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Geodesic Merging

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Abstract

We pursue an account of merging through the use of geodesic semantics, the semantics based on the length of the shortest path on a graph. This approach has been fruitful in other areas of belief change such as revision and update. To this end, we introduce three binary merging operators of propositions defined on the graph of their valuations and we characterize them with a finite set of postulates. We also consider a revision operator defined in the extended language of pairs of propositions. This extension allows us to express all merging operators through the set of revision postulates.

1 Introduction

Belief merging is an operation that combines two or more possibly inconsistent propositions into a consistent one. Such an operation yields meaning and decision support from a diverse range of opinions from multiple agents. The study of merging in its most general form provides logical foundations for several research fields such as database fusion [7], networking [14], judgment aggregation [24], and others. Merging can be based on probabilistic or statistical methods but there might be logical methods that can be equally effective, especially in discrete qualitative frameworks where rationality assumptions are transparent.

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Several logic-based studies for merging operators [4, 29, 1, 27, 26, 23, 22, 17, 5] have led to many interesting kinds of merging. Minimization is a central idea behind merging common to all approaches. In merging, as in revision, update, contraction and other belief change operators, we are seeking the closest (distance minimization) models to the given ones. Or, if merging is defined through syntax, we are syntactically transforming a set of sentences and we are seeking a minimal intervention [2, 15]. Minimization over partial or total orderings gives rise to the most general set of postulates that govern merging (most notably in the axiomatization of [22] and axiomatization of [17]).

However, most examples of merging are defined using a single distance ordering. Here's an example: Suppose we would like to merge two sets of binary valuations on three literals $A = \{(1,0,1),(1,0,0)\}$ and $B = \{(0,0,0),(0,1,0)\}$. Using the Hamming distance (number literals where two valuations differ) $(1,0,0)$ is the A-valuation closest to B and $(0,0,0)$ is the B-valuation closest to A. If we denote the operation of merge with $\otimes$, then $A \otimes B = \{(1,0,0),(0,0,0)\}$ will be a reasonable candidate for merging based on distance considerations.

This raises the question of whether there are sets of postulates that characterize merging operators defined on metric spaces. This paper gives a positive answer, but it rests on two assumptions. First, the merging operators we characterize are binary. Second, we characterize merging operators defined on geodesic metrics (shortest path on a graph). Nevertheless, geodesic metrics encompass a great deal of spaces arising in applications such as the Hamming distance (the example above), threshold distance, as well as all integer-valued metrics (see [9, 10]). Other characterization results for merging include characterizations of arbitration [22]–a form of binary merging–and integrity constraints (IC for short) merging [17]–a form of multiset merging–using orderings. Arbitration based on distance has been characterized in [28] but it was shown that those merging operators do not correspond uniquely to the distance spaces from which they were generated.

In the next section, we will introduce the basic idea of geodesic reasoning, how it applies to merging, and present the first formal definitions. In Section 3, we define three binary merging operators through sets of logical postulates and show that these are characterized by corresponding graph theoretic operators. We conclude in Section 4.

2 Geodesic Reasoning

We will define and characterize three merging operators on graphs using the metric defined on graphs, called *geodesic*. The use of geodesic metric rests on a novel view of similarity as a derived concept. Traditionally, similarity has been conceived as a primitive concept usually represented by distance; that is, the following identification is made:

$$\text{dissimilarity} \quad = \quad \text{distance}$$

Our idea [9] is that similarity is not primitive but it can be generated by a relation of indistinguishability. This idea can be summarized by the following maxim: two objects are similar when there is a context within which they are indistinguishable. Therefore, similarity can be *measured* with degrees of indistinguishability. The more contexts in which two objects appear indistinguishable, the more similar those objects are.

For example, although two similar houses might appear different in various details when we stand in front of them, they will appear identical if we observe them from a *larger* distance x. Thus, indistinguishability at distance x implies similarity. The *smaller* the distance x at which they appear indistinguishable, the more similar the objects are.

A representation of indistinguishability by a reflexive symmetric non-transitive relation goes back to [25]. Such relations have been studied together with a set under various names such as tolerance spaces [30] and proximity spaces [3]. In modal logic, the indistinguishability relation can be thought of as an accessibility relation and, together with a set, form a Kripke frame. It can also be simply regarded as the adjacency relation of a graph and similarity may be defined through the distance map on the graph using the shortest path: given a relation R the distance from y to x is the least number of times we need to apply R in order to reach y from x. Traditionally, this kind of relation has been called *geodesic*. We have

$$\text{dissimilarity} \quad = \quad \text{geodesic distance (in a graph)}$$

because if a context of observation is represented with a power R^n of the relation R then objects with distance less than n appear indistinguishable. Using graphs with their geodesic metric generalizes several popular formalisms such as threshold and integer metrics, as well as Hamming distance (see [11]).

Geodesic semantics have been successfully developed for a variety of belief change operators such as revision, update, conditionalization, and contraction [11, 10, 12, 13]. This paper extends geodesic semantics to belief merging.

2.1 Tolerance spaces and their geodesic metric

We will use a reflexive and symmetric relation to model indistinguishability. A set equipped with such a relation is frequently called a tolerance space. In addition, we will assume that the space is connected:

Definition 1 Let X be a set and $R \subseteq X \times X$ a relation on X. Then (X, R) is called a *(connected) tolerance space* when R is reflexive, symmetric, and (X is) connected, i.e., for all $x, y \in X$ there is a non-negative integer n such that xR^ny.

In the above definition, we assume $R^0 = \mathrm{id}_X$, $R^n = R^{n-1} \circ R$ for $n > 0$.

Given a tolerance space (X, R) we can define a metric called *geodesic* with a map d_R from $X \times X$ to Z^+ (the set of non-negative integers) where

$$d_R(x, y) = \min\{n \mid xR^ny\}.$$

(We will write d instead of d_R in the context of a single R.) Note that a geodesic metric is not any integer metric. The values of the geodesic metric are determined by adjacency. The results of this paper depend heavily on this property, which can be described with: for all $x, y \in X$ such that $d(x, y) = n$ with $1 < n < \infty$ there is $z \in V$ with $z \neq x, y$ such that $d(x, y) = d(x, z) + d(z, y)$ (we use d instead of d_R if R is obvious). In particular, we can choose z so that $d(x, z) = 1$. Note here that a geodesic metric is a topological metric, that is, it satisfies identity, symmetry and triangle inequality.

The geodesic distance extends to distance between non-empty subsets with

$$d(A, B) = \min\{d(x, y) \mid x \in A, y \in B\}. \tag{1}$$

We shall also write $d(x, A)$ for $d(\{x\}, A)$. Similarly for $d(A, x)$. We will write A^c for the complement of A and A^n for the set $\{x \in X : d(A, x) \leqslant n\}$ (where $n = 0, 1, \ldots$). The proof of the following is straightforward

Lemma 2 *If A and A^c are non-empty, we have $d(A, A^c) = 1$.*

3 Merging based on a geodesic

We will define and characterize three different notions of belief merging. We will use a propositional language $\mathcal{L}$ with a finite set of atomic propositions closed under the set of classical connectives $\wedge$, $\vee$ and $\neg$ (in addition to the merging connective). An interpretation w is a function from atomic propositions to $\{T, F\}$. An interpretation extends to a map from $\mathcal{L}$ to $\{T, F\}$, using the classical interpretation of the connectives, and will be called a *model* of ϕ if it maps ϕ to T. We write W for the set of all models. If A is a set of formulas then we write $v(A)$ to denote the set of all models of A. If X is a set of models then ϕ_X denotes a formula whose set of models is X, that is, $v(\phi_X) = X$.

In the following, we will be adding a binary operator of merging, such as $\times$, to the language and we will characterize the intended interpretation of this operator using rules that govern the merging operator with respect to logical entailment. In order to interpret merging, in addition to the set of models W we mentioned above, we will assume a binary operator $\otimes$ on the subsets of W, i.e.,

$$\otimes : \mathcal{P}(W) \times \mathcal{P}(W) \to \mathcal{P}(W).$$

Then v can be extended to the rest of the language with

$$v(\phi \times \psi) = v(\phi) \otimes v(\psi).$$

In other words, the language that includes merging is interpreted on the structure $(W, \otimes)$. We will show that if we restrict the interpretation of $\times$ with an appropriate set of rules, then the set-theoretic $\otimes$ is uniquely defined from a tolerance space defined on W.

We will also assume the truth-preserving logical entailment relation and write $\phi \Rightarrow \psi$ if $v(\phi) \subseteq v(\psi)$, and we will say that ϕ implies ψ. Similarly, we write $\phi \Leftrightarrow \psi$ if $v(\phi) = v(\psi)$ and say that ϕ and ψ are are equivalent.

We say ϕ is complete, if for any propositional formula ψ, ϕ implies ψ or ϕ implies $\neg\psi$. If ϕ is complete then it can be used to syntactically represent a model, i.e. there is a model w such that $\phi_w \Leftrightarrow \phi$ (because our language is finite). A formula ϕ is consistent when $v(\phi) \neq \emptyset$, and inconsistent otherwise.

When we perform merging of two beliefs, we choose the models of the beliefs that are the most similar and therefore the closest with respect to the geodesic distance. We illustrate the process with the following examples (edges represent the reflexive symmetric tolerance relation):

Example 3 In Figure 1, let $v(A) = \{a, b\}$ and $v(B) = \{d, e\}$. Then the merging

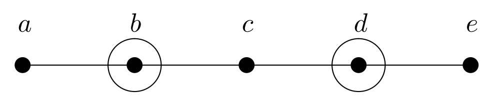

Figure 1: Non-prioritized revision merging

of A with B, denoted by $A \times B$, is interpreted by the subset $\{b, d\}$ containing those elements of $v(A)$ and $v(B)$ whose distance is the least among the elements of the two sets: the distance of b from d is 2, while the distance of a from d and e from d is 3. This form of merging is called non-prioritized revision merging, corresponds to arbitration of [22], and is a special case of the distance-based merging operator of [28].

The merging operator described has an important property, namely, it implies disjunction:

$$\phi \times \psi \Rightarrow \phi \vee \psi.$$

There are are cases, however, where this is not possible or not desirable.

Example 4 Suppose that we count the pennies saved in a jar. An initial count finds 112 pennies. A second count finds 114 pennies. It seems plausible that the merge of these two counts is the set $\{112, 113, 114\}$ as one or both counts could have been wrong. Using the propositions of the previous example, we would like that the extension of $A \times B$ is the set $\{b, c, d\}$ (see Figure 2). We will call this form of merging *convex merging*.

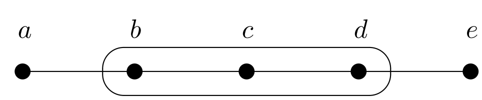

Figure 2: Convex merging

Example 5 Suppose now that a presidential candidate is characterized as left-leaning from a right-leaning pundit A, and right-leaning from a left-leaning pundit B. At least one of the pundits must be wrong and the candidate cannot be neither left-leaning or right-leaning because, presumably, it would be recognized as such from a pundit of its own kind. It seems that classifying the candidate as moderate (defined as a candidate with an equal mix of left- and right-leaning politics) is a good compromise and the result of merging A and B. We cannot risk either a left nor a right bias as both pundits are equally trusted so we have to accept the same amount of error. Also the principle of minimal change specifies that any error from both sides should not only be equal but also minimal; we need to accept the compromise that minimizes the error from both sides, so the convex merging illustrated in the previous example is not appropriate. Assuming $v(A) = \{a\}$ and $v(B) = \{e\}$, we would like that this notion of merging A with B is modeled by the set $\{c\}$ (see Figure 3). We will call this form of merging *barycentric merging*.

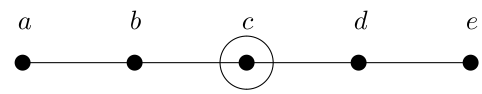

Figure 3: Barycentric merging

3.1 Non-prioritized revision merging

Non-prioritized revision merging picks the "closest" models of the propositions to be merged. It has been introduced first by [27] and its logical properties have been studied in [22]. Schlechta gave a characterization of this type of merging where closeness is defined by a general notion of distance between models [28] (see Example 3). In this section we will characterize this form of merging using the geodesic distance of a graph. Note that both Revesz and Schlechta's characterizations are based on distance notions more general than ours so their postulates are also valid in our framework. Our postulates refer explicitly neither to points (complete theories) nor to the distance, as in Schlechta's characterization. In other words we give a purely logical characterization of the underlying graph.

We begin by extending the language using the binary connective $\times$ to form $\mathcal{L}^\times$. We have that $\mathcal{L} \subseteq \mathcal{L}^\times$ and $\mathcal{L}^\times$ is closed under $\times$, $\vee$, $\wedge$ and $\neg$. Therefore, we allow arbitrary nestings of those connectives. The connective $\times$ binds stronger than $\wedge$ and $\vee$. Call the merging connective $\times$ *geodesic* it satisfies the rules of Table 1.

Rule 1 guarantees that merging returns a consistent formula if one of the formulas is consistent and Rule 2 that merging with an inconsistent formula has no effect. Using Rule 3, if two formulas are consistent together then their merging is equivalent to their conjunction. Rule 4 postulates substitution of logically equivalent formulas and Rule 5 commutativity. Notice that in Rule 6 we refer to the merging of a proposition with its negation, that is, $\phi \times \neg\phi$. We will call this merging as the *seam* of ϕ. Under this reading Rule 6 states that the part of the formula that belongs to its seam is not necessarily preserved in the seam of a weaker formula. Rule 7 implies symmetry for the underlying relation

Table 1: Geodesic merging rules

1. If ϕ is consistent then $\phi \wedge (\phi \times \psi)$ is consistent
2. If ϕ is inconsistent then $\phi \times \psi \Leftrightarrow \psi$
3. If $\phi \wedge \psi$ is consistent, then $\phi \times \psi \Leftrightarrow \phi \wedge \psi$
4. If $\phi_1 \Leftrightarrow \phi_2$ then $\phi_1 \times \psi \Leftrightarrow \phi_2 \times \psi$
5. $\phi \times \psi \Leftrightarrow \psi \times \phi$
6. If $\phi \Rightarrow \neg\psi$ then $(\psi \times \neg\psi) \wedge \phi \Rightarrow (\phi \times \neg\phi) \wedge \phi$
7. If $\phi \Rightarrow \neg\psi$ and $\phi \times \neg\phi \Rightarrow \neg\psi$ then $\psi \times \neg\psi \Rightarrow \neg\phi$
8. If $\phi \Rightarrow \neg\psi$ then $(\phi \times \psi) \wedge \psi \Leftrightarrow ((\phi \times \neg\phi) \times \psi) \wedge \psi$
9. $\phi \times \psi \Rightarrow \phi \vee \psi$

of indistinguishability. In particular, it implies that the seam of a formula and the seam of its negation are equivalent. Rule 8 allows us to define merging from less distant formulas using the seam for the induction step. Finally, Rule 9 postulates arbitration. Rule 9 implies the following

$$\phi \times \psi \Leftrightarrow ((\phi \times \psi) \wedge \phi) \vee ((\phi \times \psi) \wedge \psi) \tag{2}$$

We will characterize the class of geodesic merging operators using a merging operation on subsets based on geodesic distance, thus the use of the term "geodesic". The distance d is the geodesic distance of the tolerance space (W, R), where W is the set of models and the indistinguishability relation R is defined from the merging operator $\times$ with

$$(x, y) \in R \quad \text{if and only if} \quad \phi_y \Rightarrow \phi_x \times \neg\phi_x. \tag{3}$$

All of the results in this paper follow the same strategy: once we fix a set of rules governing a merging operator, we use (3) to define a tolerance relation and an associated geodesic distance. Then, after a series of Lemmas, we show in Theorem 12 that the set-theoretic operator $\otimes$ that interprets the merging operator is a distance based merging operator.

Lemma 6 *Suppose $\times$ is geodesic. Then, R is reflexive and symmetric.*

Proof. Reflexivity holds because, by Rule 1, $\phi_x \wedge (\phi_x \times \neg\phi_x)$ is consistent so $\phi_x \Rightarrow \phi_x \times \neg\phi_x$ because ϕ_x is complete. For symmetry, suppose $(x, y) \in R$ and $x \neq y$. We have $\phi_y \Rightarrow \phi_x \times \neg\phi_x$ therefore $\phi_y \wedge (\phi_x \times \neg\phi_x)$ is consistent which implies by (the contrapositive of) Rule 7 that $\phi_x \wedge (\phi_y \times \neg\phi_y)$ is consistent. The latter implies that $\phi_x \Rightarrow \phi_y \times \neg\phi_y$, i.e., $(y, x) \in R$, because ϕ_x is complete. ∎

The distance d between models lifts to a distance between subsets of models as described in (1). We will show that the operator $\otimes$ on subsets that interprets $\times$ is defined as follows:

$$A \otimes B = \begin{cases} \{x \in A, y \in B : d(x,y) = d(A,B)\} & \text{if } A, B \neq \varnothing \\ A \cup B & \text{otherwise.} \end{cases} \quad (4)$$

Note that both cases imply that $A \otimes B \subseteq A \cup B$.

Observe that the above definition is equivalent to the following:

$$A \otimes B = \begin{cases} (A^{d(A,B)} \cap B) \cup (B^{d(A,B)} \cap A) & \text{if } A, B \neq \varnothing \\ A \cup B & \text{otherwise.} \end{cases}$$

To see this, suppose $x \in A$ and there exists $y \in B$ such that $d(x,y) = d(A,B)$. We have that $d(x,B) = d(A,B)$, so $x \in B^{d(A,B)}$ and therefore $x \in B^{d(A,B)} \cap A$. Similarly, for $x \in B$ and the inverse direction.

The following lemma holds.

Lemma 7 *If A is non-empty then*

1. *$d(A,x) = 1$, if and only if, $x \in (A \otimes A^c) \cap A^c$.*
2. *For all $n > 1$, $d(A,x) = n$, if and only if, $d(A \otimes A^c, x) = n - 1$.*
3. *For all $n > 1$, $d(A,B) = n$, if and only if, $d(A \otimes A^c, B) = n - 1$.*

Proof. Part 1 is immediate from the Definition of $\otimes$ and Lemma 2.

For Part 2, let $n > 1$ then $d(A,x) = n$ implies that there exists $y \in A$ such that $d(y,x) = n$. Since $n > 1$, this implies that there exists z such that $d(y,z) + d(z,x) = (y,x)$ with $d(y,z) = 1$. Therefore, $d(z,x) = n - 1$ which implies that $z \notin A$ (for if not the distance of x from A would be less than n). Therefore $d(A,z) = 1$ and from above $z \in A \otimes A^c$. This implies that $d(A \otimes A^c, x) \leqslant n - 1$. Now suppose that $d(A \otimes A^c, x) = k < n - 1$. Then there would be $z' \in A \otimes A^c$ with $d(z',x) = k$. By Lemma above $d(A,z') = 1$. This implies that there is $y' \in A$ such that $d(y',z') = 1$. Hence, $d(y',x) < d(y',z') + d(z',x) = k + 1 \leqslant n - 1$ which contradicts $d(A,x) = n$.

Part 3 is a corollary of Part 2. ∎

Lemma 8 *We have*

1. *If $d(A,B) = 1$, then*

$$A \otimes B = ((A \otimes A^c) \cap B) \cup ((B \otimes B^c) \cap A).$$

2. *If $d(A,B) > 1$, then*

$$(A \otimes B) \cap B = ((A \otimes A^c) \otimes B) \cap B.$$

Proof.

Let $d(A, B) = 1$ and $x \in A \otimes B$. Then $x \in A \cup B$. Assume that $x \in A$. By the definition of $\otimes$ there exists $y \in B$ such that $d(x, y) = 1$. We have that $x \in B^c$ therefore, $x \in B \otimes B^c$. Similarly if $x \in B$.

For the other direction suppose $x \in (A \otimes A^c) \cap B$ so $x \in B$, and therefore $x \in A^c$, and $x \in A \otimes A^c$ so $d(A, x) = 1$ by Lemma 7.1. Therefore $x \in A \otimes B$.

Now let $d(A, B) > 1$ and observe that,

$$(A \otimes A^c) \otimes B = \{x \in A \otimes A^c, y \in B : d(x, y) = n - 1\},$$

therefore,

$$\begin{aligned}((A \otimes A^c) \otimes B) \cap B &= \{y \in B : d(A \otimes A^c, y) = n - 1\} \\ &= \{y \in B : d(A, y) = n\} \\ &= (A \otimes B) \cap B\end{aligned}$$

using Lemma 7.3.

∎

Lemma 9 *Suppose* $\times$ *is geodesic and* ϕ *is consistent. Then,*

$$\{x \mid d(v(\phi), x) = 1\} = v((\phi \times \neg\phi) \wedge \neg\phi).$$

Proof. For the left to right inclusion, let $d(v(\phi), x) = 1$, so there exists $y \in v(\phi)$ such that $\phi_x \Rightarrow \phi_y \times \neg\phi_y$. Notice that $x \in v(\neg\phi)$ so $\phi_x \Rightarrow (\phi_y \times \neg\phi_y) \wedge \neg\phi$. Since $\phi_y \Rightarrow \phi$, we have $\phi_x \Rightarrow (\phi \times \neg\phi) \wedge \neg\phi$, by Rule 6.

For the other direction, suppose $\phi_x \Rightarrow (\phi \times \neg\phi) \wedge \neg\phi$. Since $\phi_x \Rightarrow \neg\phi$ and $(\phi \times \neg\phi) \wedge \neg\phi_x$ is consistent, Rule 7 applies and we have that $(\phi_x \times \neg\phi_x) \wedge \phi$ is consistent. So, there exists $y \in v(\phi)$ such that $(\phi_x \times \neg\phi_x) \wedge \phi_y$ is consistent which implies $\phi_y \Rightarrow (\phi_x \times \neg\phi_x)$, because ϕ_y is complete, so $(x, y) \in R$, and by symmetry $(y, x) \in R_\times$, i.e., $d(y, x) = 1$ which implies $d(v(\phi), x) = 1$. ∎

Corollary 10

$$v(\phi \times \neg\phi) = v(\phi) \otimes v(\neg\phi).$$

Proof. If ϕ or $\neg\phi$ is inconsistent then it follows easily by Rules 1 and 2 and the definition of $\times$. If they are both consistent then we have:

$$\begin{aligned}v(\phi) \otimes v(\neg\phi) &= \{x \in v(\phi), y \in v(\neg\phi) \mid d(x, y) = 1\} \\ &= \{x \mid d(v(\phi), x) = 1\} \cup \{y \mid d(v(\neg\phi), y) = 1\} \\ &= v((\phi \times \neg\phi) \wedge \neg\phi) \cup v((\phi \times \neg\phi) \wedge \phi) \\ &= v(\phi \times \neg\phi)\end{aligned}$$

using Lemma 9. ∎

Now we can show that a geodesic generates a connected tolerance space.

Lemma 11 *If $\times$ is a geodesic merge then (W, R) is connected.*

Proof. Suppose (W, R) is not connected. Then place the (at least two) connected components of W in two subsets A and A^c. Since $A^c \neq \emptyset$ then, $\neg\phi_A$ is consistent, and by Rules 1 and 5, $(\phi_A \times \neg\phi_A) \wedge \neg\phi_A$ is consistent. Let $x \in v((\phi_A \times \neg\phi_A) \wedge \neg\phi_A)$. By Lemma 9, $d(A, x) = 1$ which implies that there exits $y \in A$ such that yRx, a contradiction. ∎

Now the following characterization theorem holds:

Theorem 12 *Let $\times$ be a geodesic merging operator. Then there exists a binary relation R such that (W, R) is a tolerance space, where W is the set of models, and the following holds*

$$v(\phi \times \psi) = v(\phi) \otimes v(\psi), \tag{5}$$

where $\otimes$ is defined by (4)*. Conversely, if an operator $\times$ satisfies* (5) *then it is a geodesic merging operator.*

Proof. We start with the first part. If ϕ or ψ is inconsistent, the first part of the theorem follows easily from Rules 1 and 2 and the definition of $\times$. If they are both consistent, then the proof of the first part of Theorem 12 is by induction on the distance $d(v(\phi), v(\psi))$.

Let $d(v(\phi), v(\psi)) = 0$ then $v(\phi) \otimes v(\psi) = v(\phi) \cap v(\psi) = v(\phi \wedge \psi)$. Since $v(\phi) \cap v(\psi) \neq \emptyset$ we have that $\phi \wedge \psi$ is consistent so by Rule 3 we have $\phi \wedge \psi \Leftrightarrow \phi \times \psi$ so $v(\phi \times \psi) = v(\phi \wedge \psi)$.

Let $d(v(\phi), v(\psi)) = 1$.

By Lemma 9, we have

$$v(\phi) \otimes v(\psi) = ((v(\phi) \otimes v(\phi)^c) \cap v(\psi)) \cup ((v(\psi) \otimes v(\psi)^c) \cap v(\phi)).$$

By Corollary 10, we have

$$v(\phi) \otimes v(\psi) = v(((\phi \times \neg\phi) \wedge \psi) \vee ((\psi \times \neg\psi) \wedge \phi)).$$

By Rule 8, we have that

$$(\phi \times \psi) \wedge \psi \Leftrightarrow ((\phi \times \neg\phi) \times \psi) \wedge \psi.$$

Using Lemma 7.3, we have that $d(v(\phi \times \neg\phi), v(\psi)) = 0$, so $\phi \times \neg\phi \wedge \psi$ is consistent so

$$(\phi \times \neg\phi) \times \psi \Leftrightarrow (\phi \times \neg\phi) \wedge \psi$$

and therefore

$$(\phi \times \psi) \wedge \psi \Leftrightarrow (\phi \times \neg\phi) \wedge \psi.$$

Similarly, we have

$$(\phi \times \psi) \wedge \phi \Leftrightarrow (\psi \times \neg\psi) \wedge \phi.$$

By Rule 9, we have

$$\phi \times \psi \Leftrightarrow ((\phi \times \neg\phi) \wedge \psi) \vee ((\psi \times \neg\psi) \wedge \phi).$$

Therefore, we have

$$v(\phi) \otimes v(\psi) = v(\phi \times \psi).$$

Assume that is true for all k, where $1 \leqslant k < n$, and let $d(v(\phi), v(\psi)) = n$. Using Rule 8, the induction hypothesis, Lemma 7.3, Corollary 10, and Lemma 8.1, we have

$$\begin{aligned} v((\phi \times \psi) \wedge \psi) &= v(((\phi \times \neg\phi) \times \psi) \wedge \psi) \\ &= ((v(\phi) \otimes v(\neg\phi)) \otimes v(\psi)) \cap v(\psi) \\ &= (v(\phi) \otimes v(\psi)) \cap v(\psi). \end{aligned}$$

Similarly,

$$v((\phi \times \psi) \wedge \phi) = (v(\phi) \otimes v(\psi)) \cap v(\phi).$$

Therefore,

$$\begin{aligned} v(\phi \times \psi) &= v(((\phi \times \psi) \wedge \phi) \vee ((\phi \times \psi) \wedge \psi)) \\ &= v(((\phi \times \psi) \wedge \phi)) \cup v(((\phi \times \psi) \wedge \psi)) \\ &= ((v(\phi) \otimes v(\psi)) \cap v(\psi)) \cup ((v(\phi) \otimes v(\psi)) \cap v(\phi)) \\ &= v(\phi) \otimes v(\psi). \end{aligned}$$

The converse amounts to verifying the rules in Table 1 for the set-theoretic merging operator $\otimes$ which is straightforward (the case of Rule 8 is Lemma 8.2). ∎

The above proposition shows that the set of rules of Table 1 characterizes the class of geodesic metrics.

Now the question that arises is in what sense this is a non-prioritized revision. The answer is that the definition of merge can be based on the definition of revision:

Definition 13 A merge operator $\times$ will be called *non-prioritized revision operator* if there exists an operator $*$ such that

$$\phi \times \psi \Leftrightarrow (\phi * \psi) \vee (\psi * \phi).$$

It is not hard to show that a revision geodesic merge operator is a non-prioritized revision. It suffices to define the operator $*$. Simply let (as in [18])

$$\phi * \psi \Leftrightarrow (\phi \times \psi) \wedge \phi. \tag{6}$$

We now have the following

Proposition 14 *Let $*$ be the operator defined as above then*

$$\phi \times \psi \Leftrightarrow (\phi * \psi) \vee (\psi * \phi).$$

Table 2: Geodesic revision rules

1. $\phi * \psi \Rightarrow \psi$
2. If ψ is consistent, then $\phi * \psi$ is consistent
3. If ϕ is inconsistent, then $\phi * \psi \Leftrightarrow \psi$
4. If $\phi \wedge \psi$ is consistent, then $\phi * \psi \Leftrightarrow \phi \wedge \psi$
5. If $\psi_1 \Leftrightarrow \psi_2$ and $\phi_1 \Leftrightarrow \phi_2$ then $\phi_1 * \psi_1 \Leftrightarrow \phi_2 * \psi_2$
6. If $\psi \Rightarrow \neg\phi$ then $\phi * \psi \Leftrightarrow (\phi * \neg\phi) * \psi$
7. If $\psi \Rightarrow \neg\phi$ then $\phi * \psi \Leftrightarrow (\neg\phi * \phi) * \psi$
8. If $\phi * \psi \Leftrightarrow \chi * \psi$ then $\phi * \psi \Leftrightarrow (\phi \vee \chi) * \psi$
9. If $\psi \Rightarrow \neg\phi$ then $\phi * \psi \Rightarrow \neg\psi * \psi$
10. If $\phi \Rightarrow \neg\psi$ then $\phi * \neg\phi \Rightarrow \neg\psi$ iff $\psi * \neg\psi \Rightarrow \neg\phi$

A characterization of the operator $*$ can be given in terms of distance. Let

$$A \circledast B = \begin{cases} \{y \in B : d(A, y) = d(v(A), v(B)\} & \text{if } A, B \neq \emptyset \\ B & \text{otherwise} \end{cases}$$

or, equivalently,

$$A \circledast B = \begin{cases} A^{d(A,B)} \cap B & \text{if } A, B \neq \emptyset \\ B & \text{otherwise.} \end{cases}$$

Corollary 15 *If $*$ is defined by* (6) *then*

$$v(\phi * \psi) = v(\phi) \circledast v(\psi). \tag{7}$$

The operator $*$ is justifiably considered a revision operator because it selects the models of the formula we revise with through distance minimization as in [21] and [11]. It is useful to know what properties exactly this revision operator satisfies. This question has been answered in [11]. Call a revision operator *geodesic* if it satisfies the properties of Table 2.

The following has been proved in [11]

Proposition 16 *Given a geodesic revision operator $*$, then there exists a binary relation R, defined by* (3)*, such that (W, R) is a tolerance space, where W is the set of models, and $*$ satisfies* (7)*. Conversely, if $*$ is an operator that satisfies* (7)*, then $*$ is a geodesic revision operator.*

Table 3: Convex merging rules

1. If ϕ is consistent then $\phi \wedge (\phi \times_{\mathsf{c}} \psi)$ is consistent
2. If ϕ is inconsistent then $\phi \times_{\mathsf{c}} \psi \Leftrightarrow \psi$
3. If $\phi \wedge \psi$ is consistent, then $\phi \times_{\mathsf{c}} \psi \Leftrightarrow \phi \wedge \psi$
4. If $\phi_1 \Leftrightarrow \phi_2$ then $\phi_1 \times_{\mathsf{c}} \psi \Leftrightarrow \phi_2 \times_{\mathsf{c}} \psi$
5. $\phi \times_{\mathsf{c}} \psi \Leftrightarrow \psi \times_{\mathsf{c}} \phi$
6. If $\phi \Rightarrow \neg\psi$ then $(\psi \times_{\mathsf{c}} \neg\psi) \wedge \phi \Rightarrow (\phi \times_{\mathsf{c}} \neg\phi) \wedge \phi$
7. If $\phi \Rightarrow \neg\psi$ and $\phi \times_{\mathsf{c}} \neg\phi \Rightarrow \neg\psi$ then $\psi \times_{\mathsf{c}} \neg\psi \Rightarrow \neg\phi$
8. If $\phi \Rightarrow \neg\psi$ then $\phi \times_{\mathsf{c}} \psi \Leftrightarrow ((\phi \times_{\mathsf{c}} \neg\phi) \times_{\mathsf{c}} \psi) \vee ((\psi \times_{\mathsf{c}} \neg\psi) \times_{\mathsf{c}} \phi)$

3.2 Convex merging

The second binary operator $\times_{\mathsf{c}}$ we will consider is convex merging. This form of merging is illustrated by Example 4. The idea is not only picking the closest worlds modeling the propositions to be merged, but also including in-between worlds not necessarily belonging to the propositions to be merged.

We proceed as in the previous section: we add $\times_{\mathsf{c}}$ in the language and seek a representation of the corresponding semantic operator $\otimes_{\mathsf{c}}$. Call an operator (geodesic) *convex merging* if it satisfies the rules of Table 3. Notice that the only changes are the omission of the arbitration rule and the replacement of the induction rule with the more appropriate Rule 8.

Define a tolerance relation R on the set of models using (3) and denote the induced geodesic distance with d. Now define an operator $\otimes_{\mathsf{c}}$ on subsets with

$$A \otimes_{\mathsf{c}} B = \begin{cases} \{x : d(A,x) + d(B,x) = d(A,B)\} & \text{if } A, B \neq \emptyset \\ A \cup B & \text{otherwise.} \end{cases} \tag{8}$$

The convex merging operator behaves the same on subsets that intersect or border each other as the geodesic merging operator defined in the previous section:

Lemma 17 *For all A, B such that $d(A,B) = 0$ or 1 we have*

$$A \otimes B = A \otimes_{\mathsf{c}} B.$$

Proof. If $d(A, B) = 0$ then we have

$$A \otimes B = A \otimes_{\mathsf{c}} B = A \cap B.$$

Let $d(A, B) = 1$ and $x \in A \otimes B$. Assume that $x \in A$. We have $d(B, x) = 1$ so $d(A, x) + d(B, x) = d(A, B)$ and therefore $x \in A \otimes_{\mathsf{c}} B$. Similarly if $x \in B$. Now let $x \in A \otimes_{\mathsf{c}} B$. Since $d(A, x) + d(B, x) = 1$ then either $d(A, x) = 0$ and $d(B, x) = 1$ or $d(A, x) = 1$ and $d(B, x) = 0$. Using Lemmas 7.1 and 8.1 we have $x \in A \otimes B$ in both cases. ∎

As a result of this common behavior we have that Lemmas 7 and 8.1 hold for the convex merging operator. Lemma 9, Lemma 11, and Corollary 10 are valid for the convex merging operator as the rules used to prove those results are common to $\times$ and $\times_{\mathsf{c}}$. Further, we have the following

Lemma 18 *If $d(A, B) > 1$, then*

$$A \otimes_{\mathsf{c}} B = ((A \otimes_{\mathsf{c}} A^c) \otimes_{\mathsf{c}} B) \cup ((B \otimes_{\mathsf{c}} B^c) \otimes_{\mathsf{c}} A).$$

Proof. By the definition of $\times_{\mathsf{c}}$ we have

$$A \times_{\mathsf{c}} B = \{x : d(A, x) + d(B, x) = d(A, B)\}.$$

If $x \notin A$ then $d(A, x) = m > 0$ is equivalent to $d(A \otimes_{\mathsf{c}} A^c, x) = m - 1$. This implies that $d(A, x) + d(B, x) = d(A, B)$ is equivalent to $d(A \times_{\mathsf{c}} A^c, x) + d(B, x) = d(A, B) - 1$. Therefore,

$$x \in A \otimes_{\mathsf{c}} B \text{ is equivalent to } x \in (A \otimes_{\mathsf{c}} A^c) \otimes_{\mathsf{c}} B.$$

Similarly, if $x \notin B$

$$x \in A \otimes_{\mathsf{c}} B \text{ is equivalent to } x \in (B \otimes_{\mathsf{c}} B^c) \otimes_{\mathsf{c}} A.$$

So,

$$A \otimes_{\mathsf{c}} B = ((A \otimes_{\mathsf{c}} A^c) \otimes_{\mathsf{c}} B) \cup ((B \otimes_{\mathsf{c}} B^c) \otimes_{\mathsf{c}} A).$$

∎

The following characterization theorem holds:

Theorem 19 *Let $\times_{\mathsf{c}}$ be a geodesic convex merging operator. Then there exists a binary relation R, defined by (3), such that (W, R) is a tolerance space and the following holds*

$$v(\phi \times_{\mathsf{c}} \psi) = v(\phi) \otimes_{\mathsf{c}} v(\psi), \tag{9}$$

where $\otimes_{\mathsf{c}}$ is defined by (8). Conversely, if an operator satisfies (9) then it is a convex merging operator.

Proof. We proceed by induction exactly as in the proof of Theorem 12.

If ϕ or ψ is inconsistent or $d(v(\phi), v(\psi)) = 0$ then the proof proceeds as in the proof of Theorem 12. There are two cases left:

Let $d(v(\phi), v(\psi)) = 1$. By Lemma 8.1 (adjusted for $\times_{\mathsf{c}}$), we have

$$v(\phi) \otimes_{\mathsf{c}} v(\psi) = ((v(\phi) \otimes_{\mathsf{c}} v(\neg\phi)) \cap v(\psi)) \cup ((v(\psi) \otimes_{\mathsf{c}} v(\neg\psi)) \cap v(\phi)).$$

We have

$$(v(\phi) \otimes_{\mathsf{c}} v(\neg\phi)) \cap v(\psi) = v(\phi \times_{\mathsf{c}} \neg\phi) \cap v(\psi) = v((\phi \times_{\mathsf{c}} \neg\phi) \wedge \psi)$$

by Corollary 10 and

$$v((\phi \times_{\mathsf{c}} \neg\phi) \wedge \psi) = v((\phi \times_{\mathsf{c}} \neg\phi) \times_{\mathsf{c}} \psi)$$

using Rule 3. Similarly,

$$(v(\psi) \otimes_{\mathsf{c}} v(\neg\psi)) \cap v(\phi) = v((\psi \times_{\mathsf{c}} \neg\psi) \times_{\mathsf{c}} \phi).$$

So

$$v(\phi) \otimes_{\mathsf{c}} v(\psi) = v((\phi \times_{\mathsf{c}} \neg\phi) \times_{\mathsf{c}} \psi) \vee v((\psi \times_{\mathsf{c}} \neg\psi) \times_{\mathsf{c}} \phi)$$

and, therefore

$$v(\phi) \otimes_{\mathsf{c}} v(\psi) = v(\phi \times_{\mathsf{c}} \psi),$$

using Rule 8.

Assume that is true for all k, where $1 \leqslant k < n$, and let $d(v(\phi), v(\psi)) = n$. We have that

$$v(\phi \times_{\mathsf{c}} \psi) = v((\phi \times_{\mathsf{c}} \neg\phi) \times_{\mathsf{c}} \psi) \vee ((\psi \times_{\mathsf{c}} \neg\psi) \times_{\mathsf{c}} \phi))$$

using Rule 8 and

$$v(\phi \times_{\mathsf{c}} \psi) = ((v(\phi) \otimes_{\mathsf{c}} v(\neg\phi)) \otimes_{\mathsf{c}} v(\psi)) \cup ((v(\psi) \otimes_{\mathsf{c}} v(\neg\psi)) \otimes_{\mathsf{c}} v(\phi))$$

using the induction hypothesis from Lemma 7.3 and Corollary 10. By Lemma 18,

$$v(\phi) \otimes_{\mathsf{c}} v(\psi) = v(\phi \times_{\mathsf{c}} \psi).$$

The converse is straightforward (the case of Rule 8 is Lemma 18). ∎

Just as a contraction operator can generate a revision operator using the Levi identity, a convex merging operator generates a revision merging operator. To see that notice that we have

$$A \otimes B = (A \otimes_{\mathsf{c}} B) \cap (A \cup B).$$

However, there is no obvious analogue of the Harper identity, because $A \cup A \otimes B$ is not necessarily convex.

3.3 Barycentric merging

In this section, we will characterize the third notion of binary merging called barycentric. This notion of merging corresponds to Example 5. The elements of the barycentric merge fall between and are spaced equally from the merged subsets. The need for barycentric merging has been early identified:

Example 20 This is a modified version of Example 3.1 in [27]. Suppose two students are tutored in programming using examples in Datalog (D), SQL (S) or Query-by-example (Q). One student prefers examples in all three languages ($D \wedge S \wedge Q$) while the other prefers examples only in Query-by-example ($\neg D \wedge \neg S \wedge Q$). If we need to merge the preferences of the students, that is

$$(D \wedge S \wedge Q) \otimes (\neg D \wedge \neg S \wedge Q)$$

then non-prioritized revision merging would pick their disjunction $(D \wedge S \wedge Q) \vee (\neg D \wedge \neg S \wedge Q)$ (using as distance between two worlds the number of atoms they differ). Barycentric merging, in contrast, would pick $(\neg D \wedge S \wedge Q) \vee (D \wedge \neg S \wedge Q)$. Revesz points out that in this example barycentric merging should be chosen over geodesic merging, as the former satisfies both students (albeit not completely), while the latter, should the tutor chooses to teach all three, might result to one student dropping out of the tutoring sessions.

Again we proceed as in the previous section. We add the barycentric merging operator $\cdot$ and seek a logical characterization of a semantic barycentric operator $\odot$.

Definition 21 We call an operator (geodesic) barycentric merging if it satisfies the rules of Table 4.

Observe that Rule 9 of Table 1 does not hold in barycentric merging because the equidistant elements might not belong to the merged subsets.

None of the merging operators introduced is associative. A counterexample of association for barycentric merging appears in Figure 4:

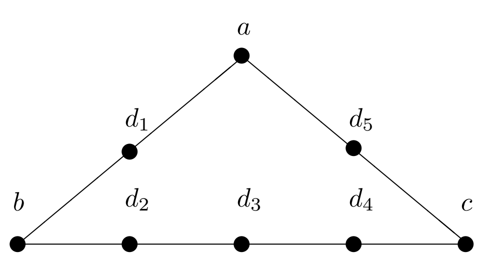

Figure 4: Barycentric revision is not associative

$$\{a\} \cdot (\{b\} \cdot \{c\}) = \{a\} \cdot \{a, d_3\} = \{a\}$$

Table 4: Barycentric merging rules

1. If ϕ is inconsistent, then $\phi \cdot \psi \Leftrightarrow \psi$
2. If $\phi \wedge \psi$ is consistent, then $\phi \cdot \psi \Leftrightarrow \phi \wedge \psi$
3. If $\phi_1 \Leftrightarrow \phi_2$ then $\phi_1 \cdot \psi \Leftrightarrow \phi_2 \cdot \psi$
4. $\phi \cdot \psi \Leftrightarrow \psi \cdot \phi$
5. If $\phi \Rightarrow \neg\psi$ then $(\psi \cdot \neg\psi) \wedge \phi \Rightarrow (\phi \cdot \neg\phi) \wedge \phi$
6. If $\phi \Rightarrow \neg\psi$ and $\phi \cdot \neg\phi \Rightarrow \neg\psi$ then $\psi \cdot \neg\psi \Rightarrow \neg\phi$
7. If $\phi \Rightarrow \neg\psi$ and $(\phi \cdot \neg\phi) \wedge \psi$ is consistent, then $\phi \cdot \psi \Leftrightarrow ((\phi \cdot \neg\phi) \cdot \psi) \vee ((\psi \cdot \neg\psi) \cdot \phi)$
8. If $\phi \vee \phi \cdot \neg\phi \Rightarrow \neg\psi$, then $\phi \cdot \psi \Leftrightarrow ((\phi \cdot \neg\phi) \cdot \psi) \cdot ((\psi \cdot \neg\psi) \cdot \phi)$

whereas

$$(\{a\} \cdot (\{b\}) \cdot \{c\} = \{d_1\} \cdot \{c\} = \{a, d_5\}.$$

Define a tolerance space (W, R) using (3) and denote its geodesic distance with d. Define the following operator on subsets of W:

$$A \odot B = \begin{cases} \{x \in \mathrm{mid}(A, B) : d(A, x) + d(B, x) = d(A, B)\} & \text{if } A, B \neq \varnothing \\ A \cup B & \text{otherwise.} \end{cases} \tag{10}$$

where

$$\mathrm{mid}(A, B) = A^k \cap B^k, \quad k = \min\{l : A^l \cap B^l \neq \varnothing\}.$$

Note that in case $B = A^c$ we get the seam of A and $A \odot A^c = \mathrm{mid}(A, A^c)$. The barycentric merging arises by selecting those elements of the convex merging that are "midway" between the merged subsets.

We can recast the above definition in such a way that the midway becomes more explicit:

Lemma 22 *Let $d(A, B) = n$. Then we have:*

1. *If $n = 2k$, then*

$$A \odot B = \{x : d(A, x) = k,\ d(B, x) = k\}.$$

2. *If $n = 2k + 1$, then*

$$A \odot B = \{x : d(A, x) = k,\ d(B, x) = k+1\} \cup \{x : d(A, x) = k+1,\ d(B, x) = k\}.$$

Proof. If $n = 2k$, then observe that

$$\mathrm{mid}(A, B) = A^k \cap B^k = \{x : d(A, x) = k,\ d(B, x) = k\}.$$

If $n = 2k + 1$, then

$$\begin{aligned}\mathrm{mid}(A, B) = A^{k+1} \cap B^{k+1} = &\{x : d(A, x) = k,\ d(B, x) = k + 1\} \\ &\cup \{x : d(A, x) = k + 1,\ d(B, x) = k\} \\ &\cup \{x : d(A, x) = k + 1,\ d(B, x) = k + 1\},\end{aligned}$$

and only the first two subsets satisfy the definition of merging. ∎

Corollary 23 *Let $d(A, B) = n > 1$. Then we have:*

1. $d(((A \odot A^c) \odot B), ((B \odot B^c) \odot A)) \leqslant 1$.
2. *If $n = 2k$, then*

$$(A \odot A^c) \odot B = \{x : d(A \odot A^c, x) = k,\ d(B, x) = k - 1\} \cup A \odot B$$

3. *If $n = 2k + 1$, then*

$$(A \odot A^c) \odot B = \{x : d(A, x) = k + 1,\ d(B, x) = k\}.$$

4. $((A \odot A^c) \odot B) \odot ((B \odot B^c) \odot A) = A \odot B$.

In case $d(A, B) = 0$ we have $\mathrm{mid}(A, B) = A \cap B$ and therefore $A \odot B = A \cap B$. If $d(A, B) = 1$ we have $\mathrm{mid}(A, B) = A^1 \cap B^1$ and therefore $A \odot B = \{x : d(A, x) + d(B, x) = 1\}$. This implies the following lemma

Lemma 24 *For all A, B such that $d(A, B) = 0$ or 1 we have*

$$A \odot B = A \otimes B = A \otimes_{\mathsf{c}} B.$$

Because of the above Lemma we have that Lemmas 7 and 8 hold for $\cdot$ as well. Moreover, the rules used to prove Lemma 9, Lemma 11, and Corollary 10 are also present in Table 4, so the results hold for $\odot$, as well. Now the following characterization theorem holds:

Theorem 25 *Let $\cdot$ be a geodesic barycentric merging operator. Then there exists a binary relation R, defined by* (3)*, such that (W, R), where W is the set of models, is a tolerance space and the following holds*

$$v(\phi \cdot \psi) = v(\phi) \odot v(\psi), \tag{11}$$

where $\odot$ is defined by (10)*. Conversely, if an operator satisfies* (11) *then it is a barycentric merging operator.*

Proof. Again, we proceed by induction exactly as in the proof of Theorems 12 and 19.

As in Theorem 19, there are two cases left:

Let $d(v(\phi), v(\psi)) = 1$. This implies that $\psi \Rightarrow \neg\phi$ and $\phi \cdot \neg\phi \wedge \psi$ is consistent. Using Rule 7 we have

$$v(\phi \cdot \psi) = v(((\phi \cdot \neg\phi) \cdot \psi) \vee ((\psi \cdot \neg\psi) \cdot \phi)).$$

Again, as in Theorem 19, we have

$$v(\phi) \odot v(\psi) = v((\phi \cdot \neg\phi) \cdot \psi) \vee v((\psi \cdot \neg\psi) \cdot \phi).$$

and, therefore

$$v(\phi) \odot v(\psi) = v(\phi \cdot \psi).$$

Assume that is true for all k, where $1 \leqslant k < n$, and let $d(v(\phi), v(\psi)) = n$. We have that

$$v(\phi \cdot \psi) = v((\phi \cdot \neg\phi) \cdot \psi) \cdot ((\psi \cdot \neg\psi) \cdot \phi)$$

using Rule 8 and

$$v(\phi \cdot \psi) = v((\phi \cdot \neg\phi) \cdot \psi) \cdot v((\psi \cdot \neg\psi) \cdot \phi)$$

by Corollary 23.1 and the induction hypothesis. Again by the induction hypothesis we have

$$v(\phi \cdot \psi) = (v(\phi) \odot v(\neg\phi)) \odot v(\psi) \cup (v(\psi) \odot v(\neg\psi)) \odot v(\phi).$$

Therefore by Corollary 23.4

$$v(\phi \cdot \psi) = v(\phi) \odot v(\psi).$$

The converse is straightforward (Rule 8 is verified in Corollary 23.4).

∎

4 Conclusion and further work

Merging was first studied as a binary operator. This is the line of research we extend in this paper. We have introduced three classes of distance-based binary merging operators that can be iterated, and we characterized them using graph-based metric spaces. All three operators arise naturally. The first, non-prioritized revision merging, is the standard method of merging that was tailored to the geodesic framework. The other two methods of merging – convex and barycentric – take advantage of the geodesic framework to define merging areas that lie between the sets to be merged. Both convexity and barycenter are geometrical concepts that seem appropriate to express in-between areas. The geodesic framework also holds the potential for discovering other concepts, geometrical or topological, that may correspond to other useful merging or, more generally, belief change operations.

Geodesic modeling can be an important tool when numerical information (discrete or continuous) is available. All examples of merging use the Hamming distance or some other integer metric to describe the similarity of propositional models. By contrast, continuous metrics are usually intractable and often reduced to a discrete metric using threshold values that have been shown to be embedded in a geodesic space.

The exact applicability range of each method is still unclear. In fact, most merging proposals are not accompanied by a description of the context in which they do apply. It is not enough to know how the available information is structured. We may be able to organize worlds in a tolerance space but, as we showed, we still need to decide which form of merging is appropriate for received information. Such problems are of great importance and we believe the logical characterization we presented in this paper will help delineate the application range of each formalism.

Further, the simplicity of our approach can become a stepping stone for more ambitious, more expressive formalisms such as Dynamic Epistemic Logic (DEL) [6]. DEL makes use of other more basic logics (e.g., modal logic) to express changes on their models. For example, announcements are modeled by restricting the domain of an S5 model. In a similar fashion, epistemic modal logic may be used to describe changes of the underlying geodesic space while being equipped with merging and revision operators that such spaces allow.

Recent research on this area [19] is focused on the case of multiset merging, and therefore the relevance of our results to this more general area is worth special mention. In the case of multiple agents, a multiset represents the beliefs of each individual. As two or more agents may have the same belief, a belief can appear more than once in the group of their epistemic states and therefore the group of epistemic states should be a multiset in order to express this multiplicity. This observation is the basis of IC merging [17] where multisets are used for representing groups of belief states. Notice that a multiset representation is not sensitive to order, so beliefs in a multiset cannot be indexed. Therefore, the set of multiple agents is anonymous. If, instead, the set of agents is eponymous, then a sequence representation is more appropriate.

A straightforward, if not simplistic, definition of merging for multisets of formulas, or rather sequences of formulas, from binary merging is to use iteration. Let $\otimes(\phi_1, \phi_2, \ldots, \phi_n)$ denote the merging of the sequence ϕ_1,ϕ_2,. . .,ϕ_n. Using a binary merging operator we have

$$\otimes(\phi_1, \ldots, \phi_n) = ((\ldots(\phi_1 \otimes \phi_2) \cdots) \otimes \phi_n).$$

Therefore, three different operators can be defined depending on the basic binary operator used. Unfortunately, the order of operations needs to be specified as none of the merging operators introduced is associative. Nevertheless, fixing a specific order of evaluation to sequences can be of interest. For example, we can choose to favor the more recently received information. One may argue that this is revision, rather than merging, but it does not have to be a belief revision strategy either, as we may allow the recently received information to influence our beliefs, but not necessarily adopt it. In short, more general belief change strategies can arise. Although we can simulate an iteration of merging with a multiset of cardinality three using IC merging:

$$(\phi \otimes \psi) \otimes \chi \Leftrightarrow \Delta_{\top}\{\phi, \psi, \chi\},$$

this simulation would not be faithful to left hand side merging because it both ignores the order, and requires a running list of the propositions we are merging. More importantly, merging is to be thought of as an operation that can be iterated in the sense that it can be applied to its outcome. For example, we expect that when merging two databases we create a third database we may have to merge further later.

Finally, there is no characterization of distance-based merging operators on multisets. We know many classes of useful distance-based operators [16, 8, 20] and their properties but we do not know their characterization. We believe that the employment of geodesic semantics is a step towards representation results of this kind.

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