_{L_2} I'$, that $I \models \psi$, and that $I' \not\models \psi$.

2. Suppose that $d : L_2 \to \mathbb{N}$ is a depth function for L_2. Then there is a formula $\psi \in L_1$ such that for each $k \in \mathbb{N}$, we have $I, I' \in \mathcal{I}$ such that $I \equiv_{L_2^{<k}} I'$, that $I \models \psi$, and that $I' \not\models \psi$.

Proof. We show each item implies $L_1 \not\hookrightarrow_{\mathcal{I}} L_2$.

1. Given a function $u : L_1 \to L_2$, it follows from the assumption $I \equiv_{L_2} I'$ that $I \models \psi^u$ iff $I' \models \psi^u$. Yet, by assumption, we have that $I \models \psi$ and $I' \not\models \psi$. It therefore cannot be the case both that $I \models \psi$ iff $I \models \psi^u$ and that $I' \models \psi$ iff $I' \models \psi^u$, and so for the formula $\psi \in L_1$ it is not the case that $I'' \models \psi$ iff $I'' \models \psi^t$ for all $I'' \in \mathcal{I}$. But then we have shown that $L_1 \not\hookrightarrow_{\mathcal{I}} L_2$.

2. Given a function $u : L_1 \to L_2$, there is a $k \in \mathbb{N}$ such that $\psi^u \in L_2^{<k}$. We may then apply the assumptions that $I \equiv_{L_2^{<k}} I'$, that $I \models \psi$, and that $I' \not\models \psi$ as in Item 1 to show that $L_1 \not\hookrightarrow_{\mathcal{I}} L_2$. □

Remark 3.10. Suppose that $\mathcal{I}$ is a class of pointed models for a modal language L. Then to show that $(M, \Gamma) \equiv_L (N, \Delta)$ for $(M, \Gamma) \in \mathcal{I}$ and $(N, \Delta) \in \mathcal{I}$, it is sufficient to show that (M, Γ) and (N, Δ) are bisimilar for L. So L-bisimulation is often used in Method 1 of Lemma 3.9 to show that $(M, \Gamma) \equiv_L (N, \Delta)$.

If $d : L \to \mathbb{N}$ is a depth function for this modal language L (typically d counts the maximum nested modal depth of L), then to show that $(M, \Gamma) \equiv_{L^{<k}} (N, \Delta)$ for some $k \in \mathbb{N}$, it is enough to show that (M, Γ) and (N, Δ) are k-bisimilar for L. So *finite* L-bisimulation [10] is often used in Method 2 of Lemma 3.9 to show that $(M, \Gamma) \equiv_{L^{<k}} (N, \Delta)$.

Making use of our notion of bisimulation for Justification Logic, we will apply Method 1 of Lemma 3.9 in order to show that adding public announcements to JL makes the language more expressive (for the class of all pointed Fitting models).

4 Public Announcements

A public announcement of the formula φ operates on a possible worlds model by deleting all those worlds in which φ does not hold [18, 15, 23]. The public announcement of φ should be thought of as a truthful public communication that φ is the case, which has the intuitive effect of making it common knowledge that the $\neg\varphi$ worlds are inconsistent and so should be eliminated from consideration. It is in this sense that the public announcement of φ communicates the proposition φ to all the agents in an n-agent possible worlds model.

4.1 Public Announcements Defined

Public announcements appear as labeled modalities: $[\varphi]\psi$ means that ψ holds after the public announcement of φ. The intuitive reading of $[\varphi]\psi$ is "ψ after announcing φ."

Definition 4.1 ($L[]$)**.** If L is a language, then the language L *with public announcements*, written $L[]$, is the extension of L obtained by adding brackets (for formation of public announcement formulas) and admitting an additional rule of formula formation for public announcement formulas: if φ and ψ are formulas, then so is $[\varphi]\psi$.

We now define the truth of a public announcement formula in a Fitting model.

Definition 4.2. If Γ is a world in the Fitting model $M = (W, \{R_i\}_{i=1}^n, V, \mathcal{E}, R_e)$, then $M, \Gamma \models [\varphi]\psi$ means that $M, \Gamma \models \varphi$ implies $M|\varphi, \Gamma \models \psi$, where $M|\varphi$ is the submodel of M defined by φ. That is, if there is an $\Omega \in W$ such that $M, \Omega \models \varphi$, then we define the model $M|\varphi$ as follows.

$$M|\varphi := (W^\varphi, \{R_i^\varphi\}_{i=1}^n, V^\varphi, \mathcal{E}^\varphi, R_e^\varphi)$$

where

- $W^\varphi := \{\Delta \in W : M, \Delta \models \varphi\}$
- $R_i^\varphi := R_i \cap (W^\varphi \times W^\varphi)$ for each positive integer $i \leq n$
- $V^\varphi(\Delta) := V(\Delta)$ for each $\Delta \in W^\varphi$
- $\mathcal{E}^\varphi(\Delta, t) := \mathcal{E}(\Delta, t)$ for each $\Delta \in W^\varphi$ and each term t
- $R_e^\varphi := R_e \cap (W^\varphi \times W^\varphi)$

This definition also works for the public announcement of φ in a Kripke model M by omitting both the evidence function $\mathcal{E}$ and the relation R_e. We will write $\langle\varphi\rangle\psi$ as an abbreviation for $\neg[\varphi]\neg\psi$. To understand the semantics of $\langle\varphi\rangle\psi$, observe that the scheme $\langle\varphi\rangle\psi \equiv \big(\varphi \wedge [\varphi]\psi\big)$ is valid.

The following correctness lemma follows by induction on the construction of JL-formulas.

Lemma 4.3 (Correctness)**.** Let φ be a JL-formula. If there is a world Γ in a Fitting model M such that $M, \Gamma \models \varphi$, then $M|\varphi$ is a Fitting model.

Public announcements are so-named by way of analogy: if p is a propositional letter and C is the common knowledge modal,[5] then $[p]Cp$ is valid; that is, after the announcement of p, it is common knowledge that p is true. But this is only an analogy, since neither $[\varphi]C\varphi$ nor $[\varphi]\varphi$ are valid for every formula φ in the language of modal logic.[6] Nonetheless, formulas φ for which $[\varphi]\varphi$ is in fact valid will play a special role in the next section, so we provide these formulas with a name.

Definition 4.4 ([22, 23])**.** Let T be a theory for a language L. To say that a formula $\varphi \in L$ is *successful (for T)* means that the formula $[\varphi]\varphi$ in the language of $T[]$ is valid.

[5] The modal C behaves as reachability: $M, \Gamma \models C\varphi$ means that φ holds at each world reachable from Γ by a path in $(\bigcup_{i=1}^n R_i)^*$, the reflexive transitive closure of $\bigcup_{i=1}^n R_i$.

[6] A well-known counterexample: $p \wedge \neg K_1 p$. See [22] for details.

4.2 JL with Public Announcements

The Plaza-Gerbrandy Theorem shows that adding public announcements to epistemic logic does not make the language more expressive.[7]

Definition 4.5 ($\mathsf{EL}^{P,n}$)**.** Let P be a set of propositional letters, let $\bot$ be the propositional constant for falsity, let $n \geq 1$ be an integer, and let $\bar{n} := \{i \in \mathbb{N} : i \leq n\}$. The *language of (n-agent) epistemic logic (over P)*, written $\mathsf{EL}^{P,n}$, is given by the following grammar.

$$\mathsf{EL}^{P,n} \text{ is } \quad \varphi \ ::= \ p \mid \bot \mid \varphi_1 \supset \varphi_2 \mid K_i\varphi \quad \text{for } p \in P \text{ and } i \in \bar{n}$$

We will omit one or both of P or n when writing $\mathsf{EL}^{P,n}$ when doing so ought not cause confusion.

Theorem 4.6 (Plaza-Gerbrandy [18, 15])**.** $\mathsf{EL}^{P,n}[]$ and $\mathsf{EL}^{P,n}$ are equally expressive (for any class of pointed Kripke models).

The following theorem shows that we cannot hope for a Plaza-Gerbrandy result for JL.

Theorem 4.7. Let P be a nonempty set of propositional letters. Then $\mathsf{JL}^{P,n}[]$ is more expressive than $\mathsf{JL}^{P,n}$ for the class $\mathcal{I}$ of all pointed Fitting models.

Proof. That $\mathsf{JL}^{P,n} \hookrightarrow_{\mathcal{I}} \mathsf{JL}^{P,n}[]$ is obvious. Now observe that for the models M_1 and M_2 of Example 2.19, we have that $(M_1, \Gamma_1) \simeq (M_2, \Gamma_2)$, from which it follows by Theorem 2.18 that $(M_1, \Gamma_1) \equiv_{\mathsf{JL}^{P,n}} (M_2, \Gamma_2)$. But notice that $M_1, \Gamma_1 \models [p]x\!:\!p$ and $M_1, \Gamma_2 \not\models [p]x\!:\!p$ for a propositional letter $p \in P$. It follows by Method 1 of Lemma 3.9 that $\mathsf{JL}^{P,n}[] \not\hookrightarrow_{\mathcal{I}} \mathsf{JL}^{P,n}$. □

Corollary 4.8. Let P be a nonempty set of propositional letters. Then $\mathsf{LP}^{P}[]$ is more expressive than the language of LP^{P} for the class of all pointed Fitting models.

Proof. Dropping the relations R_i and S_i in the models M_1 and M_2 of Example 2.19 gives us bisimilar Fitting models for LP, and so we may then follow the proof of Theorem 4.7. □

As Evan Goris observed, the formula $[p]x\!:\!p$ has an interesting interpretation, which we explain by way of the following proposition.

Proposition 4.9 (Evan Goris, personal communication)**.** Let Γ be a world in a Fitting model $M = (W, \{R_i\}_{i=1}^n, V, \mathcal{E}, R_e)$. Then for a propositional letter p, we have that

$$M, \Gamma \models [p]x\!:\!p \text{ iff } \Big(M, \Gamma \models p \text{ implies } p \in \mathcal{E}(\Gamma, x) \Big).$$

[7] The Plaza-Gerbrandy Theorem breaks down in multi-agent epistemic logic with common knowledge [9, 23], though public announcements are again definable if the language of epistemic logic with common knowledge is extended by introducing a notion of *relativized* common knowledge [21, 23]. Whereas the (unary) common knowledge modality C behaves as reachability along a path in $(\bigcup_{i=1}^n R_i)^*$, relativized common knowledge is a binary modality C^r that behaves as restricted reachability: $C^r(\varphi, \psi)$ holds iff ψ holds in each world reachable from Γ via a path in $(\bigcup_{i=1}^n R_i)^*$ whose worlds all satisfy φ. Then $C\varphi \equiv C^r(\top, \varphi)$ is valid, and it can be shown that public announcements are definable in the extended language containing C^r [21, 23].

Proof. Suppose that both $M, \Gamma \models [p]x\!:\!p$ and $M, \Gamma \models p$. It then follows from the definition of truth of a public announcement in a pointed model that $M|p, \Gamma \models x\!:\!p$. This implies $p \in \mathcal{E}^p(\Gamma, x)$ by the definition of truth of a formula of the form $x\!:\!p$. (Here $\mathcal{E}^p$ is as in Definition 4.2.) But then $p \in \mathcal{E}(\Gamma, x)$ because $\mathcal{E}^p(\Gamma, x) = \mathcal{E}(\Gamma, x)$ by definition.

Conversely, suppose $M, \Gamma \models p$ implies $p \in \mathcal{E}(\Gamma, x)$. In the case $M, \Gamma \not\models p$, we have $M, \Gamma \models [p]x\!:\!p$ trivially, so assume $M, \Gamma \models p$ and thus that $p \in \mathcal{E}(\Gamma, x)$. To see that $M|p, \Gamma \models x\!:\!p$, it remains to be shown that p is knowable at Γ in $M|p$. But this follows immediately: by our definition of the public announcement operation and the fact that p is a propositional letter, we have that p holds at every world of $M|p$. □

So JL with public announcements can describe evidence admissibility for certain formulas, though evidence admissibility is usually a strictly semantic notion. Noting that p is successful, we are led to the following generalization of Proposition 4.9.

Theorem 4.10. Let P be a nonempty set of propositional letters and let Γ be a world in a Fitting model $M = (W, \{R_i\}_{i=1}^n, V, \mathcal{E}, R_e)$. If a formula φ is successful, then we have for each term t that

$$M, \Gamma \models [\varphi]t\!:\!\varphi \text{ iff } \Big(M, \Gamma \models \varphi \text{ implies } \varphi \in \mathcal{E}(\Gamma, t)\Big).$$

Proof. $M, \Gamma \models [\varphi]t\!:\!\varphi$ means that if $M, \Gamma \models \varphi$, then we have both that (a) $\varphi \in \mathcal{E}^\varphi(\Gamma, t)$ and also that (b) $M|\varphi, \Delta \models \varphi$ for each $\Delta \in W^\varphi$ with $\Gamma R^\varphi \Delta$. (Here $\mathcal{E}^\varphi$, W^φ, and R^φ are defined as in Definition 4.2.) But $M, \Gamma \models \varphi$ and φ successful together imply that $M|\varphi, \Omega \models \varphi$ for all $\Omega \in W^\varphi$, which in particular implies (b). Since we assumed that $M, \Gamma \models \varphi$ implies $\varphi \in \mathcal{E}(\Gamma, t)$ and we have that $M, \Gamma \models \varphi$ also implies $\mathcal{E}(\Gamma, t) = \mathcal{E}^\varphi(\Gamma, t)$, then $M, \Gamma \models \varphi$ and φ successful together imply both (a) and (b), which gives us the desired result. □

4.3 Letterless JL with Public Announcements

In the previous section, we considered expressivity for $\mathsf{JL}^{P,n}$ with $P \neq \emptyset$. In this section we look at expressivity for *letterless* JL.

Definition 4.11. For each language L introduced in this paper, there is a set P of propositional letters such that every L-formula contains propositional letters all in P; we will say that this P *corresponds to* L. Now to say that one of these in-paper languages L is *letterless* means that for the P corresponding to L, we have that $P = \emptyset$. And to say that one of these in-paper languages L is *lettered* means that L is not letterless.

Letterless JL formulas have a certain monotonicity property in reflexive Fitting models.

Lemma 4.12. Let Γ be a world in a reflexive Fitting model $M = (W, \{R_i\}_{i=1}^n, V, \mathcal{E}, R_e)$. Then for each $\varphi \in \mathsf{JL}^{\emptyset,n}$, we have that $M, \Gamma \models \varphi$ implies that $M, \Delta \models \varphi$ for all $\Delta \in W$ with $\Gamma R_e \Delta$.

Proof. By induction on the construction of $\mathsf{JL}^{\emptyset,n}$-formulas. If such a formula φ contains neither terms nor modals—so φ is in the language of letterless propositional logic—then either φ or its negation is valid. So the base cases and inductive cases for Boolean connectives are straightforward. The case for formulas of the form $t{:}\psi$ follows by the meaning of truth for formulas of this form combined with the property of Evidence Monotonicity for $\mathcal{E}$. Finally, suppose that $M,\Gamma \models K_i\chi$, from which it follows by reflexivity that $M,\Gamma \models \chi$. Applying the induction hypothesis, we have that $M,\Delta \models \chi$ for all $\Delta \in W$ with $\Gamma R_e \Delta$. Using the transitivity of R_e and the fact that $R_i \subseteq R_e$, it follows that $M,\Delta \models K_i\chi$, as desired. □

We now use the monotonicity property of Lemma 4.12 to show that public announcements do not add expressive power either to letterless JL or to letterless LP for a certain class of reflexive Fitting models.

Theorem 4.13. Define the function $u : \mathsf{JL}^{\emptyset,n}[] \to \mathsf{JL}^{\emptyset,n}$ by induction on formula construction as follows:

$$\begin{array}{lcl} \bot^u & := & \bot \\ (\varphi \supset \psi)^u & := & \varphi^u \supset \psi^u \\ (t{:}\varphi)^u & := & t{:}(\varphi^u) \\ (K_i\varphi)^u & := & K_i(\varphi^u) \\ ([\varphi]\psi)^u & := & \varphi^u \supset \psi^u \end{array}$$

Let $\mathcal{I}$ be the collection of all reflexive pointed Fitting models (M,Γ) such that for $M = (W,\{R_i\}_{i=1}^n, V, \mathcal{E}, R_e)$, we have that $\mathcal{E}$ is *compatible with* u, which means that $\varphi \in \mathcal{E}(\Delta,t)$ iff $\varphi^u \in \mathcal{E}(\Delta,t)$ for each $\Delta \in W$ and each term t. Then $\mathsf{JL}^{\emptyset,n}[]$ and $\mathsf{JL}^{\emptyset,n}$ are equally expressive for $\mathcal{I}$. In addition, $\mathsf{LP}^{\emptyset}[]$ and $\mathsf{LP}^{\emptyset}$ are equally expressive for $\mathcal{I}$.

Proof. It suffices for us to show that u is a translation function. This amounts to proving by induction on formula construction that for each $\varphi \in \mathsf{JL}^{\emptyset,n}[]$ and each $(M,\Gamma) \in \mathcal{I}$, we have that $M,\Gamma \models \varphi$ iff $M,\Gamma \models \varphi^u$. We will handle only two of the inductive cases, leaving the other cases to the reader.

Let us first show that $M,\Gamma \models t{:}\psi$ iff $M,\Gamma \models t{:}\psi^u$. Notice that the induction hypothesis implies that we have $M,\Delta \models \psi$ for all $\Delta \in W$ with $\Gamma R_e \Delta$ iff $M,\Delta \models \psi^u$ for all such Δ. Since $\mathcal{E}$ is compatible with u, the result follows.

Let us now show that $M,\Gamma \models [\psi]\chi$ iff $M,\Gamma \models \psi^u \supset \chi^u$. Now it follows from the induction hypothesis that $M,\Gamma \models \psi$ iff $M,\Gamma \models \psi^u$, so let us assume that $M,\Gamma \models \psi$ iff we have that $M,\Gamma \models \psi^u$. Applying Lemma 4.12 to our hypothesis $M,\Gamma \models \psi^u$, we have that $M,\Delta \models \psi^u$ for all $\Delta \in W$ with $\Gamma R_e \Delta$ and thus that $M,\Delta \models \psi$ for all such Δ by the induction hypothesis. But then

$$\{\Delta \in W^\psi : \Gamma R_e^\psi \Delta\} = \{\Delta \in W : \Gamma R_e \Delta\} \ ,$$

and so the tree model generated from $(M|\psi,\Gamma)$ is isomorphic to the tree model generated from (M,Γ), from which it follows by a straightforward adaptation of a standard argument in modal logic that $(M|\psi,\Gamma) \equiv_{\mathsf{JL}^{\emptyset,n}} (M,\Gamma)$.[8] Since it follows from the induction hypothesis that $M|\psi,\Gamma \models \chi$ iff $M|\psi,\Gamma \models \chi^u$, it then follows from $(M|\psi,\Gamma) \equiv_{\mathsf{JL}^{\emptyset,n}} (M,\Gamma)$ that $M|\psi,\Gamma \models \chi$

[8]See Proposition 2.6 in §2.1 of [10] for details on the standard argument.

iff $M, \Gamma \models \chi^u$. So we have shown that each of $M, \Gamma \models \psi$ and $M, \Gamma \models \psi^u$ individually implies that we have $M|\psi, \Gamma \models \chi$ iff $M, \Gamma \models \chi^u$, which means that $M, \Gamma \models [\psi]\chi$ iff $M, \Gamma \models \psi^u \supset \chi^u$, as desired. □

We conclude this section with some results on the relative expressivity of various lettered and letterless logics.

Theorem 4.14. Let P be a nonempty set of propositional letters and let $\mathcal{I}$ be the class of all pointed Fitting models.

- LP^P is more expressive than $\mathsf{LP}^\emptyset$ for $\mathcal{I}$.
- $\mathsf{JL}^{P,n}[]$ is more expressive than $\mathsf{JL}^{\emptyset,n}[]$ for $\mathcal{I}$.
- $\mathsf{JL}^{P,n}[]$ is more expressive than $\mathsf{JL}^{\emptyset,n}$ for $\mathcal{I}$.
- $\mathsf{JL}^{P,n}[]$ is more expressive than $\mathsf{LP}^P[]$ for $\mathcal{I}$.
- $\mathsf{JL}^{\emptyset,n}$ is more expressive than $\mathsf{LP}^\emptyset$ for $\mathcal{I}$.

Proof. We prove the first three items simultaneously. For each language L appearing to the left of "is more expressive than" in one of the first three items, let L' be the language appearing the the right of "is more expressive than" in that same item. Now it is clear that we have $L' \hookrightarrow_{\mathcal{I}} L$ for each such pair (L, L'), so what remains is to show $L \not\hookrightarrow_{\mathcal{I}} L'$.

Since P is nonempty, there is a propositional letter $p \in P$. Let $W := \{\Gamma\}$, let R_i be a binary relation on W for each positive integer $i \leq n$, let $V_1(\Gamma) := \{p\}$, let $V_2(\Gamma) := \emptyset$, let $R_e := \{(\Gamma, \Gamma)\}$, and let $\mathcal{E}$ be an evidence function based on $(W, \{R_i\}_{i=1}^n, V_1)$. Finally, define the Fitting models

$$M_1 := (W, \{R_i\}_{i=1}^n, V_1, \mathcal{E}, R_e) \text{ and } M_2 := (W, \{R_i\}_{i=1}^n, V_2, \mathcal{E}, R_e) \ .$$

Notice that $(M_1, \Gamma) \in \mathcal{I}$ and $(M_2, \Gamma) \in \mathcal{I}$. It is not difficult to show by an induction on the construction of L'-formulas that $(M_1, \Gamma) \equiv_{L'} (M_2, \Gamma)$ (notice that $[\varphi]\psi \equiv (\varphi \supset \psi)$ is valid in one-world Fitting models). But for the L-formula p, we have $M_1, \Gamma \models p$ and $M_2, \Gamma \not\models p$, and so $L \not\hookrightarrow_{\mathcal{I}} L'$ by Method 1 of Lemma 3.9.

We now simultaneously prove the last two items, letting L be one of the languages to the left of "is more expressive than" in one of the last two items and letting L' be the language to the right of "is more expressive than" in that same item. First, it is clear that $L' \hookrightarrow_{\mathcal{I}} L$. To see that $L \not\hookrightarrow_{\mathcal{I}} L'$, we assume the notation from the previous paragraph. Now define $R_1^1 := \emptyset$ and $R_1^2 := \{(\Gamma, \Gamma)\}$. We define the Fitting models N_1 and N_2 as follows:

$$\begin{aligned} N_1 &:= (W, \{R_1^1\} \cup \{R_i\}_{i=2}^n, V_2, \mathcal{E}, R_e) \\ N_2 &:= (W, \{R_1^2\} \cup \{R_i\}_{i=2}^n, V_2, \mathcal{E}, R_e) \end{aligned}$$

Observe that $(N_1, \Gamma) \in \mathcal{I}$ and $(N_2, \Gamma) \in \mathcal{I}$. It is not difficult to see by an induction on the construction of L'-formulas that $(N_1, \Gamma) \equiv_{L'} (N_2, \Gamma)$. But notice that for the L-formula $K_1\bot$, we have $N_1, \Gamma \models K_1\bot$ and $N_2, \Gamma \not\models K_1\bot$, and so $L \not\hookrightarrow_{\mathcal{I}} L'$ by Method 1 of Lemma 3.9. □

Our expressivity results are summarized in Figure 2 (on Page 30).

5 Extending JL for Announcement Definability

While we have seen that public announcements do not add expressivity to letterless JL for a certain class of reflexive Fitting models, public announcements do indeed add expressivity to JL with propositional letters, so there is no hope for a Plaza-Gerbrandy Theorem for JL with propositional letters.

Our task of the present section is to extend the language JL in a way that brings the language closer to its Fitting semantics. We will see that this extension, called E, has the property that adding announcements does not increase language expressivity (for a certain class of Fitting models), so this extension may be considered a Plaza-Gerbrandy analog of modal logic, in the sense that we can reason about knowledge (justified true belief) in a language in which public announcements are definable.

5.1 Syntax

Definition 5.1 ($\mathsf{E}^{P,n}$)**.** Let P be a set of propositional letters, let n be a positive integer, let $\bar{n} := \{i \in \mathbb{N} : i \leq n\}$, and let $\bot$ be the propositional constant for falsity. Then the language of $\mathsf{E}^{P,n}$ is given by the following grammar.

$$\mathsf{E}^{P,n} \text{ is } \begin{cases} \varphi ::= p \mid \bot \mid \varphi_1 \supset \varphi_2 \mid t{:}\varphi \mid K_i\varphi \mid J\varphi \mid t \gg \varphi & \text{for } p \in P \text{ and } i \in \bar{n} \\ t ::= c_i \mid x_j \mid t_1 + t_2 \mid t_1 \cdot t_2 \mid\, !t & \text{for } i, j \in \mathbb{N} \end{cases}$$

The meanings of $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \equiv \psi$, *constant*, *variable*, *atomic term*, and *term* are as in Definition 2.1. The intended reading of $J\varphi$ is "φ is knowable" (see Definition 2.15), and the intended reading of $t \gg \varphi$ is "t is admissible (as evidence) for φ" (see the footnote on Page 7). We will omit one or both of the P or n when writing $\mathsf{E}^{P,n}$ when doing so ought not cause confusion.

Definition 5.2. $\mathsf{E}^{P,0}$ is the language obtained from $\mathsf{E}^{P,n}$ by deleting the rule of formula formation for formulas of the form $K_i\varphi$.

So $\mathsf{E}^{P,0}$ extends the language LP^P by adding new formulas of the form $J\varphi$ and $t \gg \varphi$ with the intent that $t{:}\varphi \equiv J\varphi \wedge (t \gg \varphi)$, thereby allowing us to express the truth conditions for the formula $t{:}\varphi$ directly in the language.

5.2 Theories for E^n

As in the case of JL^n, we will study a number of theories for E^n, with the individual theory given by the choice for each positive integer $i \leq n$ of a modal logic governing the behavior of the modal K_i. For present purposes, we will choose $\Lambda_i \in \{\mathsf{K}, \mathsf{T}, \mathsf{S4}, \mathsf{S5}\}$ for each positive integer $i \leq n$. So for some example theories: the theory $\mathsf{E}(\otimes_{i=1}^n \mathsf{S4})$ is the theory where each modal K_i is governed by $\mathsf{S4}$; the theory $\mathsf{E}(\otimes_{i=1}^n \mathsf{S5})$ is like $\mathsf{E}(\otimes_{i=1}^n \mathsf{S4})$ except that each K_i is $\mathsf{S5}$; the theory $\mathsf{E}(\mathsf{S4} \otimes \mathsf{S5})$ is the theory where $n = 2$, the modal K_1 is $\mathsf{S4}$, and the modal K_2 is $\mathsf{S5}$. Let us now give a precise definition of such theories.

Definition 5.3 (Theory $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)^{P,n}$)**.** Let P be a set of propositional letters and let n be a non-negative integer. For each positive integer $i \leq n$, let $\Lambda_i \in \{\mathsf{K}, \mathsf{T}, \mathsf{S4}, \mathsf{S5}\}$ be a in the language L^{P,K_i} (Definition 2.5). Then the theory $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)^{P,n}$ is given by the following axiom schemes and rules of inference.

- *Classical propositional logic*

  A. Finite number of axiom schemes for classical propositional logic

  RA. Modus Ponens: infer ψ from φ and $\varphi \supset \psi$

- *Fusion of* $\{\Lambda_i\}_{i=1}^n$*:* for each positive integer $i \leq n$, the axiom schemes and rules of inference for Λ_i

- $\mathsf{S4}$ *for the modal* J

  J1. $J(\varphi \supset \psi) \supset (J\varphi \supset J\psi)$

  J2. $J\varphi \supset JJ\varphi$

  J3. $J\varphi \supset \varphi$

  RJ. J-Necessitation: infer $J\varphi$ from φ

- *Evidence admissibility*

  E1. $(t \gg (\varphi \supset \psi)) \supset (s \gg \varphi \ \supset \ (t \cdot s) \gg \psi)$

  E2. $(t \gg \varphi) \supset (!t \gg (t{:}\varphi))$

  E3. $(t \gg \varphi) \supset ((t + s) \gg \varphi)$
  $(s \gg \varphi) \supset ((t + s) \gg \varphi)$

  E4. $(t \gg \varphi) \supset J(t \gg \varphi)$

  RE. Infer $c \gg A$ whenever c is a constant and A is an axiom

- *Connection principles*

  C1. **a.** $t{:}\varphi \supset J\varphi$
  b. $t{:}\varphi \supset (t \gg \varphi)$
  c. $(t \gg \varphi) \supset (J\varphi \supset t{:}\varphi)$

  C2. $J\varphi \supset K_i\varphi$ for each positive integer $i \leq n$

The *theory of* $\mathsf{E}^{P,0}$ is obtained by omitting both the fusion of $\{\Lambda_i\}_{i=1}^n$ and also the connection principle **C2**. Naming convention: since the schemes **C1a**, **C1b**, and **C1c** are together equivalent to $t{:}\varphi \equiv J\varphi \wedge (t \gg \varphi)$, we will use the common name **C1** when referring to any one of these three schemes. We may omit one or both of the P or n when writing $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)^{P,n}$ when doing so ought not cause confusion.

Note that the rule **RE** has us infer $c \gg A$ whenever c is a constant and A is an axiom, where A is an instance of a scheme **A** of classical propositional logic, an instance of an axiom scheme of Λ_i for any positive integer $i \leq n$, an instance of a scheme **E1** through **E4** of evidence management, or an instance of a connection principle **C1** or **C2**.

5.3 The Internalization Property

Like LP, the theory $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ also has the Internalization Property.

Theorem 5.4 (Internalization)**.** If φ is a theorem of $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, then there is a term t such that $t\!:\!\varphi$ is also a theorem of $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$.

Proof. By induction on the length of the derivation of φ in $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. In case φ is an axiom, then $c \gg \varphi$ is a theorem by **RE** and $J\varphi$ is a theorem by J-Necessitation. It then follows from **C1** that $c\!:\!\varphi$ is a theorem. In the case that φ is not an axiom, then the last step in the derivation of φ is Modus Ponens, **RE**, J-Necessitation, or K_i-necessitation. We consider each case separately.

Suppose φ was derived by Modus Ponens from $\psi \supset \varphi$ and ψ. It then follows from the induction hypothesis that there are terms t and s such that both $t\!:\!(\psi \supset \varphi)$ and $s\!:\!\psi$ are also theorems. Applying **C1**, we have that both $t \gg (\psi \supset \varphi)$ and $s \gg \psi$ are theorems, and so $(t \cdot s) \gg \varphi$ is a theorem by **E1**. Since J is $\mathsf{S4}$, it follows from the theorems $\psi \supset \varphi$ and φ that $J\varphi$ is a theorem. Applying **C1** to the theorems $(t \cdot s) \gg \varphi$ and $J\varphi$, we then have that $(t \cdot s)\!:\!\varphi$ is a theorem.

Suppose φ was derived by **RE**, which implies that φ is of the form $c \gg \psi$ for a constant c and an axiom ψ. Applying **E2**, we then have that $!c \gg (c\!:\!\psi)$ is a theorem. The formula $c\!:\!\psi \supset (c \gg \psi)$ is an instance of **C1**, so it follows by **RE** that $d \gg (c\!:\!\psi \supset (c \gg \psi))$ is a theorem for a constant d. Applying **E1** to the theorems $d \gg (c\!:\!\psi \supset (c \gg \psi))$ and $!c \gg (c\!:\!\psi)$ gives us the theorem $(d \cdot !c) \gg (c \gg \psi)$. Applying J-Necessitation to the theorem $c \gg \psi$, we obtain the theorem $J(c \gg \psi)$, which we combine with the theorem $(d \cdot !c) \gg (c \gg \psi)$ using **C1** to get the theorem $(d \cdot !c)\!:\!(c \gg \psi)$.

Suppose φ was derived by J-Necessitation, which implies that φ is of the form $J\psi$ for a theorem ψ. By the induction hypothesis, there is a term t such that $t\!:\!\psi$ is a theorem and so $t \gg \psi$ is a theorem by **C1**. Applying **E2** to the theorem $t \gg \psi$, we obtain the theorem $!t \gg (t\!:\!\psi)$. The formula $t\!:\!\psi \supset J\psi$ is an instance of **C1**, so it follows from **RE** that $c \gg (t\!:\!\psi \supset J\psi)$ is a theorem for a constant c. Applying **E1** to the theorems $c \gg (t\!:\!\psi \supset J\psi)$ and and $!t \gg (t\!:\!\psi)$ gives us the theorem $(c \cdot !t) \gg (J\psi)$. Applying J-Necessitation to the theorem $J\psi$ implies that $JJ\psi$ is a theorem. It then follows from the theorems $(c \cdot !t) \gg (J\psi)$ and $JJ\psi$ by **C1** that $(c \cdot !t)\!:\!(J\psi)$ is a theorem.

Suppose φ was derived by K_i-Necessitation, which implies that φ is of the form $K_i\psi$ for a theorem ψ. By the induction hypothesis, there is a term t such that $t\!:\!\psi$ is a theorem, and then our argument in the previous paragraph yields the theorem $(c \cdot !t) \gg (J\psi)$ for a constant c. Since $J\psi \supset K_i\psi$ is an instance of **C2**, we have for a constant d that $d \gg (J\psi \supset K_i\psi)$ is a theorem by **RE**. It follows from the theorems $d \gg (J\psi \supset K_i\psi)$ and $(c \cdot !t) \gg (J\psi)$ by **E1** that $(d \cdot (c \cdot !t)) \gg (K_i\psi)$ is a theorem. By J-Necessitation of the theorem $K_i\psi$, we have that $J(K_i\psi)$ is a theorem. Combining the theorems $(d \cdot (c \cdot !t)) \gg (K_i\psi)$ and $J(K_i\psi)$ using **C1** gives us the theorem $(d \cdot (c \cdot !t))\!:\!(K_i\psi)$. □

So our intuitive understanding of terms as reasons (or evidence) for the formulas they label still works for the theory $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$.

5.4 Possible World Semantics

E^n-formulas are interpreted in pointed n-agent Fitting models by extending the inductive definition of truth in a pointed Fitting model to those formulas having one of the forms $J\varphi$ or $t \gg \varphi$.

Definition 5.5. Let $M = (W, \{R_i\}_{i=1}^n, V, \mathcal{E}, R_e)$ be an n-agent Fitting model. Truth of a formula $\varphi \in \mathsf{E}^n$ at the pointed model (M, Γ) is given by combining the following inductive cases with those in Definition 2.10.

- $M, \Gamma \models J\psi$ means that $M, \Delta \models \psi$ for each world Δ in M satisfying $\Gamma R_e \Delta$.
- $M, \Gamma \models t \gg \psi$ means that $\psi \in \mathcal{E}(\Gamma, t)$.

For an extension T of $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, to say that $\varphi \in E^n$ is *valid (for T)* means that for each pointed Fitting model (M, Γ) for theory T, we have that $M, \Gamma \models \varphi$.

Theorem 5.6. Let P be a set of propositional letters and for each positive integer $i \leq n$, let $\Lambda_i \in \{\mathsf{K}, \mathsf{T}, \mathsf{S4}, \mathsf{S5}\}$. Then a formula $\varphi \in \mathsf{E}^{P,n}$ is provable in $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ if and only if φ is valid for $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$.

Proof. In this proof, all of our Fitting models are Fitting models for $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. Now an easy induction on the length of the derivation of $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$-theorems shows that $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ is sound for pointed Fitting models: notice in particular that the axioms and rule of evidence admissibility (**E1**–**E4** and **RE**) are exactly the conditions on evidence functions and that **C1** ensures that $t{:}\varphi \equiv J\varphi \wedge (t \gg \varphi)$. So let us focus on the completeness argument.

To show that $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ is complete for Fitting models, we will give a canonical model construction. Terminology: a *set* is a set of $\mathsf{E}^{P,n}$-formulas; the *conjunction* of a finite set S, written $\bigwedge S$, is the conjunction of the formulas making up that set; a set S_c is *consistent* iff there is no finite subset $S \subseteq S_c$ such that $\bigwedge S \supset \bot$ is provable in $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$; a set is *inconsistent* iff it is not consistent; a consistent set is *maximal* iff adding any $\mathsf{E}^{P,n}$-formula not already in the set would make the resulting set inconsistent. Any consistent set can be extended to a maximal consistent set by a Lindenbaum argument (enumerate through the formulas, adding a formula to the set iff doing so would not make the resulting set inconsistent; after enumerating all formulas, the resulting set is maximal consistent). It follows from that definition of maximal consistency that that each maximal consistent set Γ satisfies the following properties: Γ contains every $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$-theorem, Γ is closed under **RA** (Modus Ponens), Γ is closed under **RE**, and Γ contains either φ or $\neg\varphi$ for each $\varphi \in \mathsf{JL}^{P,n}$. If Γ is a set, we define three subsets of Γ, as follows.

$$\begin{aligned}
\Gamma^{K_i} &:= \{\varphi : K_i\varphi \in \Gamma\} \\
\Gamma^{J} &:= \{\varphi : J\varphi \in \Gamma\} \\
\Gamma^{t} &:= \{\varphi : t \gg \varphi \in \Gamma\}
\end{aligned}$$

The *canonical model* for $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ is the tuple $M = (W, \{R_i\}_{i=1}^n, V, \mathcal{E}, R_e)$ whose components are defined as follows: W is the collection of all maximal consistent sets; for each $\Gamma, \Delta \in W$ and each positive integer $i \leq n$, we have $\Gamma R_i \Delta$ iff $\Gamma^{K_i} \subseteq \Delta$ and we have $\Gamma R_e \Delta$ iff $\Gamma^J \subseteq \Delta$; we define $V(\Gamma) := \Gamma \cap P$; and for each $\Gamma \in W$ and each term t, we define $\mathcal{E}(\Gamma, t) := \Gamma^t$. We must verify that the structure we just called the *canonical model* is in fact a Fitting model. This verification requires us to check four items, which we now state and prove in turn.

1. For each positive integer $i \leq n$, the frame (W, R_i) is a frame for Λ_i, meaning that every Λ_i-theorem is valid on (W, R_i).

   (a) Case: Λ_i is K.

   K is sound for the class of all Kripke frames.

   (b) Case: Λ_i is T.

   T is sound for the class of all reflexive Kripke frames, so it suffices for us to show that R_i is reflexive. Suppose that $K_i\varphi \in \Gamma$ for some $\Gamma \in W$. Since Λ_i contains the T-axiom for K_i, the formula $K_i\varphi \supset \varphi$ is a theorem of $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. Therefore, $K_i\varphi \supset \varphi \in \Gamma$ and hence $\varphi \in \Gamma$. So we have shown that $\Gamma^{K_i} \subseteq \Gamma$, which is what it means to say $\Gamma R_i \Gamma$. Thus R_i is reflexive.

   (c) Case: Λ_i is $\mathsf{S4}$.

   $\mathsf{S4}$ is sound for the class of all reflexive and transitive Kripke frames, so it suffices for us to show that R_i is reflexive and transitive. The argument for reflexivity is as in the previous case, so let us focus on transitivity. Suppose that $\Gamma R_i \Delta$ and $\Delta R_i \Omega$ for some $\Gamma, \Delta, \Omega \in W$. This means that $\Gamma^{K_i} \subseteq \Delta$ and $\Delta^{K_i} \subseteq \Omega$. Now suppose that $K_i\varphi \in \Gamma$. Since Λ_i contains the $\mathsf{4}$-axiom for K_i, the formula $K_i\varphi \supset K_iK_i\varphi$ is a theorem of $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. Therefore, $K_i\varphi \supset K_iK_i\varphi \in \Gamma$ and hence $K_iK_i\varphi \in \Gamma$. Since $\Gamma^{K_i} \subseteq \Delta$ and $\Delta^{K_i} \subseteq \Omega$, we then have $\varphi \in \Omega$. Thus $\Gamma^{K_i} \subseteq \Omega$, which means that $\Gamma R_i \Omega$. Thus R_i is also transitive.

   (d) Case: Λ_i is $\mathsf{S5}$.

   $\mathsf{S5}$ is sound for the class of all reflexive, transitive, and Euclidean Kripke frames, so it suffices for us to show that R_i is reflexive, transitive, and Euclidean. The first two of these were shown above, so let us focus on the Euclidean property. Suppose that $\Gamma R_i \Delta$ and that $\Gamma R_i \Omega$, which means that $\Gamma^{K_i} \subseteq \Delta$ and $\Gamma^{K_i} \subseteq \Omega$. Were it not the case that $\Delta R_i \Omega$, then it would follow from the definition of R_i that we would have $K_i\varphi \in \Delta$ with $\varphi \notin \Omega$. It would then follow by maximal consistency that $\neg\varphi \in \Omega$. Since $\Gamma^{K_i} \subseteq \Omega$, it would follow that $K_i\varphi \notin \Gamma$. Applying maximal consistency, we would have $\neg K_i\varphi \in \Gamma$. Since Λ_i contains the $\mathsf{5}$-axiom for K_i, the formula $\neg K_i\varphi \supset K_i\neg K_i\varphi$ is a theorem of $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. Therefore, $\neg K_i\varphi \supset K_iK_i\varphi \in \Gamma$, from which it would follow that $K_i\neg K_i\varphi \in \Gamma$ and thus that $\neg K_i\varphi \in \Delta$ because $\Gamma^{K_i} \subseteq \Delta$. But then we would have both $K_i\varphi \in \Delta$ and $\neg K_i\varphi \in \Delta$, contradicting the consistency of Δ. Therefore it must have been the case that $\Delta R_i \Omega$, and so R_i is indeed Euclidean.

2. R_e is reflexive and transitive.

   Delete the first sentence from the case $\Lambda_i = \mathsf{T}$ and the first two sentences from the case $\Lambda_i = \mathsf{S4}$, combine the resulting paragraphs (in order), and then replace each occurrence of Λ_i with $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, each occurrence of K_i with J, and each occurrence of R_i with R_e.

3. $\bigcup_{i=1}^n R_i \subseteq R_e$.

   It suffices for us to show that $\Gamma R_i \Delta$ implies $\Gamma R_e \Delta$ for each $\Gamma, \Delta \in W$. So suppose that $\Gamma R_i \Delta$, which means that $\Gamma^{K_i} \subseteq \Delta$. To show that $\Gamma^J \subseteq \Delta$, notice that since $J\varphi \supset K_i\varphi$ is a theorem of $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, we have that $\Gamma^J \subseteq \Gamma^{K_i} \subseteq \Delta$ by the maximal consistency of Γ and of Δ. Thus $\Gamma R_e \Delta$, as desired.

4. $\mathcal{E}$ is an evidence function on $(W, \{R_i\}_{i=1}^n, V)$ for $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$.

   By our definition of $\mathcal{E}$, for a $\Gamma \in W$ and a term t, we have that $\varphi \in \mathcal{E}(\Gamma, t)$ iff $t \gg \varphi \in \Gamma$. So we see that the evidence function conditions follow almost immediately from the axioms and rule for evidence admissibility (**E1**–**E4** and **RE**). We thus provide detailed arguments only for *Application* and *Evidence Monotonicity*.

   (a) $\mathcal{E}$ satisfies the conditions of Evidence Closure.

   - *Application.* If $\varphi \supset \psi \in \mathcal{E}(\Gamma, t)$ and $\varphi \in \mathcal{E}(\Gamma, s)$, then $\psi \in \mathcal{E}(\Gamma, t \cdot s)$.

     $\varphi \supset \psi \in \mathcal{E}(\Gamma, t)$ and $\varphi \in \mathcal{E}(\Gamma, s)$ mean that $t \gg (\varphi \supset \psi) \in \Gamma$ and $s \gg \varphi \in \Gamma$. By **E1** and the maximal consistency of Γ, it follows that $(t \cdot s) \gg \psi \in \Gamma$, which means that $\psi \in \mathcal{E}(\Gamma, t \cdot s)$.
   - *Verification.* If $\varphi \in \mathcal{E}(\Gamma, t)$, then $t{:}\varphi \in \mathcal{E}(\Gamma, !t)$. (Apply **E2**.)
   - *Sum.* $\mathcal{E}(\Gamma, t) \cup \mathcal{E}(\Gamma, s) \subseteq \mathcal{E}(\Gamma, t + s)$. (Apply **E3**.)
   - *Constant Specification.* $\mathcal{C}(c) \subseteq \mathcal{E}(\Gamma, c)$ for each constant c. (Apply **RE**.)

   (b) $\mathcal{E}$ satisfies the condition of Evidence Monotonicity: if $\varphi \in \mathcal{E}(\Gamma, t)$ and $\Gamma R_e \Delta$, then $\varphi \in \mathcal{E}(\Delta, t)$.

   $\varphi \in \mathcal{E}(\Gamma, t)$ means that $t \gg \varphi \in \Gamma$. By **E4** and the maximal consistency of Γ, we have $J(t \gg \varphi) \in \Gamma$. Since $\Gamma R_e \Delta$ means that $\Gamma^J \subseteq \Delta$, we then have $t \gg \varphi \in \Delta$, which means that $\varphi \in \mathcal{E}(\Delta, t)$, as desired

So the canonical model is indeed a Fitting model. What remains is for us to prove that the canonical model satisfies the *Truth Lemma*: for each $\Gamma \in W$ and for each $\varphi \in \mathsf{JL}^{P,n}$, we have $\varphi \in \Gamma$ iff $M, \Gamma \models \varphi$. We prove the Truth Lemma by induction on formula construction. Since the base and Boolean inductive cases are straightforward, we focus on the remaining inductive cases.

- $K_i\varphi \in \Gamma$ iff $M, \Gamma \models K_i\varphi$.

  Suppose $K_i\varphi \in \Gamma$. If $\Gamma R_i \Delta$, which means $\Gamma^{K_i} \subseteq \Delta$, then $\varphi \in \Delta$. By the induction hypothesis, $M\Delta \models \varphi$. Since we chose $\Delta \in W$ satisfying $\Gamma R_i \Delta$ arbitrarily, we have shown $M, \Gamma \models K_i\varphi$.

Now suppose $K_i\varphi \notin \Gamma$. We claim that $\Gamma^{K_i} \cup \{\neg\varphi\}$ is consistent. Were it not, then there would be a finite subset $\{\psi_j : j \in I\} \subseteq \Gamma^{K_i}$, where I is a finite index set, such that $\bigwedge_{j\in I} \psi_j \supset \varphi$ is provable in $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. Since Λ_i is normal, we would then have that $\bigwedge_{j\in I} K_i\psi_j \supset K_i\varphi$ is provable in $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. Now $\{\psi_j : j \in I\} \subseteq \Gamma^{K_i}$ implies $K_i\psi_j \in \Gamma$ for each $j \in I$, so the maximal consistency of Γ would then give us $\bigwedge_{j\in I} K_i\psi_j \in \Gamma$, from which it would follow that $K_i\varphi \in \Gamma$, contradicting our assumption $K_i\varphi \notin \Gamma$. Thus $\Gamma^{K_i} \cup \{\neg\varphi\}$ is in fact consistent and so may be extended to a maximal consistent $\Delta \in W$. We then have $\Gamma^{K_i} \subseteq \Delta$ and thus that $\Gamma R_i \Delta$. Since $\neg\varphi \in \Delta$, it follows from the induction hypothesis that $M, \Delta \not\models \varphi$. We have thus shown that $M, \Gamma \not\models K_i\varphi$.

- $J\varphi \in \Gamma$ iff $M, \Gamma \models J\varphi$.

  Take the two paragraphs from the previous case and then replace each occurrence of K_i with J and each occurrence of R_i with R_e.

- $t \gg \varphi \in \Gamma$ iff $M, \Gamma \models t \gg \varphi$.

  $t \gg \varphi \in \Gamma$ means that $\varphi \in \mathcal{E}(\Gamma, t)$, which is the meaning of $M, \Gamma \models t \gg \varphi$.

- $t{:}\varphi \in \Gamma$ iff $M, \Gamma \models t{:}\varphi$.

  $M, \Gamma \models t{:}\varphi \equiv J\varphi \wedge (t \gg \varphi)$ follows by the meaning of truth in a pointed Fitting model. Since $t{:}\varphi \equiv J\varphi \wedge (t \gg \varphi)$ is also a theorem of $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, the result follows by the induction hypothesis.

This completes the proof of the Truth Lemma. Now for completeness: if φ is not provable in $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, then $\{\neg\varphi\}$ is consistent and so may be extended to a maximal consistent set $\Gamma \in W$. By the Truth Lemma, $M, \Gamma \not\models \varphi$, which completes the proof. □

Corollary 5.7 (Conservativity)**.** For each positive integer $i \leq n$, let $\Lambda_i \in \{\mathsf{K}, \mathsf{T}, \mathsf{S4}, \mathsf{S5}\}$. Then $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ is a conservative extension of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$.

Proof. The proof follows almost immediately from the soundness of $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ (Theorem 5.6) and the completeness of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ (Theorem 2.13) once we have shown that for each pointed Fitting model (M, Γ) for $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, there is a pointed Fitting model (M', Γ') for $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ such that $M, \Gamma \models \psi$ iff $M', \Gamma' \models \psi$ for each ψ in the language of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. So we take up this task first.

Let M be a Fitting model for $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, where $\mathcal{E}$ is the evidence function in M. An *extension* of $\mathcal{E}$ is any evidence function $\mathcal{F}$ based on the Kripke model underlying M and satisfying $\mathcal{F}(\Gamma, t) \supseteq \mathcal{E}(\Gamma, t)$ for all worlds Γ in M and for all terms t; we write $\mathcal{F} \supseteq \mathcal{E}$ whenever $\mathcal{F}$ is an extension of $\mathcal{E}$. $\mathcal{F} \supseteq \mathcal{E}$ is a *nontrivial extension*, written $\mathcal{F} \supsetneq \mathcal{E}$, iff there is a world Γ in M and a term t such that $\mathcal{F}(\Gamma, t) \supsetneq \mathcal{E}(\Gamma, t)$. To say that $\mathcal{F} \supseteq \mathcal{E}$ is a *minimal (extension) having property* $\mathfrak{P}$ means that $\mathcal{F}$ has property $\mathfrak{P}$ and that no $\mathcal{G}$ satisfying both $\mathcal{F} \supsetneq \mathcal{G}$ and $\mathcal{G} \supseteq \mathcal{E}$ has property $\mathfrak{P}$. The *intersection* of a nonempty collection C of extensions of $\mathcal{E}$ is the function $\bigcap C$ defined by setting $(\bigcap C)(\Gamma, t) := \bigcap_{\mathcal{F}\in C} \mathcal{F}(\Gamma, t)$ for each world Γ in

M and each term t. It can be shown that the intersection of any nonempty collection of extensions of $\mathcal{E}$ is itself an extension of $\mathcal{E}$.

Let us now define a property $\mathfrak{P}$ of evidence functions. The evidence function $\mathcal{F}$ is said to have property $\mathfrak{P}$ exactly when $\mathcal{F} \supseteq \mathcal{E}$ and the Fitting model $M(\mathcal{F})$ obtained from M by replacing the evidence function $\mathcal{E}$ with the evidence function $\mathcal{F}$ is a Fitting model for $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. It can be shown that the intersection $\mathcal{E}'$ of the class of all extensions of $\mathcal{E}$ having property $\mathfrak{P}$ is a minimal extension of $\mathcal{E}$ having property $\mathfrak{P}$. We let M' denote $M(\mathcal{E}')$.

It can be shown by induction on the construction of a formula ψ in the language of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ that $M, \Gamma \models \psi$ iff $M', \Gamma \models \psi$. Verifying this fact boils down to proving by induction on the construction of a term t that for every formula φ in the language of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, we have $\varphi \in \mathcal{E}(\Gamma, t)$ iff $\varphi \in \mathcal{E}'(\Gamma, t)$. The most difficult part of this is the right-to-left direction, and it will be instructive for us to handle a few cases in detail.

So we are to show by induction on the construction of a term t that for each formula φ in the language of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, we have that $\varphi \in \mathcal{E}'(\Gamma, t)$ implies $\varphi \in \mathcal{E}(\Gamma, t)$.

- Base case: t is the constant c.

  Suppose $\varphi \in \mathcal{E}'(\Gamma, c)$. Since every axiom of $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ that is not an axiom of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ is itself not in the language of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, it cannot be the case that φ is an axiom of $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. So $\varphi \in \mathcal{E}(\Gamma, c)$ then follows from the minimality of $\mathcal{E}'$.

- Base case: t is the variable x.

  It follows from the minimality of $\mathcal{E}'$ that $\mathcal{E}'(\Gamma, x) = \mathcal{E}(\Gamma, x)$.

- Inductive case: t is the term $s_1 \cdot s_2$.

  If for $\psi \supset \varphi$ in the language of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ we have that $\psi \supset \varphi \in \mathcal{E}'(\Gamma, t)$ and that $\psi \in \mathcal{E}'(\Gamma, s)$, then it follows from the induction hypothesis that $\psi \supset \varphi \in \mathcal{E}(\Gamma, t)$ and that $\psi \in \mathcal{E}(\Gamma, s)$, from which we have $\varphi \in \mathcal{E}(\Gamma, t \cdot s)$ by the Application property of evidence functions. So assume that for no $\psi \supset \varphi$ in the language of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ do we have both $\psi \supset \varphi \in \mathcal{E}'(\Gamma, t)$ and $\psi \in \mathcal{E}'(\Gamma, s)$. By the minimality of $\mathcal{E}'$, we then have that $\varphi \in \mathcal{E}'(\Gamma, t)$ implies $\varphi \in \mathcal{E}(\Gamma, t)$.

The remaining inductive cases—where t is $!s$ or $s_1 + s_2$—are quite similar to the case $s_1 \cdot s_2$.

So we have hopefully convinced the reader of the following: for every pointed Fitting model (M, Γ) for $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, there is a pointed Fitting model (M', Γ) for $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ such that $M, \Gamma \models \psi$ iff $M', \Gamma \models \psi$ for every formula ψ in the language of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. With this result in hand, we proceed with the proof of conservativity.

Suppose φ is a theorem of $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ in the language of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. By the soundness of $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ (Theorem 5.6), we have that φ is valid for $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, which means that φ is true in every pointed Fitting model for $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. Now for any pointed Fitting model (M, Γ) for $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ we may construct the pointed Fitting model (M', Γ) for $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ as above, and then φ is true at (M', Γ)—since φ is valid for $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$—which implies that φ is also true at (M, Γ) by what we proved above. So φ

is also valid for $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. By the completeness of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ (Theorem 2.13), we then have that φ is a theorem of $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. □

5.5 Relative Expressivity of $\mathsf{E}^n[]$

We now show that adding public announcements to the language of E^n does not increase language expressivity (for a certain class of Fitting models), so E^n may be considered a Plaza-Gerbrandy analog of EL^n (Definition 4.5) that we can use to reason about justified true belief via the theory $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$. Our method of proof will be to define by induction on the depth of $\mathsf{E}^n[]$-formulas a translation function u that assigns to each $\mathsf{E}^n[]$-formula φ an E^n-formula φ^u with the property that for a certain class of pointed Fitting models $\mathcal{I}$, we have $M, \Gamma \models \varphi \equiv \varphi^u$ for each $(M, \Gamma) \in \mathcal{I}$. So in order to proceed, we first introduce an appropriate notion of depth for the language $\mathsf{E}^{P,n}[]$.

Definition 5.8. For each term t, let $|t|$ denote the number of symbols occurring in t. We then define the depth function $d : \mathsf{E}^{P,n}[] \to \mathbb{N}$ as follows.

$$\begin{array}{lcl}
d(p) & := & 1, \text{ for each } p \in P \\
d(\bot) & := & 1 \\
d(\varphi \supset \psi) & := & 1 + \max\{d(\varphi), d(\psi)\} \\
d(t{:}\varphi) & := & 5 + |t| + d(\varphi) \\
d(K_i\varphi) & := & 1 + d(\varphi) \\
d(J\varphi) & := & 1 + d(\varphi) \\
d(t \gg \varphi) & := & 1 + |t| + d(\varphi) \\
d([\varphi]\psi) & := & (4 + d(\varphi)) \cdot d(\psi)
\end{array}$$

It follows from the meaning of $\varphi \wedge \psi$ (Definition 2.1) that

$$d(\varphi \wedge \psi) = 2 + \max\{d(\varphi), 1 + d(\psi)\} \ .$$

In Definition 5.8, the integers 5 and 4 were chosen in order to obtain the following lemma, whose proof is straightforward (though tedious).

Lemma 5.9. Let d be as in Definition 5.8. Then each of the following holds.

$$\begin{array}{lcl}
d(\varphi) & > & \psi, \text{ for each proper subformula } \psi \text{ of } \varphi \\
d(t{:}\varphi) & > & d\big(J\varphi \wedge (t \gg \varphi)\big) \\
d\big([\varphi]p\big) & > & d(\varphi \supset p), \text{ for each } p \in P \\
d\big([\varphi]\bot\big) & > & d(\varphi \supset \bot) \\
d\big([\varphi](\psi \supset \chi)\big) & > & d\big([\varphi]\psi \supset [\varphi]\chi\big) \\
d\big([\varphi]t{:}\psi\big) & > & d\big([\varphi]J\psi \wedge [\varphi](t \gg \psi)\big) \\
d\big([\varphi]K_i\psi\big) & > & d\big(\varphi \supset K_i([\varphi]\psi)\big) \\
d\big([\varphi]J\psi\big) & > & d\big(\varphi \supset J([\varphi]\psi)\big) \\
d\big([\varphi](t \gg \psi)\big) & > & d\big(\varphi \supset (t \gg \psi)\big) \\
d\big([\varphi][\psi]\chi\big) & > & d\big([\varphi \wedge [\varphi]\psi]\chi\big)
\end{array}$$

We may now show that adding public announcements to E^n does not increase language expressivity over an appropriate class of Fitting models.

Theorem 5.10. Let P be a possibly empty set of propositional letters. Let d be as in Definition 5.8. Define the function $u : \mathsf{E}^{P,n}[] \to \mathsf{E}^{P,n}$ by induction on $d(\varphi)$ of $\varphi \in \mathsf{E}^{P,n}[]$ as follows.

$$\begin{array}{lcl}
p^u & := & p, \text{ for each } p \in P \\
\bot^u & := & \bot \\
(\varphi \supset \psi)^u & := & \varphi^u \supset \psi^u \\
(t{:}\varphi)^u & := & (J\varphi)^u \wedge (t \gg \varphi)^u \\
(K_i\varphi)^u & := & K_i(\varphi^u) \\
(J\varphi)^u & := & J(\varphi^u) \\
(t \gg \varphi)^u & := & t \gg \varphi^u \\
\big([\varphi]p\big)^u & := & \varphi^u \supset p, \text{ for each } p \in P \\
\big([\varphi]\bot\big)^u & := & \varphi^u \supset \bot \\
\big([\varphi](\psi \supset \chi)\big)^u & := & \big([\varphi]\psi\big)^u \supset \big([\varphi]\chi\big)^u \\
\big([\varphi]t{:}\psi\big)^u & := & \big([\varphi]J\psi\big)^u \wedge \big([\varphi](t \gg \psi)\big)^u \\
\big([\varphi]K_i\psi\big)^u & := & \varphi^u \supset K_i\big([\varphi]\psi\big)^u \\
\big([\varphi]J\psi\big)^u & := & \varphi^u \supset J\big([\varphi]\psi\big)^u \\
\big([\varphi](t \gg \psi)\big)^u & := & \varphi^u \supset (t \gg \psi^u) \\
\big([\varphi][\psi]\chi\big)^u & := & \big([\varphi \wedge [\varphi]\psi]\chi\big)^u
\end{array}$$

That this is a correct definition by induction follows from Lemma 5.9. Now let $\mathcal{I}$ be the collection of all pointed Fitting models (M, Γ) such that for

$$M = (W, \{R_i\}_{i=1}^n, V, \mathcal{E}, R_e) \ ,$$

we have that $\mathcal{E}$ is *compatible with* u, which means that $\varphi \in \mathcal{E}(\Delta, t)$ iff $\varphi^u \in \mathcal{E}(\Delta, t)$ for each $\Delta \in W$ and each term t. Then $\mathsf{E}^{P,n}[]$ and $\mathsf{E}^{P,n}$ are equally expressive for $\mathcal{I}$.

Proof. It is sufficient for us to show that u is a translation function. To show this, we prove by induction on $d(\varphi)$ that for each $(M, \Gamma) \in \mathcal{I}$, we have $M, \Gamma \models \varphi \equiv \varphi^u$. In case $\varphi^u = \varphi$, the result is obvious, so we only need address the inductive cases where $\varphi^u \neq \varphi$. Many of these cases are by now well-known [23], so we will only check those new cases introduced in the present paper.

- Case: $M, \Gamma \models (J\varphi) \equiv (J\varphi)^u$.

  By the meaning of truth for formulas of the form $J\varphi$ and the induction hypothesis, we have $M, \Gamma \models (J\varphi) \equiv (J\varphi^u)$ and thus that $M, \Gamma \models (J\varphi) \equiv (J\varphi)^u$ by the meaning of u.

- Case: $M, \Gamma \models (t \gg \varphi) \equiv (t \gg \varphi)^u$.

  By the meaning of truth for formulas of the form $s \gg \psi$ and the compatibility of $\mathcal{E}$ with u, we have that $M, \Gamma \models (t \gg \varphi) \equiv (t \gg \varphi^u)$ and thus that $M, \Gamma \models (t \gg \varphi) \equiv (t \gg \varphi)^u$ by the definition of u.

- Case: $M, \Gamma \models (t{:}\varphi) \equiv (t{:}\varphi)^u$.

  We have $M, \Gamma \models (t{:}\varphi) \equiv \big(J\varphi \wedge (t \gg \varphi)\big)$ by the definition of truth. Note that $(\chi_1 \wedge \chi_2)^u$ is $\chi_1^u \wedge \chi_2^u$ by the definition of u. Now we have seen in the two previous cases that $M, \Gamma \models (J\varphi) \equiv (J\varphi)^u$ and that $M, \Gamma \models (t \gg \varphi) \equiv (t \gg \varphi)^u$. It follows that $M, \Gamma \models (t{:}\varphi) \equiv \big(J\varphi \wedge (t \gg \varphi)\big)^u$.

- Case: $M, \Gamma \models \big([\varphi]J\psi\big) \equiv \big([\varphi]J\psi\big)^u$.

  $\big([\varphi]J\psi\big)^u$ is $\varphi^u \supset J\big([\varphi]\psi\big)^u$ by the definition of u. By the induction hypothesis, we have $M, \Gamma \models \varphi \equiv \varphi^u$. Now in case $M, \Gamma \not\models \varphi$ and $M, \Gamma \not\models \varphi^u$, then the result follows trivially. So let us assume both that $M, \Gamma \models \varphi$ and that $M, \Gamma \models \varphi^u$. Then $M, \Gamma \models [\varphi]J\psi$ is equivalent to $M|\varphi, \Gamma \models J\psi$, which means that $M|\varphi, \Delta \models \psi$ for each $\Delta \in W^\varphi$ with $\Gamma R_e^\varphi \Delta$. But notice that this is equivalent to $M, \Delta \models [\varphi]\psi$ for each $\Delta \in W$ with $\Gamma R_e \Delta$, which, by the induction hypothesis, is itself equivalent to $M, \Delta \models ([\varphi]\psi])^u$ for all $\Delta \in W$ with $\Gamma R_e \Delta$. But this is the meaning of $M, \Gamma \models J\big([\varphi]\psi\big)^u$, which is equivalent to $M, \Gamma \models \varphi^u \supset J\big([\varphi]\psi\big)^u$ under our assumption $M, \Gamma \models \varphi^u$.

- Case: $M, \Gamma \models \big([\varphi](t \gg \psi)\big) \equiv \big([\varphi](t \gg \psi)\big)^u$.

  $\big([\varphi](t \gg \psi)\big)^u$ is $\varphi^u \supset (t \gg \psi^u)$ by the definition of u. By the induction hypothesis, we have $M, \Gamma \models \varphi \equiv \varphi^u$. Now in case $M, \Gamma \not\models \varphi$ and $M, \Gamma \not\models \varphi^u$, the result follows trivially. So let us assume both that $M, \Gamma \models \varphi$ and that $M, \Gamma \models \varphi^u$. Then $M, \Gamma \models [\varphi](t \gg \psi)$ is equivalent to $M|\varphi, \Gamma \models t \gg \psi$, which means that $\psi \in \mathcal{E}^\varphi(\Gamma, t)$. By the definition of $\mathcal{E}^\varphi$ (Definition 4.2), we have $\psi \in \mathcal{E}(\Gamma, t)$. By the compatibility of $\mathcal{E}$ with u, we have that $\psi \in \mathcal{E}(\Gamma, t)$ is equivalent to $\psi^u \in \mathcal{E}(\Gamma, t)$, and the latter means that $M, \Gamma \models t \gg \psi^u$. Since we assumed $M, \Gamma \models \varphi^u$, we have that $M, \Gamma \models t \gg \psi^u$ is equivalent to $M, \Gamma \models \varphi^u \supset (t \gg \psi^u)$.

- Case: $M, \Gamma \models \big([\varphi]t{:}\psi\big) \equiv \big([\varphi]t{:}\psi\big)^u$.

  $M, \Gamma \models \big([\varphi]t{:}\psi\big) \equiv \big([\varphi]J\psi \wedge [\varphi](t \gg \psi)\big)$ by the definition of truth. Note that $(\chi_1 \wedge \chi_2)^u$ is $\chi_1^u \wedge \chi_2^u$ by the definition of u. Now have seen in the previous two cases that $M, \Gamma \models \big([\varphi]J\psi\big) \equiv \big([\varphi]J\psi\big)^u$ and that $M, \Gamma \models \big([\varphi](t \gg \psi)\big) \equiv \big([\varphi](t \gg \psi)\big)^u$. So the result follows. □

We now compare the expressivity of our new language $\mathsf{E}^{P,n}$ with that of $\mathsf{JL}^{P,n}[]$, the latter of which we already know to be more expressive than $\mathsf{JL}^{P,n}$ for the class of all pointed Fitting models (Theorem 4.7).

Theorem 5.11. Let P be a possibly empty set of propositional letters and let $\mathcal{I}$ be the class of all pointed Fitting models.

- $\mathsf{E}^{P,n}[]$ is more expressive than $\mathsf{JL}^{P,n}[]$ for $\mathcal{I}$.
- $\mathsf{E}^{P,n}$ is more expressive than $\mathsf{JL}^{\emptyset,n}$ for $\mathcal{I}$.

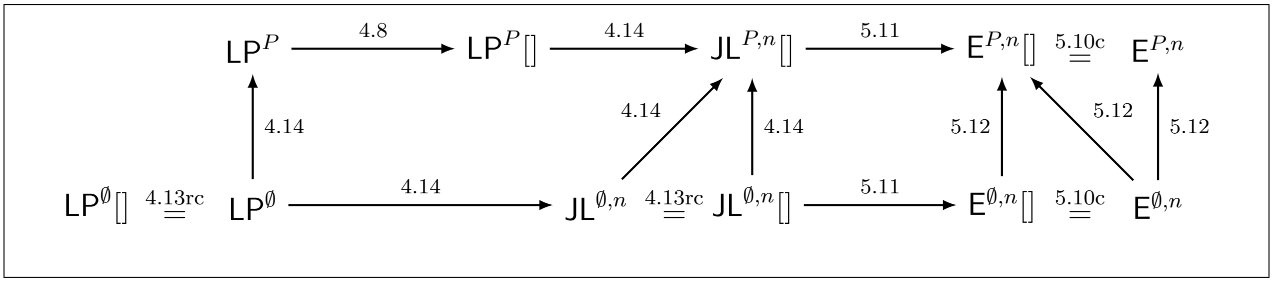

Figure 2. A summary of our expressivity results (with references) for each $n \in \mathbb{N}$ and each $P \neq \emptyset$. The results hold for the class of all pointed Fitting models unless indicated by an 'r' (for the reflexive pointed Fitting models) or a 'c' (for the pointed Fitting models *compatible with* a certain translation function—see both the referenced result and Section 6 for details).

Proof. Consider the models M_1 and M_2 from Example 2.20. It is not hard to see that $(M_1, \Gamma) \equiv_{\mathsf{JL}^{P,n}[]} (M_2, \Gamma)$. This proof, by induction on the construction of $\mathsf{JL}^{P,n}[]$-formulas, is straightforward once we observe that $[\psi]\chi \equiv (\psi \supset \chi)$ is valid in any one-world Fitting model. But observe that for $x \gg \bot$, which is both an $\mathsf{E}^{P,n}[]$-formula and an $\mathsf{E}^{P,n}$-formula, we have $M_1, \Gamma \models x \gg \bot$ and yet $M_2, \Gamma \not\models x \gg \bot$. The result thus follows from Method 1 of Lemma 3.9. This proves the first item. The second follows by the same proof because $(M_1, \Gamma) \equiv_{\mathsf{JL}^{P,n}[]} (M_2, \Gamma)$ implies $(M_1, \Gamma) \equiv_{\mathsf{JL}^{P,n}} (M_2, \Gamma)$. $\square$

We conclude this section with a theorem that reports the relative expressivity of various lettered versions of E with their corresponding letterless versions. The proof of this theorem is nearly identical to the proof of the first three items of Theorem 4.14.

Theorem 5.12. Let P be a nonempty set of propositional letters and let $\mathcal{I}$ be the class of all pointed Fitting models.

- $\mathsf{E}^{P,n}[]$ is more expressive than $\mathsf{E}^{\emptyset,n}[]$ for $\mathcal{I}$.
- $\mathsf{E}^{P,n}$ is more expressive than $\mathsf{E}^{\emptyset,n}$ for $\mathcal{I}$.
- $\mathsf{E}^{P,n}[]$ is more expressive than $\mathsf{E}^{\emptyset,n}$ for $\mathcal{I}$.

We summarize our expressivity results in Figure 2.

6 Evidence Function Compatibility

Both for letterless JL and for E, we have seen that adding public announcements does not increase language expressivity *for a certain class of pointed Fitting models* (one class for JL and another class for E). We now wish to comment on the emphasized proviso in the previous sentence.

6.1 Choosing Unnatural Interpretations

Our discussion in Section 3 suggests that we align the meaning of the intuitive statement "L_1 and L_2 express the same concepts" with the conjunction of $L_1 \hookrightarrow_{\mathcal{I}} L_2$ and $L_2 \hookrightarrow_{\mathcal{I}} L_1$ (Definition 3.5). But notice that whether $L \hookrightarrow_{\mathcal{I}} L'$ depends on the choice of a fixed class $\mathcal{I}$ of interpretations. This choice can sometimes cause our formal notion of *equally expressive* (Definition 3.5) to diverge from our informal understanding of what it means to "express the same concepts."

Example 6.1. Let P be a nonempty set of propositional letters. We define two propositional languages according to the following grammars.

$$\begin{array}{llll} L_\vee^P \text{ is} & \psi & ::= & p \mid \psi_1 \vee \psi_2 \quad \text{for } p \in P \\ L_\wedge^P \text{ is} & \chi & ::= & q \mid \chi_1 \wedge \chi_2 \quad \text{for } q \in P \end{array}$$

Let $\mathcal{I}$ be the class consisting of the (two) truth assignments over P that assign a unique truth value to every letter in P. Then $L_\vee^P$ and $L_\wedge^P$ are equally expressive for $\mathcal{I}$.

In Example 6.1, we chose a class $\mathcal{I}$ of interpretations relative to which the propositional languages of conjunction and disjunction are equally expressive. So for this class of interpretations, conjunction and disjunction "express the same concepts." But this goes against our intuition: disjunction and conjunction are different!

The problem, of course, is that we chose an "unnatural" class $\mathcal{I}$ of interpretations, which then allows us to show that the the languages $L_\vee^P$ and $L_\wedge^P$, which intuitively express different concepts, are nonetheless equally expressive. Said informally: we so severely limited the concepts (that is, the propositions) about which these languages could form sentences that the languages could then no longer describe the ways in which they differ.

So whether a formal result about equal expressivity corresponds to our intuitions about "expressing the same concepts" depends on whether the class $\mathcal{I}$ of interpretations our intuitions would have us choose is the same as the class of interpretations chosen for the formal result.

6.2 The Naturalness of Our Choices for JL

We now wish to discuss the naturalness of our choice of interpretations for our equal expressivity result in Theorem 4.13 (on $\mathsf{JL}^{\emptyset,n}[]$ and $\mathsf{JL}^{\emptyset,n}$) and for our equal expressivity result in Theorem 5.10 (on $\mathsf{E}^{P,n}[]$ and $\mathsf{E}^{P,n}$). It will be sufficient for us to restrict our comments to the latter theorem.

Now in proving $\mathsf{E}^{P,n}[] \hookrightarrow_{\mathcal{I}} \mathsf{E}^{P,n}$ in Theorem 5.10, we chose the class $\mathcal{I}$ in such a way that for each $(M,\Gamma) \in \mathcal{I}$, the evidence function $\mathcal{E}$ of M is *compatible with* the translation function $u : \mathsf{E}^{P,n}[] \to \mathsf{E}^{P,n}$, which means that $\varphi \in \mathcal{E}(\Gamma, t)$ iff $\varphi^u \in \mathcal{E}(\Gamma, t)$ for each world Γ in M. This compatibility was necessary in order to ensure that u is indeed a translation function: since u sets $(t \gg \varphi)^u := \big(t \gg (\varphi^u)\big)$, the term t in the formula $t \gg \varphi$ does not change when we apply u to $t \gg \varphi$. Thus for us to have $M,\Gamma \models t \gg \varphi$ iff $M,\Gamma \models (t \gg \varphi)^u$, the evidence function must be compatible with u.

We might like to eliminate this requirement of evidence function compatibility and then try to define a translation function $u : \mathsf{E}^{P,n}[] \to \mathsf{E}^{P,n}$ over the class of *all* pointed Fitting models. Our discussion in the previous paragraph suggests that the trick might be to allow u to change both t and φ when we apply u to $t \gg \varphi$. Unfortunately, this trick cannot work.

Theorem 6.2. Let $u : \mathsf{E}^{P,n}[] \to \mathsf{E}^{P,n}$ be a function. There is no function f mapping each term s and each formula ψ to a term $f(s,\psi)$ such that for each pointed Fitting model (M,Γ), for each formula $\varphi \in \mathsf{E}^{P,n}[]$, and for each term t, we have

$$M,\Gamma \models t \gg \varphi \text{ iff } M,\Gamma \models f(t,\varphi) \gg (\varphi^u) \ .$$

Proof. Choose $\varphi \in \mathsf{E}^{P,n}[]$ such that $\varphi^u \neq \varphi$ (example: $[\bot]\bot$). Let $W := \{\Gamma\}$, let $R_i := \emptyset$ for each positive integer $i \leq n$, let $V(\Gamma) := \emptyset$, and let $R_e := \{(\Gamma,\Gamma)\}$. Let $\mathcal{C}$ be the *empty* constant specification, which is defined by setting $\mathcal{C}(c) := \emptyset$ for each constant c. Let $\mathcal{E}$ be the evidence function based on $N := (W, \{R_i\}_{i=1}^n, V)$ and $\mathcal{C}$ such that $\mathcal{E}$ has the smallest graph subject to the restriction that $\varphi \in \mathcal{E}(\Gamma, x)$ for the variable x. By induction on term construction, we can show that for each term t and each $\psi \in \mathcal{E}(\Gamma,t)$, we have that φ is a subformula of ψ. Since φ is not a subformula of φ^u, it follows that $\varphi^u \notin \mathcal{E}(\Gamma,t)$ for each term t. Now define the Fitting model $M := (W, \{R_i\}_{i=1}^n, V, \mathcal{E}, R_e)$. Then for each term t, we have that $M,\Gamma \models x \gg \varphi$ and yet $M,\Gamma \not\models t \gg (\varphi^u)$. ☐

This leads us to the following theorem, which says that in order for $u : \mathsf{E}^{P,n}[] \to \mathsf{E}^{P,n}$ to be a translation function from $\mathsf{E}^{P,n}[]$ to $\mathsf{E}^{P,n}$ over a nonempty class $\mathcal{I}$ of pointed Fitting models, each $(M,\Gamma) \in \mathcal{I}$ must satisfy a certain general compatibility condition with respect to u.

Theorem 6.3. Suppose $u : \mathsf{E}^{P,n}[] \to \mathsf{E}^{P,n}$ is a translation function over a nonempty class $\mathcal{I}$ of pointed Fitting models. Then each $(M,\Gamma) \in \mathcal{I}$ satisfies the following condition of *generalized compatibility for* u: if $\mathcal{E}$ is the evidence function of M, then we have that $\varphi \in \mathcal{E}(\Gamma,t)$ iff $M,\Gamma \models (t \gg \varphi)^u$ for each formula $\varphi \in \mathsf{E}^{P,n}[]$ and each term t.

Proof. $M,\Gamma \models (t \gg \varphi)^u$ is equivalent to $M,\Gamma \models t \gg \varphi$ because u is a translation function. By the definition of truth, $M,\Gamma \models t \gg \varphi$ means that $\varphi \in \mathcal{E}(\Gamma,t)$. ☐

Corollary 6.4. Let f be a function mapping each term s and each formula $\psi \in \mathsf{E}^{P,n}[]$ to a term $f(s,\psi)$ such that there is a translation function $u : \mathsf{E}^{P,n}[] \to \mathsf{E}^{P,n}$ over a nonempty class $\mathcal{I}$ of pointed Fitting models that sets $(t \gg \varphi)^u := f(t,\varphi) \gg (\varphi^u)$ for each $\varphi \in \mathsf{E}^{P,n}[]$. Then each $(M,\Gamma) \in \mathcal{I}$ is *compatible with* u, which means that for the evidence function $\mathcal{E}$ of M we have $\varphi \in \mathcal{E}(\Gamma,t)$ iff $\varphi^u \in \mathcal{E}(\Gamma, f(t,\varphi))$ for each term t and each formula $\varphi \in \mathsf{E}^{P,n}[]$.

Proof. By Theorem 6.3, we have that $\varphi \in \mathcal{E}(\Gamma,t)$ iff $M,\Gamma \models (t \gg \varphi)^u$ for the evidence function $\mathcal{E}$ of an arbitrary $(M,\Gamma) \in \mathcal{I}$. Since $(t \gg \varphi)^u := f(t,\varphi) \gg (\varphi^u)$, we have that $\varphi \in \mathcal{E}(\Gamma,t)$ iff $\varphi^u \in \mathcal{E}(\Gamma, f(t,\varphi))$ by the meaning of truth for the formula $f(t,\varphi) \gg (\varphi^u)$. ☐

So for a function $u : \mathsf{E}^{P,n}[] \to \mathsf{E}^{P,n}$ to be a translation function over a nonempty class $\mathcal{I}$ of pointed Fitting models, (generalized) compatibility of each $(M,\Gamma) \in \mathcal{I}$ with u is required. This tells us that our choice of $\mathcal{I}$ in Theorem 5.10 is quite natural.

7 Conclusion

We began this paper by looking at a number of theories for Justification Logic. Using the schematic name $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ for these theories, we recalled Artemov's *Internalization Property*: for each $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ theorem φ, there is a term t such that $t{:}\varphi$ is also a $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ theorem. Internalization was central to our understanding of terms as reasons for the formulas they label, and, combined with Fitting's possible world semantics for Justification Logic, led us to the reading of $t{:}\varphi$ as "φ is known for reason t." These considerations led us to suggest that $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ is an appropriate theory for reasoning about *justified true belief*.

After studying a notion of bisimulation for the language of Justification Logic, we introduced public announcements, which are a form of truthful public communication. Using our notion of bisimulation, we saw that adding public announcements to the language of Justification Logic strictly increases language expressivity. This stands in contrast to the Plaza-Gerbrandy Theorem, which states that public announcements do not add expressivity to the language of multi-modal logic, a language whose logics are used to reason about *true belief*.

We then introduced the language E as an extension of the language of Justification Logic. Using the schematic name $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ for a conservative extension in the language E of the theory $\mathsf{LP}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$, we showed that adding public announcements to $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ does not increase language expressivity. It is in this sense that we said that the language E is a Plaza-Gerbrandy analog of multi-modal logic. Since $\mathsf{E}(\Lambda_1 \otimes \cdots \otimes \Lambda_n)$ also satisfies Internalization, we may again consider this theory appropriate for reasoning about justified true belief.

Much of our work in the paper was spent studying the relative expressivity of various letterless or lettered extensions of Justification Logic, both with and without public announcements. We summarized our results in Figure 2 on Page 30. We have also seen that many of these expressivity results were given relative to a certain restricted class of pointed Fitting models consisting of those models that are *compatible with* a given translation function. We saw at the end of the paper that this restriction was a necessary consequence of our working with Justification Logic.

Stepping back a bit from the details of our work, this paper is really a first-step in a larger project whose goal is to merge the areas of Justification Logic and Dynamic Epistemic Logic. The latter is an area that generalizes public announcements to a broader class of communications collectively called the *BMS Updates* [8, 19], though this work is done in a language extending multi-modal logic. Thus the next step in our project is to generalize the work in this paper to the case of BMS Updates. Such a generalization will provide us with systems for reasoning about very general forms of communication alongside *justified true belief*.

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