2008

TR-2008012: Product-Free Lambek Calculus is NP-Complete

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Product-free Lambek Calculus is NP-complete

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Abstract

In this paper we prove that the derivability problems for product-free Lambek calculus and product-free Lambek calculus allowing empty premises are NP-complete. Also we introduce a new derivability characterization for these calculi.

Introduction

Lambek calculus L was first introduced in [3]. Lambek calculus uses syntactic types that are built from primitive types using three binary connectives: multiplication, left division, and right division. Natural fragments of Lambek calculus are the product-free Lambek calculus $L(\backslash, /)$, which does not use multiplication, and the unidirectional Lambek calculi, which have only one connective left: a division (left or right).

For the non-associative variant of Lambek calculus the derivability can be checked in polynomial time as shown in [2] (for the product-free fragment of the non-associative Lambek calculus this was proved already in [1]).

In [4] NP-completeness was proved for the derivability problem for full associative Lambek calculus. In [5] there was presented a polynomial algorithm for its unidirectional fragments.

We show that the classical satisfiability problem *SAT* is polynomial time reducible to the $L(\backslash, /)$-derivability problem and thus $L(\backslash, /)$ is NP-complete.

*This research was supported in part by the Russian Foundation for Basic Research grant 08-01-00399.

1 Product-free Lambek Calculus

Product-free Lambek calculus L(\, /) can be constructed as follows. Let $\mathbf{P} = \{p_0, p_1, \ldots\}$ be a countable set of what we call *primitive types*. Let Tp be the set of *types* constructed from primitive types with two binary connectives /, \. We will denote primitive types by small letters (p, q, $r, \ldots$) and types by capital letters (A, B, $C, \ldots$). By capital greek letters (Π, Γ, Δ,...) we will denote finite (possibly empty) sequences of types. Expressions like $\Pi \to A$, where Π is not empty, are called *sequents*.

Axioms and rules of L(\, /):

$$A \to A, \qquad \frac{\Phi \to B \quad \Gamma B \Delta \to A}{\Gamma \Phi \Delta \to A}\ (\mathrm{CUT}),$$

$$\frac{\Pi A \to B}{\Pi \to (B/A)}\ (\to /), \qquad \frac{\Phi \to A \quad \Gamma B \Delta \to C}{\Gamma (B/A) \Phi \Delta \to C}\ (/ \to),$$

$$\frac{A \Pi \to B}{\Pi \to (A \backslash B)}\ (\to \backslash), \qquad \frac{\Phi \to A \quad \Gamma B \Delta \to C}{\Gamma \Phi (A \backslash B) \Delta \to C}\ (\backslash \to),$$

(Here Γ and Δ can be empty.)

In this paper we will consider two calculi — L(\, /) and L*(\, /), called product-free Lambek calculus allowing empty premises. In L*(\, /) we allow the antecedent of a sequent to be empty.

It can be shown that in these calculi every derivable sequent has a cut-free derivation where all instances of the axiom are of the form $p \to p$ where $p \in \mathbf{P}$.

2 Reduction from *SAT*

Let $c_1 \wedge \ldots \wedge c_m$ be a Boolean formula in conjunctive normal form with clauses $c_1 \ldots c_m$ and variables $x_1 \ldots x_n$. The reduction maps the formula to a sequent, which is derivable in L(\, /) (and in L*(\, /)) if and only if the formula $c_1 \wedge \ldots \wedge c_m$ is satisfiable.

For any Boolean variable x_i let $\neg_0 x_i$ stand for the literal $\neg x_i$ and $\neg_1 x_i$ stand for the literal x_i.

Note that $\langle t_1, \ldots, t_n \rangle \in \{0, 1\}^n$ is a satisfying assignment for the Boolean formula $c_1 \wedge \ldots \wedge c_m$ if and only if for every $j \leq m$ there exists $i \leq n$ such that the literal $\neg_{t_i} x_i$ appears in the clause c_j (as usual, 1 stands for "true" and 0 stands for "false").

Let $p_i^j, q_i^j, a_i^j, b_i^j, c_i^j, d_i^j; 0 \leq i \leq n, 0 \leq j \leq m$ be distinct primitive types from $\mathbf{P}$.

We define the following families of types:

$$
\begin{aligned}
G^0 &\rightleftharpoons (p_0^0 \backslash p_n^0) \\
G^j &\rightleftharpoons (q_n^j / ((q_0^j \backslash p_0^j) \backslash G^{j-1})) \backslash p_n^j \\
G &\rightleftharpoons G^m \\
A_i^0 &\rightleftharpoons (a_i^0 \backslash p_i^0) \\
A_i^j &\rightleftharpoons (q_i^j / ((b_i^j \backslash a_i^j) \backslash A_i^{j-1})) \backslash p_i^j \\
A_i &\rightleftharpoons A_i^m \\
C_i^0 &\rightleftharpoons (c_i^0 \backslash p_i^0) \\
C_i^j &\rightleftharpoons (q_i^j / ((d_i^j \backslash c_i^j) \backslash C_i^{j-1})) \backslash p_i^j \\
C_i &\rightleftharpoons C_i^m \\
E_i^0(t) &\rightleftharpoons p_{i-1}^0 \\
E_i^j(t) &\rightleftharpoons \begin{cases} q_i^j / (((q_{i-1}^j / E_i^{j-1}(t)) \backslash p_{i-1}^j) \backslash p_i^{j-1}), & \text{if } \neg_t x_i \text{ appears in } c_j \\ (q_{i-1}^j / (q_i^j / (E_i^{j-1}(t) \backslash p_i^{j-1}))) \backslash p_{i-1}^j, & \text{if } \neg_t x_i \text{ does not appear in } c_j \end{cases} \\
F_i^j(t) &\rightleftharpoons (E_i^j(t) \backslash p_i^j) \\
F_i(t) &\rightleftharpoons F_i^m(t) \\
H_i^0 &\rightleftharpoons p_{i-1}^0 \backslash p_i^0 \\
H_i^j &\rightleftharpoons ((q_{i-1}^j / (q_i^j / H_i^{j-1})) \backslash p_{i-1}^j) \backslash p_i^j \\
H_i &\rightleftharpoons H_i^m \\
B_i^0 &\rightleftharpoons a_i^0 \\
B_i^j &\rightleftharpoons q_{i-1}^j / (((b_i^j / B_i^{j-1}) \backslash a_i^j) \backslash p_{i-1}^{j-1}) \\
B_i &\rightleftharpoons B_i^m \backslash p_{i-1}^m \\
D_i^0 &\rightleftharpoons c_i^0 \\
D_i^j &\rightleftharpoons q_{i-1}^j / (((d_i^j / D_i^{j-1}) \backslash c_i^j) \backslash p_{i-1}^{j-1}) \\
D_i &\rightleftharpoons D_i^m \backslash p_{i-1}^m.
\end{aligned}
$$

Let Π_i denote the following sequences of types:

$$(F_i(0)/(B_i \backslash A_i)) \; H_i \; ((D_i \backslash C_i) \backslash F_i(1)).$$

Theorem 2.1. *The following statements are equivalent:*

1. $c_1 \wedge \ldots \wedge c_m$ *is satisfiable.*
2. $\mathrm{L}(\backslash, /) \vdash \Pi_1 \ldots \Pi_n \to G$
3. $\mathrm{L}^*(\backslash, /) \vdash \Pi_1 \ldots \Pi_n \to G.$

This theorem will be proven in section 4.

3 Derivability Characterization

Let At be the set of *atoms* or *primitive types with superscripts*, $\{p^{(i)} | p \in \mathbf{P}, i \in \mathbb{Z}\}$. Let FS be the free monoid (the set of all finite strings) generated by elements of At. We will denote elements of FS by $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and so on, by ε we will denote the empty string.

Consider two mappings:

$$t : \mathrm{FS} \to \mathbf{P}, t(\mathbb{A}p^{(i)}) = p; \quad d : \mathrm{FS} \to \mathbb{Z}, d(\mathbb{A}p^{(i)}) = i.$$

Let $\mathbb{A} \sqsubset \mathbb{B}$ denote that $\mathbb{A}$ is a strict prefix of $\mathbb{B}$ (i.e. there is $\mathbb{C} \neq \varepsilon \in \mathrm{FS}$ such that $\mathbb{B} = \mathbb{A}\mathbb{C}$). We will denote such $\mathbb{C}$ as $\mathbb{A} \diagdown \mathbb{B}$. By $\mathbb{A} \sqsubseteq \mathbb{B}$ we will denote that either $\mathbb{A} \sqsubset \mathbb{B}$ or $\mathbb{A} = \mathbb{B}$. We can define in the usual way the following notions: $\min_{\sqsubset}$, $\max_{\sqsubset}$, $\inf_{\sqsubset}$, $\sup_{\sqsubset}$, $[\mathbb{A}, \mathbb{B}]_{\sqsubset}$, and $(\mathbb{A}, \mathbb{B}]_{\sqsubset}$.

For $\mathbb{A} \in \mathrm{FS}, \mathbb{A} \neq \varepsilon$ let $\mathcal{P}_{\mathbb{A}} = \{\mathbb{B} \mid \mathbb{B} \sqsubseteq \mathbb{A}, \mathbb{B} \neq \varepsilon\}$. The relation $\sqsubseteq$ is a total order on $\mathcal{P}_{\mathbb{A}}$.

Let α be a partial function on $\mathcal{P}_{\mathbb{A}}$. For each such function we can define the following:

$$\begin{gathered}
\mathbb{B} <_\alpha \mathbb{C} \Leftrightarrow \exists n \geq 1, \alpha^n(\mathbb{B}) = \mathbb{C}, \\
\mathbb{B} \leq_\alpha \mathbb{C} \Leftrightarrow \mathbb{B} <_\alpha \mathbb{C} \vee \mathbb{B} = \mathbb{C}, \\
\mu_\alpha^-(\mathbb{B}) = \min_{\sqsubset}(\mathbb{B}, \alpha(\mathbb{B})), \\
\mu_\alpha^+(\mathbb{B}) = \max_{\sqsubset}(\mathbb{B}, \alpha(\mathbb{B})), \\
\mathcal{F}_\alpha(\mathbb{B}) = \{\mathbb{C} \mid \mathbb{C} \leq_\alpha \mathbb{B}\}, \\
\nu_\alpha^-(\mathbb{B}) = \inf_{\sqsubset}(\mathcal{F}_\alpha(\mathbb{B})), \\
\nu_\alpha^+(\mathbb{B}) = \sup_{\sqsubset}(\mathcal{F}_\alpha(\mathbb{B})).
\end{gathered}$$

Consider two antiendomorphisms $(\cdot)^{\leftarrow}$ and $(\cdot)^{\rightarrow}$ on FS defined by

$$\begin{gathered}
(p^{(0)})^{\leftarrow} = p^{(-1)}, \quad (p^{(0)})^{\rightarrow} = p^{(1)}, \\
(p^{(i)})^{\leftarrow} = (p^{(i)})^{\rightarrow} = p^{(-i-sgn(i))}, \text{for } i \neq 0.
\end{gathered}$$

(A function $f \colon X \to X$ is an antiendomorphism if $\forall a, b \in X, f(ab) = f(b)f(a)$. In a free monoid it can be defined by its actions on the generators).

Consider $[\![\cdot]\!] \colon \mathrm{Tp} \to \mathrm{FS}$, a mapping from Lambek types to elements of the free monoid defined by

$$[\![p]\!] = p^{(0)}, \quad [\![(A/B)]\!] = [\![B]\!]^{\rightarrow}[\![A]\!], \quad [\![(A\backslash B)]\!] = [\![B]\!][\![A]\!]^{\leftarrow}.$$

Let us define φ — the partial function on $\mathcal{P}_{[\![A]\!]}$ that reflects the structure of the Lambek type A:

$$\varphi(\mathbb{A}) = \begin{cases} \inf_{\sqsubset}\{\mathbb{B} \mid \mathbb{A} \sqsubset \mathbb{B}, |d(\mathbb{B})| = |d(\mathbb{A})| - 1\}, & \text{if } d(\mathbb{A}) > 0; \\ \sup_{\sqsubset}\{\mathbb{B} \mid \mathbb{B} \sqsubset \mathbb{A}, |d(\mathbb{B})| = |d(\mathbb{A})| - 1\}, & \text{if } d(\mathbb{A}) < 0. \end{cases}$$

It can be easily shown that the following facts hold:

1. There is a unique $\mathbb{A}_0 \in \mathcal{P}_{[\![A]\!]}$ such that $d(\mathbb{A}_0) = 0$.

2. $\varphi(\mathbb{A})$ is defined for every $\mathbb{A} \neq \mathbb{A}_0$.

3. $\leq_\varphi$ is a partial order on $\mathcal{P}_{[\![A]\!]}$.

4. For every $i \in \mathbb{N}$ such that $i < |d(\mathbb{A})|$ there exists $\mathbb{B}$ such that $|d(\mathbb{B})| = i$ and $\mathbb{A} <_\varphi \mathbb{B}$, for instance $\mathbb{A} \leq_\varphi \mathbb{A}_0$.

5. If $\mathbb{A} \in [\mu^-_\varphi(\mathbb{B}), \mu^+_\varphi(\mathbb{B})]_{\sqsubset}$, then $\mathbb{A} \leq \varphi(\mathbb{B})$.

Suppose $\mathbb{A}, \mathbb{B} \in \mathcal{P}_{[\![A]\!]}$. There exists $\mathbb{C} \in \mathcal{P}_{[\![A]\!]}$ such that $\mathbb{A} \leq_\varphi \mathbb{C}$, $\mathbb{B} \leq_\varphi \mathbb{C}$, and for all $\mathbb{C}' \in \mathcal{P}_{[\![A]\!]}$ such that $\mathbb{A} <_\varphi \mathbb{C}'$ and $\mathbb{A} \leq_\varphi \mathbb{C}'$, $\mathbb{C} \leq_\varphi \mathbb{C}'$. Such $\mathbb{C}$ is called the φ-join of $\mathbb{A}$ and $\mathbb{B}$.

A set $\mathcal{G} \subset \mathcal{P}_{[\![A]\!]}$ is called φ-closed if there is no $\mathbb{A} \notin \mathcal{G}$ such that $\varphi(\mathbb{A}) \in \mathcal{G}$.

Let $\mathcal{N}_{\mathbb{A}} = \{\mathbb{B} \in \mathcal{P}_{\mathbb{A}} \mid d(\mathbb{B}) = 2i + 1, i \in \mathbb{Z}\}$.

Suppose we have a Lambek sequent $A_1 \ldots A_n \to B$. Let

$$\mathbb{W} = [\![(\ldots(B/A_n)/\ldots)/A_1]\!] = [\![A_1]\!]^{\to} \ldots [\![A_n]\!]^{\to}[\![B]\!].$$

Let π be a function on $\mathcal{P}_{\mathbb{W}}$, and ψ be a partial function defined by

$$\psi(\mathbb{A}) = \begin{cases} \pi(\mathbb{A}), & \text{if } \mathbb{A} \in \mathcal{N}_{\mathbb{W}}; \\ \varphi(\mathbb{A}), & \text{if } \mathbb{A} \notin \mathcal{N}_{\mathbb{W}} \text{ and } d(\mathbb{A}) \neq 0. \end{cases}$$

To characterize derivability of the sequent $A_1 \ldots A_n \to B$ we shall use the following conditions, which we call proof conditions.

1. If $\mathbb{A} \in \mathcal{N}_{\mathbb{W}}$, then $\pi(\mathbb{A}) \notin \mathcal{N}_{\mathbb{W}}$ and $\pi^2(\mathbb{A}) = \mathbb{A}$ for all $\mathbb{A} \in \mathcal{P}_{\mathbb{W}}$.

2. $t(\pi(\mathbb{A})) = t(\mathbb{A})$.

3. $\mu^-_\pi(\mathbb{A}) \sqsubset \mu^-_\pi(\mathbb{B}) \Rightarrow \mu^+_\pi(\mathbb{A}) \sqsubset \mu^-_\pi(\mathbb{B}) \vee \mu^+_\pi(\mathbb{B}) \sqsubset \mu^+_\pi(\mathbb{A})$.

4. $\mathbb{A} \in \mathcal{N}_{\mathbb{W}} \Longrightarrow \mathbb{A} <_\psi \varphi(\mathbb{A})$ or equivalently $\forall \mathbb{A} \in \mathcal{P}_{\mathbb{W}}, \mathcal{F}_\varphi(\mathbb{A}) \subset \mathcal{F}_\psi(\mathbb{A})$.

5. $\mathbb{A} \notin \mathcal{N}_{\mathbb{W}} \wedge \mathbb{A} \neq \mathbb{A}_0 \Longrightarrow \exists \mathbb{B}(\mathbb{B} <_\psi \mathbb{A} \wedge \mathbb{B} \not<_\varphi \mathbb{A})$.

We will call $\mathcal{G} \subset \mathcal{P}_{\mathbb{W}}$ π-closed if for all $\mathbb{A} \in \mathcal{G}, \pi(\mathbb{A}) \in \mathcal{G}$. It is readily seen that if π satisfies proof conditions (1) and (3), then for every $\mathbb{A} \in \mathcal{N}_{\mathbb{W}}$, $[\mu_\pi^-(\mathbb{A}), \mu_\pi^+(\mathbb{A})]_\sqsubset$ and $\mathcal{P}_{\mathbb{W}} \setminus [\mu_\pi^-(\mathbb{A}), \mu_\pi^+(\mathbb{A})]_\sqsubset$ are π-closed. If π satisfies proof conditions (1) and (2), then $\mathcal{G}$ cannot be π-closed if for given $p \in \mathbf{P}$ there are odd number of $\mathbb{A} \in \mathcal{G}$ such that $t(\mathbb{A}) = p$.

Lemma 3.1. *If π satifies proof condition (4), then $\leq_\psi$ is a partial order on $\mathcal{P}_{\mathbb{W}}$.*

Proof. Reflexivity and transitivity directly follow from the definition of $\leq_\psi$.

Now lets prove antisymmetry. Suppose that there are $\mathbb{B}, \mathbb{C} \in \mathcal{P}_{\mathbb{W}}$ such that $\mathbb{B} \leq_\psi \mathbb{C}$ and $\mathbb{C} \leq_\psi \mathbb{B}$. If $\mathbb{B} \neq \mathbb{C}$ then there is $i > 0$ such that $\psi^i(\mathbb{B}) = \mathbb{B}$ and thus for all $j > 0$, $\psi^j(\mathbb{B})$ is defined.

If π satisfy proof condition (4) then if $\mathbb{A} \leq_\varphi \mathbb{B}$ then $\mathbb{A} \leq_\psi \mathbb{B}$. There is $\mathbb{A}_0 \in \mathcal{P}_{\mathbb{W}}$ such that $d(\mathbb{A}_0) = 0$, and for all $\mathbb{A} \in \mathcal{P}_{\mathbb{W}}$, $\mathbb{A} \leq_\varphi \mathbb{A}_0$. This means that $\mathbb{B} \leq_\varphi \mathbb{A}_0$ and thus $\mathbb{B} \leq_\psi \mathbb{A}_0$. The function ψ is not defined on $\mathbb{A}_0$. Contradiction. □

Lemma 3.2. *$\mathrm{L}^*(\backslash, /) \vdash A_1 \dots A_n \to B$ if and only if there exists π satisfying proof conditions (1)-(4).*

$\mathrm{L}(\backslash, /) \vdash A_1 \dots A_n \to B$ if and only if $n > 0$ and there exists π satisfying proof conditions (1)-(5).

Proof. Suppose that $\mathrm{L}^{(*)}(\backslash, /) \vdash A_1 \dots A_n \to B$. Induction on the length of the derivation.

If the sequent is of the form $p \to p$, then $\mathbb{W} = p^1p^0$, $\mathcal{P}_{\mathbb{W}} = \{p^1, p^1p^0\}$, $\mathcal{N}_{\mathbb{W}} = \{p^1\}$ and π such that $\pi(p^1) = p^1p^0$ and $\pi(p^1p^0) = p^1$ satisfies all necessary proof conditions.

Suppose that the last step in the derivation of $A_1 \dots A_n \to B$ was an application of the rule $(\to /)$. Then $B = (C/D)$, $\mathrm{L}^{(*)}(\backslash, /) \vdash A_1 \dots A_n D \to C$ and for $\mathcal{P}_{\mathbb{W}'}$, where $\mathbb{W}' = [\![A_1]\!]^{\to} \dots [\![A_n]\!]^{\to}[\![D]\!]^{\to}[\![C]\!]$ there exists π' satisfying all necessary proof conditions. But in this case $\mathbb{W} = \mathbb{W}'$, and therefore this π' works for the sequent $A_1 \dots A_n \to B$ too.

Suppose that the last step in the derivation of $A_1 \dots A_n \to B$ was an application of the rule $(\to \backslash)$. Then $B = (C \backslash D)$, $\mathbb{W} = [\![A_1]\!]^{\to} \dots [\![A_n]\!]^{\to}[\![D]\!][\![C]\!]^{\leftarrow}$, $\mathrm{L}^{(*)}(\backslash, /) \vdash C A_1 \dots A_n \to D$, and by induction hypothesis for $\mathcal{P}_{\mathbb{W}'}$, where

$$\mathbb{W}' = [\![C]\!]^{\to}[\![A_1]\!]^{\to} \dots [\![A_n]\!]^{\to}[\![D]\!]$$

there exists π' satisfying all necessary proof conditions. Consider

$$\beta : \mathcal{P}_{\mathbb{W}'} \to \mathcal{P}_{\mathbb{W}}, \beta(\mathbb{A}) = \begin{cases} [\![A_1]\!]^{\to} \dots [\![A_n]\!]^{\to}[\![D]\!](\mathbb{A}^{\to^{-1}})^{\leftarrow}, & \text{if } \mathbb{A} \sqsubseteq [\![C]\!]^{\to}; \\ [\![C]\!]^{\to} \diagdown \mathbb{A}, & \text{if } [\![C]\!]^{\to} \sqsubset \mathbb{A}. \end{cases}$$

Let $\pi(\mathbb{A}) = \beta(\pi'(\beta^{-1}(\mathbb{A})))$. Such π satisfies all necessary proof conditions.

Suppose that the last step in the derivation of $A_1 \dots A_n \to B$ was an application of the rule $(/ \to)$. Then $A_1 \dots A_n \to B$ is of the form

$$C_1 \dots (C_i/D) D_1 \dots D_k C_{i+1} \dots C_l \to C$$

so that $\mathrm{L}^{(*)}(\backslash, /) \vdash C_1 \dots C_l \to C$ and $\mathrm{L}^{(*)}(\backslash, /) \vdash D_1 \dots D_k \to D$.

Consider $\mathbb{W}' = [\![C_1]\!]^{\to} \dots [\![C_l]\!]^{\to} [\![C]\!]$ and $\mathbb{W}'' = [\![D_1]\!]^{\to} \dots [\![D_k]\!]^{\to} [\![D]\!]$. By induction hypothesis there are π' and π'' — functions on $\mathcal{P}_{\mathbb{W}'}$ and $\mathcal{P}_{\mathbb{W}''}$ respectively, satisfying all necessary proof conditions.

Let $\mathbb{C} = [\![C_1]\!]^{\to} \dots [\![C_i]\!]^{\to}$ and $\mathbb{D} = [\![D_1]\!]^{\to} \dots [\![D_k]\!]^{\to}$. Consider

$$\beta' \colon \mathcal{P}_{\mathbb{W}'} \to \mathcal{P}_{\mathbb{W}}, \beta'(\mathbb{A}) = \begin{cases} \mathbb{A} & \text{, if } \mathbb{A} \sqsubseteq \mathbb{C}; \\ \mathbb{C}([\![D]\!]^{\to})^{\to} \mathbb{D} (\mathbb{C} \diagdown \mathbb{A}) & \text{, if } \mathbb{C} \sqsubset \mathbb{A}; \end{cases}$$

$$\text{and } \beta'' \colon \mathcal{P}_{\mathbb{W}''} \to \mathcal{P}_{\mathbb{W}}, \beta''(\mathbb{A}) = \begin{cases} \mathbb{C}([\![D]\!]^{\to})^{\to} \mathbb{A} & \text{, if } \mathbb{A} \sqsubseteq \mathbb{D}; \\ \mathbb{C}((\mathbb{D} \diagdown \mathbb{A})^{\to})^{\to} & \text{, if } \mathbb{D} \sqsubset \mathbb{A}; \end{cases}.$$

$$\text{Let } \pi(\mathbb{A}) = \begin{cases} \beta'(\pi'(\beta'^{-1}(\mathbb{A}))) & \text{, if } \mathbb{A} \sqsubseteq \mathbb{C} \text{ or } \mathbb{C}([\![D]\!]^{\to})^{\to} \mathbb{D} \sqsubset \mathbb{A}; \\ \beta''(\pi''(\beta''^{-1}(\mathbb{A}))) & \text{, if } \mathbb{C} \sqsubset \mathbb{A} \sqsubseteq \mathbb{C}([\![D]\!]^{\to})^{\to} \mathbb{D}; \end{cases}.$$

Such π satisfies all necessary proof conditions.

Suppose that the last step in the derivation of $A_1 \dots A_n \to B$ was an application of the rule $(\backslash \to)$. Then $A_1 \dots A_n \to B$ is of the form

$$C_1 \dots C_{i-1} D_1 \dots D_k (D \backslash C_i) \dots C_l \to C$$

so that $\mathrm{L}^{(*)}(\backslash, /) \vdash C_1 \dots C_l \to C$ and $\mathrm{L}^{(*)}(\backslash, /) \vdash D_1 \dots D_k \to D$.

Consider $\mathbb{W}' = [\![C_1]\!]^{\to} \dots [\![C_l]\!]^{\to} [\![C]\!]$ and $\mathbb{W}'' = [\![D_1]\!]^{\to} \dots [\![D_k]\!]^{\to} [\![D]\!]$. By induction hypothesis there are π' and π'' — functions on $\mathcal{P}_{\mathbb{W}'}$ and $\mathcal{P}_{\mathbb{W}''}$ respectively, satisfying all necessary proof conditions.

Let $\mathbb{C} = [\![C_1]\!]^{\to} \dots [\![C_{i-1}]\!]^{\to}$ and $\mathbb{D} = [\![D_1]\!]^{\to} \dots [\![D_k]\!]^{\to}$. Consider

$$\beta' \colon \mathcal{P}_{\mathbb{W}'} \to \mathcal{P}_{\mathbb{W}}, \beta'(\mathbb{A}) = \begin{cases} \mathbb{A} & \text{, if } \mathbb{A} \sqsubseteq \mathbb{C}; \\ \mathbb{C}\mathbb{D}([\![D]\!]^{\leftarrow})^{\to} (\mathbb{C} \diagdown \mathbb{A}) & \text{, if } \mathbb{C} \sqsubset \mathbb{A}; \end{cases}$$

$$\text{and } \beta'' \colon \mathcal{P}_{\mathbb{W}''} \to \mathcal{P}_{\mathbb{W}}, \beta''(\mathbb{A}) = \begin{cases} \mathbb{C}\mathbb{A} & \text{, if } \mathbb{A} \sqsubseteq \mathbb{D}; \\ \mathbb{C}\mathbb{D}((\mathbb{D} \diagdown \mathbb{A})^{\leftarrow})^{\to} & \text{, if } \mathbb{D} \sqsubset \mathbb{A}; \end{cases}.$$

$$\text{Let } \pi(\mathbb{A}) = \begin{cases} \beta'(\pi'(\beta'^{-1}(\mathbb{A}))) & \text{, if } \mathbb{A} \sqsubseteq \mathbb{C} \text{ or } \mathbb{C}\mathbb{D}([\![D]\!]^{\leftarrow})^{\to} \sqsubset \mathbb{A}; \\ \beta''(\pi''(\beta''^{-1}(\mathbb{A}))) & \text{, if } \mathbb{C} \sqsubset \mathbb{A} \sqsubseteq \mathbb{C}\mathbb{D}([\![D]\!]^{\leftarrow})^{\to}; \end{cases}.$$

Such π satisfies all necessary proof conditions.

Thus we proved one side of the lemma.

Now suppose that for the given sequent $A_1 \dots A_n \to B$, for $\mathcal{P}_{\mathbb{W}}$ there exists π satisfying proof conditions (1)-(4).

Induction on total number of connectives in the sequent.

If there are no connectives, the sequent is of the form $p_1 \dots p_n \to q$ and $\mathbb{W} = p_1^{(1)} \dots p_n^{(1)} q^{(0)}$. The function π satisfies proof condition (1), thus $|\mathcal{N}_{\mathbb{W}}| = |\mathcal{P}_{\mathbb{W}} \setminus \mathcal{N}_{\mathbb{W}}|$. This means that $n = 1$. So $\mathcal{P}_{\mathbb{W}} = \{p_1^{(1)}, p_1^{(1)} q^{(0)}\}$ and $\mathcal{N}_{\mathbb{W}} = \{p_1^{(1)}\}$. The function π satisfies proof condition (2), therefore $p_1 = q$, and the sequent is an axiom.

If $B = (C/D)$, then the sequent $A_1 \dots A_n D \to C$ has less connectives then the original sequent, but $[\![A_1]\!]^{\to} \dots [\![A_n]\!]^{\to} [\![D]\!]^{\to} [\![C]\!] = \mathbb{W}$, and therefore π satisfies all necessary proof conditions for the new sequent. By inductional hypothesis this means that $\mathrm{L}^*(\backslash, /) \vdash A_1 \dots A_n D \to C$ and by applying the rule $(\to /)$ we get $\mathrm{L}^*(\backslash, /) \vdash A_1 \dots A_n \to B$.

If $B = (C \backslash D)$, then the sequent $C A_1 \dots A_n \to D$ has less connectives then the original sequent.

Let $\mathbb{W}' = [\![C]\!]^{\to} [\![A_1]\!]^{\to} \dots [\![A_n]\!]^{\to} [\![D]\!]$. Consider

$$\beta : \mathcal{P}_{\mathbb{W}'} \to \mathcal{P}_{\mathbb{W}}, \beta(\mathbb{A}) = \begin{cases} [\![A_1]\!]^{\to} \dots [\![A_n]\!]^{\to} [\![D]\!] (\mathbb{A}^{\to^{-1}})^{\leftarrow} & \text{, if } \mathbb{A} \sqsubseteq [\![C]\!]^{\to}; \\ [\![C]\!]^{\to} \diagdown \mathbb{A} & \text{, if } [\![C]\!]^{\to} \sqsubset \mathbb{A}; \end{cases}.$$

Let $\pi'(\mathbb{A}) = \beta^{-1}(\pi(\beta(\mathbb{A})))$. Such π' satisfies all necessary proof conditions. By induction hypothesis this means that $\mathrm{L}^*(\backslash, /) \vdash C A_1 \dots A_n \to D$, and by applying the rule $(\to \backslash)$ we get $\mathrm{L}^*(\backslash, /) \vdash A_1 \dots A_n \to B$.

Now we can only consider sequents of the form $A_1 \dots A_n \to p$. This means theat $\mathbb{W} = [\![A_1]\!]^{\to} \dots [\![A_n]\!]^{\to} p^{(0)}$. Let $\mathbb{A}_1 = \pi(\mathbb{W})$. Since π satisfies proof condition (4) and ψ is not defined on $\mathbb{W}$, $\varphi(\mathbb{A}_1) = \mathbb{W}$. Therefore $d(\mathbb{A}_1) = 1$. Let $\mathbb{A}_1 = [\![A_1]\!]^{\to} \dots [\![A_{i-1}]\!]^{\to} \mathbb{A}''$.

Suppose that $A_i = (C/D)$. This means that $[\![A_i]\!]^{\to} = [\![C]\!]^{\to}([\![D]\!]^{\to})^{\to}$. There exists a unique $\mathbb{D}_0 \in \mathcal{P}_{[\![D]\!]}$ such that $d(\mathbb{D}_0) = 0$. Consider

$$\mathbb{A}_2 = [\![A_1]\!]^{\to} \dots [\![A_{i-1}]\!]^{\to} [\![C]\!]^{\to} (\mathbb{D}_0^{\to})^{\to} \in \mathcal{P}_{\mathbb{W}}.$$

$d(\mathbb{A}_2) = -2$, $\varphi(\mathbb{A}_2) = \mathbb{A}_1$, $\psi^2(\mathbb{A}_2) = \mathbb{W}$, and there is no $\mathbb{B} \in \mathcal{P}_{\mathbb{W}}$ such that $\mathbb{A}_2 \sqsubset \mathbb{B}$ and $\varphi(\mathbb{B}) = \mathbb{A}_1$.

Also $\mathcal{F}_{\psi}(\mathbb{A}_2) = [\nu_{\psi}^-(\mathbb{A}_2), \nu_{\psi}^+(\mathbb{A}_2)]_{\sqsubset}$. Let us prove this statement. Consider $\mathbb{B} \in [\nu_{\psi}^-(\mathbb{A}), \nu_{\psi}^+(\mathbb{A})]_{\sqsubset}, \mathbb{B} \neq \mathbb{A}_2$. There exists $\mathbb{C} \in \mathcal{F}_{\psi}(\mathbb{A}_2)$ such that $\mathbb{B} \sqsubseteq \mathbb{C}$ and $\psi(\mathbb{C}) \sqsubseteq \mathbb{B}$. If $\mathbb{C} \in \mathcal{P}_{\mathbb{W}} \setminus \mathcal{N}_{\mathbb{W}}$, then $\mathbb{B} \in [\mu_{\varphi}^-(\mathbb{C}), \mu_{\varphi}^+(\mathbb{C})]_{\sqsubset}$, and thus $\mathbb{B} \leq_{\psi} \varphi(\mathbb{C}) \leq_{\psi} \mathbb{A}_2$. If $\mathbb{C} \in \mathcal{P}_{\mathbb{W}}$, then $\pi(\mathbb{B}) \in [\mu_{\varphi}^-(\mathbb{C}), \mu_{\varphi}^+(\mathbb{C})]_{\sqsubset} \subset [\nu_{\psi}^-(\mathbb{A}), \nu_{\psi}^+(\mathbb{A})]_{\sqsubset}$. Since $\mathbb{B} <_{\psi} \mathbb{A}_1$, this means that $\mathbb{B} <_{\psi} \mathbb{A}_2$.

Let $\nu^{+}_{\psi}(\mathbb{A}_2) = [\![A_1]\!]^{\rightarrow} \dots [\![A_l]\!]^{\rightarrow}$. Consider

$$\mathbb{W}' = [\![A_1]\!]^{\rightarrow} \dots [\![A_{i-1}]\!]^{\rightarrow}[\![C]\!]^{\rightarrow}[\![A_{l+1}]\!]^{\rightarrow} \dots [\![A_n]\!]^{\rightarrow}p^{(0)}$$

and $\mathbb{W}'' = [\![A_{i+1}]\!]^{\rightarrow} \dots [\![A_l]\!]^{\rightarrow}[\![D]\!]$. Let $\mathbb{C} = [\![A_1]\!]^{\rightarrow} \dots [\![A_{i-1}]\!]^{\rightarrow}[\![C]\!]^{\rightarrow}$ and $\mathbb{D} = [\![A_{i+1}]\!]^{\rightarrow} \dots [\![A_l]\!]^{\rightarrow}$. Consider

$$\beta' \colon \mathcal{P}_{\mathbb{W}'} \to \mathcal{P}_{\mathbb{W}}, \beta'(\mathbb{A}) = \begin{cases} \mathbb{A} & \text{, if } \mathbb{A} \sqsubseteq \mathbb{C}; \\ \mathbb{C}([\![D]\!]^{\rightarrow})^{\rightarrow}\mathbb{D}(\mathbb{C}\diagdown\mathbb{A}) & \text{, if } \mathbb{C} \sqsubset \mathbb{A}; \end{cases},$$

$$\text{and } \beta'' \colon \mathcal{P}_{\mathbb{W}''} \to \mathcal{P}_{\mathbb{W}}, \beta''(\mathbb{A}) = \begin{cases} \mathbb{C}([\![D]\!]^{\rightarrow})^{\rightarrow}\mathbb{A} & \text{, if } \mathbb{A} \sqsubseteq \mathbb{D}; \\ \mathbb{C}((\mathbb{D}\diagdown\mathbb{A})^{\rightarrow})^{\rightarrow} & \text{, if } \mathbb{D} \sqsubset \mathbb{A}; \end{cases}.$$

Functions $\pi' = \beta'^{-1}\pi\beta'$ and $\pi'' = \beta''^{-1}\pi\beta''$ satisfy all necessary proof conditions. By induction hypothesis this means that

$$\mathrm{L}^*(\backslash, /) \vdash A_1 \dots A_{i-1} C A_{l+1} \dots A_n \to p$$

and $\mathrm{L}^*(\backslash, /) \vdash A_{i+1} \dots A_l \to D$. By applying the rule $(/ \to)$ we get

$$\mathrm{L}^*(\backslash, /) \vdash A_1 \dots A_n \to p.$$

Suppose that $A_i = (D \backslash C)$. This means that $[\![A_i]\!]^{\rightarrow} = ([\![D]\!]^{\leftarrow})^{\rightarrow}[\![C]\!]^{\rightarrow}$. There exists a unique $\mathbb{D}_0 \in \mathcal{P}_{[\![D]\!]}$ such that $d(\mathbb{D}_0) = 0$. Consider

$$\mathbb{A}_2 = [\![A_1]\!]^{\rightarrow} \dots [\![A_{i-1}]\!]^{\rightarrow}(\mathbb{D}_0^{\leftarrow})^{\rightarrow} \in \mathcal{P}_{\mathbb{W}}.$$

$d(\mathbb{A}_2) = 2$, $\varphi(\mathbb{A}_2) = \mathbb{A}_1$, $\psi^2(\mathbb{A}_2) = \mathbb{W}$, and there is no $\mathbb{B} \in \mathcal{P}_{\mathbb{W}}$ such that $\mathbb{B} \sqsubset \mathbb{A}_2$ and $\varphi(\mathbb{B}) = \mathbb{A}_1$. Like in previous case we can say that $\mathcal{F}_{\psi}(\mathbb{A}_2) = [\nu^{-}_{\psi}(\mathbb{A}_2), \nu^{+}_{\psi}(\mathbb{A}_2)]_{\sqsubset}$.

Let $\nu^{-}_{\psi}(\mathbb{A}_2) = [\![A_1]\!]^{\rightarrow} \dots [\![A_l]\!]^{\rightarrow}q^{(j)}$ for some $q^{(j)} \in \mathrm{At}$. Consider

$$\mathbb{W}' = [\![A_1]\!]^{\rightarrow} \dots [\![A_l]\!]^{\rightarrow}[\![C]\!]^{\rightarrow}[\![A_{i+1}]\!]^{\rightarrow} \dots [\![A_n]\!]^{\rightarrow}p^{(0)}$$

and $\mathbb{W}'' = [\![A_{l+1}]\!]^{\rightarrow} \dots [\![A_{i-1}]\!]^{\rightarrow}[\![D]\!]$. Let $\mathbb{C} = [\![A_1]\!]^{\rightarrow} \dots [\![A_l]\!]^{\rightarrow}$ and $\mathbb{D} = [\![A_{l+1}]\!]^{\rightarrow} \dots [\![A_{i-1}]\!]^{\rightarrow}$. Consider

$$\beta' \colon \mathcal{P}_{\mathbb{W}'} \to \mathcal{P}_{\mathbb{W}}, \beta'(\mathbb{A}) = \begin{cases} \mathbb{A} & \text{, if } \mathbb{A} \sqsubseteq \mathbb{C}; \\ \mathbb{C}\mathbb{D}([\![D]\!]^{\leftarrow})^{\rightarrow}(\mathbb{C}\diagdown\mathbb{A}) & \text{, if } \mathbb{C} \sqsubset \mathbb{A}; \end{cases},$$

$$\text{and } \beta'' \colon \mathcal{P}_{\mathbb{W}''} \to \mathcal{P}_{\mathbb{W}}, \beta''(\mathbb{A}) = \begin{cases} \mathbb{C}\mathbb{A} & \text{, if } \mathbb{A} \sqsubseteq \mathbb{D}; \\ \mathbb{C}\mathbb{D}((\mathbb{D}\diagdown\mathbb{A})^{\leftarrow})^{\rightarrow} & \text{, if } \mathbb{D} \sqsubset \mathbb{A}; \end{cases}.$$

Functions $\pi' = \beta'^{-1}\pi\beta'$ and $\pi'' = \beta''^{-1}\pi\beta''$ satisfy all necessary proof conditions. By induction hypothesis this means that

$$\mathrm{L}^*(\backslash, /) \vdash A_1 \dots A_l C A_{i+1} \dots A_n \to p$$

and $\mathrm{L}^*(\backslash, /) \vdash A_{l+1} \dots A_{i-1} \to D$. By applying the rule $(\backslash \to)$ we get

$$\mathrm{L}^*(\backslash, /) \vdash A_1 \dots A_n \to p.$$

Thus we fully proved the lemma for $\mathrm{L}^*(\backslash, /)$.

Suppose we have π that satisfies proof conditions (1)-(5). We already proved that $\mathrm{L}^*(\backslash, /) \vdash A_1 \dots A_n \to B$. The construction given provides us with possible last step of the derivation. Hence we can construct a derivation. If π satisfies proof condition (5) this means that there will be no $\mathbb{A}_2$ such that $\mathcal{F}_\psi(\mathbb{A}_2) = \mathcal{F}_\varphi(\mathbb{A}_2)$, and thus there will be no steps in derivation that require sequents of the form $\to A$. Thus $\mathrm{L}(\backslash, /) \vdash A_1 \dots A_n \to B$.

The lemma is fully proven. □

Lemma 3.3. *Suppose we have two sequents $A_1 \dots A_n \to B$ and $C_1 \dots C_m \to D$. $\mathrm{L}^*(\backslash, /) \vdash A_1 \dots A_n \to B$. Let $\mathbb{W} = [\![A_1]\!]^{\to} \dots [\![A_n]\!]^{\to}[\![B]\!]$ and $\mathbb{W}' = [\![C_1]\!]^{\to} \dots [\![C_m]\!]^{\to}[\![D]\!]$. Suppose that there is a mapping $\beta\colon \mathcal{P}_{\mathbb{W}'} \to \mathcal{P}_{\mathbb{W}}$ such that the following holds:*

1. *β is injective,*
2. *For all $\mathbb{A} \in \mathcal{P}_{\mathbb{W}'}$, $t(\beta(\mathbb{A})) = t(\mathbb{A})$, $d(\beta(\mathbb{A})) = d(\mathbb{A})$,*
3. *For all $\mathbb{A}, \mathbb{B} \in \mathbb{P}_{\mathbb{W}'}$, $\mathbb{A} \sqsubset \mathbb{B}$ if and only if $\beta(\mathbb{A}) \sqsubset \beta(\mathbb{B})$.*

Let $\mathcal{G} = \{\mathbb{A} \in \mathcal{P}_{\mathbb{W}} \mid \neg\exists\mathbb{B} \in \mathcal{P}_{\mathbb{W}'}, \beta(\mathbb{B}) = \mathbb{A}\}$. If $\mathcal{G}$ is π-closed and φ-closed, then $\mathrm{L}^*(\backslash, /) \vdash C_1 \dots C_n \to D$.

Proof. Let φ' be φ for $\mathcal{P}_{\mathbb{W}'}$. Since $\mathcal{G}$ is φ-closed, for all $\mathbb{A} \in \mathcal{P}_{\mathbb{W}'}$, $\varphi'(\mathbb{A}) = \beta^{-1}(\varphi(\beta(\mathbb{A})))$. Since $\mathcal{G}$ is π-closed, π' defined as $\beta^{-1}\pi\beta$ is defined on all $\mathcal{P}_{\mathbb{W}'}$ and satisfies proof conditions (1)-(4). Therefore by lemma 3.2

$$\mathrm{L}^*(\backslash, /) \vdash C_1 \dots C_n \to D.$$

□

Lemma 3.4. *Suppose $\mathbb{A} \in \mathcal{N}_{\mathbb{W}}$ and $\mathbb{B}$ is the φ-join of $\mathbb{A}$ and $\pi(\mathbb{A})$. If π satisfies proof conditions (1)-(4), then $\mathbb{B} \notin \mathcal{N}_{\mathbb{W}}$.*

Proof. Suppose that $\mathbb{B}_i \in \mathcal{N}_{\mathbb{W}}$. There is $\mathbb{B}_1$ such that $\mathbb{A} <_\psi \mathbb{B}_1$ and $\varphi(\mathbb{B}_1) = \mathbb{B}$. There is $\mathbb{B}_2$ such that $\pi(\mathbb{A}) <_\psi \mathbb{B}_2$ and $\varphi(\mathbb{B}_2) = \mathbb{B}$. This means that $\mathbb{A} \leq_\psi \mathbb{B}_1$ and $\pi(\mathbb{A}) \leq_\psi \mathbb{B}_2$ and since $\psi(\mathbb{A}) = \pi(\mathbb{A})$, either $\mathbb{B}_1 <_\psi \mathbb{B}_2$ or $\mathbb{B}_2 <_\psi \mathbb{B}_1$. But since $\psi(\mathbb{B}_1) = \psi(\mathbb{B}_2) = \mathbb{B}$, we get $\mathbb{B} <_\psi \mathbb{B}$. Contradiction. □

4 Proof of the Main Theorem

Consider $\mathbb{W} = [\![F_1(t_1)]\!]^{\rightarrow} \dots [\![F_n(t_n)]\!]^{\rightarrow}[\![G]\!]$.

If a primitive type occurs in the sequent $F_1(t_1) \dots F_n(t_n) \rightarrow G$ it occurs exactly twice. Let $\mathbb{P}_i^{j+}$ be the element of $\mathcal{N}_{\mathbb{W}}$ such that $t(\mathbb{P}_i^{j+}) = p_i^j$ and $\mathbb{P}_i^{j-}$ be the element of $\mathcal{P}_{\mathbb{W}} \setminus \mathcal{N}_{\mathbb{W}}$ such that $t(\mathbb{P}_i^{j-}) = p_i^j$. In the same way we define $\mathbb{Q}_i^{j+}$ and $\mathbb{Q}_i^{j-}$.

The function π can only satisfy proof conditions (1) and (2) if for every i and j, $\pi(\mathbb{P}_i^{j+}) = \mathbb{P}_i^{j-}$ and $\pi(\mathbb{Q}_i^{j+}) = \mathbb{Q}_i^{j-}$. If it is so then π satisfies proof conditions (3) and (5).

The following facts hold:

1. $d(P_i^{m-}) = 0$.

2. If $\neg_{t_i} x_i$ does not appear in clause c_j, then $\varphi^3(\mathbb{P}_i^{j-1+}) = \varphi^2(\mathbb{Q}_i^{j-}) = \varphi(\mathbb{Q}_{i-1}^{j+}) = \mathbb{P}_{i-1}^{j-}$.

3. If $\neg_{t_i} x_i$ appears in clause c_j, then $\varphi^3(\mathbb{Q}_{i-1}^{j+}) = \varphi^2(\mathbb{P}_{i-1}^{j-}) = \varphi(\mathbb{P}_i^{j-1+}) = \mathbb{Q}_i^{j-}$.

4. $\varphi^4(\mathbb{Q}_0^{j-}) = \varphi^3(\mathbb{P}_0^{j+}) = \varphi^2(\mathbb{P}_n^{j-1-}) = \varphi(\mathbb{Q}_n^{j+}) = \mathbb{P}_n^{j-}$.

Lemma 4.1. *For every* $0 < i \leq n$ *and* $j > 0$, $\mathbb{P}_i^{j-1+} <_\psi \mathbb{Q}_i^{j-}$.

Proof. For $i = n$ this is true, because

$$\psi^3(\mathbb{P}_n^{j-1+}) = \pi\varphi\pi(\mathbb{P}_n^{j-1+}) = \pi\varphi(\mathbb{P}_n^{j-1-}) = \pi(\mathbb{Q}_n^{j+}) = \mathbb{Q}_n^{j-}.$$

Now suppose that for all $i' > i$ this was already proven. There are four possibilities:

1. If $\neg_{t_{i+1}} x_{i+1}$ does not appear in clauses c_{j-1} and c_j, then $\psi^2(\mathbb{P}_i^{j-1+}) = \mathbb{P}_{i+1}^{j-1+}$, $\psi^2(\mathbb{Q}_{i+1}^{j-}) = \mathbb{Q}_i^{j-}$, and $\mathbb{P}_{i+1}^{j-1+} <_\psi \mathbb{Q}_{i+1}^{j-}$. Thus $\mathbb{P}_i^{j-1+} <_\psi \mathbb{Q}_i^{j-}$.

2. If $\neg_{t_{i+1}} x_{i+1}$ does not appear in the clause c_{j-1}, but appears in c_j, then $\psi^3(\mathbb{P}_i^{j-1+}) = \pi\varphi\pi(\mathbb{P}_i^{j-1+}) = \pi\varphi(\mathbb{P}_i^{j-1-}) = \pi(\mathbb{Q}_i^{j+}) = \mathbb{Q}_i^{j-}$.

3. If $\neg_{t_{i+1}} x_{i+1}$ appears in the clause c_{j-1}, but does not appear in c_j, then $\psi^2(\mathbb{P}_i^{j-1+}) = \mathbb{P}_{i+1}^{j-2+}$, $\psi^2(\mathbb{Q}_{i+1}^{j}) = \mathbb{Q}_i^{j-}$, $\varphi(\mathbb{Q}_{i+1}^{j-1+}) = \mathbb{P}_{i+1}^{j-1+}$, $\mathbb{P}_{i+1}^{j-2+} <_\psi \mathbb{Q}_{i+1}^{j-1-}$, and $\mathbb{P}_{i+1}^{j-1+} <_\psi \mathbb{Q}_{i+1}^{j-}$. Thus $\mathbb{P}_i^{j-1+} <_\psi \mathbb{Q}_i^{j-}$.

4. If $\neg_{t_{i+1}} x_{i+1}$ appears in both clauses c_{j-1} and c_j, then $\psi^2(\mathbb{P}_i^{j-1+}) = \mathbb{P}_{i+1}^{j-2+}$, $\psi^2(\mathbb{Q}_{i+1}^{j-1-}) = \mathbb{Q}_i^{j-}$, and $\mathbb{P}_{i+1}^{j-2+} <_\psi \mathbb{Q}_{i+1}^{j-1-}$. Thus $\mathbb{P}_i^{j-1+} <_\psi \mathbb{Q}_i^{j-}$.

□

Lemma 4.2. *For every $0 \le i < n$ and $j > 0$, $\mathbb{Q}_i^{j+} <_\psi \mathbb{P}_i^{j-}$.*

Proof. For $i = 0$ this is true, because

$$\psi^3(\mathbb{Q}_0^{j+}) = \pi\varphi\pi(\mathbb{Q}_0^{j+}) = \pi\varphi(\mathbb{Q}_0^{j-}) = \pi(\mathbb{P}_0^{j+}) = \mathbb{P}_0^{j-}.$$

Now suppose that for all $i' < i$ this was already proven. There are four possibilities:

1. If $\neg_{t_i} x_i$ does not appear in clauses c_{j+1} and c_j, then $\psi^2(\mathbb{Q}_i^{j+}) = \mathbb{Q}_{i-1}^{j+}$, $\psi^2(\mathbb{P}_{i-1}^{j-}) = \mathbb{P}_i^{j-}$, and $\mathbb{Q}_{i-1}^{j+} <_\psi \mathbb{P}_{i-1}^{j-}$. Thus $\mathbb{Q}_i^{j+} <_\psi \mathbb{P}_i^{j-}$.

2. If $\neg_{t_i} x_i$ does not appear in the clause c_{j+1}, but appears in c_j, then $\psi^3(\mathbb{Q}_i^{j+}) = \pi\varphi\pi(\mathbb{Q}_i^{j+}) = \pi\varphi(\mathbb{Q}_i^{j-}) = \pi(\mathbb{P}_i^{j+}) = \mathbb{P}_i^{j-}$.

3. If $\neg_{t_i} x_i$ appears in the clause c_{j+1}, but does not appear in c_j, then $\psi^2(\mathbb{Q}_i^{j+}) = \mathbb{Q}_{i-1}^{j+}$, $\psi^2(\mathbb{P}_{i-1}^{j+1-}) = \mathbb{P}_i^{j-}$, $\varphi(\mathbb{P}_{i-1}^{j+}) = \mathbb{Q}_{i-1}^{j+1+}$, $\mathbb{Q}_{i-1}^{j+} <_\psi \mathbb{P}_{i-1}^{j-}$, and $\mathbb{Q}_{i-1}^{j+1+} <_\psi \mathbb{P}_{i-1}^{j+1-}$. Thus $\mathbb{Q}_i^{j+} <_\psi \mathbb{P}_i^{j-}$.

4. If $\neg_{t_i} x_i$ appears in both clauses c_{j+1} and c_j, then $\psi^2(\mathbb{Q}_i^{j+}) = \mathbb{Q}_{i-1}^{j+1+}$, $\psi^2(\mathbb{P}_{i-1}^{j+1-}) = \mathbb{P}_i^{j-}$, and $\mathbb{Q}_{i-1}^{j+1+} <_\psi \mathbb{P}_{i-1}^{j+1-}$. Thus $\mathbb{Q}_i^{j+} <_\psi \mathbb{P}_i^{j-}$.

□

From lemmas 4.1 and 4.2 we can conclude that if $i > 0$ and $j \le j'$ then $\mathbb{P}_i^{j+} <_\psi \mathbb{P}_i^{j'+}$.

Lemma 4.3. *If $i < i'$, then $\mathbb{P}_i^{j+} <_\psi \mathbb{P}_{i'}^{j+}$.*

Proof. If $\neg_{t_{i+1}} x_{i+1}$ appears in clause c_j, then $\psi^2(\mathbb{P}_i^{j+}) = \mathbb{P}_{i+1}^{j-1+}$ and $\mathbb{P}_{i+1}^{j-1+} <_\psi \mathbb{P}_{i+1}^{j+}$. If $\neg_{t_{i+1}} x_{i+1}$ appears in clause c_{j+1}, then $\psi(\mathbb{P}_i^{j+1-}) = \mathbb{P}_{i+1}^{j+}$ and $\mathbb{P}_i^{j-} <_\psi \mathbb{P}_i^{j+1+}$. If neither of this is the case, then $\psi^2(\mathbb{P}_i^{j+}) = \mathbb{P}_{i+1}^{j+}$. This means that $\mathbb{P}_i^{j+} <_\psi \mathbb{P}_{i+1}^{j+}$ and thus $\mathbb{P}_i^{j+} <_\psi \mathbb{P}_{i'}^{j+}$. □

Lemma 4.4. *$\langle t_1, \ldots, t_n \rangle$ is a satisfying assignment for $c_1 \wedge \ldots \wedge c_m$ if and only if $\mathrm{L}^*(\backslash, /) \vdash F_1(t_1) \ldots F_n(t_n) \to G$ and if and only if*

$$\mathrm{L}(\backslash, /) \vdash F_1(t_1) \ldots F_n(t_n) \to G.$$

Proof. Suppose that $\langle t_1, \ldots, t_n \rangle$ is a satisfying assignment for $c_1 \wedge \ldots \wedge c_m$. In view of lemmas 4.1 and 4.2 the only members of $\mathcal{N}_{\mathbb{W}}$ for which we have not proved that π satisfies proof condition (4) are $\mathbb{P}_0^{j+}$.

We now prove that for every j, $\mathbb{P}_0^{j+} <_\psi \varphi(\mathbb{P}_0^{j+}) = \mathbb{P}_n^{j-1-}$. There exist i such that $\neg_{t_i} x_i$ appears in clause c_j. This means that $\psi(\mathbb{P}_{i-1}^{j-}) = \mathbb{P}_i^{j-1+}$ and by lemma 4.3 $\mathbb{P}_0^{j+} <_\psi \mathbb{P}_i^{j+}$ and $\mathbb{P}_i^{j-1+} <_\psi \mathbb{P}_n^{j-1+}$. Thus $\mathbb{P}_0^{j+} <_\psi \varphi(\mathbb{P}_0^{j+}) = \mathbb{P}_n^{j-1-}$ and by lemma 3.2 we can now say that $\mathrm{L}^*(\backslash,/) \vdash F_1(t_1)\ldots F_n(t_n) \to G$.

Suppose that $\langle t_1,\ldots,t_n\rangle$ is not a satisfying assignment for $c_1 \wedge \ldots \wedge c_m$. There exists j such that no $\neg_{t_i} x_i$ appear in clause c_j. This means that for $i \leq n$, $\psi^{2i}(\mathbb{Q}_n^{j+}) = \mathbb{Q}_{n-i}^{j+}$, $\psi(\mathbb{P}_n^{j-1-}) = \mathbb{Q}_n^{j+}$, and $\psi(\mathbb{Q}_0^{j-}) = \mathbb{P}_0^{j+}$. Thus $\mathbb{P}_n^{j-1-} <_\psi \mathbb{P}_0^{j+}$. This means that π cannot satisfy proof condition (4). Thus by lemma 2.1 $\mathrm{L}^*(\backslash,/) \nvdash F_1(t_1)\ldots F_n(t_n) \to G$.

Since π satisfies proof condition (5),

$$\mathrm{L}(\backslash,/) \vdash F_1(t_1)\ldots F_n(t_n) \to G \Leftrightarrow \mathrm{L}^*(\backslash,/) \vdash F_1(t_1)\ldots F_n(t_n) \to G$$

and thus the lemma is fully proven. □

Lemma 4.5. $\mathrm{L}(\backslash,/) \vdash \Pi_i \to F_i(t_i)$, *where* $t_i \in \{0,1\}$.

Lemma 4.6. *If the formula* $c_1\wedge\ldots\wedge c_m$ *is satifiable, then* $\mathrm{L}(\backslash,/) \vdash \Pi_1\ldots\Pi_n \to G$.

Proof. Suppose $\langle t_1,\ldots,t_n\rangle$ is a satisfying assignment for $c_1\wedge\ldots\wedge c_m$. According to Lemma 4.4 $\mathrm{L}(\backslash,/) \vdash F_1(t_1)\ldots F_n(t_n) \to G$. Now we apply Lemma 4.5 and the cut rule n times. □

Suppose $\mathrm{L}^*(\backslash,/) \vdash \Pi_1\ldots\Pi_n \to G$. Consider

$$\begin{aligned}\mathbb{W} = &[\![F_1(0)/(B_1\backslash A_1)]\!]^\to [\![H_1]\!]^\to [\![(D_1\backslash C_1)\backslash F_1(1)]\!]^\to \ldots \\ &\ldots [\![F_n(0)/(B_n\backslash A_n)]\!]^\to [\![H_n]\!]^\to [\![(D_n\backslash C_n)\backslash F_n(1)]\!]^\to [\![G]\!]\end{aligned}$$

By Lemma 3.2 for $\mathcal{P}_\mathbb{W}$ there exists π satisfying proof conditions (1)-(4). Consider the following abbreviations:

$$\begin{aligned}
\mathbb{F}_i^0 &= [\![F_1(0)/(B_1\backslash A_1)]\!]^\to [\![H_1]\!]^\to [\![(D_1\backslash C_1)\backslash F_1(1)]\!]^\to \ldots [\![F_i(0)]\!]^\to \\
\mathbb{A}_i &= \mathbb{F}_i^0([\![A_i]\!]^\to)^\to \\
\mathbb{B}_i &= \mathbb{A}_i(([\![B_i]\!]^\leftarrow)^\to)^\to \\
\mathbb{H}_i &= \mathbb{B}_i[\![H_i]\!]^\to \\
\mathbb{C}_i &= \mathbb{H}_i([\![C_i]\!]^\leftarrow)^\to \\
\mathbb{D}_i &= \mathbb{C}_i(([\![D_i]\!]^\leftarrow)^\leftarrow)^\to \\
\mathbb{F}_i^1 &= \mathbb{D}_i[\![F_i(1)]\!]^\to
\end{aligned}$$

Lemma 4.7. *If* $\mathrm{L}^*(\backslash,/) \vdash \Pi_1\ldots\Pi_i F_{i+1}(t_{i+1})\ldots F_n(t_n) \to G$, *then there is* $t_i \in \{0,1\}$ *such that* $\mathrm{L}^*(\backslash,/) \vdash \Pi_1\ldots\Pi_{i-1}F_i(t_i)\ldots F_n(t_n) \to G$

Proof. Consider $\mathbb{W}' = \mathbb{F}_i^1\mathbb{W}''$, where $\mathbb{W}'' = [\![F_{i+1}(t_{i+1})]\!]^{\rightarrow} \ldots [\![F_n(t_n)]\!]^{\rightarrow}[\![G]\!]$. By Lemma 3.2 for $\mathcal{P}_{\mathbb{W}'}$ there exists π satisfying proof conditions (1)-(5).

Let $\mathbb{W}'_0 = \mathbb{F}_{i-1}^1[\![F_i(0)]\!]^{\rightarrow}\mathbb{W}''$ and $\mathbb{W}'_1 = \mathbb{F}_{i-1}^1[\![F_i(1)]\!]^{\rightarrow}\mathbb{W}''$.

For each j there are only two elements of $\mathcal{P}_{\mathbb{W}'}$ such that $t(\mathbb{A}) = a_i^j$, two elements such that $t(\mathbb{A}) = b_i^j$, two elements such that $t(\mathbb{A}) = c_i^j$, and two elements such that $t(\mathbb{A}) = d_i^j$. This means that these pairs of elements are π-closed.

For each j there are six elements of $\mathcal{P}_{\mathbb{W}'}$ such that $t(\mathbb{A}) = p_i^0$. Let us denote them by $\mathbb{P}_1, \ldots, \mathbb{P}_6$ so that $\mathbb{P}_1 \sqsubset \ldots \sqsubset \mathbb{P}_6$. The following holds:

$$\mathbb{F}_{i-1}^1 \sqsubset \mathbb{P}_1 \sqsubseteq \mathbb{F}_i^0 \sqsubset \mathbb{P}_2 \sqsubseteq \mathbb{A}_i \sqsubset \mathbb{B}_i \sqsubset \mathbb{P}_3 \sqsubseteq \mathbb{H}_i \sqsubset \mathbb{P}_4 \sqsubseteq \mathbb{C}_i \sqsubset \mathbb{D}_i \sqsubset \mathbb{P}_5 \sqsubseteq \mathbb{F}_i^1 \sqsubset \mathbb{P}_6.$$

$\{\mathbb{P}_1, \ldots, \mathbb{P}_6\}$ is π-closed. $\mathbb{P}_1, \mathbb{P}_3, \mathbb{P}_5 \in \mathcal{N}_{\mathbb{W}}$. $[\mathbb{P}_1, \mathbb{P}_2]_{\sqsubset}$,$[\mathbb{P}_3, \mathbb{P}_6]_{\sqsubset}$, and $[\mathbb{P}_4, \mathbb{P}_5]_{\sqsubset}$ cannot be π-closed, therefore there are only two possibilities: either $\pi(\mathbb{P}_1) = \mathbb{P}_4$, $\pi(\mathbb{P}_3) = \mathbb{P}_2$, and $\pi(\mathbb{P}_5) = \mathbb{P}_6$, or $\pi(\mathbb{P}_1) = \mathbb{P}_6$, $\pi(\mathbb{P}_3) = \mathbb{P}_4$, and $\pi(\mathbb{P}_5) = \mathbb{P}_2$.

Suppose that $\pi(\mathbb{P}_1) = \mathbb{P}_4$, $\pi(\mathbb{P}_3) = \mathbb{P}_2$, and $\pi(\mathbb{P}_5) = \mathbb{P}_6$. Since $[\mathbb{P}_1, \mathbb{P}_4]_{\sqsubset}$ is π-closed, in $(\mathbb{F}_{i-1}^1, \mathbb{D}_i]_{\sqsubset}$ the only elements for which we had not determined $\pi(\mathbb{A})$ are elements in $(\mathbb{F}_{i-1}^1, \mathbb{F}_i^0]_{\sqsubset}$ and in $(\mathbb{C}_i, \mathbb{D}_i]_{\sqsubset}$ with $t(\mathbb{A}) = p_{i-1}^j$ and with $t(\mathbb{A}) = q_{i-1}^j$. Notice that $t(\mathbb{D}_i) = p_{i-1}^m$ and $\mathbb{D}_i \in \mathcal{N}_{\mathbb{W}'}$.

If $i = 1$, then there are only two variants for $\pi(\mathbb{D}_i)$: one is $p_0^{m(l)}$ and the other one is $\mathbb{D}_1 p_0^{m(l)}$, where $l = 2$ or $l = 4$. Therefore, since the φ-join of $\mathbb{D}_1$ and $\mathbb{D}_1 p_0^{m(l)}$ is $\mathbb{F}_1^1 \in \mathcal{N}_{\mathbb{W}'}$, $\pi(\mathbb{D}_1) = p_0^{m(l)}$ and $[p_0^{m(l)}, \mathbb{D}_1]_{\sqsubset}$ is π-closed.

If $i > 1$, then there are four variants for $\pi(\mathbb{D}_i)$: $\mathbb{F}_{i-1}^1 p_{i-1}^{m(l)}$, $\mathbb{D}_i p_{i-1}^{m(l)}$, where $l = 2$ or $l = 4$, $\mathbb{H}_{i-1} p_{i-1}^{m(2)}$, and $\mathbb{F}_{i-1}^0 p_{i-1}^{m(-2)}$. The second variant is ruled out. If $\pi(\mathbb{D}_i) = \mathbb{H}_{i-1} p_{i-1}^{m(2)}$, then $\pi(\mathbb{D}_{i-1}) = \mathbb{D}_{i-1} p_{i-2}^{m(l)}$, where $l = 2$ or $l = 4$, and the φ-join of $\mathbb{D}_{i-1}$ and $\mathbb{D}_{i-1} p_{i-2}^{m(l)}$ is $\mathbb{F}_{i-1}^1 \in \mathcal{N}_{\mathbb{W}'}$. If $\pi(\mathbb{D}_i) = \mathbb{F}_{i-1}^0 p_{i-1}^{m(-2)}$ then since the segment $(\mathbb{F}_{i-1}^0, \mathbb{D}_i]_{\sqsubset}$ is φ-closed and π-closed, $\mathbb{G} \not\leq_{\psi} \mathbb{F}_{i-1}^0 p_{i-1}^{m(-2)}$ for all $\mathbb{G} \notin (\mathbb{F}_{i-1}^0, \mathbb{D}_i]_{\sqsubset}$. But $\psi^2(\mathbb{D}_i) = \varphi(\pi(\mathbb{D}_i)) = \varphi(\mathbb{F}_{i-1}^0 p_{i-1}^{m(-2)}) = \mathbb{F}_{i-1}^0 \notin (\mathbb{F}_{i-1}^0, \mathbb{D}_i]_{\sqsubset}$. Therefore $\mathbb{D}_i \not<_{\psi} \varphi(\mathbb{D}_i) = \mathbb{H}_i p_i^{m(2)}$ and proof condition (4) is not satisfied. Therefore $\pi(\mathbb{D}_i) = \mathbb{F}_{i-1}^1 p_{i-1}^{m(l)}$ and $(\mathbb{F}_{i-1}^1, \mathbb{D}_i]_{\sqsubset}$ is π-closed.

Therefore since $(\mathbb{F}_{i-1}^1, \mathbb{D}_i]_{\sqsubset}$ is π-closed and φ-closed, by Lemma 3.3 for $\mathbb{W}'_1$ there is π' satisfying proof conditions (1)-(4) and

$$\mathrm{L}^*(\backslash, /) \vdash \Pi_1 \ldots \Pi_{i-1} F_i(1) \ldots F_n(t_n) \to G.$$

Suppose that $\pi(\mathbb{P}_1) = \mathbb{P}_6$, $\pi(\mathbb{P}_3) = \mathbb{P}_4$, and $\pi(\mathbb{P}_5) = \mathbb{P}_2$.

Here in $(\mathbb{F}_i^0, \mathbb{F}_i^1]_{\sqsubset}$ the only elements for which we had not determined $\pi(\mathbb{A})$ are elements in $(\mathbb{D}_i, \mathbb{F}_i^1]_{\sqsubset}$ and in $(\mathbb{F}_i^0, \mathbb{A}_i]_{\sqsubset}$ with $t(\mathbb{A}) = p_{i+1}^j$ and with $t(\mathbb{A}) = q_{i+1}^j$. Let $\mathbb{E} = \mathbb{F}_i^0 p_{i+1}^{m(-2)} \in \mathcal{P}_{\mathbb{W}'}$.

There are only two variants for $\pi(\mathbb{E})$: one is $\mathbb{F}_i^0$ and the other one is $\mathbb{F}_i^1$. The φ-join of $\mathbb{E}$ and $\mathbb{F}_i^0$ is $\mathbb{F}_i^0 \in \mathcal{N}_{\mathbb{W}}$. Therefore $\pi(\mathbb{E}) = \mathbb{F}_i^1$ and $(\mathbb{F}_i^0, \mathbb{F}_i^1]_{\sqsubset}$ is π-closed.

Therefore since $(\mathbb{F}_i^0, \mathbb{F}_i^1]_{\sqsubset}$ is π-closed and φ-closed, by Lemma 3.3 for $\mathbb{W}_0'$ there is π' satisfying proof conditions (1)-(4) and

$$\mathrm{L}^*(\backslash, /) \vdash \Pi_1 \dots \Pi_{i-1} F_i(0) \dots F_n(t_n) \to G.$$

$\square$

Lemma 4.8. *If* $\mathrm{L}^*(\backslash, /) \vdash \Pi_1 \dots \Pi_n \to G$, *then the formula* $c_1 \wedge \dots \wedge c_m$ *is satisfiable.*

Proof. Applying n times Lemma 4.7, we get that there exists $\langle t_1, \dots, t_n \rangle \in \{0,1\}^n$ such that $\mathrm{L}^*(\backslash, /) \vdash F_1(t_1) \dots F_n(t_n) \to G$. By Lemma 4.4 this means that $\langle t_1, \dots, t_n \rangle$ is a satisfying assignment for $c_1 \wedge \dots \wedge c_m$. $\square$

Since for all sequents $\mathrm{L}(\backslash, /) \vdash \Pi \to A \Rightarrow \mathrm{L}^*(\backslash, /) \vdash \Pi \to A$, Lemma 4.6 and Lemma 4.8 together give us Theorem 2.1.

Acknowledgements

I am most grateful to Prof. M. Pentus for his constant attention to my work.

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