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Martingales for Uniformly Quasisymmetric Circle Endomorphisms

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Abstract

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The main subject studied in this thesis is the space of all uniformly quasisymmetric circle endomorphisms preserving the Lebesgue measure. Although many of our arguments work for any degree $d \geq 2$, our proof will be mainly written for degree 2 maps. We will introduce a sequence of Markov partitions of the unit circle by using preimages of the fixed point of such circle endomorphism f. The uniform quasisymmetry condition is equivalent to the bounded nearby geometry condition of the Markov partitions. In Chapter 2 of this thesis, for each f, we use the Lebesgue invariant condition and the bounded geometry property

to construct a martingale sequence $\{X_{f,k}\}$ which has a L^1 limiting function X_f on the dual symbolic space. We also show that the limiting martingale is invariant under symmetric conjugacy. The classical Hilbert transform introduces an almost complex structure on the space of all uniformly quasisymmetric circle endomorphisms that preserve the Lebesgue measure. This is presented in Chapter 3. In Chapters 4 & 5, we study locally constant limiting martingales and the related rigidity problems. A locally constant limiting martingale is the limit of a martingale sequence $\{X_k\}$ of length n for some $n \geq 0$, i.e. the limiting martingale $X = X_n$ for some n. We prove the rigidity problem for martingale sequence of length $n \leq 4$. That is, there is a unique way to construct a sequence of Markov partitions if the given limiting martingale X_f is equal to $X_{f,n}$ for some $n \leq 4$. One of the consequences is that if two martingale sequences$\{X_{f,k}\}$ and $\{X_{g,k}\}$ have the same limit and both have length $n \leq 4$, where f and g are two uniformly quasisymmetric circle endomorphisms preserving the Lebesgue measure, then $f = g$. Another consequence is that if $\{X_{f,k}\}$ has length $n \leq 4$, then there is no other map in the symmetric conjugacy class of f that preserves Lebesgue measure. In the class of uniformly symmetric circle endomorphisms, we prove that $q(z) = z^2$, which has martingale sequence $\{X_{q,k} = 2\}$ for any k, is the only map whose limiting martingale is locally constant. Finally, we construct an analytic expanding circle endomorphism which preserves the Lebesgue measure and is a quasisymmetirc conjugate of $q(z) = z^2$, i.e. $f = hqh^{-1}$. We show that the conjugacy h is symmetric at one point but not symmetric on the whole unit circle.

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Chapter 1

Introduction

Tossing a coin is an old game in the gambling business. For a fair coin, the head and the tail of the coin have equal chances to occur. From the modern mathematical language of Kolmogorov, the probability of the head or the tail is 1/2 when one tosses a fair coin. If one tosses a fair coin n-times, there are 2^n outcome events and each event has the probability $1/2^n$. Given the result of the $(n-1)^{th}$ tossing, let X_n be the relative probability of the n^{th} tossing. Since the $(n-1)^{th}$ tossing and the n^{th} tossing are independent, the random variable X_n is equal to 1/2. The sequence of random variables $\{X_n\}_{n=1}^{\infty}$ thus forms an infinite martingale sequence. The limiting martingale exists and is a random variable

$X = 1/2$ defined on the space of all infinite sequences of head and tail. Therefore, from the long term perspective, a fair coin is predictable.

If one tosses an infinite sequence of unfair coins, the story will be completely different. The long term predictability could fail. The main purpose of the first two chapters in this thesis is to show that the long term predictability is still valid if we propose a bounded geometry condition and a Lebesgue measure invariant condition. Under these two conditions, a uniformly bounded infinite martingale sequence $\{X_n\}$ can be constructed whose limiting martingale X exists.

1.1 Uniformly quasisymmetric circle endomorphisms

Markov map and dual symbolic space has been used to study geometric structures of dynamical systems [15], [17]. A circle endomorphism of degree two is a special Markov map with a standard Markov partition. In [3], Cui, Gardiner, and Jiang considered the scaling functions on dual symbolic space for uniformly symmetric circle endomorphisms. In this thesis, we study a larger family of circle endomorphisms, which are called uniformly quasisymmetric circle endomorphisms.

Let $T = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle in the complex plane $\mathbb{C}$. Suppose

$$f : T \to T$$

is an orientation-preserving covering map of degree $d \geq 2$. We call it a *circle endomorphism*. Suppose

$$h : T \to T$$

is an orientation-preserving homeomorphism. We call it a *circle homeomorphism*.

For a circle endomorphism f, it has a fixed point. By conjugating a rotation of the circle, we will always assume that 1 is a fixed point of f, that is, $f(1) = 1$.

The universal cover of T is the real line $\mathbb{R}$ with a covering map

$$\pi(x) = e^{2\pi i x} : \mathbb{R} \to T.$$

Then every circle endomorphism f can be lifted to an orientation-preserving homeomorphism

$$F : \mathbb{R} \to \mathbb{R}, \quad F(x+1) = F(x) + d, \quad \forall x \in \mathbb{R}.$$

We will assume that $F(0) = 0$. Then there is a one-to-one correspondence between f and F. Therefore, we also call such an F a circle endomorphism.

Every orientation-preserving circle homeomorphism h can be lifted to an orientation-preserving homeomorphism

$$H : \mathbb{R} \to \mathbb{R}, \quad H(x+1) = H(x) + 1, \quad \forall x \in \mathbb{R}.$$

We will assume throughout this thesis that $0 \le H(0) < 1$. Then there is a one-to-one correspondence between h and H. Therefore, we also call such an H a circle homeomorphism.

Definition 1. *A circle homeomorphism h is called quasisymmetric if there is a constant $M \ge 1$ such that*

$$M^{-1} \le \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \le M, \quad \forall x \in \mathbb{R}, \ \forall t > 0.$$

Definition 2. *A circle endomorphism f is called uniformly qua-*

sisymmetric if there is a constant $M > 0$ such that

$$M^{-1} \leq \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq M \tag{1.1}$$

for all $x \in \mathbb{R}$ and $t > 0$ and any $n > 0$.

The following example of a uniformly quasisymmetric circle endomorphism can be found in [17, 20].

A circle endomorphism f is C^1 if the first derivative F' exists and is continuous. And, furthermore, it is called $C^{1+\alpha}$ for some $0 < \alpha \leq 1$ if F' is α-Hölder continuous, that is,

$$\sup_{x \neq y \in \mathbb{R}} \frac{|F'(x) - F'(y)|}{|x-y|^{\alpha}} = \sup_{x \neq y \in [0,1]} \frac{|F'(x) - F'(y)|}{|x-y|^{\alpha}} < \infty.$$

A C^1 circle endomorphism f is called expanding if there are constants $C > 0$ and $\lambda > 1$ such that

$$(F^n)'(x) \geq C\lambda^n, \quad n = 1, 2, \cdots .$$

Example 1. *A $C^{1+\alpha}$, for some $0 < \alpha \leq 1$, circle expanding endomorphism f is uniformly quasisymmetric.*

Consider the map $q(z) = z^d$ for $d > 1$. Then it is a circle endomorphism of degree d. Let

$$f = h \circ q \circ h^{-1}$$

where h is a circle homeomorphism.

Theorem 1. *The circle endomorphism f is uniformly quasisymmetric if and only if h is a quasisymmetric homeomorphism.*

Refer to [20, 21] (or refer to [16, 17]) for this theorem. Thus, we consider two spaces

$$\mathcal{F} = \{f \mid f \text{ is a uniformly quasisymmetric circle endomorphism with } f(1) = 1\}$$

and

$$\mathcal{H} = \{h \mid h \text{ is a quasisymmetric circle homeomorphism with } h(1) = 1\}.$$

The map

$$\beta : \mathcal{H} \to \mathcal{F}; \qquad \beta(h) = h \circ q \circ h^{-1}$$

is a bijective map (see [21]).

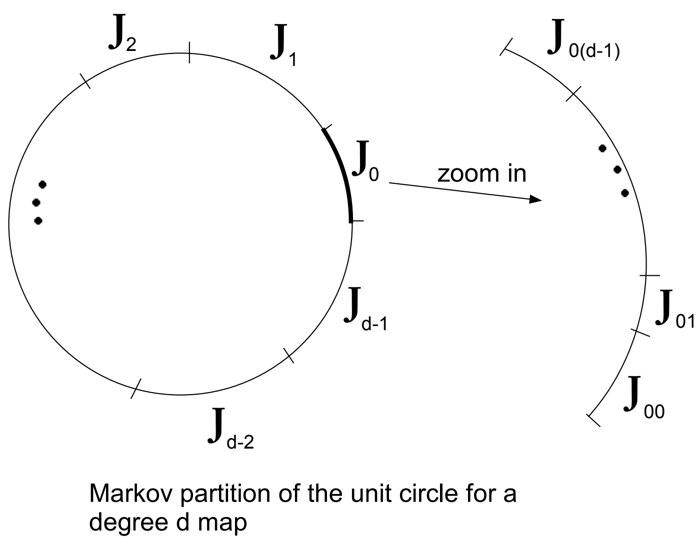

Figure 1.1: Markov partition

1.2 Circle endomorphisms and their symbolic representation

Suppose f is a circle endomorphism in $\mathcal{F}$. Consider the preimage $f^{-1}(1)$. Then $f^{-1}(1)$ cuts T into d closed intervals J_0, J_1, $\cdots$, J_{d-1}, ordered by the counter-clockwise order of T. Suppose J_0 has an endpoint 1. Then J_{d-1} also has an endpoint 1.

Let

$$\varpi_0 = \{J_0, J_1, \cdots, J_{d-1}\}.$$

Then it is a Markov partition, that is,

i. $T = \cup_{k=0}^{d-1} J_k$,

ii. the restriction of f to the interior of J_i is injective for every $0 \leq i \leq d-1$,

iii. $f(J_i) = T$ for every $0 \leq i \leq d-1$.

Let I_0, I_1, $\cdots$, I_{d-1} be the lifts of J_0, J_1, $\cdots$, J_{d-1} in $[0,1]$. Then we have that

i) $[0,1] = \cup_{k=0}^{d-1} I_k$,

ii) $F(I_i) = [i, i+1]$ for every $0 \leq i \leq d-1$.

Let

$$\eta_0 = \{I_0, I_1, \cdots, I_{d-1}\}.$$

Then it is a partition of $[0,1]$.

Consider the pull-back partition $\varpi_n = f^{-n}\varpi_0$ of ϖ_0 by f^n. It contains d^{n+1} intervals and is also a Markov partition of T.

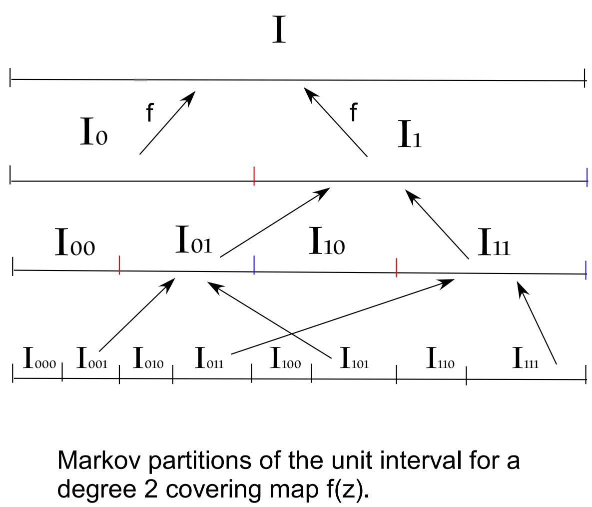

Figure 1.2: Markov partitions of unit interval

Intervals J in ϖ_n can be labeled as follows. Let $w_n = i_0 i_1 \cdots i_{n-1}$ be a word of length n of $0's$, $1's$, $\cdots$, and $(d-1)'s$. Then $J_{w_n} \in \varpi_n$ if $f^k(J_{w_n}) \subset J_{i_k}$ for $0 \leq k \leq n-1$. Then

$$\varpi_n = \{J_{w_n} \mid w_n = i_0 i_1 \cdots i_{n-1}, i_k \in \{0, 1, \cdots, d-1\}, k = 0, 1, \cdots, d-1\}.$$

Let η_n be the corresponding lift partition of ϖ_n in $[0, 1]$ with the same labelings. Then

$$\eta_n = \{I_{w_n} \mid w_n = i_0 i_1 \cdots i_{n-1}, i_k \in \{0, 1, \cdots, d-1\}, k = 0, 1, \cdots, d-1\}.$$

Adding a digit at the end, we get a subinterval, and we get a preimage by adding a digit in the front.

Consider the space

$$\Sigma = \prod_{n=0}^{\infty} \{0, 1, \cdots, d-1\}$$

$$= \{w = i_0 i_1 \cdots i_k \cdots i_{n-1} \cdots \mid i_k \in \{0, 1, \cdots, d-1\}, \ k = 0, 1, \cdots\}$$

with the product topology. It is a compact topological space. A cylinder for a fixed word $w_n = i_0 i_1 \cdots i_{n-1}$ of length n is

$$[w_n] = \{w' = i_0 i_1 \cdots i_{n-1} i'_n i'_{n+1} \cdots \mid i'_{n+k} \in \{0, 1, \cdots, d-1\}, k = 0, 1, \cdots\}$$

All left cylinders form a topological basis of Σ. We call it the left topology. The space Σ with this left topology is called the symbolic space.

For any $w = i_0 i_1 \cdots i_{n-1} i_n \cdots$, let

$$\sigma(w) = i_1 \cdots i_{n-1} i_n \cdots$$

be the shift map. Then (Σ, σ) is called a symbolic dynamical system.

For a point $w = i_0 \cdots i_{n-1} i_n \cdots \in \Sigma$, let $w_n = i_0 \cdots i_{n-1}$. Then

$$\cdots \subset J_{w_n} \subset J_{w_{n-1}} \subset \cdots J_{w_1} \subset T.$$

Since each J_{w_n} is compact,

$$J_w = \cap_{n=1}^{\infty} J_{w_n} \neq \emptyset.$$

If every $J_w = \{x_w\}$ contains only one point, then we define the projection π_f from Σ onto T as

$$\pi_f(w) = x_w.$$

The projection π_f is $1-1$ except for a countable set

$$B = \{w = i_0 i_1 \cdots i_{n-1} 1000 \cdots, i_0 i_1 \cdots i_{n-1} 0(d-1)(d-1)(d-1) \cdots\}.$$

From our construction, one can check that

$$\pi_f \circ \sigma(w) = f \circ \pi_f(w), \quad w \in \Sigma.$$

In this case, we say that f is semi-conjugate to σ by the semi-conjugacy π_f.

For any interval $I = [a, b]$ in $[0, 1]$, we use $|I| = b - a$ to mean its Lebesgue length. Let

$$\iota_{n,f} = \max_{w_n} |I_{w_n}|,$$

where w_n runs over all words of $\{0, 1, \cdots, d-1\}$ of length n.

Two circle endomorphisms f and g are topologically conjugate if there is an orientation-preserving circle homeomorphism h of T such that

$$f \circ h = h \circ g.$$

The following result was first proved by Shub for C^2 expanding circle endomorphisms 1960's by using the contracting mapping theorem.

Theorem 2. *Let f and g be two circle endomorphisms such that both $\iota_{n,f}$ and $\iota_{n,g}$ tend to zero as $n \to \infty$. Then f and g are* *topologically conjugate if and only if their topological degrees are the same.*

Refer to [20, 21] for a proof.

From the bounded nearby geometry property for any $f \in \mathcal{F}$ which we will discuss in the next section, we have that $\iota_{n,f}$ tends to zero as $n \to \infty$ for any $f \in \mathcal{F}$. And from Theorem 1, we have that

Theorem 3. *For any $f, g \in \mathcal{F}$, f and g are topologically conjugate. And the conjugacy h (that is, $h \circ f = g \circ h$) is quasisymmetric. Furthermore, f is always semi-conjugate to σ by the semi-conjugacy π_f.*

From the above theorem the symbolic dynamical system (Σ, σ) is the topological representation for all maps in $\mathcal{F}$.

1.3 Bounded nearby geometry

In this section, we show that the uniformly quasisymmetric condition is equivalent to the bounded nearby geometry defined in [15–17].

Definition 3. *The sequence* $\{\varpi_n\}_{n=0}^{\infty}$ *of nested partitions of* T *is said to have bounded geometry if there is a constant* $C > 0$ *such that*

$$\frac{|J_{\sigma(w_n)}|}{|J_{w_n}|} \leq C, \quad \forall\, J \in \varpi_n; \quad \forall\, n \geq 0. \tag{1.2}$$

Definition 4. *The sequence* $\{\varpi_n\}_{n=0}^{\infty}$ *of nested partitions of* T *is said to have bounded nearby geometry if there is a constant* $C > 0$ *such that*

$$\frac{|J|}{|J'|} \geq C, \quad \forall\, J, J' \in \varpi_n \textit{ with a common endpoint}; \quad \forall\, n \geq 0. \tag{1.3}$$

Theorem 4. *Suppose* f *is a circle endomorphism. Then* f *is uniformly quasisymmetric if and only if the sequence* $\{\varpi_n\}_{n=0}^{\infty}$ *of nested partitions of* T *has the bounded nearby geometry.*

Proof. We first prove the "only if" part. Let F with $F(0) = 0$ be the lift of f. Define

$$G_k(x) = F^{-1}(x+k) : [0,1] \to [0,1], \quad \text{for} \quad k = 0, 1, \cdots, n-1.$$

For any word $w_n = i_0 i_1 \cdots i_{n-1}$, define

$$G_{w_n} = G_{i_0} \circ G_{i_1} \circ \cdots \circ G_{i_{n-1}}.$$

Then

$$I_{w_n} = G_{w_n}([0,1]) = F^{-n}([m, m+1]),$$

where $m = i_{n-1} + i_{n-2}d + \cdots + i_0 d^{n-1}$. Suppose I'_{w_n} is an interval in η_n having a common endpoint with I_{w_n} modulo 1. Then

$$I'_{w_n} = F^{-n}([m+1, m+2]) \quad \text{or} \quad F^{-n}([m, m+1]).$$

Thus

$$C^{-1} \leq \frac{|I_{w_n}|}{|I'_{w_n}|} \leq C$$

where $C > 0$ is the constant in Definition 2. Since I_{w_n} and I'_{w_n} are just lifts of J and J'. We have

$$C^{-1} \leq \frac{|J|}{|J'|} \leq C \tag{1.4}$$

for any intervals $J, J' \in \varpi_n$ with a common endpoint and $n = 0, 1, \cdots$.

The "if" part follows the exact argument in [16, Theorem B, pp.645-646] (also, refer to [17, pp. 90-91]). We give a outline of the proof here. Consider $f = h \circ q \circ h^{-1}$. Recall that $q(z) = z^d$ where $d > 1$ is the degree of f. Then from Theorem 1, f is uniformly quasisymmetric if and only if h is quasisymmetric. Let $\varpi_{n,q}$ be the n^{th}-partition for q and ϖ_n be the n^{th}-partition for f. For any intervals $J, J' \in \varpi_{n,q}$ with a common endpoint and any $n = 0, 1, \cdots$, we have $|J|/|J'| = 1$ and $h(J), h(J') \in \varpi_n$ with a common endpoint. The bounded nearby geometry says that we have a constant $C > 0$, independent of J, J', and n, such that

$$C^{-1} \leq \frac{|h(J)|}{|h(J')|} \leq C. \tag{1.5}$$

This says that h is quasisymmetric at the common endpoint of J and J' with a fixed quasisymmetric constant C. But these common endpoints form a dense subset of the unit circle. This implies the "if" part. □

Remark 1. *From Theorem 4, for any* $f \in \mathcal{F}$, $\iota_{n,f} = \max_{w_n} |I_{w_n}|$ *tends to zero exponentially as* n *goes to infinity.*

Chapter 2

Martingales

2.1 Martingales in probability theory

In this section, we give a brief review of the theory of martingales in probability theory. The standard reference which we used is [27]. Suppose that we have a probability space $(\Omega, \mathcal{B}, P)$ consisting of a space Ω, a $\sigma-$field $\mathcal{B}$ of subsets of Ω, and a probability measure P on the $\sigma-$field $\mathcal{B}$.

Definition 5. *A random variable or measurable function with respect to the $\sigma-$field $\mathcal{B}$ is a map $f : \Omega \to R$, i.e., a real-valued function $f(w)$ on Ω, such that for every Borel set $B \subset R$, $f^{-1}(B) \in \mathcal{B}$.*

Definition 6. *Suppose $\mathcal{B}'$ is a sub $\sigma-$field of $\mathcal{B}$. Suppose that f and g are L^1 integrable random variables with respect to $\mathcal{B}$ and*

$\mathcal{B}'$ respectively. We call g the conditional expectation of f for the given sub $\sigma-$field $\mathcal{B}'$ and denote it by $g = E[f|\mathcal{B}']$ if

$$\int_A g(w)dP = \int_A f(w)dP$$

for all $A \in \mathcal{B}'$.

The Radon-Nikodym theorem shows the existence of conditional expectation for any L^1 integrable random variable with respect to $\mathcal{B}$ for any given sub $\sigma-$field of $\mathcal{B}$ (refer to [27]). The conditional expectation is unique up to measure zero set (also refer to [27]).

Definition 7. *Suppose $(\Omega, \mathcal{B}, P)$ is a probability space. Suppose that $\{\mathcal{B}_n\}_{n=1}^{\infty}$ is a sequence of sub $\sigma-$fields of $\mathcal{B}$ and suppose that $\{X_n\}_{n=1}^{\infty}$ is a sequence of functions. Then $\{(X_n, \mathcal{B}_n)\}_{n=1}^{\infty}$ is called an infinite martingale sequence if*

1. *X_n is a L^1 integrable random variable with respect to $\mathcal{B}_n$ for all $n \geq 1$;*

2. *$\mathcal{B}_n \subset \mathcal{B}_{n+1}$ for all $n \geq 1$;*

3. $X_n = E[X_{n+1}|\mathcal{B}_n]$ *almost everywhere with respect to the probability measure* P.

An infinite martingale sequence $\{(X_n, \mathcal{B}_n)\}_{n=1}^{\infty}$ is called uniformly integrable if

$$\lim_{M\to\infty}(\sup_i E(|X_i| : |X_i| > M)) = 0.$$

Theorem 5 (Martingale Convergence Theorem)**.** *Suppose* $\{(X_n, \mathcal{B}_n)\}_{n=1}^{\infty}$ *is an uniformly integrable infinite martingale sequence. Then there is a random variable* X *with respect to* $\mathcal{B}$ *such that*

$$X_n = E[X|B_n],$$

and then, of course, $X_n \to X$ *in* L^1 *as* n *goes to* ∞.

The main purpose of this chapter is to construct a uniformly bounded infinite martingale sequence $\{(X_n, \mathcal{B}_n)\}_{n=1}^{\infty}$ from a uniformly quasisymmetric circle endomorphism f and then apply the above theorem to construct the limiting martingale X.

2.2 Dual symbolic representation and quasisymmetric invariant probability measures

Suppose f is a circle endomorphism in $\mathcal{F}$. Suppose

$$\varpi_n = \{J_{w_n} \mid w_n = i_0 i_1 \cdots i_{n-1}, i_k \in \{0, 1, \cdots, d-1\}, k = 0, 1, \cdots, n-1\},$$

for $n = 0, 1, \cdots$, and

$$\eta_n = \{I_{w_n} \mid w_n = i_0 i_1 \cdots i_{n-1}, i_k \in \{0, 1, \cdots, d-1\}, k = 0, 1, \cdots, n-1\},$$

for $n = 0, 1, \cdots$, are the corresponding sequences of nested Markov partitions on T and $[0, 1]$, respectively.

For any $w_n = i_0 i_1 \cdots i_{n-1}$, let

$$w_n^* = j_{n-1} j_{n-2} \ldots j_0$$

where $j_{n-1} = i_0$, $\cdots$, $j_0 = i_{n-1}$.

For any fixed $w_n^* = j_{n-1} j_{n-2} \ldots j_0$ and any $n \geq 0$, define the right cylinder

$$[w_n^*] = \{w^* = \ldots j'_n j'_{n-1} j'_{n-2} \ldots j'_0 \mid j'_{n-1} = j_{n-1}, j'_{n-2} = j_{n-2}, \ldots, j'_0 = j_0\}$$

Define

$$\Omega = \prod_{-\infty}^{0} \{0, \cdots, d-1\} = \{w^* = \cdots j_{n-1} \cdots j_0\}.$$

For any fixed $w_n^* = j_{n-1}j_{n-2}...j_0$ and any $n \geq 0$, define the right cylinder

$$[w_n^*] = \{w^* = ...j'_n j'_{n-1} j'_{n-2}...j'_0 \mid j'_{n-1} = j_{n-1}, j'_{n-2} = j_{n-2}, ..., j'_0 = j_0\}$$

Consider

$$\mathcal{G} = \Big\{ \bigcup_{k=1}^{N} [w_{n_k}^*] \Big\} \cup \{\emptyset\}$$

The following proposition is easy.

Proposition 1. *If $n < m$, then either $[w_n^*] \cap [w_m^*] = \emptyset$ or $[w_n^*] \cap [w_m^*] = [w_m^*]$*

Following this, we have that

Proposition 2. *The set $\mathcal{G}$ is a field.*

Proof. From Proposition 1, we can assume every element in $\mathcal{G}$ is a disjoint union of finitely many right cylinders $\{[w_{n_k}^*]\}_{k=1}^{N}$.

First

$$\Omega = \bigcup_{k=0}^{d-1} [k^*] \in \mathcal{G}$$

and $\emptyset \in \mathcal{G}$.

If $A, B \in \mathcal{G}$, then $A \cup B$ and $A \cap B$ are in $\mathcal{G}$. If $w_n^* = j_{n-1}j_{n-2}...j_0$, then

$$[w_n^*]^c = \bigcup_{0\leq k\neq j_{n-1}\leq d-1} [kw_{n-1}^*] \bigcup \bigcup_{\widetilde{w}_{n-1}^* \neq w_{n-1}^*} [\widetilde{w}_{n-1}^*] \in \mathcal{G}$$

and

$$(\bigcup_{k=1}^{N} [w_{n_k}^*])^c = \bigcap_{k=1}^{N} [w_{n_k}^*]^c \in \mathcal{G}$$

Thus $\mathcal{G}$ is a field. □

Let $\mathcal{B}$ be the σ-field generated by the field $\mathcal{G}$. Then we have a space

$$(\Omega, \mathcal{B}).$$

Now we are going to associate a non-atomic probability measure P_f on this space with some uniformly quasisymmetric circle endomorphism f. Before we are able to do this, we need to have the following important invariant condition. We use $Leb(\cdot)$ to denote the Lebesgue measure on T.

Definition 8. *We say that a circle endomorphism f preserves the Lebesgue measure if for any Lebesgue measurable set A in the* *unit circle T,*

$$Leb(f^{-1}(A)) = Leb(A). \tag{2.1}$$

This is equivalent to the following condition, for any interval I in the unit interval $[0,1]$,

$$\sum_{k=0}^{d-1} |F^{-1}(I+k)| = |I| \tag{2.2}$$

One easy example of a circle endomorphism preserving the Lebesgue measure is $q(z) = z^d$ for any $d > 1$. Actually, there are many circle endomorphisms preserving the Lebesgue measure. A well-known example among experts is a Blaschke product fixing zero and mapping the unit disk onto itself (for example, see [23]). More precisely, consider the Blaschke product

$$f(z) = z^j \prod_{1\le i\le d-j} \frac{z-a_i}{1-\overline{a}_i z} : T \to T \tag{2.3}$$

where $1 \le j \le d$ and $|a_i| < 1$, $1 \le i \le d-j$. Then it is a circle endomorphism of degree $d > 1$.

Example 2. *Every f in the form of (2.3) preserves the Lebesgue measure.*

Proof. For the reader's convenience, we include a detailed proof. The equality (2.1) is equivalent to

$$\int_T \phi(f(z))dz = \int_T \phi(z)dz \tag{2.4}$$

for all continuous functions ϕ on T.

Now let ϕ be a continuous function on T. Consider the harmonic partial differential equation on the unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$:

$$\begin{cases} \Delta u = 0; \\ u|T = \phi \end{cases}$$

Then it has a unique solution u which is a harmonic function on D. Since f is analytic on $\mathbb{C}$, $u \circ f$ is also a harmonic function on D with a continuous extension to the boundary T of D. By the mean value theorem in harmonic analysis,

$$\int_T \phi(f(z))|dz| = 2\pi i u(f(0))$$

and

$$\int_T \phi(z)|dz| = 2\pi i u(0).$$

But $f(0) = 0$, this proves (2.4). □

Remark 2. *There are also other circle endomorphisms preserving the Lebesgue measure. It is known that for any $C^{1+\alpha}$ expanding circle endomorphism $\tilde{f}$, there is a $C^{1+\alpha}$-diffeomorphism h of T such that $f = h \circ \tilde{f} \circ h^{-1}$ preserves the Lebesque measure (see, for example, [22]).*

Remark 3. *In addition, a circle endomorphism f is called uniformly symmetric if there is a bounded function $\epsilon(t) > 0$ such that $\epsilon(t) \to 0$ as $t \to 0^+$ and such that*

$$\frac{1}{1+\epsilon(t)} \leq \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq 1 + \epsilon(t)$$

for all $x \in \mathbb{R}$ and $t > 0$ and any $n > 0$. A $C^{1+\alpha}$ expanding circle endomorphism is uniformly symmetric (see [20, 21]). It is clear that a uniformly symmetric circle endomorphism is uniformly quasisymmetric. It has been proved in [20] that for any uniformly symmetric circle endomorphism $\tilde{f}$, there is a symmetric homeomorphism h of T with $h(1) = 1$ such that $f = h \circ \widetilde{f} \circ h^{-1}$ is still uniformly symmetric and preserves the Lebesgue measure.

Remark 4. *However, for arbitrary uniformly quasisymmetric circle endomorphism $\tilde{f}$, we do not have a similar statement. The reason is that if f preserves the Lebesgue measure then the local quasiconformal dilatations of f at the grand orbit*

$$GO(x) = \cup_{m=0}^{\infty} \cup_{n=0}^{\infty} f^{-n}(f^{m}(x))$$

of any point x must be the same. This property will not be changed by conjugating by a symmetric circle homeomorphism. But it is easy to construct a counter-example f of a uniformly quasisymmetric circle endomorphism such that the local quasiconformal dilatations at x and one point of $f^{-1}(x)$ are different. So the following problem is interesting for us. Find conditions on a uniformly quasisymmetric circle endomorphism $\tilde{f}$ such that there is a symmetric circle homeomorphism h and $f = h \circ \tilde{f} \circ h^{-1}$ preserves the Lebesgue measure. (Refer to [20, 21] for the relevant materials to study this problem.)

Suppose $f \in \mathcal{F}$ preserves the Lebesgue measure. We define a $P = P_f$ on $\mathcal{G}$ as

$$P(\emptyset) = 0,$$

$$P([w_n^*]) = |I_{w_n^*}|,$$

and

$$P(\bigcup_{k=1}^{N}[w_{n_k}^*]) = \sum_{k=1}^{N}|I_{w_{n_k}^*}|$$

where $\bigcup_{k=1}^{N}[w_{n_k}^*]$ is a disjoint union. Then $P(\Omega) = |I_0| + |I_1|... + |I_{d-1}| = 1$.

For any right cylinder $[w_n^*]$, $[kw_n^*]$, $k = 0, \cdots, d-1$, are all the sub-cylinders of one level lower and

$$[w_n^*] = \cup_{k=0}^{d-1}[kw_n^*].$$

This implies that

$$f^{-1}(J_{w_n^*}) = \cup_{k=0}^{d-1}J_{kw_n^*}$$

Since f preserves the Lebesgue measure, we have that

$$|I_{w_n^*}| = \sum_{k=0}^{d-1}|I_{kw_n^*}|$$

This implies that

$$P([w_n^*]) = \sum_{k=0}^{d-1}P([kw_n^*]).$$

So P is a probability measure on $\mathcal{G}$. The Kolmogorov extension principle (or the Carathéodory Theorem) implies that P can be extended to a probability measure on the σ-field $\mathcal{B}$, which we still denote as P_f or simply P if there is no confusion. The probability measure P is non-atomic. Thus we construct a probability space

$$(\Omega, \mathcal{B}, P)$$

for every $f \in \mathcal{F}$ preserving the Lebesgue measure.

Define a right shift map σ^* as

$$\sigma^*(w_n^*) = j_{n-1}j_{n-2}...j_1$$

for $w_n^* = j_{n-1}j_{n-2}...j_1j_0$. It is extended to the right shift, we still denote as $\sigma^* : \Omega \to \Omega$, as

$$\sigma^*(w^*) = \cdots j_{n-1}j_{n-2}...j_1$$

for $w^* = \cdots j_{n-1}j_{n-2}...j_1j_0$.

Define the adding machine as

$$\text{add}(j_{n-1}\cdots j_1j_0) = j_{n-1}\cdots(j_k+1)0\cdots 0$$

if $0 \leq k < n-1$ is the first integer such that $0 \leq j_k < d-1$ and

$$add((d-1)\cdots(d-1)) = 0\cdots 0.$$

From Theorem 4, we have the following result.

Theorem 6. *The probability measure P is σ^*-invariant, that is,*

$$P((\sigma^*)^{-1}(A)) = P(A)$$

for any $A \in \mathcal{B}$. Moreover, there is a constant $C > 0$ such that

$$C^{-1} \leq \frac{P([w_n^*])}{P(add([w_n^*]))} \leq C \tag{2.5}$$

for all $w_n^* = j_{n-1} \cdots j_1 j_0$.

Following the proof of Theorem 4 and the construction of P_f, we also have:

Theorem 7. *For any non-atomic σ^*-invariant probability measure P on $(\Omega, \mathcal{B})$ satisfying Condition (2.5), there is an $f \in \mathcal{F}$ preserving the Lebesgue measure such that $P_f = P$.*

Define the space

$$\mathcal{F}_{inv} = \{f \in \mathcal{F} \mid f \text{ preserves the Lebesgue measure}\}$$

Then the corresponding space in $\mathcal{H}$ is

$$\mathcal{H}_{inv} = \{h \in \mathcal{H} \mid Leb(h(q^{-1}(A))) = Leb(h(A)) \text{ for any measurable set } A\}$$

The Leb-invariant condition in the definition of $\mathcal{H}_{inv}$ can be translated into the following condition for the lift H of h:

$$\sum_{k=0}^{k=d-1} \Big(H(\frac{x+k}{d}) - H(\frac{k}{d}) \Big) = H(x). \tag{2.6}$$

Define the probability measure space

$$\mathcal{M}_{inv} = \{P \mid P \text{ is a non-atomic } \sigma^*\text{-invariant probability measure } P \text{ on } (\Omega, \mathcal{B}) \\ \text{satisfying the condition (2.5)}\}.$$

We call a measure in $\mathcal{M}_{inv}$ a quasisymmetric invariant probability measure.

Then

$$\beta : \mathcal{H}_{inv} \to \mathcal{F}_{inv}$$

is bijective. Define

$$\gamma : \mathcal{F}_{inv} \to \mathcal{M}_{inv}$$

by $\gamma(f) = P_f$. Then it is also surjective. We are still interested in the injectivity of γ. This is equivalent to the following conjecture.

Conjecture 1. *Given any two $f, g \in \mathcal{F}_{inv}$. Then $f = g$ if and only if there is a sequence $\epsilon(n) \to 0$ as $n \to \infty$ such that*

$$\Big| \frac{P_f([w_n^*])}{P_f(add([w_n^*]))} - \frac{P_g([w_n^*])}{P_g(add([w_n^*]))} \Big| \leq \epsilon(n)$$

for all $n > 0$.

A related conjecture is:

Conjecture 2. *Suppose* $f, g \in \mathcal{F}_{inv}$ *are conjugate by a symmetric homeomorphism* h*, that is,* $h \circ f = g \circ h$*. Then* h *must be the identity.*

The reader who is interested in Conjecture 2 can refer to [23] for the proof of this conjecture under the smooth assumption and for the proof of this conjecture under the assumption that one of f and g is $z \mapsto z^d$ (for the later theorem, the reader can find two simpler proofs in Chapter 4 in this thesis). The reader who is interested in the equivalence between Conjecture 1 and Conjecture 2 can refer to [5] for some metric property of symmetric homeomorphisms.

2.3 Martingales for dynamical systems with bounded geometry

For any fixed $n \geq 0$, define $\mathcal{B}_n$ as the σ-field generated by all right cylinders $\{[w_s^*], 0 \leq s \leq n\}$. Then $\mathcal{B}_n$ is a sub-σ-field and we have a filter

$$\cdots \subset \mathcal{B}_n \subset \mathcal{B}_{n+1} \subset \cdots \subset \mathcal{B}.$$

Consider $P = P_f$ for any $f \in \mathcal{F}_{inv}$ Define

$$X_n(w^*) = \frac{P([\sigma^*(w_n^*)])}{P([w_n^*])}$$

and define

$$BX_n(w^*) = \frac{P([add(w_n^*)])}{P([w_n^*))}$$

for any $w^* = \cdots w_n^* \in [w_n^*]$. Then $\{X_n\}_{n=0}^{\infty}$ and $\{BX_n\}_{n=0}^{\infty}$ are two sequences of random variables defined on Ω. We have the following theorem.

Theorem 8. *Suppose f is a uniformly quasisymmetric circle endomorphism preserving the Lebesgue measure. Both sequences $\{(X_n, \mathcal{B}_n)\}_{n=0}^{\infty}$ and $\{(BX_n, \mathcal{B}_n)\}_{n=0}^{\infty}$ are uniformly bounded infinite Martingale sequences and bounded away from 0 uniformly. Therefore, there are two bounded L^1 functions $X(w)$ and $BX(w)$ bounded away from 0 defined on $(\Omega, \mathcal{B}, P)$ such that*

$$X_n(w^*) \to X(w^*) \quad \textit{and} \quad BX_n(w^*) \to BX(w^*)$$

in the L^1-norm.

Proof. First, on every right cylinder $[w_n^*]$,

$$X_n(w^*) = \frac{P([\sigma^*(w_n^*)])}{P([w_n^*])} = \frac{|I_{\sigma^*(w_n^*)}|}{|I_{w_n^*}|}$$

is a constant. So $\{X_n(w^*) \leq x\}$ is the union of some open set $[w_{n_k}*] \in \mathcal{B}_n$ on which the function $X_n(w^*) = \frac{|I_{\sigma^*(w_{n_k}^*)}|}{|I_{w_{n_k}^*}|}$ is a constant less than or equal to x for every k. Hence $X_n(w^*)$ is $\mathcal{B}_n$-measurable.

Second, $X_n(w^*)$ is uniformly integrable as follows. The expectation

$$E[X_n] = \sum_{w_n^*} \frac{|I_{\sigma^*(w_n^*)}|}{|I_{w_n^*}|} \cdot P([w_n^*]) = \sum_{w_n^*} \frac{|I_{\sigma^*(w_n^*)}|}{|I_{w_n^*}|} \cdot |I_{w_n^*}| = \sum_{w_n^*} |I_{\sigma^*(w_n^*)}| \leq d < \infty.$$

Third, we have the following important relation for the conditional expectation $E[X_n|\mathcal{B}_{n-1}]$:

$$X_{n-1} = E[X_n|\mathcal{B}_{n-1}]$$

almost everywhere with respect to P. That is equivalent to

$$\int_{[w_{n-1}^*]} X_{n-1}(w^*) dP(w^*) = \int_{[w_{n-1}^*]} X_n(w^*) dP(w^*)$$

for every right cylinder $[w_{n-1}^*] \in \mathcal{B}_{n-1}$. The reason is as follows.

Let

$$RHS = \int_{[w_{n-1}^*]} X_n(w^*) dP(w^*)$$

and

$$LHS = \int_{[w^*_{n-1}]} X_{n-1}(w^*) dP(w^*).$$

Then

$$RHS = \int_{[w^*_{n-1}]} \frac{|I_{\sigma^*(w^*_n)}|}{|I_{w^*_n}|} dP(w^*) = \sum_{k=0}^{d-1} \int_{[kw^*_{n-1}]} \frac{|I_{\sigma^*(w^*_n)}|}{|I_{w^*_n}|} dP(w^*).$$

So

$$RHS = \sum_{k=0}^{d-1} \frac{|I_{\sigma^*(kw^*_{n-1})}|}{|I_{kw^*_{n-1}}|} \cdot |I_{kw^*_{n-1}}| = \sum_{k=0}^{d-1} |I_{\sigma^*(kw^*_{n-1})}| = |I_{\sigma^*(w^*_{n-1})}|.$$

The last equality is because f preserves the Lebesgue measure.

Now

$$LHS = \int_{[w^*_{n-1}]} \frac{|I_{\sigma^*(w^*_{n-1})}|}{|I_{w^*_{n-1}}|} dP(w^*) = \frac{|I_{\sigma^*(w^*_{n-1})}|}{|I_{w^*_{n-1}}|} \cdot |I_{w^*_{n-1}}| = |I_{\sigma^*(w^*_{n-1})}|.$$

Hence $RHS = LHS$.

From the bounded nearby geometry in §4, we have constants $0 < m < M$ such that

$$m < X_n(w^*) \leq M$$

for all $w^* \in \Omega$ and $n \geq 0$. Therefore, $\{(X_n, \mathcal{B}_n)\}_{n=0}^{\infty}$ is a bounded martingale and bounded away from 0 uniformly.

From the martingale convergence theorem, we have a bounded L^1 function $X(w^*)$ on the probability space $(\Omega, \mathcal{B}, P)$ such that

$$X_n(w^*) \to X(w^*)$$

in the L^1-norm. We completed the proof for $\{(X_n, \mathcal{B}_n)\}_{n=0}^{\infty}$.

The proof for $\{(BX_n, \mathcal{B}_n)\}_{n=0}^{\infty}$ is similar. We leave it as an exercise for the reader. □

We call the random variable (or function) $X(w^*)$ in the above theorem a limiting martingale and the random variable (or function) $BX(w^*)$ in the above theorem a limiting nearby martingale.

In Theorem 8, the condition f being uniformly quasisymmetric implies that the corresponding Markov partition has bounded nearby geometry. The bounded nearby geometry implies the bounded geometry. However the other direction doesn't work (see the following example). Note that, for the existence of $X(w^*)$, the Markov partition only needs to have bounded geometry which is a weaker condition.

Example 3. *Assume* $F(x) = 3x$ *for* $x \in [0, \frac{1}{3}]$ *and* $F(x) = \frac{3}{2}(x - \frac{1}{3})$

for $x \in [\frac{1}{3}, 1]$*. Then* $F(x)$ *has bounded geometry, but not bounded nearby geometry.*

Remark 5. *If* $\widetilde{f}$ *is a* $C^{1+\alpha}$ *expanding map, then it is uniformly quasisymmetric (see [20]) and there is a unique* $C^{1+\alpha}$*-diffeomorphism* h *of* T *with* $h(1) = 1$ *such that* $f = h \circ \widetilde{f} \circ h^{-1}$ *preserves the Lebesgue measure (see Remark 3) and the martingales* $\{(X_n, \mathcal{B}_n)\}_{n=0}^{\infty}$ *and* $\{(BX_n, \mathcal{B}_n)\}_{n=0}^{\infty}$ *induced from* f *converge to the limiting martingale* $X(w^*)$ *and the limiting nearby martingale* $BX(w^*)$ *exponentially. That is, there is a constant* $C > 0$ *and* $0 < \tau < 1$ *such that*

$$\max_{w^* \in \Omega} |X_n(w^*) - X(w^*)| \leq C\tau^n, \quad \forall n \geq 0$$

and

$$\max_{w^* \in \Omega} |BX_n(w^*) - BX(w^*)| \leq C\tau^n, \quad \forall n \geq 0.$$

Moreover, the limiting martingale $X(w^*)$ *and the limiting nearby martingale* $BX(w^*)$ *are Hölder continuous functions in the meaning that there is a constant* $C > 0$ *and* $0 < \tau < 1$ *such that*

$$|X(w^*) - X(\widetilde{w}^*)| \leq C\tau^n$$

and

$$|BX(w^*) - BX(\widetilde{w}^*)| \le C\tau^n$$

for all $w^* = \cdots j_n w_n^*$ *and* $\widetilde{w}^* = \cdots \widetilde{j}_n w_n^*$ *and* $w_n^* = j_{n-1} \cdots j_0$. *In [20],* $X(w^*)$ *is called the dual derivative and is used in the study of dual Gibbs measure theory for Hölder continuous potentials. The reader who is interested in this result can refer to [20].*

Remark 6. *The space* Ω *is also a metric space with a standard metric*

$$d(w^*, \widetilde{w}^*) = \sum_{n=1}^{\infty} \frac{|j_{n-1} - \widetilde{j}_{n-1}|}{d^n}$$

where $w^* = \cdots j_{n-1} \cdots j_0$ *and* $\widetilde{w}^* = \cdots \widetilde{j}_{n-1} \cdots \widetilde{j}_0$. *Suppose* $\tilde{f}$ *is a uniformly symmetric circle endomorphism (see Remark 3). Then there is a symmetric homeomorphism* h *of* T *with* $h(1) = 1$ *such that* $f = h \circ \widetilde{f} \circ h^{-1}$ *preserves the Lebesgue measure (see [20]) and the martingales* $\{X_n, \mathcal{B}_n\}_{n=0}^{\infty}$ *and* $\{BX_n, \mathcal{B}_n\}_{n=0}^{\infty}$ *induced from* f *converge to the limiting martingale* $X(w^*)$ *the limiting nearby martingale* $BX(w^*)$ *uniformly with respect to the metric* $d(\cdot,\cdot)$. *That is,*

$$\max_{w^* \in \Omega} |X_n(w^*) - X(w^*)| \to 0, \quad as \quad n \to \infty$$

and

$$\max_{w^* \in \Omega} |BX_n(w^*) - BX(w^*)| \to 0, \quad as \quad n \to \infty.$$

Moreover, the limiting martingale $X(w^*)$ and the limiting nearby martingale $BX(w^*)$ are continuous functions. In [20], $X(w^*)$ is also called the dual derivative and is used in the study of dual Gibbs measure theory for continuous potentials. The reader who is interested in this result can go to [20].

Remark 7. *The limiting martingale X and the limiting nearby martingale BX are symmetric invariants. This means that if f_1 and f_2 are both uniformly quasisymmetric circle endomorphisms and conjugated by a symmetric homeomorphism h, that is, $f_1 = h \circ f_2 \circ h^{-1}$ and if X_1 and X_2 are limiting martingales and BX_1 and BX_2 are limiting nearby martingales, then $X_1 = X_2$ and $BX_1 = BX_2$. Actually, in [21] (see also [20]), it has been proved that*

$$\lim_{n \to \infty} \|X_{n,1} - X_{n,2}\| = 0$$

and

$$\lim_{n \to \infty} \|BX_{n,1} - BX_{n,2}\| = 0$$

for any circle endomorphisms f_1 and f_2 conjugated by a symmetric circle homeomorphism h, does not matter the limits exist or not for $X_{n,1}$ or $X_{n,2}$ or $BX_{n,1}$ or $BX_{n,2}$, where $\|\cdot\|$ means the maximum norm.

We conclude that $\mathcal{F}_{inv}$ is the space where we can define limiting martingales and limiting nearby martingales on the space $(\Omega, \mathcal{B})$. If two maps in $\mathcal{F}_{inv}$ are symmetrically conjugate, then their limiting martingales are the same and their limiting nearby martingales are the same. Define $\mathcal{M}T$ as the space of all limiting martingales X_f with the probability measure P_f for $f \in \mathcal{F}_{inv}$ and define $\mathcal{B}MT$ as the space of all limiting nearby martingales BX_f with the probability measure P_f for $f \in \mathcal{F}_{inv}$. We are still interested in the following problem.

Problem 1. *Given a characterization of X_f for any $f \in \mathcal{F}_{inv}$ such that for any $P \in \mathcal{M}_{inv}$ and any L^1 function X with respect to $(\Omega, \mathcal{B}, P)$ satisfying this characterization, there is an $f \in \mathcal{F}_{inv}$ such that $X_f = X$.*

Furthermore, we have the following conjecture.

Conjecture 3. *Suppose the limiting martingales or the limiting nearby martingales corresponding to f_1 and f_2 in $\mathcal{F}_{inv}$ are the same, that is, $X_1 = X_2$ and $P_1 \sim P_2$ or $BX_1 = BX_2$ and $P_1 \sim P_2$ where $P_1 \sim P_2$ means that P_1 is absolutely continuous with respect to P_2 and P_2 is absolutely continuous with respect to P_1. Then $f_1 = f_2$*

This conjecture is closely related to Conjecture 1.

Remark 8. *If both of f_1 and f_2 are $C^{1+\alpha}$, $0 < \alpha \leq 1$, expanding circle endomorphisms and preserving the Lebesgue measure, it has been proved in [18,19,23] that if the limiting martingales or the limiting nearby martingales corresponding to f_1 and f_2 are the same, that is, $X_1 = X_2$ or $BX_1 = BX_2$, then f_1 and f_2 are smoothly conjugate. More precisely, there is a $C^{1+\alpha}$ diffeomorphism h of T such that*

$$f_1 = h \circ f_2 \circ h^{-1}.$$

Furthermore, h is the identity. Thus $f_1 = f_2$ and $P_1 = P_2$.

When f_1 and f_2 are both uniformly symmetric circle endomorphisms, the reader can refer to [20, 21, 23] for some related results.

Chapter 3

Almost Complex Structure

3.1 Hilbert transform and almost complex structure

In this chapter, we would like to discuss an almost complex structure and complex manifold structure on $\mathcal{F}_{inv}$ (as well as $\mathcal{H}_{inv}$). The relation between the classical Hilbert transform and the complex structure on the universal Teichmüller space has been studied deeply (refer to [8], [11], [24]). We will use this idea to study an almost complex structure and complex structure on $\mathcal{F}_{inv}$. In this section, we will prove that the Hilbert transform defines a natural almost complex structure on $\mathcal{F}_{inv}$.

The space $\mathcal{F}_{inv}$ is a subspace of $\mathcal{F}$. There is a bijective map

β from $\mathcal{H}$ to $\mathcal{F}$ such that it is also a bijective map from $\mathcal{H}_{inv}$ to $\mathcal{F}_{inv}$. We will prove that the Hilbert transform defined on $\mathcal{H}$ preserves the space $\mathcal{H}_{inv}$. Thus the Hilbert transform defines a natural almost complex structure on $\mathcal{H}_{inv}$ as well as on $\mathcal{F}_{inv}$.

Following the study of the universal Teichmüller space (refer to [8]), the tangent space $\mathcal{V}$ of $\mathcal{H}_{Inv}$ at the identity is the space of all Zygmund functions on the real line satisfying that

$$V(0) = 0 \quad \text{and} \quad V(x+1) = V(x) \tag{3.1}$$

and

$$\sum_{k=0}^{d-1} \Big(V(\frac{x+k}{d}) - V(\frac{k}{d}) \Big) = V(x) \tag{3.2}$$

Example 4. *When the degree $d = 2$,*

$$V(x) = \sum_{n=1}^{\infty} 2^{-n} \sin(2^n \pi x)$$

is a vector in the tangent space $\mathcal{V}$ of $\mathcal{H}_{inv}$ since

$$V(0) = 0 \quad \textit{and} \quad V(x+1) = V(x)$$

and

$$V(\frac{x}{2}) + V(\frac{1+x}{2}) - V(\frac{1}{2}) = V(x).$$

The Hilbert transform $\mathcal{J}$ on $\mathcal{V}$ is defined by

$$\mathcal{J}V(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} V(y)R(x,y)dy$$

where

$$R(x,y) = \frac{1}{y-x} - \frac{x}{y-1} + \frac{x-1}{y}.$$

Let $\zeta = \xi + i\eta$ be a complex number. Then $\overline{\zeta} = \xi - i\eta$. By Stokes' formula,

$$\mathcal{J}V(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} V(\zeta)R(x,\zeta)d\zeta = \frac{2i}{\pi}\int\int_{\mathbb{H}} \overline{\partial}V(\zeta)R(x,\zeta)d\xi d\eta + iV(x),$$

where $\mathbb{H}$ is the upper-half plane and $\overline{\partial}V = \partial V/\partial\overline{\zeta}$.

Let $\mu(\zeta) = \overline{\partial}V(\zeta)$, then

$$\begin{aligned}\mathcal{J}V(x) &= \frac{2i}{\pi}\int\int_{\mathbb{H}} \mu(\zeta)R(x,\zeta)d\xi d\eta + iV(x)\\ &= \frac{-2i}{\pi}\int\int_{\mathbb{L}} \mu(\zeta)R(x,\zeta)d\xi d\eta - iV(x)\end{aligned}$$

where $\mathbb{L}$ is the lower-half plane.

Define

$$\widetilde{\mu}(\zeta) = \begin{cases} -i\mu(\zeta), & \zeta \in \mathbb{H};\\ i\mu(\zeta), & \zeta \in \mathbb{L}.\end{cases}$$

Then

$$\mathcal{J}V(x) = \frac{-1}{\pi}\int\int_{\mathbb{C}} \widetilde{\mu}(\zeta)R(x,\zeta)d\xi d\eta.$$

And we also have

$$\overline{\partial}(\mathcal{J}V) = -i\mu \quad \text{and} \quad \overline{\partial}(H^2V) = i^2\mu = -\mu.$$

This implies

$$\mathcal{J}^2V = -V.$$

Hence the Hilbert transform $\mathcal{J}$ gives an almost complex structure on the space of all Zygmund functions on the real line.

The following theorem shows that the Hilbert transform $\mathcal{J}$ also gives an almost complex structure on the space $\mathcal{H}_{Inv}$. Let $\mathcal{I}$ denote the identity map on $\mathcal{H}_{inv}$.

Theorem 9. *Let $\mathcal{J}V$ be the Hilbert transform of $V \in \mathcal{V}$, then $\mathcal{J}V$ satisfies the equations (3.1) and (3.2). Therefore, $\mathcal{J}V \in \mathcal{V}$. Since $\mathcal{J} : \mathcal{V} \to \mathcal{V}$ and $\mathcal{J}^2 = -\mathcal{I}$, it is an almost complex structure on $\mathcal{H}_{inv}$.*

Proof. Note that

$$\mathcal{J}V(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} V(y)\Big(\frac{1}{y-x} - \frac{x}{y-1} + \frac{x-1}{y}\Big)dy.$$

From $V(x+1) = V(x)$, we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{xV(y)}{y-1}dy = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{xV(y+1)}{y}dy = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{xV(y)}{y}dy.$$

Hence

$$\mathcal{J}V(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{V(y)}{y-x}dy - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{V(y)}{y}dy$$
$$= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{V(y+x)}{y}dy - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{V(y)}{y}dy.$$

From this form of the Hilbert transform, it is easy to check $\mathcal{J}V(0) = 0$ and $\mathcal{J}V(x+1) = \mathcal{J}V(x)$ since $V(y+1) = V(y)$. These are the equations in (3.1).

To prove the equation (3.2) for $\mathcal{J}V$, we only need to show

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sum_{k=0}^{d-1}(V(y+\frac{x+k}{d}) - V(y+\frac{k}{d})) + V(y) - V(y+x)}{y}dy = 0.$$

Since

$$\sum_{k=0}^{d-1}\Big(V(\frac{x+k}{d}) - V(\frac{k}{d})\Big) = V(x),$$

we have that

$$\sum_{k=0}^{d-1}\Big(V(y+\frac{x+k}{d}) - V(\frac{k}{d})\Big) = V(yd+x)$$

and

$$\sum_{k=0}^{d-1}\Big(V(y+\frac{k}{d}) - V(\frac{k}{d})\Big) = V(yd).$$

The difference of these two equations is

$$\sum_{k=0}^{d-1}\Big(V(y+\frac{x+k}{d}) - V(y+\frac{k}{d})\Big) = V(yd+x) - V(yd).$$

Hence the numerator of the fraction inside the equation in the previous integral is

$$\sum_{k=0}^{d-1}\Big(V(y+\frac{x+k}{d})-V(y+\frac{k}{d})\Big)+V(y)-V(y+x)$$

$$=V(y)-V(y+x)+V(yd+x)-V(yd).$$

So we need to show

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{V(y)+V(yd+x)-V(yd)-V(y+x)}{y}dy$$

$$=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{V(y)-V(yd)}{y}dy+\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{V(yd+x)-V(y+x)}{y}dy=0$$

It is easy to see that

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{V(y)}{y}dy=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{V(yd)}{yd}d(yd)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{V(yd)}{y}dy.$$

So

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{V(y)-V(yd)}{y}dy=0.$$

For the term

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{V(yd+x)-V(y+x)}{y}dy,$$

we have

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{V(y+x)}{y}dy=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{V(y)}{y-x}dy$$

$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{V(yd+x)}{yd} d(yd+x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{V(yd+x)}{y} dy.$$

Thus

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{V(yd+x) - V(y+x)}{y} dy = 0.$$

We have proved the theorem. □

3.2 Complex manifold structure

In this section, we will discuss the complex manifold structure on $\mathcal{H}_{inv}$, therefore, on $\mathcal{F}_{inv}$. The complex manifold structure which we will discuss is the integration of the almost complex structure $\mathcal{J}$ which we discussed in the previous section.

Consider

$$\mathcal{H} = \{h \mid h \text{ is a quasisymmetric circle homeomorphism with } h(1) = 1\}.$$

Since $\mathcal{H}_{inv}$ is the subspace of all $h \in \mathcal{H}$ satisfying the linear equation (2.6), it is a smooth submanifold of $\mathcal{H}$. The almost complex structure $\mathcal{J} : \mathcal{H}_{inv} \to \mathcal{H}_{inv}$ is the restriction of the almost complex structure $\mathcal{J} : \mathcal{H} \to \mathcal{H}$. Thus, to have our original goal, we can discuss the complex manifold structure on $\mathcal{H}$ which is the integration of the almost complex structure $\mathcal{J} : \mathcal{H} \to \mathcal{H}$.

In the following, we will discuss the Teichmüller structure on $\mathcal{H}$ and its embedding in the space of all holomorphic functions on the unit disk. This will give a complex manifold structure on $\mathcal{H}$ (as well as $\mathcal{F}$). Furthermore, we will prove that the almost complex structure induced from this complex manifold structure is the Hilbert transform $\mathcal{J}$. Thus, the induced complex manifold structure on $\mathcal{H}_{inv}$ (as well as on $\mathcal{F}_{inv}$) is the integral of the almost complex structure on $\mathcal{H}_{inv}$ (as well as $\mathcal{F}_{inv}$) defined by the Hilbert transform $\mathcal{J}$.

The covering map from the real line $\mathbb{R}$ to the unit circle T is

$$\pi(x) = e^{2\pi ix} : \mathbb{R} \to T.$$

So any map in $\mathcal{H}$ can be lifted to a periodic quasisymmetric homeomorphism H of $\mathbb{R}$. We denote the space of all lifting map to be $\widetilde{\mathcal{H}}$ i.e.

$$\widetilde{\mathcal{H}} = \{H \mid H \text{ is a quasisymmetric homeomorphism of } \mathbb{R}$$

$$\text{with } H(x+1) = H(x) + 1 \text{ and } H(0) = 0\}.$$

Since h and H are quasisymmetric, we can extend h and H to quasiconformal homeomorphisms of $\overline{\mathbb{C}}$. Let $f(z)$ be the Beurling-Ahlfors extension of H (see [1]), then $f(z+1) = f(z) + 1$ and the corresponding $\mu = f_{\overline{z}}/f_z$ satisfies the condition $\mu(z) = \mu(z+1)$. The converse of the previous argument is also true. Suppose $\mu(z)$ is a measurable function defined on $\overline{\mathbb{C}}$ with $\|\mu\|_\infty < 1$. Such a function is called a Beltrami coefficient. Consider the Beltrami equation

$$f_{\overline{z}} = \mu f_z. \tag{3.3}$$

A solution of the Beltrami equation (3.3) is called normalized if it fixes 0, 1, and ∞. The normalized solution is unique for any given Beltrami coefficient μ.

Lemma 1. *Suppose μ is a Beltrami coefficient satisfying that $\mu(z+1) = \mu(z)$ for all $z \in \mathbb{C}$. Suppose $f(z)$ is the normalized solution of the corresponding Beltrami equation (3.3). Then*

$$f(z+1) = f(z) + 1, \quad \forall z \in \mathbb{C}.$$

Proof. The following function $\mathcal{P}$ and $\mathcal{T}$ are two operators defined in [1].

$$P\mu(\zeta) = -\frac{1}{\pi}\int\int \mu(z)(\frac{1}{z-\zeta} - \frac{1}{z})dxdy$$

and

$$T\mu(\zeta) = \lim_{\epsilon \to 0} -\frac{1}{\pi} \int\int_{|z-\zeta|>\epsilon} \frac{\mu(z)}{(z-\zeta)^2} dxdy.$$

Since $\mu(z+1)=\mu(z)$, $\mathcal{P}(\mu(z)) = \mathcal{P}(\mu(z+1))$ and $\mathcal{T}(\mu(z)) = \mathcal{T}(\mu(z+1))$. We have that both $f(z)$ and $f(z+1)-1$ are the normalized solution of the corresponding Beltrami equation (3.3), so $f(z+1) = f(z)+1$. We proved the lemma. □

Consider the space $\mathcal{M}_r$ of all measurable functions μ defined on the Riemann sphere $\overline{\mathbb{C}}$ satisfying

- $\mu(z+1) = \mu(z)$;
- $\mu(\overline{z}) = \overline{\mu(z)}$ for all $z \in \mathbb{C}$;
- $\|\mu\|_\infty < 1$.

For any $\mu \in \mathcal{M}_r$, the restriction of the normalized solution f of the corresponding Betrami equation (3.3) to the real line $\mathbb{R}$ is in $\tilde{\mathcal{H}}$. For any $H \in \tilde{\mathcal{H}}$, let $f(z)$ be the Beurling-Ahlfors extension of H (see [1]), then $f(z+1) = f(z)+1$ and the corresponding $\mu = f_{\overline{z}}/f_z$ is in $\mathcal{M}_r$. Thus every map in $\tilde{\mathcal{H}}$ can be thought as the restriction of the normalized solution f of a Betrami equation (3.3) for some Beltrami coefficient $\mu \in \mathcal{M}_r$.

Consider the space $\mathcal{M}_c$ of all measurable functions μ defined on the Riemann sphere $\overline{\mathbb{C}}$ satisfying

- $\mu(z) = \frac{z^2}{\overline{z}^2}\overline{\mu(1/\overline{z})}$ for any $z \in \mathbb{C}$;
- $\|\mu\|_\infty < 1$.

For any $\mu \in \mathcal{M}_c$, the restriction of the normalized solution g of the corresponding Betrami equation (3.3) to the unit circle T is in $\mathcal{H}$. Similarly, every map in $\mathcal{H}$ can be thought as the restriction of the normalized solution g of a Betrami equation (3.3) for some Beltrami coefficient $\mu \in \mathcal{M}_c$.

For any $\mu \in \mathcal{M}_r$, we use f_μ to denote the normalized solution of the corresponding Beltrami equation (3.3). Then f_μ maps the real line $\mathbb{R}$ onto itself. For any $\mu \in \mathcal{M}_c$, we use g_μ to denote the normalized solution of the Beltrami equation (3.3). Then g_μ maps the unit circle T onto itself.

Definition 9 (Teichmüller Equivalence)**.** *Suppose μ and ν in $\mathcal{M}_r$.*

We say that they are equivalent, denoted as $\mu \sim_r \nu$ *if* $f_\mu(x) = f_\nu(x)$ *for all* $x \in \mathbb{R}$. *We use* $[\mu]_r$ *to denote the* $\sim_r$*-equivalent class of* $\mu \in \mathcal{M}_r$. *We define the Teichmüller space*

$$\mathcal{T}_r = \{[\mu]_r \mid \mu \in \mathcal{M}_r\}$$

as the space of all $\sim_r$*-equivalence classes.*

Suppose μ *and* ν *in* $\mathcal{M}_c$. *We say that they are equivalent, denoted as* $\mu \sim_c \nu$ *if* $g_\mu(z) = g_\nu(z)$ *for all* $z \in T$. *We use* $[\mu]_c$ *to denote the* $\sim_c$*-equivalent class of* $\mu \in \mathcal{M}_c$. *We define the Teichmüller space*

$$\mathcal{T}_c = \{[\mu]_c \mid \mu \in \mathcal{M}_c\}$$

as the space of all $\sim_c$*-equivalence classes.*

From the definition and the standard theory for quasiconformal mappings (see [1]), there is a one-to-one and onto correspondence between $\tilde{\mathcal{H}}$ and $\mathcal{T}_r$. Therefore, $\tilde{\mathcal{H}}$ can be thought as another representation of the Teichmüller space $\mathcal{T}_r$. There is a one-to-one and onto correspondence between $\mathcal{H}$ and $\mathcal{T}_c$. Therefore, $\mathcal{H}$ can be thought as another representation of the Teichmüller space $\mathcal{T}_c$. We also know that there is a one-to-one and onto correspondence between $\tilde{\mathcal{H}}$ and $\mathcal{H}$. Therefore, there is a one-to-one and onto correspondence

$$\tau : \mathcal{T}_c \to \mathcal{T}_r. \tag{3.4}$$

3.2.1 Complex structures on $\mathcal{T}_c$, $\mathcal{T}_r$ and $\mathcal{T}(\Delta^*)$.

The space $\mathcal{T}_r$ is a subspace of the universal Teichmüller space. From Bers' embedding (refer to [7]), there is a natural complex manifold structure on $\mathcal{T}_r$ which is given by considering the Schwarzian derivative

$$S(f^\mu) = \Big(\frac{(f^\mu)''}{f^\mu)'}\Big)' - \frac{1}{2}\Big(\frac{(f^\mu)''}{(f^\mu)'}\Big)^2$$

on the lower-half plane H^*, where f^μ is the normalized solution of the Beltrami equation (3.3) with the Beltrami coefficient $\tilde{\mu} = \mu$ on the upper-half plane and $\tilde{\mu} = 0$ on the lower-half plane.

Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disk and $\Delta^* = \Delta \setminus \{0\}$ be the punctured disk. Let $T(\Delta^*)$ be the Teichmüller space of Riemann surfaces with the basepoint Δ^* i.e.

$$\mathcal{T}(\Delta^*) = \{[\mu]_{\Delta^*} \mid \mu \in \mathcal{M}\},$$

where $\mathcal{M}$ is the unit ball of $L_\infty(\Delta^*)$, is the space of all Teichmüller equivalence classes $[\mu]_{\Delta^*}$. Here μ and ν are said to be Teichmüller equivalent, denoted as $\mu \sim_{\Delta^*} \nu$, if two unique solutions f and g of the Beltrami equation (3.3) with the Beltrami coefficients μ and ν in Δ and $1/\overline{\mu(1/\overline{z})}$ and $1/\overline{\nu(1/\overline{z})}$ in $\overline{\mathbb{C}} \setminus \Delta$ fixing $-1, 1, i$ agree on the unit circle T and on 0, that is, $f|T = g|T$ and $f(0) = g(0)$.

On $\mathcal{T}_c$ we can define the Teichmüller metric by

$$d_{\mathcal{T}}(x, y) = \frac{1}{2} \inf_{\mu \in x, \nu \in y} \log K(f_\mu \circ f_\nu^{-1})$$

where f_μ and f_ν are normalized solutions of Beltrami equation (3.3) with beltrami coefficients μ and ν and $K(f_\mu \circ f_\nu^{-1})$ is the quasiconformal dilatation of $f_\mu \circ f_\nu^{-1}$.

Let $d^*(\cdot, \cdot)$ be the Teichmüller distance on $T(\Delta^*)$.

Theorem 10. *The space $(\mathcal{T}_c, d_{\mathcal{T}})$ is isometric to the space $(\mathcal{T}(\Delta^*), d^*)$.*

Proof. For any $x = [\mu]_c \in \mathcal{T}_c$, $h = f_\mu|T$ is in $\mathcal{H}$. The map h fixes 1. There exists a Möbius map

$$M_a(z) = \frac{z + a}{1 + \overline{a}z} \cdot \frac{1 + \overline{a}}{1 + a}$$

such that $M_a \circ h$ is a homeomorphism of T fixing 1, i and -1. Thus $M_a \circ h$ represents the equivalent class of the Beltrami coefficient of

$M_a \circ f_\mu$ in $\mathcal{T}(\Delta^*)$. Moreover, $M_a^{-1} \circ h$ will map a point $h \in T(\Delta^*)$ to a point in $\mathcal{T}_c$. This correspondence preserves the metrics $d_{\mathcal{T}}$ and d^*. $\square$

The pull back of the complex structure, given by the Schwarzian derivative, on $\mathcal{T}_r$ by τ gives a complex complex structure on $\mathcal{T}_c$. From Theorem 8, this also gives a complex structure on $\mathcal{T}(\Delta^*)$. Therefore, we have a complex manifold structure on $\mathcal{H}$.

3.2.2 Almost complex structure.

The tangent vector V in the tangent space of $\mathcal{T}_r$ at the identity has the form

$$V(x) = \frac{-1}{\pi} \int \int_{\mathbb{C}} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-x)} d\xi d\eta,$$

for any real number x and for some μ which is symmetric, periodic and in $L_\infty(H)/N(H)$ (see [7]). So multiplication by $-i$ on Beltrami coefficients μ determines the standard almost complex structure on Teichmuller space $\mathcal{T}_r$. From the calculation in Section 7, the Hilbert transform gives the same almost complex structure on $\mathcal{V}$. This observation for the universal Teichmüller space is due to Steven Kerckhoff (see, for example, [26]).

So the pull back of Hilbert transform by $\tau : \mathcal{T}_c \to \mathcal{T}_r$ gives an almost complex structure on the tangent space of $\mathcal{T}_c$ at the identity. From Theorem 7 in Section 7, the Hilbert transform keeps the condition (9) and (10) which implies the pull back of Hilbert transform by τ gives an almost complex structure on the tangent space $\mathcal{F}_{inv}$ at identity, whose integration is the complex manifold we just discussed in Section 8.

3.3 The Teichmüller metric and Kobayashi's metric

Since $\mathcal{T}_c$ is a complex manifold, we can define Kobayashi's metric on $\mathcal{T}_c$ (see [10]). Since the Teichmüller metric is equal to Kobayashi's metric on $\mathcal{T}(\Delta^*)$ (see [10]), from Theorem 8, the Teichmüller metric is equal to Kobayashi's metric on $\mathcal{T}_c$.

The restriction of a Möbius transformation

$$M_a(z) = \frac{z+a}{1+\overline{a}z} \cdot \frac{1+\overline{a}}{1+a}$$

to the unit circle T maps T onto T and fixes 1. It is the boundary map of a quasiconformal map f_μ of $\overline{\mathbb{C}}$ fixing 0, 1, and ∞ for a

$[\mu] \in \mathcal{T}_c$. Note that μ is not $\sim_c$-equivalent to $\mu_0 \equiv 0$. But μ is the Teichmüller equivalent to $\mu_0 \equiv 0$ in the universal Teichmüller space $\mathcal{T}(\Delta)$.

Let $\mathcal{P} : \mathcal{T}(\Delta^*) \to \mathcal{T}(\Delta)$ be the forgetful map since $\Delta^* \subset \Delta$. It is a holomorphic split submersion, that means that for every point $x \in \mathcal{T}(\Delta)$, there is a neighborhood U about x and a holomorphic map $s : U \to \mathcal{T}(\Delta^*)$ such that $\mathcal{P} \circ s = id$.

Suppose $[id] = [0]_\Delta$ is the basepoint of $\mathcal{T}(\Delta)$. Let $\mathcal{K} = \mathcal{P}^{-1}([id])$. Then

$$\mathcal{K} = \{M_a(z) | a \in \Delta\}$$

is a one dimensional complex manifold of $\mathcal{T}(\Delta^*)$ conformally equivalent to the hyperbolic disk Δ. Therefore, we have Kobayashi's metric on $\mathcal{K}$ and the restriction of Teichmüller metric on $\mathcal{K}$, which we denote as d^*. Comparing these two metrics is an interesting problem. Gardiner and Lakic [8, 9] have studied this problem infinitesimally in a more general setting as follows:

Definition 10. *Suppose Ω is a domain contained in $\overline{\mathbb{C}}$ with three* *or more boundary points. Its Poincare density is denoted as*

$$\rho_\Omega(p) = \frac{\rho_\Delta(z)}{|\pi'(z)|}$$

where $\rho_\Delta(z) = \frac{|dz|}{1-|z|^2}$ *and* $\pi : \Delta \to \Omega$ *is the universal covering and* $p = \pi(z)$.

Definition 11. *The Teichmüller density* $\lambda_\Omega(p)$ *evaluated at a point* p *in* Ω *is*

$$\lambda_\Omega(p) = \inf\{||\overline{\partial}(\widetilde{V})||_\infty\}$$

where the infimum is taken over all continuous vector fields $\widetilde{V}(z)\frac{\partial}{\partial z}$ *for which* $\widetilde{V}(p) = 1$ *and* $\widetilde{V}(z) = 0$ *for all* z *on the boundary of* Ω.

Suppose $\mathcal{P} : \mathcal{T}(\Omega - p) \to \mathcal{T}(\Omega)$ is the forgetful map, which is a holomorphic split submersion, and suppose $[id]$ is the basepoint of $\mathcal{T}(\Omega)$. Consider the fiber $\mathcal{K} = \mathcal{P}^{-1}([id])$. In [2], Bers shows that when Ω is of finite analytic type, $\mathcal{K}$ is conformally equivalent to the universal covering of Ω. In [8], Gardiner and Lakic showed that the infinitesimal Kobayashi's metric at the basepoint for $\mathcal{K}$ is equal to the Poincaré density $\rho_\Omega(p)$ at p in Ω. The infinitesimal Teichmüller's metric at the basepoint for $\mathcal{K}$ is the Teichmüller density

$\lambda_\Omega(p)$. Furthermore, they have the following comparison.

Theorem 11 (Gardiner and Lakic [8]). *The Teichmüller density $\lambda_\Omega(p)$ and the Kobayashi's density $\rho_\Omega(p)$ are equivalent. In particular,*

$$\frac{1}{2}\rho_\Omega(p) \le \lambda_\Omega(p) \le \rho_\Omega(p).$$

The right hand side of the inequality is an easy corollary of Slodkowski's extension theorem [25] in theory of the holomorphic motion (see also [10]). And they use the Poincare theta series of a quadratic differential to show the left hand side. The reader who is interested in this theorem can refer to [8] for more details.

For $\Omega = \Delta$ and $p = 0$, Gardiner and Lakic have proved the following result (see [9]). We give a proof of this result in this section.

Theorem 12 (Gardiner and Lakic [9]). *Then*

$$\lambda_\Delta(0) = \frac{1}{2}\rho_\Delta(0) = \frac{1}{2}.$$

Proof. Since $\frac{1}{2}\rho_\Delta(0) = \frac{1}{2}$, from Theorem 9, we have that

$$\frac{1}{2}\rho_\Delta(0) \le \lambda_\Delta(0).$$

So we only need to show that

$$\lambda_\Delta(0) \leq \frac{1}{2}$$

as follows.

Since the fiber $\mathcal{K}$ is the hyperbolic disk Δ conformally, we consider the holomorphic motion

$$h(t,z) = \begin{cases} z, & |z| \geq 1; \\ z + t(1-|z|), & |z| \leq 1 \end{cases} : \Delta \times \mathcal{C} \to \overline{\mathcal{C}}.$$

The tangent vector of this motion on $|z| \leq 1$ is

$$V(z) = \frac{d(h(t,z))}{dt}|_{t=0} = 1 - |z| = 1 - z^{\frac{1}{2}}\overline{z}^{\frac{1}{2}}.$$

So $V(0) = 1$ and $V(z) = 0$ for $|z| = 1$.

For any $z \in \Delta$,

$$|\overline{\partial}V(z)| = \left|\frac{1}{2}\frac{z^{\frac{1}{2}}}{\overline{z}^{\frac{1}{2}}}\right| = \frac{1}{2}.$$

Therefore, $||\overline{\partial}V||_\infty = \frac{1}{2}$ and

$$\lambda_\Delta(0) = \inf\{||\overline{\partial}(\widetilde{V})||_\infty\} \leq \frac{1}{2}$$

where the infimum is taken over all continuous vector fields $\widetilde{V}(z)\frac{\partial}{\partial z}$ for which $\widetilde{V}(0) = 1$ and $\widetilde{V}(z) = 0$ for all z on the boundary of Δ. □

The above theorems are about the estimation of the Kobayashi's density and the Teichmüller density on the fiber $\mathcal{K}$ at the basepoint for the fiber $\mathcal{K}$. In this section, we prove that these estimations will not hold in the global. That is, we will prove that from the global point of view, Theorems 9 and 10 will not hold on the fiber $\mathcal{K}$. More precisely, we prove that

Theorem 13. *On the fiber $\mathcal{K}$, the global Teichmüller metric d_{Tei} and the global Kobayashi's metric d_{Kob} on $\mathcal{K}$ are not quasi-equivalent.*

Proof. We use the upper-half plane model for the universal Teichmüller space $\mathcal{T}(H)$. Then $p = i$. Denote $H^* = H - \{i\}$. Then we have the forgetful map $\mathcal{P} : \mathcal{T}(H^*) \to \mathcal{T}(H)$. Let $[id]$ be the basepoint for $\mathcal{T}(H)$. Then the fiber

$$\mathcal{K} = \mathcal{P}^{-1}([id]) = \Big\{ f_w(z) | f(x) = x, \forall x \in R, f(i) = w \Big\} = \Big\{ M_w(z) = \eta z + \xi \Big\} = H$$

where $w = \xi + \eta i \in H$. Note that $M_w(i) = w$ and $M_w(\infty) = \infty$.

Let $K > 1$. Suppose f_K is a quasiconformal self homeomorphism of H such that $f_K(x) = x$ for any real $x \in \mathbb{R}$ and $f_K(i) =$

Ki. Consider the Beltrami coefficient $\mu_{f_K}(z) = \overline{\partial} f_K(z)/\partial f_K(z)$. Let $\tau_K = [\mu_{f_K}]_{H^*}$ be a point in $\mathcal{T}(H^*)$. That is, τ_K is the equivalence class of all Beltrami coefficient ν on H such that $f_\nu|\mathbb{R} = id$ and $f_\nu(i) = Ki$. Clearly, τ_K is a point in the fiber $\mathcal{K}$. Let $[id]_{H^*}$ be the basepoint of $\mathcal{T}(H^*)$. The Kobayashi's distance $d_{Kob}([id]_{H^*}, \tau_K)$ on the fiber $\mathcal{K}$ is equal to the Poincaré distance $d_{Hyp}(i, Ki) = \log K$ on the hyperbolic upper-half plane H.

We will use Strebel's extremal example to get a upper bound of the Teichmüller distance $d_{Tei}([id]_{H^*}, \tau_K)$ on the fiber $\mathcal{K}$. Let

$$z = \pi(w) = \frac{2}{\pi}\log r + 4i\theta, \quad w = re^{2\pi i\theta},\ r > 0,\ 0 \leq \theta \leq \frac{1}{2}.$$

Then it maps the upper-half plane H to the horizontal strip

$$A = \{z = x + yi \mid -\infty < x < \infty, 0 \leq y \leq 2\}.$$

It maps the positive real axis to the real line; the positive imaginary axis to the line $y = 1$; the negative real line to the line $y = 2$.

Let $g : A \to A$ be a quasiconformal self-homeomorphism of A

defined as

$$g(z) = \begin{cases} z + \frac{2}{\pi}\frac{z-\bar{z}}{2i}\log K, & 0 \le y \le 1 \\ z + \frac{2}{\pi}\log K(2 - \frac{z-\bar{z}}{2i}), & 1 \le y \le 2. \end{cases}$$

Then the Beltrami coefficient is

$$\mu_g = \frac{-\frac{1}{\pi i}\log K}{1 + \frac{1}{\pi i}\log K}.$$

The quasiconformal dilatation is

$$K(g) = (\sqrt{1 + (\frac{1}{\pi}\log K)^2} + \frac{1}{\pi}\log K)^2.$$

Define $f = \pi^{-1} \circ g \circ \pi$. It is a quasiconformal self homeomorphism of the upper-half plane H. We have that $K(f) = K(g)$. Since $f|\mathbb{R} = id$ and $f(i) = Ki$, the Beltrami coefficient $\mu_f \in \tau_K$. This gives us that the Teichmüller distance

$$d_{Tei}([id]_{H^*}, \tau_K) \le \log K(f) = 2\log(\sqrt{1 + (\frac{1}{\pi}\log K)^2} + \frac{1}{\pi}\log K).$$

Hence

$$d_{Tei}([id]_{H^*}, \tau_K) \approx 2\log(\frac{2}{\pi}\log K), \quad \text{as } K \to \infty.$$

Thus we have that

$$\frac{d_{Tei}([id]_{H^*}, \tau_K)}{d_{Kob}([id]_{H^*}, \tau_K)} \approx \frac{2\log(\frac{2}{\pi}\log K)}{\log K} \to 0 \quad \text{as } K \to \infty.$$

This proves our theorem. □

Chapter 4

Finite Martingales

4.1 Conditions for Markov partitions

Given a circle endomorphism f, we have constructed a sequence of Markov partitions $\{\{I_{w_n}, n \geq 1\}\}$. Then the uniform quasisymmetry condition can be translated as the bounded nearby geometry condition, which implies the bounded geometry condition. Preserving Lebesgue measure condition can also be translated into a condition on the sequence of Markov partitions. In this chapter, we investigate that how to construct back, that is, we construct f by constructing a sequence of Markov partitions satisfying the above conditions.

Suppose f is a circle endomorphism preserving the Lebesgue measure and the corresponding Markov partitions of f have bounded geometry. We use $w_n = i_0 i_1 \cdots i_{n-1}$ to denote a string of $0's$ and $1's$ of length n. Suppose $\eta_n = \{I_{w_n}\}$, $n = 1, 2, \cdots$, is a nested sequence of partitions of the interval $[0, 1]$ such that

$$I_{w_n} = I_{w_n 0} \cup I_{w_n 1}.$$

We call η_n level n. Given all intervals in level n and intervals $I_{0 i_1 \cdots i_{n-1} 0}$ in level $n+1$, we can construct all other intervals in level $n+1$ by the following conditions:

$$|I_{w_n}| = |I_{0 w_n}| + |I_{1 w_n}|$$

and

$$|I_{w_n}| = |I_{w_n 0}| + |I_{w_n 1}|.$$

The first condition is equivalent to the Lebesgue invariant condition and the the second condition is equivalent to the sub-interval condition. The other intervals in level $n+1$ can be calculated by the following:

$$|I_{0 i_1 i_2 \ldots i_{n-1} 1}| = |I_{0 i_1 i_2 \ldots i_{n-1}}| - |I_{0 i_1 i_2 \ldots i_{n-1} 0}|,$$

$$|I_{1i_1i_2...i_{n-1}0}| = |I_{i_1i_2...i_{n-1}0}| - |I_{0i_1i_2...i_{n-1}0}|,$$

and

$$|I_{1i_1i_2...i_{n-1}1}| = |I_{i_1i_2...i_{n-1}1}| - |I_{0i_1i_2...i_{n-1}}| + |I_{0i_1i_2...i_{n-1}0}|.$$

To have a sequence of Markov partitions with bounded geometry with universal upper bound C, $I_{0i_1i_2...i_{n-1}0}$ has to satisfy the following conditions:

$$\max\{0, |I_{0i_1i_2...i_{n-1}}| - |I_{i_1i_2...i_{n-1}1}|\} < |I_{0i_1i_2...i_{n-1}0}| < \min\{|I_{0i_1i_2...i_{n-1}}|, |I_{i_1i_2...i_{n-1}0}|\} \tag{4.1}$$

and for any $i_1i_2...i_{n-1}$,

$$\frac{1}{C}|I_{0i_1i_2...i_{n-1}}| \leq |I_{0i_1i_2...i_{n-1}0}| \leq \frac{C-1}{C}|I_{0i_1i_2...i_{n-1}}|$$

and

$$\frac{1}{C}|I_{1i_1i_2...i_{n-1}}| \leq |I_{1i_1i_2...i_{n-1}0}| \leq \frac{C-1}{C}|I_{1i_1i_2...i_{n-1}}|$$

which is equivalent to

$$|I_{i_1i_2...i_{n-1}0}| - \frac{C-1}{C}|I_{1i_1i_2...i_{n-1}}| \leq |I_{0i_1i_2...i_{n-1}0}| \leq |I_{i_1i_2...i_{n-1}0}| - \frac{1}{C}|I_{1i_1i_2...i_{n-1}}| \tag{4.2}$$

Condition (4.1) is to ensure that all these intervals have positive lengths. And the second Condition (4.2) is equivalent to the bounded geometry condition.

Definition 12. *We say a martingale sequence is of length n if $X_k = X_n$ a.e. for all $k \geq n$.*

This means, for some fixed n, $X_f = X_n$ is a constant function almost everywhere on every open set $[w_n^*] = \{w^* = \cdots w_n\}$ for a fixed string of $0's$ and $1's$ of length n. We call a martingale X_f locally constant if there is an integer $n > 0$ such that $X_f(\mu^*) = X_f(\nu^*)$ for almost all $\mu^*, \nu^* \in [w_n^*]$. .

Proposition 3. *If X_f is locally constant, then $\{X_k\}$ is a sequence of length n and $X_f = X_n$ almost everywhere.*

Proof. By definition, X_n is a constant on $[w_n^*]$, and

$$\int_{[w_n^*]} X_n dp = \int_{[w_n^*]} X_f dp.$$

So $X_n([w_n^*])P([w_n^*]) = X_f([w_n^*])P([w_n^*])$. Since P is a non-atomic measure, this implies $X_n = X_f$ a.e. on $[w_n^*]$.

□

The map $q(z) = z^2$ is the only map whose corresponding martingale sequence is of length 1. We give two examples of martingale sequences of length 3 with bounded geometry with upper bound $C = 4$. The first example has only bounded geometry but not bounded nearby geometry. Thus the corresponding circle endomorphism f_1 is not uniformly quasisymmetric. The second example has both bounded and bounded nearby geometry. Thus the corresponding circle endomorphism f_2 is uniformly quasisymmetric.

Corresponding values of interval lengths:

Since $X_2 = 4 = C$ on I_{00} and $X_2 = \frac{4}{3} = \frac{1}{1-\frac{1}{c}} = \frac{c}{c-1}$ on I_{01}, to satisfy Condition (4.2), X_3 has to be 4 and $\frac{3}{4}$ respectively on its preimage. One can check that this example has only bounded geometry but not bounded nearby geometry. Actually, the nearby geometry goes to infinity at every partition point in this example. Thus the map defined by these partitions is not uniformly quasisymmetric.

This example has bounded geometry and bounded nearby ge-

Example 5.

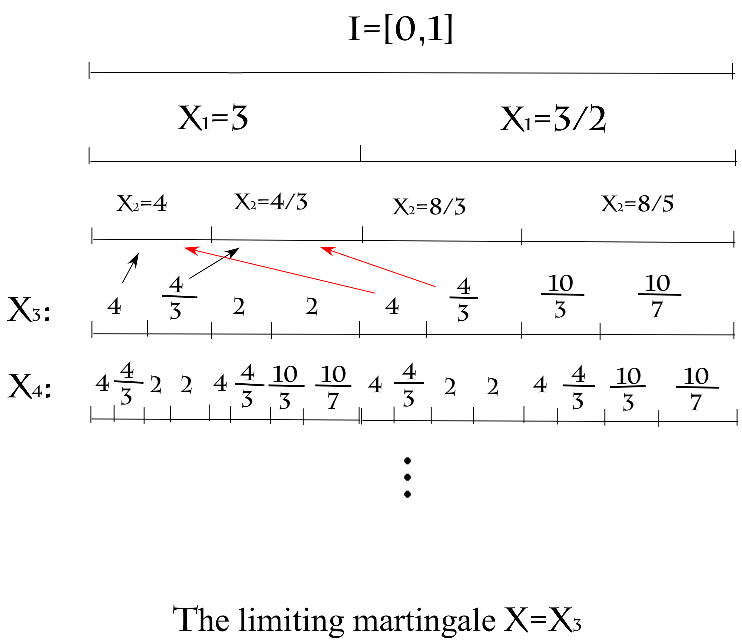

Figure 4.1: Finite Martingale of length 3

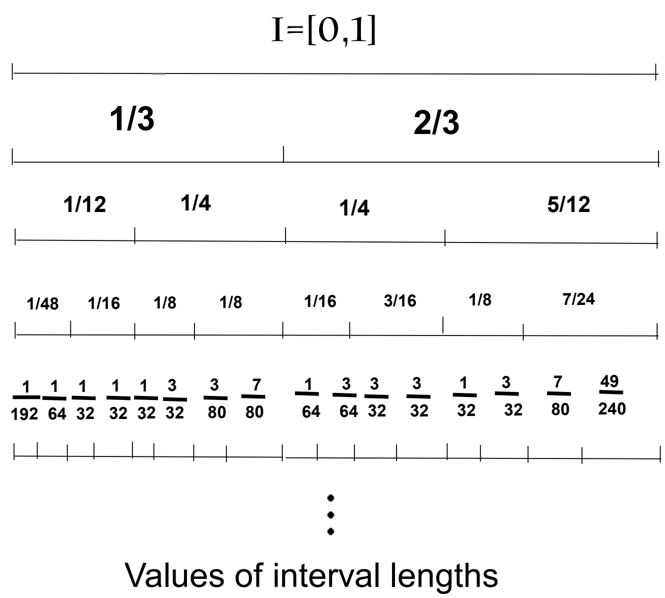

Figure 4.2: Finite martingale and interval lengths

Example 6.

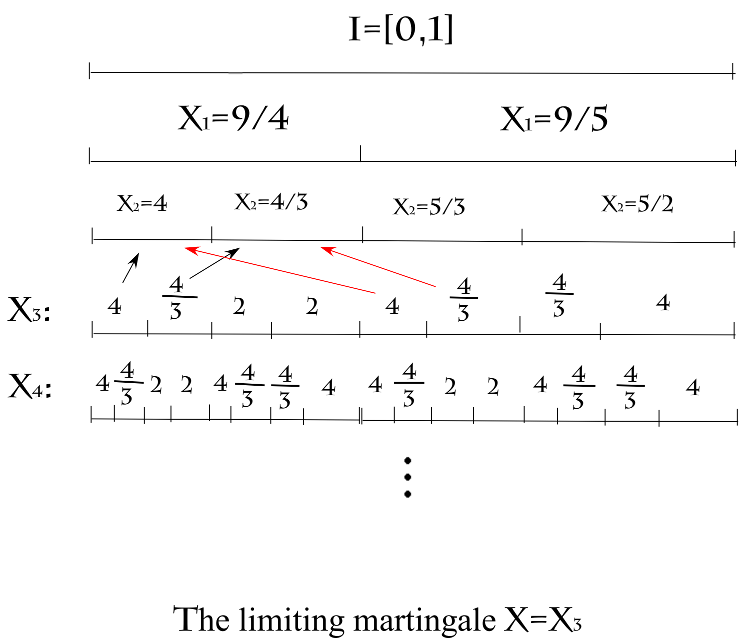

Figure 4.3: Finite Martingale with bounded nearby geometry

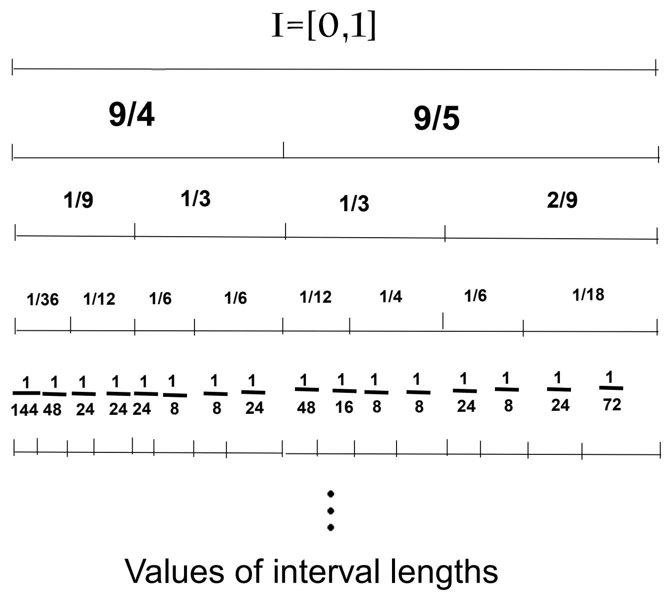

Figure 4.4: Corresponding values of interval lengths

ometry since the martingale $X = X_3$ and $X_3 = 4$ on I_{000} and I_{111} by the following proposition.

Proposition 4. *If* $\{X_k\}, k \geq 1$ *has finite length* n*, then* f *is uniformly quasisymmetric if and only if* $X_n([0 \cdots 0^*]) = X_n([1 \cdots 1^*])$.

Proof. By definition of martingale,

$$X_k([\underbrace{0 \cdots 0^*}_{k}]) = \frac{|I_{\underbrace{0 \cdots 0}_{k-1}}|}{|I_{\underbrace{0 \cdots 0}_{k}}|}$$

and

$$X_k([\underbrace{1 \cdots 1^*}_{k}]) = \frac{|I_{\underbrace{1 \cdots 1}_{k-1}}|}{I_{\underbrace{1 \cdots 1}_{k}}|}.$$

Since X_f has length n,

$$X_k([\underbrace{0 \cdots 0^*}_{k}]) = X_n([\underbrace{0 \cdots 0^*}_{n}])$$

and

$$X_k([\underbrace{1 \cdots 1^*}_{k}]) = X_n([\underbrace{1 \cdots 1^*}_{n}])$$

for any $k \geq n$.

Hence

$$\frac{|I_{\underbrace{0\cdots0}_{n+k}}|}{|I_{\underbrace{1\cdots1}_{n+k}}|} = \frac{|I_{\underbrace{0\cdots0}_{n}}|}{|I_{\underbrace{1\cdots1}_{n}}|}\Big(\frac{X_n([\underbrace{1\cdots1^*}_{n}])}{X_n([\underbrace{0\cdots0^*}_{n}])}\Big)^k.$$

Since the map f is uniformly quasisymmetric, the corresponding partition has bounded nearby geometry. So $X_n([0\cdots0^*]) = X_n([1\cdots1^*])$.

For the other direction, if $X_n([0\cdots0^*]) = X_n([1\cdots1^*])$, then the ratio of nearby intervals stays the same after n levels. So there are only finitly possible nearby ratios, which implies that the Markov partitions have bounded nearby geometry. By Theorem 4 in Chapter 1, f is uniformly quasisymmetric.

□

Suppose f preserves the Lebesgue measure and the partitions of f have bounded geometry. We can get a martingale sequence $\{X_k\}$. If we keep the first n level of the partitions and get a new partition by letting $X_k = X_n$ for each $k > n$, then we will get a new sequence of Markov partitions with bounded geometry. We will call the map corresponding to these new partitions f_n. Since f_n is defined on all partition points which is a dense subset of the interval $[0,1]$, f_n is well defined on the whole interval. The map f_n also preserves Lebesgue measure since it is generated by a martingale.

Theorem 14. *Suppose f is uniformly quasisymmetric with a martingale X_f. Let f_n be the map generated by length n martingale $X_{f,n}$, i.e. $X_{f_n} = X_{f,n}$, then f_n tends to f uniformly on S^1 as n goes to infinity. If $f_n = h_n q h_n^{-1}$ where h_n is a homeomorphism of S^1, then h_n tends to h uniformly where h is the conjugacy between f and q.*

Proof. Since f_n is the function generated by length n martingale $X_{f,n}$, the first n levels of partitions of f_n are same as of f. Hence $|f(x) - f_{n+1}(x)|$ is less than $\max_{w_n}\{|I_{w_n}|\}$ for fixed n.

The length of any interval at level n goes to zero uniformly as n goes to zero because f has bounded geometry, so f_n goes to f uniformly on S^1. h_n maps a partition point to the corresponding partition point, so the similar proof works for h_n and h. □

Remark 9. *f_n is not necessarily uniformly quasisymmetric, because $X_n([0\cdots 0^*])$ and $X_n([1\cdots 1^*])$ might be different.*

4.2 Uniqueness for a finite martingale of length $n \leq 4$

Theorem 15. *If f and g are two maps generated by the same martingale of length $n \leq 4$, then $f = g$*

Proof. Let $C_{w_n 0}$ be $\frac{1}{X([w_n 0^*])} = \frac{I_{w_n 0}}{I_{w_n}}$, then $C_{w_n 0} < 1$.

From Condition (4,2) (sub-interval condition) and Condition (4.1) (Lebesgue invariant condition), we have the following system of equations.

$$\begin{cases} C_{000}|I_{000}| + C_{100}|I_{100}| = |I_{000}| \\ C_{001}|I_{001}| + C_{101}|I_{101}| = |I_{010}| \\ C_{010}|I_{010}| + C_{110}|I_{110}| = |I_{100}| \\ C_{011}|I_{011}| + C_{111}|I_{111}| = |I_{110}| \\ |I_{000}| + |I_{100}| = |I_{000}| + |I_{001}| \\ |I_{010}| + |I_{110}| = |I_{100}| + |I_{101}| \\ |I_{000}| + |I_{001}| + |I_{010}| + |I_{011}| = |I_{000}| + |I_{010}| + |I_{100}| + |I_{110}| \\ |I_{000}| + |I_{001}| + |I_{010}| + |I_{011}| + |I_{100}| + |I_{101}| + |I_{110}| + |I_{111}| = 1 \end{cases}$$

Taking the coefficients we get,

$$\begin{pmatrix} C_{000}-1 & 0 & 0 & 0 & C_{100} & 0 & 0 & 0 \\ 0 & C_{001} & -1 & 0 & 0 & C_{101} & 0 & 0 \\ 0 & 0 & C_{010} & 0 & -1 & 0 & C_{110} & 0 \\ 0 & 0 & 0 & C_{011} & 0 & 0 & -1 & C_{111} \\ 1-1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 1-1 & -1 & 1-1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

we are going to show the determinant is not zero which implies that the system of equations has a unique solution. That means if we are given only X_4, then the partitions in the first four levels are uniquely determined. In other words, if f and g are two circle endomorphisms preserving the Lebesgue measure which have the same X_4, and $X_k = X_4$ for all $k \geq 4$, then $f = g$.

$$\begin{vmatrix} C_{000}-1 & 0 & 0 & 0 & C_{100} & 0 & 0 & 0 \\ 0 & C_{001} & -1 & 0 & 0 & C_{101} & 0 & 0 \\ 0 & 0 & C_{010} & 0 & -1 & 0 & C_{110} & 0 \\ 0 & 0 & 0 & C_{011} & 0 & 0 & -1 & C_{111} \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix}$$

Add column 5 to column 2, add column 6 to column 3, and add column 7 to column 4

$$\begin{vmatrix} C_{000}-1 & C_{100} & 0 & 0 & C_{100} & 0 & 0 & 0 \\ 0 & C_{001} & C_{101}-1 & 0 & 0 & C_{101} & 0 & 0 \\ 0 & -1 & C_{010} & C_{110} & -1 & 0 & C_{110} & 0 \\ 0 & 0 & 0 & C_{011}-1 & 0 & 0 & -1 & C_{111} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \end{vmatrix}$$

Delete row and column 5, then row and column 6 in the new determinant, we have

$$\begin{vmatrix} C_{000}-1 & C_{100} & 0 & 0 & 0 & 0 \\ 0 & C_{001} & C_{101}-1 & 0 & C_{101} & 0 \\ 0 & -1 & C_{010} & C_{110} & 0 & 0 \\ 0 & 0 & 0 & C_{011}-1 & 0 & C_{111} \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 1 & 2 & 2 & 2 & 1 & 1 \end{vmatrix}$$

$$= (C_{000}-1)\begin{vmatrix} C_{001} & C_{101}-1 & 0 & C_{101} & 0 \\ -1 & C_{010} & C_{110} & 0 & 0 \\ 0 & 0 & C_{011}-1 & 0 & C_{111} \\ -1 & 0 & 1 & -1 & 0 \\ 2 & 2 & 2 & 1 & 1 \end{vmatrix}$$

$$+(-1)\begin{vmatrix} C_{100} & 0 & 0 & 0 & 0 \\ C_{001} & C_{101}-1 & 0 & C_{101} & 0 \\ -1 & C_{010} & C_{110} & 0 & 0 \\ 0 & 0 & C_{011}-1 & 0 & C_{111} \\ -1 & 0 & 1 & -1 & 0 \end{vmatrix}$$

Now I want to show these two products are both positive.

The second product

$$(-1)\begin{vmatrix} C_{100} & 0 & 0 & 0 & 0 \\ C_{001} & C_{101}-1 & 0 & C_{101} & 0 \\ -1 & C_{010} & C_{110} & 0 & 0 \\ 0 & 0 & C_{011}-1 & 0 & C_{111} \\ -1 & 0 & 1 & -1 & 0 \end{vmatrix}$$

$$= (-1)C_{100}(-C_{111})\begin{vmatrix} C_{101}-1 & 0 & C_{101} \\ C_{010} & C_{110} & 0 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= C_{100}C_{111}[C_{101}C_{010} + (1-C_{101})C_{110}] > 0$$

The first product

$$(C_{000}-1)\begin{vmatrix} C_{001} & C_{101}-1 & 0 & C_{101} & 0 \\ -1 & C_{010} & C_{110} & 0 & 0 \\ 0 & 0 & C_{011}-1 & 0 & C_{111} \\ -1 & 0 & 1 & -1 & 0 \\ 2 & 2 & 2 & 1 & 1 \end{vmatrix}$$

$$= (C_{000}-1)C_{111}\begin{vmatrix} C_{001} & C_{101}-1 & 0 & C_{101} \\ -1 & C_{010} & C_{110} & 0 \\ -1 & 0 & 1 & -1 \\ 2 & 2 & 2 & 1 \end{vmatrix}$$

$$+(C_{000}-1)\begin{vmatrix} C_{001} & C_{101}-1 & 0 & C_{101} \\ -1 & C_{010} & C_{110} & 0 \\ 0 & 0 & C_{011}-1 & 0 \\ -1 & 0 & 1 & -1 \end{vmatrix}$$

$$= (C_{000}-1)C_{111}\begin{vmatrix} C_{001} & C_{101}-1 & 0 & C_{101} \\ -1 & C_{010} & C_{110} & 0 \\ -1 & 0 & 1 & -1 \\ 2 & 2 & 2 & 1 \end{vmatrix}$$

$$+(C_{000}-1)(C_{011}-1)[C_{101}C_{010} - C_{001}C_{010} - C_{101} + 1]$$

The second part in this sum

$$(C_{000}-1)(C_{011}-1)[C_{101}C_{010}-C_{001}C_{010}-C_{101}+1]$$

is larger than

$$(C_{000}-1)(C_{011}-1)(1-C_{101})(1-C_{010})>0.$$

So we only need to show

$$(C_{000}-1)C_{111}\begin{vmatrix} C_{001} & C_{101}-1 & 0 & C_{101} \\ -1 & C_{010} & C_{110} & 0 \\ -1 & 0 & 1 & -1 \\ 2 & 2 & 2 & 1 \end{vmatrix}$$

$$=(C_{000}-1)C_{111}\begin{vmatrix} C_{001} & C_{101}-1 & 0 & C_{101} \\ C_{110}-1 & C_{010} & C_{110} & C_{110} \\ 0 & 0 & 1 & 0 \\ 4 & 2 & 2 & 3 \end{vmatrix}>0$$

i.e.

$$=\begin{vmatrix} C_{001} & C_{101}-1 & 0 & C_{101} \\ C_{110}-1 & C_{010} & C_{110} & C_{110} \\ 0 & 0 & 1 & 0 \\ 4 & 2 & 2 & 3 \end{vmatrix}<0$$

The deteminale is equal to

$$3C_{001}C_{010}-2C_{001}C_{110}+C_{101}+3C_{101}C_{110}-3-C_{110}-4C_{010}C_{101}$$

$$< 3C_{001}C_{010}-2C_{001}C_{110}+C_{101}+3C_{101}C_{110}-C_{110}-C_{010}C_{101}-3C_{010}-3C_{101}$$

$$= 3C_{010}(C_{001}-1)-2C_{001}C_{110}+C_{110}(C_{101}-1)+2C_{101}(C_{110}-1)-C_{010}C_{101} < 0$$

□

One of applications of Theorem 15 is to show the symmetric rigidity for uniformly quasisymmetric circle endomorphisms whose corresponding martingale sequences are of length 4. That is, if f and g are two circle endomorphisms whose corresponding martingale sequences are of length $n \leq 4$, and if f and g are symmetrically conjugate, then $f = g$ since $X_f = X_g$. Thus we have that

Theorem 16. *Suppose f is a circle endomorphism preserving the Lebesgue measure generated by a martingale sequence of length $n \leq 4$. Then f is the only one in its symmetric conjugacy class preserving the Lebesgue measure.*

More generally, we conjecture that

Conjecture 4. *(a) If f and g are two maps generated by the same martingale of length n, then $f = g$. (b) Suppose f is a circle endomorphism preserving the Lebesgue measure generated* *by a martingale sequence of length* n*. Then* f *is the only one in its symmetric conjugacy class preserving the Lebesgue measure.*

In the next chapter, we will study more on the symmetric conjugacy class of f.

Chapter 5

Symmetric Conjugacy Class

5.1 Symmetric conjugacy

We first give the following definition from Gardiner and Sullivan's paper [12].

Definition 13. *A circle homeomorphism h is called symmetric if there is a bounded function $\epsilon(t) > 0$ such that $\epsilon(t) \to 0$ as $t \to 0^+$ and such that*

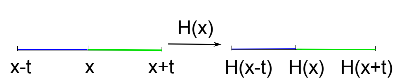

Figure 5.1: Symmetric homeomorphism

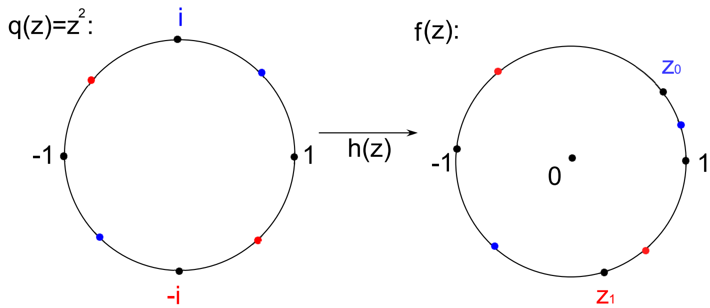

Figure 5.2: Conjugacy map h

$$(1+\epsilon(t))^{-1} \leq \frac{|H(x+t)-H(x)|}{|H(x)-H(x-t)|} \leq 1+\epsilon(t), \quad \forall x \in \mathbb{R},\ \forall t>0,$$

where the homeomorphism $H:\mathbb{R}\to\mathbb{R}$ *is the lift of* h *and* $H(0)=0$.

For an equivalent definition of a symmetric circle endomorphism in terms of complex analysis, the reader can refer to [12]. Assume h is the symmetric conjugacy between f, which is a quasisymmetric circle endomorphism preserving the Lebesgue measure, and $q(z)=z^2$, $f=hqh^{-1}$. Then h maps partition points of q to partition points of f on each level. Suppose f and g are uniformly quasisymmetric circle endomorphisms preserving the Lebesgue measure and $f=hgh^{-1}$, then h maps partitions of g

to partitions of f for each level n.

Theorem 17. *Suppose f and g are uniformly quasisymmetric circle endomorphisms preserving the Lebesgue measure. If $f = hgh^{-1}$ and h is a symmetric homeomorphism, then $X_f = X_g$.*

Proof. For any fixed $w^* \in [0^*]$, let $w_n 0$ be the first $n+1$ digits of w^* from the right.

Then

$$X_f(w^*) - 1 = \lim_{n\to\infty} \frac{I^f_{w_n 1}}{I^f_{w_n 0}}, \quad X_g(w^*) - 1 = \lim_{n\to\infty} \frac{I^g_{w_n 1}}{I^g_{w_n 0}}$$

So we need to show that if $\lim_{n\to\infty} \frac{I^g_{w_n 1}}{I^g_{w_n 0}} = a$ and h is symmetric, then $\lim_{n\to\infty} \frac{I^f_{w_n 1}}{I^f_{w_n 0}} = a$.

For any $q \in \mathbb{Z}$, there exits an integer p such that $\frac{p-1}{q} < a < \frac{p+1}{q}$. Hence there exits an N such that for any $n \geq N$, $\frac{p-1}{q} < \frac{I^g_{w_n 1}}{I^g_{w_n 0}} < \frac{p+1}{q}$.

Now let us cut $I^g_{w_n 0}$ into q equivalent subintervals and the length of each is δ_n. Then the length of $I^g_{w_n 1}$ is bounded by $(p-1)\delta_n$ and $(p+1)\delta_n$ for any $n \geq N$.

Since h is symmetric,

$$\frac{(1-\epsilon(\delta_n)) + (1-\epsilon(\delta_n))^2 ... + (1-\epsilon(\delta_n))^{p-1}}{1 + (1+\epsilon(\delta_n)) + (1+\epsilon(\delta_n))^2 ... + (1+\epsilon(\delta_n))^{q-1}} < \frac{I^f_{w_n 1}}{I^f_{w_n 0}}$$

$$< \frac{(1+\epsilon(\delta_n)) + (1+\epsilon(\delta_n))^2 ... + (1+\epsilon(\delta_n))^{p+1}}{1 + (1-\epsilon(\delta_n)) + (1-\epsilon(\delta_n))^2 ... + (1-\epsilon(\delta_n))^{q-1}}.$$

When $n \to \infty$, $\delta_n \to 0$ and $\epsilon(\delta_n) \to 0$. So

$$\lim_{n\to\infty} \frac{(1-\epsilon(\delta_n)) + (1-\epsilon(\delta_n))^2 ... + (1-\epsilon(\delta_n))^{p-1}}{1 + (1+\epsilon(\delta_n)) + (1+\epsilon(\delta_n))^2 ... + (1+\epsilon(\delta_n))^{q-1}} = \frac{p-1}{q}$$

and

$$\lim_{n\to\infty} \frac{(1+\epsilon(\delta_n)) + (1+\epsilon(\delta_n))^2 ... + (1+\epsilon(\delta_n))^{p+1}}{1 + (1-\epsilon(\delta_n)) + (1-\epsilon(\delta_n))^2 ... + (1-\epsilon(\delta_n))^{q-1}} = \frac{p+1}{q}.$$

Hence there exist a $N_1 > N$ such that $\frac{p-2}{q} < \frac{I^f_{w_n 1}}{I^f_{w_n 0}} < \frac{p+2}{q}$ for any $n \geq N_1$. So $|\frac{I^f_{w_n 1}}{I^f_{w_n 0}} - a| < \frac{3}{q}$ for any $n \geq N_1$ where N_1 is a function of q. Now let q goes to infinity, then $\lim_{n\to\infty} \frac{I^f_{w_n 1}}{I^f_{w_n 0}} = a$

□

The following theorem was first proved in [17]. Here we give a simpler proof.

Theorem 18 (Jiang)**.** *If $f = hqh^{-1}$ and h is symmetric, then $h = id$ i.e., $f = q$ where $q(z) = z^2$.*

Proof. Since $f = hqh^{-1}$ and h is symmetric, then $X_f = 2$ almost everywhere by Theorem 17. By the definition of martingale,

$$\int_{[w_n^*]} X_n dP = \int_{[w_n^*]} X_f dP,$$

which implies $X_n([w_n^*])P([w_n^*]) = X_f([w_n^*])P([w_n^*])$, so $X_n([w_n^*]) = 2$, i.e. $|I_{\sigma(w_n)}| = 2|I_{w_n}|$ for all w_n. This implies that $|I_{w_n}| = 1/2^n$ for all w_n. Thus $h = id$.

□

Theorem 16, in Chapter 4, is a more general result of the previous theorem when q is replaced by any circle endomorphism which has a martingale sequence of length $n \leq 4$. Gardiner and Jiang are working on the symmetric rigidity problem for general uniformly symmetric circle endomorphism. That is, they would like to show that Theorem 18 is still true if q is replaced by any uniformly symmetric circle endomorphism preserving the Lebesgue measure (see their work in progress [6]). It is conjectured by Jiang in [17], [21] that Theorem 18 is still true when q is replaced by any uniformly quasisymmetric circle endomorphism that preserves the Lebesgue measure. In the next section, we will study uniformly symmetric circle endomorphisms. Theorem 19, in the next section, shows that if f is a uniformly symmetric circle endomorphism preserving the Lebesgue measure and its martingale sequence X_f is of length

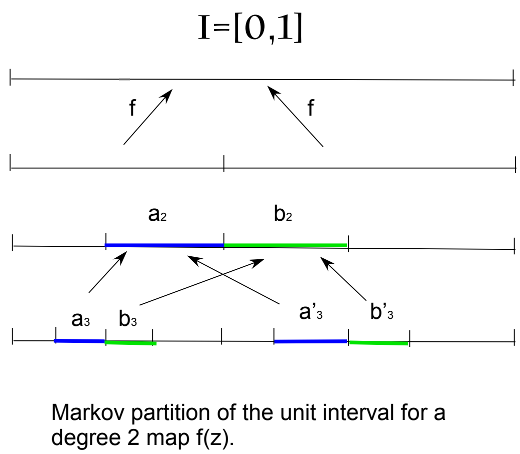

Figure 5.3: Conditions of Lebesgue invariant

n, then $f(z) = q(z) = z^2$.

In another proof of Theorem 18, we will only use the following property of Markov partitions.

Proposition 5. *If f preserves Lebesgue measure , then* $\min_i\{\frac{a_n^i}{b_n^i}\}$ *is decreasing and* $\max_i\{\frac{a_n^i}{b_n^i}\}$ *is increasing, where a_n^i is the length of i^{th} interval on level n partition and b_n^i is the length of interval to the right of a_n^i in the same level.*

Proof. We choose one pair of adjacent intervals with length a_2 and b_2 on level 2. Consider the two pairs of preimages of this pair.

They are on level 3 with lengths a_3 and b_3 and a_3' and b_3'.

$$a_2 = a_3 + a_3', \quad b_2 = b_3 + b_3'$$

$$\min\{\frac{a_3}{b_3}, \frac{a_3'}{b_3'}\} \leq \frac{a_3 + a_3'}{b_3 + b_3'} \leq \max\{\frac{a_3}{b_3}, \frac{a_3'}{b_3'}\}$$

$$\min\{\frac{a_3}{b_3}, \frac{a_3'}{b_3'}\} \leq \frac{a_2}{b_2} \leq \max\{\frac{a_3}{b_3}, \frac{a_3'}{b_3'}\}$$

In general, we choose one pair of adjacent intervals with length a_n and b_n on level n. The preimages of this pair are on level $n+1$ with lengths a_{n+1} and b_{n+1} and a_{n+1}' and b_{n+1}'. Using the fact that f preserves the Lebesgue measure and the previous argument, we get

$$\min_i\{\frac{a_{n+1}^i}{b_{n+1}^i}\} \leq \min_i\{\frac{a_n^i}{b_n^i}\}$$

$$\max_i\{\frac{a_{n+1}^i}{b_{n+1}^i}\} \geq \max_i\{\frac{a_n^i}{b_n^i}\}$$

□

Proof. [Another proof of Theorem 18]

Since f is Lebesgue invariant and h is symmetric, $\min_i\{\frac{a_n^i}{b_n^i}\} \downarrow 1$ and $\max_i\{\frac{a_n^i}{b_n^i}\} \uparrow 1$.

So for any n,

$$\min_i\{\frac{a_n^i}{b_n^i}\} = \max_i\{\frac{a_n^i}{b_n^i}\} = 1.$$

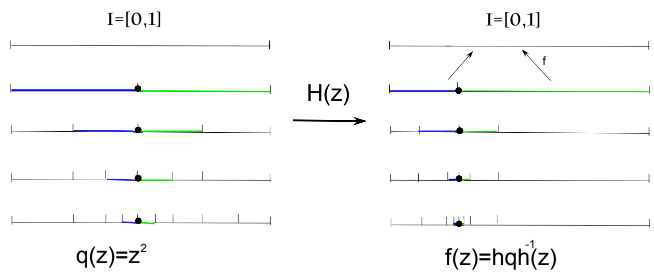

Figure 5.4: Conjugacy map H

This implies that $\frac{a_n^i}{b_n^i} = 1$ for all n and all i and that $h = id$.

□

5.2 Finite martingales for uniformly symmetric circle endomorphisms

In this section, we will study uniformly symmetric circle endomorphisms which preserve the Lebesgue measure.

Definition 14. *A circle endomorphism f is called uniformly symmetric if there is a bounded function $\epsilon(t) > 0$ such that $\epsilon(t) \to 0$ as $t \to 0^+$ and such that*

$$\frac{1}{1+\epsilon(t)} \leq \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq 1+\epsilon(t)$$

for all $x \in \mathbb{R}$ and $t > 0$ and any $n > 0$.

For an equivalent definition of a uniformly symmetric circle endomorphism in terms of complex analysis, the reader can refer to [5].

In any Markov partitions of a martingale of length n, the cutting ratios $\frac{|I_{w_k 0}|}{|I_{w_k 1}|}$ are constants for $k \geq n$ on any dynamical paths of w_n, i.e. $[w_k^*]$ is a subset of $[w_n^*]$. Suppose the cutting ratios on level n are r_1, r_2,...,r_{2^n}, then the cutting ratios on level $n+1$ are r_1, r_2,...,r_{2^n}, r_1, r_2,...,r_{2^n}. On any level $n+k$, the sequence will repeat 2^k times.

For $x \in \{f^{-n}(1)\}$, let $I^n_{x^-}$ denote the interval, on level n, to the left of point x, i.e. x is the right endpoint of $I^n_{x^-}$ and $I^n_{x^+}$ denote the interval , on level n, to the right of x, i.e. x is the left endpoint of $I^n_{x^+}$. We will use C^n_x for the ratio of the length of these two intervals, and $C^n_x = \frac{|I^n_{x^+}|}{|I^n_{x^-}|}$.

Theorem 19. *Suppose f is a uniformly symmertic circle endomorphism preserving the Lebesgue measure. Suppose its martingale sequence is of length n. Then $f = q$.*

Proof. Suppose x and y are two points in $f^{-n}(1)$, such that $f(x) =$

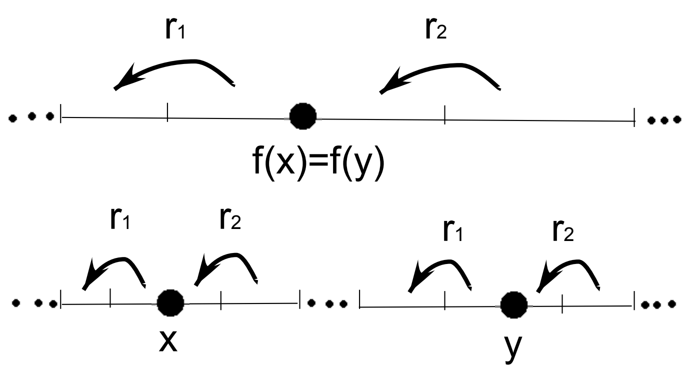

Figure 5.5: Uniformly symmetric circle endomorphism

$f(y)$. Let $r = \frac{I^n_{x_+}}{I^n_{x_-}}$ and $r' = \frac{I^n_{y_+}}{I^n_{y_-}}$. We want to show that $r = r'$.

On level $n+1$, $I^{n+1}_{x-} : I^{n+1}_{x+} = \frac{r_1}{1+r_1} : \frac{r}{1+r_2}$ and $I^{n+1}_{y-} : I^{n+1}_{y+} = \frac{r_1}{1+r_1} : \frac{r'}{1+r_2}$ since $\{X_k\}$ has length n where r_1 and r_2 are cutting ratios to the left and right of point $f(x) = f(y)$. So

$$\frac{I^{n+1}_{x_+}}{I^{n+1}_{x_-}} / \frac{I^{n+1}_{y_+}}{I^{n+1}_{y_-}} = \frac{r}{r'}$$

Similarly, on level $n+k$ for any $k \geq 1$,

$$\frac{I^{n+k}_{x_+}}{I^{n+k}_{x_-}} / \frac{I^{n+k}_{y_+}}{I^{n+k}_{y_-}} = \frac{r}{r'}$$

.

Let $I^m_{1_-}$ and $I^m_{1_+}$ be two intervals in level m adjacent to 1 and on the left and right of 1. Since the martingale sequence of f has length n,

$$X_n([0\cdots 0^*]) = X_n([1\cdots 1^*]) = X_k([0\cdots 0^*]) = X_k([1\cdots 1^*])$$

for any $k \leq n$ by Proposition 4. So we have

$$\lim_{m\to\infty} \frac{|I^m_{1_+}|}{|I^m_{1_-}|} = \frac{|I^k_{1_+}|}{|I^k_{1_-}|} = a$$

for any $k \leq n$ exists. Since $f^n(x) = f^n(y) = 1$ and f is uniformly symmetric, we have that

$$\lim_{k\to\infty} \frac{|I^{n+k}_{z+}|}{|I^{n+k}_{z-}|} = a$$

for both $z = x$ or y.

$$\lim_{k\to\infty} \frac{I^{n+k}_{x_+} I^{n+k}_{y_-}}{I^{n+k}_{x_1} I^{n+k}_{y_+}} = 1,$$

so $r = r'$. This implies that $\{X_k\}$ is a martingale of length $n-1$, since $f(x) = f(y)$. Repeating the previous argument, we see that $\{X_k\}$ is a martingale of length $n-2$, of length $n-3$,...finally of length 1. Because the partitions have bounded nearby geometry, $X_f = 2$ a.e. and $f = q$.

$\square$

5.3 Symmetric at a point

An interesting problem in the study of rigidity is to show that one-point property implies the global property.

Theorem 20 (Sullivan, 1986)**.** *Suppose f and g are two $C^{1+Lipschitz}$ circle expanding endomorphisms of the same degree. Let h be the conjugacy between f and g, that is, $f \circ h = h \circ g$. Then the conjugacy h is $C^{1+Lipschitz}$ if and only if it is differentiable at one point with nonzero derivative.*

Theorem 21 (Jiang 1990)**.** *Suppose f and g are two $C^{1+\alpha}$ circle expanding endomorphism of the same degree for $0 < \alpha < 1$. Let h be the conjugacy between f and g, that is, $f \circ h = h \circ g$. Then the conjugacy h is $C^{1+\alpha}$ if and only if it is differentiable at one point with uniform bound.*

Here h is differentiable at p with uniform bound means that there is a constant $C > 0$ and a small neighborhood U of p such that $C^{-1} \leq h'(q) \leq C$ for all $q \in GO(p) \cap U$, where $GO(p)$ is the grand orbit of p.

In 1997 and 2005, Yunping Jiang proved similar results for geometrically finite one-dimensional maps and sub-hyperbolic one-dimensional maps.

Then we would like to know the following problem: As a conjugacy between smooth expanding circle endomorphisms, if h is symmetric at one point, is it symmetric on the unit circle T? It is an interesting question asked in this direction. In this thesis, we give an example showing that it is false.

Theorem 22. *Take*

$$f(z) = z\frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}$$

which is an expanding Blaschke product. The conjugacy between f and $q(z) = z^2$ is symmetric at 1, but not symmetric on T.

Before we prove the theorem, we want to show that f preserves the Lebesgue measure and is expanding by using the following picture. We know that

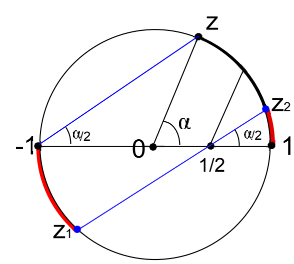

Figure 5.6: Lebesgue invariant condition for f

$$f(z) = z\frac{z-\frac{1}{2}}{1-\frac{1}{2}z} = \frac{z-\frac{1}{2}}{\bar{z}-\frac{1}{2}} = e^{i2arg(z-\frac{1}{2})},$$

so $f^{-1}(e^{2\pi i\alpha}) = \{z_1, z_2\}$ where $arg(z_1 - \frac{1}{2}) = \frac{\alpha}{2} + \pi$ and $arg(z_2 - \frac{1}{2}) = \frac{\alpha}{2}$.

The picture shows that $f(z)$ is Lebesgue invariant. The line through -1 and z is parallel to the line through z_1 and z_2. So the arc between -1 and z_1 is equal to the arc between z and z_2 which implies the Lebesgue invariant condition.

$$f'(z) = \frac{-\frac{1}{2}z^2 + 2z - \frac{1}{2}}{1 + \frac{1}{4}z^2 - z}$$

Let $z = \cos\theta + i\sin\theta$, then

$$f'(z) = \frac{(\cos\theta + i\sin\theta)(2-\cos\theta)}{\frac{1}{2}\cos^2\theta + \frac{3}{4} - \cos\theta + i\sin\theta(\frac{1}{2}\cos\theta - 1)}$$

We are going to show that f is expanding, which is equivalent to $|f'(z)| \geq C > 1$. So we need

$$(2-\cos\theta)^2 \geq C^2 \times [(\frac{1}{2}\cos^2\theta + \frac{3}{4} - \cos\theta)^2 + (\sin\theta(\frac{1}{2}\cos\theta - 1))^2],$$

that is

$$\frac{39}{16} \geq C^2 \times \frac{3}{2}\cos\theta$$

.

So we can choose $C^2 = \frac{13}{8}$. Then $|f'(z)|$ is greater than or equal to the square root of $\frac{13}{8}$, so it is expanding.

By the following proposition, f is uniformly quasisymmetric.

Proposition 6 (Jiang). *If f is $C^{1+\alpha}$ and it is expanding, then $f(z)$ is uniformly quasisymmetric.*

And in Chapter 2, we showed that a sequence of Markov partitions of uniformly quasisymmetric circle endomorphism has bounded and bounded nearby geometry. So the Markov partition of

$$f(z) = z\frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}$$

is Lebesgue invariant and it has bounded and bounded nearby geometry.

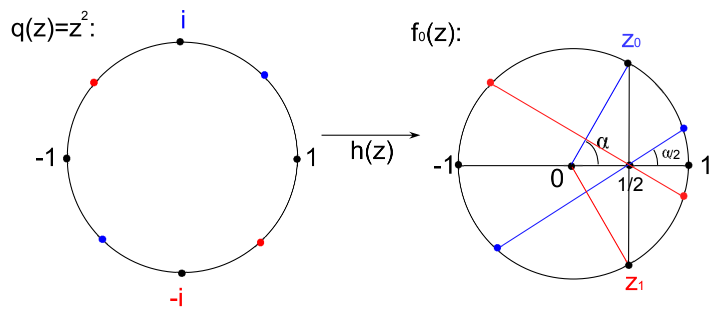

Figure 5.7: Conjugacy map between q and f

Now we are ready to prove Theorem 22.

Proof. Inductively, it is easy to see that $f^{-n}(1)$ is symmetric with respect to the real line. So the quasisymmetric conjugacy h is symmetric at 1, but it is not a symmetric homeomorphism by Theorem 18. □

We also can construct an h which is symmetric at a point but not on T by constructing the Markov partitions.

Example 7. *Corresponding values of interval lengths:*

All the partition points are symmetric with respect to the midpoint 1/2, since the martingale is symmetric. So h is symmetric at 1/2 but it is not a global symmetric circle homeomorphism. It

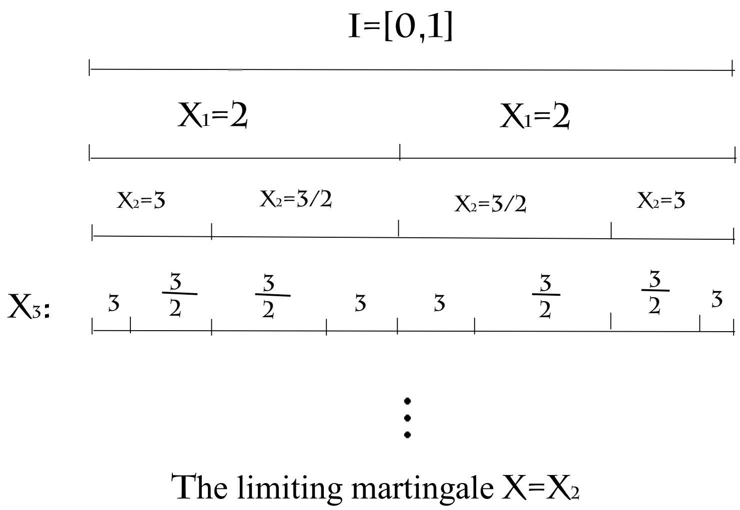

Figure 5.8: Symmetric at a point

is easy to see that h is not symmetric at the whole track $f^{-n}(1)$. on some partition points, the ratios are $1:2$.

Suppose f and g are two uniformly symmetric circle endomorphisms. Suppose h is the conjugacy between f and g and it is symmetric at point p.

Definition 15. *We say h is symmetric at p with uniform bound if there is a neighborhood U about p such that*

$$\sup_{q \in GO(p) \bigcap U} \epsilon_q(t) \to 0+$$

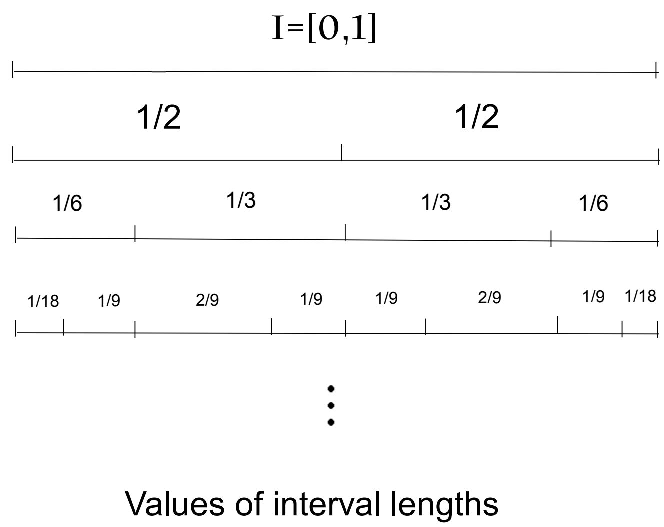

Figure 5.9: Symmetric at a point and interval lengths

as $t \to 0+$ *where* $GO(p)$ *is the grant orbits of* p*.*

Problem 2. *Suppose* h *is symmetric at one point* p *with uniform bound. Is it symmetric on* T*?*

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