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Abstract

A generalization of the Curry-Howard-Lambek isomorphism for cartesian closed categories and typed lambda calculi is given for the LP categories with weak natural numbers object, which correspond to the positive conjunction fragment of the intuitionistic Logic of Proofs **LP** of Artemov, and LP-typed lambda calculi with natural numbers type.

1 Introduction

The Logic of Proofs **LP** of Artemov is a Hilbert-style logical system extending classical and intuitionistic propositional logic, with additional propositions of the form $(t : A)$, read as "term t is justification for A" [Art01a, Art01b]. **LP** and the related broader class of justification logics are, in a precise sense, refinements of epistemic and modal logics such as **K**, **K4**, **K45**, **KD45**, **T**, **S4** and **S5** [Art08]. Semantics for these systems include the Kripke–Fitting models, the Mkrtychev models, and arithmetical provability semantics [Fit05, Mkr97, Art01a]. In addition to its applications in epistemic logic, modal logic and proof theory, **LP** has been proposed as a logic for certified mobile computation [BF09].

Developing a categorical semantics for the system **LP** remains an open problem; here we provide a categorical semantics for the positive intuitionistic conjunction fragment of **LP** by generalizing the Curry-Howard-Lambek isomorphism [LS86]. In place of cartesian closed categories and typed lambda calculi, we extend the Curry-Howard-Lambek isomorphism to the category of LP categories and corresponding categories of lambda calculi, called LP-typed lambda calculi. An LP category is a cartesian closed category with additional object terms of the form $(t : A)$ for terms t belonging to a proof term algebra acting on the object terms of the category, and with additional arrow terms corresponding to the evidence axioms of **LP**. An LP category (LP-typed lambda calculus) may include a weak natural numbers object (natural numbers type).

In [Art01b], Artemov constructed the natural deduction system $\mathcal{ILPN}$ for intuitionistic **LP**, and a corresponding typed lambda calculus $\mathcal{ILPN}\lambda$; the Curry-Howard isomorphism was generalized to these systems for types of the form $(t : A)$ (the reflective fragment of **LP**). Other related lambda calculi were developed by analogy with **LP**, which can internalize its own proofs. The reflective lambda calculus λ^∞ of Alt and Artemov is an extension of typed λ-calculus that can internalize its own derivations [AA01]. In [AB07], Artemov and Bonelli presented a natural deduction formulation of the Logic of Proofs, together with an associated calculus $\lambda^{\mathbf{I}}$, the intensional lambda calculus, which can internalize its own computations.

Our approach to the Curry-Howard-Lambek isomorphism will borrow extensively from the presentation of Lambek and Scott [LS86]. We have not checked the relation between λ^∞, $\lambda^{\mathbf{I}}$ and the LP-typed lambda calculus defined here.

1.1 Definitions

A *directed graph* $\mathcal{G} = \langle \mathcal{O}, \mathcal{A}, s, t \rangle$ is a structure with a set $\mathcal{O}$ of elements called objects, a set $\mathcal{A}$ of elements called arrows, and a pair of maps $s, t : \mathcal{A} \to \mathcal{O}$, which associate to each arrow $f \in \mathcal{A}$ its source $s(f) \in \mathcal{O}$ and its target $t(f) \in \mathcal{O}$, respectively. We will write $f \mid A \vdash B$ to assert that $f \in \mathcal{A}$, $s(f) = A \in \mathcal{O}$ and $t(f) = B \in \mathcal{O}$. The vertical bar '|' is used instead of the colon ':' since the colon appears in the notation $(t : A)$ for the new type from the Logic of Proofs.

A *deductive system* is a directed graph $\mathcal{G} = \langle \mathcal{O}, \mathcal{A}, s, t \rangle$ with a specified identity arrow $1_A \mid A \vdash A$ for each object $A \in \mathcal{O}$, together with an operation of composition

$$\frac{f \mid A \vdash B \qquad g \mid B \vdash C}{(g \circ f) \mid A \vdash C}$$

such that $\mathcal{A}$ is closed under composition (of composable arrows).

1.2 LP deductive systems

An LP deductive system is a positive, conjunction deduction system with additional structure; such a system corresponds to the positive conjunction, implication fragment of intuitionistic **LP** [Art01b]. We describe the language of an LP-deductive system and use this to define its proof terms, object terms and arrow terms. It will be helpful to define in tandem related algebraic structures that arise in categorical constructions occurring in the proof of the Curry–Howard–Lambek correspondence.

Symbols occurring in proof terms

- *proof constants* $c, c_1, c_2, \ldots$
- *proof variables* $u, v, w, x, y, z, x_1, x_2, \ldots$.
- *unary operation* ! and *binary operations* $\cdot, \&, \overline{\lambda}$.

Symbols occurring in object terms (propositions)

- *propositional variables* $A, B, C, \ldots, X, Y, Z, A_1, A_2, \ldots$,

- *propositional constant* $\top$ and *binary operations* $\wedge$ and $\rightarrow$
- *punctuation* :

Symbols occurring in arrow terms (proofs)

- *arrows (names of proofs)* $f, g, h, \ldots, f^{t_1,\ldots,t_n}_{A_1,\ldots,A_m}, \ldots$
- *turnstile, punctuation* $\vdash, |$
- *unary operation* $(-)^*$ and *binary operations* $\langle -, - \rangle$, $\circ$

1.2.1 Proof terms of an LP deductive system

A *proof term* t is an expression of the form

$$t ::= c_i \,|\, x_j \,|\, (t \cdot t) \,|\, (!t) \,|\, (t \,\&\, t) \,|\, (\overline{\lambda} x_j . t)$$

where c_i is a proof constant and x_j is a proof variable. The set of proof terms is denoted by $\mathcal{P}$. A *proof-term algebra* $\mathcal{P} = \langle c_i, x_j, !, \cdot, \&, \overline{\lambda} \rangle$ is the term algebra obtained by closing a collection of *proof constants* $\{c_i\}$ and *proof variables* $\{x_j\}$ under the unary operation !, the binary operations $\cdot$ and & and the partial binary operation $\overline{\lambda}(-).(-)$, the left argument of which is always a proof variable. A *morphism* $\mathcal{P} \rightarrow \mathcal{P}'$ of proof term algebras is a mapping that commutes with all operations, sends constants to constants and variables to variables. Proof terms are called proof polynomials in the parlance of LP. However, the phrase 'proof term' is used in type theory for similar objects, and we follow Lambek and Scott's usage of the term 'polynomial' to refer to certain kinds of arrow terms and categories [LS86].

1.2.2 Object terms of an LP deductive system

An *object term* A of an LP category is an expression of the form

$$A ::= \top \,|\, A_i \,|\, (A \wedge A) \,|\, (A \rightarrow A) \,|\, (t : A)$$

where t is a proof term and where A_i is a propositional variable.

The set of object terms is denoted by $\mathcal{O}$. An *object term algebra* $\mathcal{O} = \langle \top, A_i, \wedge, \rightarrow, (\ :\ ), \mathcal{P} \rangle$ is the term algebra obtained by closing the constant $\top$ and a collection $\{A_i\}$ of propositions under the binary operations $\wedge$ and $\rightarrow$, and under the action of a free proof term algebra $\mathcal{P}$ on $\mathcal{O}$. This means there is a map $\mathcal{P} \times \mathcal{O} \rightarrow \mathcal{O}$ given by $(t, A) \longmapsto (t : A)$. The action is free: whenever $(s : A) = (t : B)$, $s = t$ and $A = B$. A *morphism* of an object term algebra is a pair consisting of a proof term algebra morphism Φ on $\mathcal{P}$ and a morphism, also denoted Φ, of $\mathcal{O}$ that preserves the nullary operation $\top$; the binary operations $\wedge$ and $\rightarrow$; and the action: for $A \in \mathcal{O}$, $\Phi\,(t : A) = (\Phi(t) : \Phi(A))$.

1.2.3 Arrow terms of an LP deductive system

An *arrow term* of an LP deductive system has the form

$$f^{t_1,\ldots,t_n}_{A_1,\ldots,A_m} \mid A \vdash B$$

where f is an arrow name, $t_1, \ldots, t_n$ are proof terms, $A_1, \ldots, A_m$ are object terms, and where where A and B are object terms in which some or all of $t_1, \ldots, t_n$ and $A_1, \ldots, A_m$ occur.

There are two types of arrow terms: *CCC arrow terms* and *evidence arrow terms*. The set of arrow terms is denoted by $\mathcal{A}$.

The CCC arrow terms are those of a cartesian closed category. We follow the notation of Lambek and Scott [LS86].

- *identity*
$$1_A \mid A \vdash A$$
- *truth*
$$\bigcirc_A \mid A \vdash \top$$
- *left projection*
$$\pi^1_{A,B} \mid A \wedge B \vdash A$$
- *right projection*
$$\pi^2_{A,B} \mid A \wedge B \vdash B$$
- *evaluation*
$$\varepsilon_{A,B} \mid (B \to A) \wedge B \vdash A$$

Evidence arrow terms reflect the operations of composition, pairing and curry available in an LP deductive system. The evidence arrow terms are motivated by the evidence axioms of Artemov's **LP**, and by the desire to prove an analog of the internalization theorem (Proposition 7 in the sequel) [Art01a]. Differences between the evidence arrow terms of an LP category and the evidence axioms of **LP** reflect differences between deductive systems and Hilbert systems.

In the following, $c, v, s, t \in \mathcal{P}$, where c is a proof constant, v is a proof variable; and $A, B, C \in \mathcal{O}$.

- *evidence necessitation*
$$\mathrm{N}^{s,t}_{B,C} \mid (t : (B \to C)) \wedge (s : B) \vdash (t \cdot s : C)$$
- *evidence pairing*
$$\mathrm{P}^{s,t}_{A,B} \mid (s : A) \wedge (t : B) \vdash ((s \,\&\, t) : (A \wedge B))$$

- *evidence abstraction*

$$\mathrm{L}^{v,t}_{A,B} \mid (v : A) \to (t : B) \vdash \big(\overline{\lambda} v.t : (A \to B)\big)$$

  where v is a proof variable and where t is a proof term.

- *proof checker*

$$\mathrm{C}^{t}_{A} \mid (t : A) \vdash (!t : (t : A))$$

- *verification*

$$\mathrm{V}^{t}_{A} \mid (t : A) \vdash A$$

- *truth necessitation*

$$\bigcirc^{c}_{A} \mid A \vdash (c : \top)$$

  where c is a proof constant.

1.2.4 Operations on arrow terms

The set $\mathcal{A}$ of arrow terms of an LP deductive system is closed under the following operations.

$$\text{Composition: } \frac{f \mid A \vdash B \qquad g \mid B \vdash C}{g \circ f \mid A \vdash C}$$

$$\text{Pairing: } \frac{f \mid C \vdash A \qquad g \mid C \vdash B}{\langle f, g \rangle \mid C \vdash A \wedge B}$$

$$\text{Curry: } \frac{h \mid C \wedge B \vdash A}{h^* \mid C \vdash B \to A}$$

An LP *deductive system*

$$\langle \mathcal{P}, c_i, x_j, !, \cdot, \&, \overline{\lambda}; \mathcal{O}, \top, A_i, (\ :\ ), \wedge, \to; \mathcal{A}, 1, \bigcirc, \pi^1, \pi^2, \varepsilon, N, P, V, L, C, \bigcirc^{(-)} \rangle$$

is a collection of proof terms, object terms, arrow terms as in the preceding, where $\mathcal{P}$ is a proof term algebra acting on the object algebra $\mathcal{O}$, and where the collection $\mathcal{A}$ of arrow terms is closed under the arrow operations of composition, pairing and curry.

Two derived operations on arrow terms will be useful in the functional completeness of LP categories and the construction of the internal language of an LP category.

$$\text{Reference: } \frac{f \mid A \vdash B}{\ulcorner f \urcorner \mid \top \vdash A \to B}$$

where $\ulcorner f \urcorner = \left(f \pi^2_{\top, A} \right)^*$; and

$$\text{Dereference: } \frac{g \mid \top \vdash A \to B}{g_\bullet \mid A \vdash B}$$

where $g_\bullet = \varepsilon_{B,A}\langle g\bigcirc_A, 1_A\rangle$.

By comparison with the Gentzen sequent formulation $\mathcal{ILPG}$ of intuitionistic **LP** [Art01b], observe that that a derived rule

$$\text{Left Necessitation: } \frac{f \mid A \vdash B}{[f:t] \mid (t:A) \vdash B}$$

can be obtained by taking $[f:t] := f \circ \mathrm{V}^t_A$, in which $f \mid A \vdash B$ is an arrow term in $\mathcal{A}$, and in which $t \in \mathcal{P}$ is a proof term.[1]

1.3 The deduction theorem

The deduction theorem holds in an LP deductive system. Since the internalization theorem, functional completness and the definition of lambda abstraction hinge on the operation $\kappa_{x\in A}$, defined in the proof of the deduction theorem, for the convenience of the reader we give the proof in the sequel.

Let $\mathcal{D}$ be an LP deductive system. An arrow term of the form $x \mid \top \vdash A$, where $A \in \mathcal{O}$ and where x is new is called an *assumption.*

A new LP deductive system, denoted by $\mathcal{D}(x)$, is obtained from $\mathcal{D}$ by adjoining the assumption $x \mid \top \vdash A$ to $\mathcal{D}$ and closing under the operations of composition, pairing and curry.

Theorem 1 (Deduction theorem [LS86])**.** *Suppose $\mathcal{D}$ is an LP deductive system, $x \mid \top \vdash A$ is an assumption, and*

$$\varphi(x) \mid B \vdash C$$

is an arrow in $\mathcal{D}(x)$. Then there is an arrow $f \mid A \wedge B \vdash C$ in $\mathcal{D}$.

1.4 LP categories

An *LP category* is an LP deductive system $\langle \mathcal{P}, \mathcal{O}, \mathcal{A}\rangle$ in which the equations of a cartesian closed category (CCC) hold for specified arrow terms, as follows.

1. For $f \mid A \vdash B$, $g \mid B \vdash C$, $h \mid C \vdash D$,

$$1_B \circ f = f = f \circ 1_A, \text{ and } (h \circ g) \circ f = h \circ (g \circ f).$$

2. For $f \mid A \vdash \top$, $f = \bigcirc_A$.

[1] The assignment $[f:t] := f \circ \mathrm{V}^t_A$ for $f \mid A \vdash B$ defines a right action of the proof term algebra $\mathcal{P}$ on the collection $\mathcal{A}$ of arrow terms. In the LP category (defined below) associated to an LP deductive system, the equation $[g \circ f : t] = g \circ [f:t]$ holds for $g \mid B \vdash C$.

3. For $f \mid C \vdash A$, $g \mid C \vdash B$, $h \mid C \vdash A \wedge B$,

$$\pi^1_{A,B} \circ \langle f, g\rangle = f,\ \pi^2_{A,B} \circ \langle f, g\rangle = g,\ \langle \pi^1_{A,B} \circ h, \pi^2_{A,B} \circ h\rangle = h.$$

4. For $f \mid C \wedge A \vdash B$, $g \mid C \vdash A \to B$,

$$\varepsilon_{B,A} \circ \langle f^* \circ \pi^1_{C,A}, \pi^2_{C,A}\rangle = f,\ \left(\varepsilon_{B,A} \circ \langle g \circ \pi^1_{C,A}, \pi^2_{C,A}\rangle\right)^* = g.$$

One may optionally include a weak natural numbers object; see [LS86] for details. The addition of weak natural numbers objects is relevant to interpretations of the Logic of Proofs in Peano Arithmetic [Art01a].

Fix a proof polynomial term algebra $\mathcal{P}$. Let $F : \mathbb{C} \to \mathbb{D}$ be a functor between LP categories. The functor F is an *LP functor* if

(i) F is a cartesian closed functor;

(ii) F is a morphism of $\mathcal{P}$–graphs; i.e., it commutes with the action of the proof term algebra $\mathcal{P}$ on the collection of object terms $\mathcal{O}$: $F\,(t : A) = (F(t) : F(A))$;

(iii) F preserves the LP arrows.

$$F(\mathrm{N}^{s,t}_{A,B}) = \mathrm{N}^{Fs,Ft}_{FA,FB},\ F(\mathrm{P}^{s,t}_{A,B}) = \mathrm{P}^{Fs,Ft}_{FA,FB},\ F(\mathrm{L}^{v,t}_{A,B}) = \mathrm{L}^{Fv,Ft}_{FA,FB},$$

$$F(\mathrm{C}^t_A) = \mathrm{C}^{Ft}_{F(A)},\ F(\mathrm{V}^t_A) = \mathrm{V}^{Ft}_{FA},\ F(\bigcirc^t_A) = \bigcirc^{Ft}_{FA}$$

The category of small LP categories and LP functors is denoted $\mathbf{CCC}_{\mathcal{P}}$.

1.4.1 Universal constructions of LP categories

The universal constructions following are used in the statement and proof of functional completeness of LP categories, and in the construction of the internal language of an LP category. We give the statements and omit the proofs after defining related categories of directed graphs.

A $\mathcal{P}$*-graph* is a graph $\mathcal{G} = \langle \mathcal{O}, \mathcal{A}, s, t\rangle$ endowed with a free action of a free proof term algebra $\mathcal{P}$ on $\mathcal{O}$. This means there is a map $\mathcal{P} \times \mathcal{O} \longrightarrow \mathcal{O}$ given by $(t, A) \longmapsto (t : A)$; we consider this map part of the structure of the $\mathcal{P}$-graph.

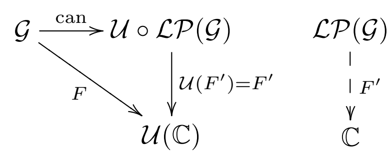
A morphism of $\mathcal{P}$-graphs is a graph morphism $\Phi : \mathcal{G} \to \mathcal{G}'$ together with a morphism (also denoted by Φ) of the proof term algebras of $\mathcal{G}$ and $\mathcal{G}'$ that commutes with the action: $A \in \mathcal{O}$, $\Phi\,(t : A) = (\Phi t : \Phi(A))$. The category of $\mathcal{P}$-graphs is denoted $\mathbf{Grph}_{\mathcal{P}}$.

An LP–*graph* is a $\mathcal{P}$-graph $\mathcal{G}$ in which the set $\mathcal{O}$ of object terms is an object term algebra, and the set $\mathcal{A}$ of arrows of $\mathcal{G}$ contains the CCC and LP arrow terms above. A morphism of LP–graphs is a $\mathcal{P}$–graph morphism that is an object term algebra morphism; i.e., commutes with the CCC and LP arrows, and with the product and implication operations on object terms. An *LP deductive system* is a $\mathcal{P}$-graph generated from an LP-graph $\mathcal{G}$ by closing under the operations of composition, pairing and curry.

Given a proof polynomial term algebra $\mathcal{P}$ and a $\mathcal{P}$–graph $\mathcal{G}$, one can construct the LP deductive system freely generated by $\mathcal{G}$, denoted $\mathcal{D}(\mathcal{G})$. All vertices of $\mathcal{G}$ are object terms, $\top$ is an object term; if A, B are object terms and if t is a proof term, then $A \wedge B$, $A \to B$ and $(t : A)$ are object terms; the arrows of $\mathcal{G}$ and 1_A, $\bigcirc_A$, $\pi^i_{A,B}$, $\varepsilon_{A,B}$, $\mathrm{N}^{s,t}_{A,B}$, $\mathrm{P}^{s,t}_{A,B}$, $\mathrm{L}^{v,t}_{A,B}$, C^t_A, V^t_A, $\bigcirc^c_A$ are arrow terms; and arrow terms are closed under $\circ$, $\langle -, - \rangle$ and $(-)^*$.

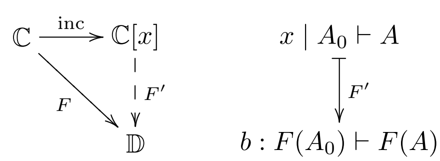
There is a functor $\mathcal{U} : \mathbf{CCC}_{\mathcal{P}} \to \mathbf{Grph}_{\mathcal{P}}$ which sends an LP category to its underlying $\mathcal{P}$-graph.

Proposition 2. *For any LP category $\mathbb{C}$ and any morphism $F : \mathcal{G} \to \mathcal{U}(\mathbb{C})$ of $\mathcal{P}$-graphs, there is a unique LP functor $F' : \mathcal{LP}(\mathcal{G}) \to \mathbb{C}$ in $\mathbf{CCC}_{\mathcal{P}}$ such that the following diagram commutes.*



Given an LP category $\mathbb{C}$, objects A_0, A of $\mathbb{C}$, one can adjoin an "indeterminate" arrow $x \mid A_0 \vdash A$ to $\mathbb{C}$ to form the LP category $\mathbb{C}[x]$, called a *polynomial LP category*. This is an LP category bearing the same relation to $\mathbb{C}$ as the deductive system $\mathcal{D}$ bears to $\mathcal{D}(x)$. One has to ensure that $\mathbb{C}[x]$ is an LP category and that the inclusion $\mathbb{C} \to \mathbb{C}[x]$ is an LP functor. This is done by forming the free LP deductive system generated by the underlying $\mathcal{P}$-graph of $\mathbb{C}$ with x adjoined, and imposing equations on proofs. We omit the details as these are immediate generalizations of those given in [LS86]. We have the following universal property.

Proposition 3. *For any LP category $\mathbb{C}$, an indeterminate $x \mid A_0 \vdash A$ over $\mathbb{C}$, any LP functor $F : \mathbb{C} \to \mathbb{D}$ and any arrow $b \mid F(A_0) \vdash F(A)$ in $\mathbb{D}$, there is a unique LP functor $F' : \mathbb{C}[x] \to \mathbb{D}$ such that $F'(x) = b$ and the following diagram commutes.*



This construction can also be obtained as the Kleisli category of a monad. There is a multivariate generalization, used at several points in the sequel (e.g., in the construction of the internal language $\mathbf{L}(\mathbb{C})$ of an LP–category $\mathbb{C}$, in Prop 6 and in the adjunction of parameters to an LP–typed lambda calculus; see also [LS86, page 61]). We also note that if the LP category $\mathbb{C}$ has a (weak) natural numbers object, the polynomial category $\mathbb{C}[x]$ has the same (weak) natural numbers object; this fact generalizes [LS86, Proposition 9.1].

1.5 Functional completeness in LP categories

Proposition 4 (Functional completeness)**.** *For every polynomial* $\varphi(x) \mid B \vdash C$ *in an indeterminate* $x \mid \top \vdash A$ *over a cartesian, cartesian closed or LP category* $\mathbb{C}$ *there is a unique arrow* $f \mid A \wedge B \vdash C$ *in* $\mathbb{C}$ *such that* $f\langle x\bigcirc_B, 1_B\rangle = \varphi(x)$ *holds in* $\mathbb{C}[x]$*.*

The proof follows that of Lambek and Scott: one verifies by induction on the length of the proof $\varphi(x)$ that $\kappa_{x\in A}\varphi(x)\langle x\bigcirc_B, 1_B\rangle = \varphi(x)$, where $\kappa_{x\in A}\varphi(x)$ is defined as in the proof of the deduction theorem.

Corollary 5. *(i) For every polynomial* $\varphi(x) \mid \top \vdash C$ *in an indeterminate* $x \mid \top \vdash A$ *over a cartesian, cartesian closed or LP category* $\mathbb{C}$*, there is a unique arrow* $g \mid A \vdash C$ *in* $\mathbb{C}$ *such that* $gx = \varphi(x)$ *holds in* $\mathbb{C}[x]$*.*

(ii) Over a cartesian closed or LP category $\mathbb{C}$*, there is a unique arrow* $h \mid \top \vdash A \to C$ *in* $\mathbb{C}$ *such that* $\varepsilon_{C,A}\langle h, x\rangle = \varphi(x)$ *holds in* $\mathbb{C}[x]$*.*

Statement (ii) is the "transposed" version of statement (i). In the definition of lambda abstraction in a typed λ-calculus, Lambek and Scott write $\lambda_{x\in A}\varphi(x)$ for the unique h such that $\varepsilon_{C,A}\langle h, x\rangle = \varphi(x)$. Note that h has type $\top \vdash A \to C$, x has type $\top \vdash A$ and $\varphi(x)$ has type $\top \vdash C$.

1.6 LP–typed lambda calculi

A *LP–typed λ-calculus* is a formal theory with proof terms, types, terms of each type, and equations between terms. We fix a proof term algebra $\mathcal{P}$. The class of types contains two basic types (the second, corresponding to a weak natural numbers object, is optional) and is closed under three operations.

(a1) 1 is a *basic* type.

(a2) $\mathbb{N}$ is a *basic* type.

(a3) If A and B are types, then so are $A \times B$ and B^A.

(a4) If $t \in \mathcal{P}$ is a proof term and A is a type, then $(t : A)$ is a type.

The class of terms is freely generated from variables and atomic constants by term forming operations. The set of variables and constants of an LP-typed lambda calculus is assumed to be disjoint from the proof variables and constants of $\mathcal{P}$. The term forming operations in a typed λ-calculus correspond to distinguished arrow terms (projections, evaluation), operations on arrow terms (pairing) and derived operations (lambda abstraction) in a cartesian closed category. Likewise, an LP-typed λ-calculus has, in addition to the term forming operations of a typed λ-calculus, term forming operations corresponding to the evidence arrow terms of an LP category.

(b1) For each type A there are countably many variables of type A:

$$x_i^A \in A \ (i = 0, 1, 2, \ldots)$$

(b2) $* \in 1$. This means that $*$ is a constant of type 1. In general, the metalinguistic phrase '$a \in A$' means that term a has type A.

(b3) If $a \in A$, $b \in B$ and $c \in A \times B$, then

$$\langle a, b\rangle \in A \times B,\ \pi^1_{A,B}(c) \in A \text{ and } \pi^2_{A,B}(c) \in B.$$

(b4) If $f \in B^A$ and $a \in A$, then $\varepsilon_{B,A}(f, a) \in B$
(b5) If $x \in A$ and $\varphi(x) \in B$ then $\lambda_{x\in A}\varphi(x) \in B^A$

If the the weak natural numbers type $\mathbb{N}$ is included, then we add the following two term formation operations.

(b6) $0 \in \mathbb{N}$; if $n \in \mathbb{N}$, then $S(n) \in \mathbb{N}$
(b7) If $a \in A$, $f \in A^A$ and $n \in \mathbb{N}$, then $I_A(a, f, n) \in A$

Evidence term forming operations ($\mathrm{N}^{s,t}_{A,B}$, $\mathrm{P}^{s,t}_{A,B}$, $\mathrm{L}^{v,t}_{A,B}$, $\mathrm{V}^{!t}_{(t:A)}$, C^t_A and $\bigcirc^c_A$), corresponding to the LP arrow terms of an LP category, are added as follows.

(b8) If $a \in \big((t : B^A) \wedge (s : A)\big)$, then $\mathrm{N}^{s,t}_{A,B}(a) \in (t \cdot s : B)$.
(b9) If $a \in ((s : A) \wedge (t : B))$, then $\mathrm{P}^{s,t}_{A,B}(a) \in (s \,\&\, t : A \wedge B)$
(b10) If $a \in \Big((t : B)^{(v:A)}\Big)$, then $\mathrm{L}^{v,t}_{A,B}(a) \in \big(\overline{\lambda} v.t : B^A\big)$.
(b11) If $a \in (!t : (t : A))$ then $\mathrm{V}^{!t}_{(t:A)}(a) \in (t : A)$.
(b12) If $a \in (t : A)$ then $\mathrm{C}^t_A \in A$.
(b13) If $a \in A$ then $\bigcirc^c_A(a) \in (c : \top)$

As in Lambek and Scott, we write $\varepsilon_{B,A}(f, a)$ (where $f \in B^A$ and $a \in A$) as $f_\bullet a$ (read 'f of a'). The λ-abstraction $\lambda_{x\in A}$ acts like a quantifier: the variable x in $\lambda_{x\in A}\varphi(x)$ is *bound*. The usual conventions for free and bound variables apply. The term a is *substitutable for x in $\varphi(x)$* if no free occurrence of a variable in the term a becomes bound in $\varphi(a)$. (This is a gloss on substitution: we assume we know how to define $\varphi(a)$ inductively.) A term is *closed* if it contains no free variables.

(c2) The binary relation $\underset{X}{=}$ between terms is an equivalence relation, and satisfies the substitution rule that if $X \subseteq Y$ and $a \underset{X}{=} a'$ holds then $a \underset{Y}{=} a'$ holds. This substitution rule, and substitution rules for application and abstraction are shown below [LS86].

$$X \subseteq Y\colon \frac{a \underset{X}{=} a'}{a \underset{Y}{=} a'} \quad \text{Appl:}\ \frac{a \underset{X}{=} a'}{f_\bullet a \underset{X}{=} f_\bullet a'} \quad \text{Abst:}\ \frac{\varphi(x) \underset{X\cup\{x\}}{=} \varphi'(x)}{\lambda_{x\in A}\varphi(x) \underset{X}{=} \lambda_{x\in A}\varphi'(x)}$$

(c3) The following equations hold:
$a \underset{X}{=} *$ for $a \in 1$
$\pi^1_{A,B}(\langle a, b\rangle) \underset{X}{=} a$ for $a \in A$, $b \in B$.
$\pi^2_{A,B}(\langle a, b\rangle) \underset{X}{=} b$ for $a \in A$, $b \in B$.
$\langle \pi^1_{A,B}(c), \pi^2_{A,B}(c)\rangle \underset{X}{=} c$ for $c \in A \times B$.
$\lambda_{x\in A}\varphi(x)_\bullet a \underset{X}{=} \varphi(a)$ for $a \in A$ substitutable for $x \in A$.
$\lambda_{x\in A}(f_\bullet x) \underset{X}{=} f$ for $f \in B^A$, provided x is not in X (so that x has no free occurrence in f).

1.7 The category of LP–typed lambda calculi

1.7.1 The internal language of an LP category

The definition of the internal language of an LP category follows the definition of the internal language of a CCC in Lambek and Scott [LS86]. We assume a fixed proof term algebra $\mathcal{P}$. The *internal language* $\mathbf{L}(\mathbb{C})$ of an LP category $\mathbb{C}$ is defined as follows. The types are the objects of $\mathbb{C}$, where object terms have their intended meanings; e.g., terminal object 1; a weak natural numbers object $\mathbb{N}$ (if it exists); product objects $A \times B$; and exponential objects B^A. In addition, the internal language $\mathbf{L}(\mathbb{C})$ inherits the proof term algebra $\mathcal{P}$ of $\mathbb{C}$, and if t is a proof term and A is a type, then so is $(t : A)$.

Terms of type A are polynomial expressions $\varphi\,(x_1, \ldots, x_n) \mid 1 \vdash A$, obtained from "indeterminate arrow terms" $x_i \mid 1 \vdash A_i$, where x_i is a variable of type A_i, and "constant arrow terms" $1 \vdash A$ in $\mathbb{C}$ by the following term forming operations.

$$\frac{a \mid 1 \vdash A \qquad b \mid 1 \vdash B}{\langle a, b\rangle \mid 1 \vdash A \times B} \qquad \frac{a \mid 1 \vdash A}{fa \mid 1 \vdash B} \qquad \frac{\varphi(x) \mid 1 \vdash B}{\lambda_{x \in A}\varphi(x) \mid 1 \vdash B^A}$$

where $f \mid A \vdash B$ and

$$\lambda_{x \in A}\varphi(x) \equiv \ulcorner \kappa_{x \in A}\varphi(x)\langle 1_A, \bigcirc_A\rangle \urcorner$$

where $\kappa_{x \in A}$ is defined in the proof of the deduction theorem.

Note that a term $\varphi\,(x_1, \ldots, x_n)$ of type A in the internal language $\mathbf{L}(\mathbb{C})$ is an arrow $\varphi\,(x_1, \ldots, x_n) \mid 1 \vdash A$ in the polynomial category $\mathbb{C}[x_1, \ldots, x_n]$.

Also note that the middle term forming rule above subsumes the term forming operations corresponding to the distinguished arrows of a CCC and LP category. If one were to write down the signature of the internal language, then it is understood that the term forming operations corresponding to the distinguished arrows of an LP category (including terminal arrow, projections, evaluation and evidence arrow terms), would be named in the signature (as well as those corresponding to operation on arrows and derived operations), as for an LP-typed lambda calculus.

If a, b are polynomial expressions (terms) with variables from the set X, then the equation

$$a \underset{X}{=} b$$

holds in $\mathbf{L}(\mathbb{C})$ provided the equation holds for polynomials in the (multivariate) polynomial LP category $\mathbb{C}[X]$. The internal language $\mathbf{L}(\mathbb{C})$ is an LP-typed lambda calculus.

1.7.2 Morphisms of LP-typed lambda calculi: translations

A morphism $\Phi : \mathcal{L} \to \mathcal{L}'$ of LP-typed lambda calculi is a translation of the underlying λ-calculi that commutes with the proof-term algebra structure and the evidence term forming operations. In detail:

(d1) Φ is a morphism of the proof term algebras $\mathcal{P}$ (and $\mathcal{P}'$) of $\mathcal{L}$ (and $\mathcal{L}'$). Φ sends types (terms) of $\mathcal{L}$ to type (terms) of $\mathcal{L}'$ so that if $a \in A$, then $\Phi(a) \in \Phi(A)$. Also, if a is a closed term, then so is $\Phi(a)$ and if x_i is the i-th variable of type A, then $\Phi(x_i)$ is the i-th variable of type $\Phi(A)$.

(d2) Φ preserves type operations; e.g., $\Phi(1) = 1$, $\Phi(A \times B) = \Phi(A) \times \Phi(B)$, $\Phi(A^B) = \Phi(A)^{\Phi(B)}$, and Φ commutes with the action of $\mathcal{P}$: if $t \in \mathcal{P}$ is a proof term, if A is a type, then $\Phi\,(t : A) = (\Phi(t) : \Phi(A))$. Also, Φ preserves λ-calculus term forming operations up to equality in $\mathcal{L}'$:

$$\Phi\left(\lambda_{x \in A}\varphi(x)_{\bullet} a\right) \underset{\Phi(X)}{=} \Phi(\varphi(a))$$

for $a \in A$ substitutable for $x \in A$;

$$\Phi\left(\lambda_{x \in A}\left(f_{\bullet} x\right)\right) \underset{\Phi(X)}{=} \Phi(f)$$

for $f \in B^A$, provided x is not in X (and so $\Phi(x)$ is not in $\Phi(X)$); etc.

(d3) Φ preserves the evidence term forming operations (b8)-(b13) up to equality in $\mathcal{L}'$ and preserves the types of evidence terms on the nose. Namely, the following equations hold in $\mathcal{L}'$.

$$\begin{aligned}
\Phi\left(\mathrm{N}^{s,t}_{A,B}(a)\right) &\underset{\Phi(X)}{=} \mathrm{N}^{\Phi(s),\Phi(t)}_{\Phi(A),\Phi(B)}(\Phi(a)) \in (\Phi(t)\cdot\Phi(s) : \Phi(B)),\\
&\text{for}\quad a \in \left(\left(t : B^A\right) \wedge (s : A)\right);\\
\Phi\left(\mathrm{P}^{s,t}_{A,B}(a)\right) &\underset{\Phi(X)}{=} \mathrm{P}^{\Phi(s),\Phi(t)}_{\Phi(A),\Phi(B)}(\Phi(a)) \in (\Phi(s)\,\&\,\Phi(t) : \Phi(A) \wedge \Phi(B)),\\
&\text{for } a \in ((s : A) \wedge (t : B));\\
\Phi\left(\mathrm{L}^{v,t}_{A,B}(a)\right) &\underset{\Phi(X)}{=} \mathrm{L}^{\Phi v,\Phi t}_{\Phi A,\Phi B}(\Phi(a)) \in \left(\overline{\lambda}\Phi v.\Phi t : \Phi B^{\Phi A}\right),\ \text{for } a \in (t : B)^{(v:A)};\\
\Phi\left(\mathrm{V}^{!t}_{(t:A)}(a)\right) &\underset{\Phi(X)}{=} \mathrm{V}^{!\Phi t}_{(\Phi t:\Phi A)}(\Phi(a)) \in (\Phi t : \Phi A),\ \text{for } a \in (!t : (t : A));\\
\Phi\left(\mathrm{C}^{t}_{A}(a)\right) &\underset{\Phi(X)}{=} \mathrm{C}^{\Phi t}_{\Phi A}(\Phi(a)) \in \Phi A,\ \text{for } a \in (t : A);\\
\Phi\left(\bigcirc^{c}_{A}(a)\right) &\underset{\Phi(X)}{=} \bigcirc^{\Phi c}_{\Phi A}(\Phi(a)) \in (\Phi c : \top),\ \text{for } a \in A.
\end{aligned}$$

(d4) Φ preserves equations in $\mathcal{L}$: If $a \underset{X}{=} b$ holds in $\mathcal{L}$, then $\Phi(a) \underset{\Phi(X)}{=} \Phi(b)$ holds in $\mathcal{L}'$.

A morphism of LP–typed lambda calculi is called a translation. Two such translations $\Phi, \Psi : \mathcal{L} \to \mathcal{L}'$ are equal if they preserve equations: $\Phi = \Psi$ if and only if $\Phi(a) \underset{\Phi(X)}{=} \Psi(b)$ holds in $\mathcal{L}'$ whenever $a \underset{X}{=} b$ holds in $\mathcal{L}$.

The LP–typed lambda calculi and translations form a category, denoted λ-$\mathbf{Calc}_{\mathcal{P}}$.

Directly generalizing Lambek and Scott's definition of the internal language functor on a cartesian closed category, we define a functor $\mathbf{L} : \mathbf{CCC}_{\mathcal{P}} \to \lambda$-$\mathbf{Calc}_{\mathcal{P}}$ from the category of LP–categories (with weak natural numbers object) to the category of LP–typed lambda calculi (with weak natural number object type) [LS86, Proposition 10.7, page 76].

Proposition 6. *Let* $\mathbf{L}(\mathbb{C})$ *be the internal language of the LP category* $\mathbb{C}$*. For any morphism* $F : \mathbb{C} \to \mathbb{C}'$ *of LP categories, let* $\mathbf{L}(F)$ *be defined by* $\mathbf{L}(F)(A) = F(A)$, $\mathbf{L}(F)(x_i) = x'_i$, $\mathbf{L}(F)(\varphi(X)) = F_X(\varphi(X))$*, where* x'_i *is the* i*-th variable of type* $F(A)$ *and* F_X *is the unique arrow in* $\mathbf{CCC}_{\mathcal{P}}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{C}' \\ \downarrow & & \downarrow \\ \mathbb{C}[X] & \overset{F_X}{\dashrightarrow} & \mathbb{C}[\mathbf{L}(F)(X)] \end{array}$$

Then $\mathbf{L}$ *is a functor from* $\mathbf{CCC}_{\mathcal{P}}$ *to* λ-$\mathbf{Calc}_{\mathcal{P}}$*.*

1.8 The functor $\mathbf{C} : \lambda\text{-}\mathbf{Calc}_{\mathcal{P}} \to \mathbf{CCC}_{\mathcal{P}}$

The functor $\mathbf{C} : \lambda\text{-}\mathbf{Calc}_{\mathcal{P}} \to \mathbf{CCC}_{\mathcal{P}}$ opposite to the functor $\mathbf{L}$ assigns to an LP-typed lambda calculus $\mathcal{L}$ an LP-category $\mathbf{C}(\mathcal{L})$ whose objects are the types of $\mathcal{L}$ and whose arrows $A \to B$ are equivalence classes of pairs $(x \in A, \varphi(x))$, where x is a variable of type A, and where $\varphi(x)$ is a term of $\mathcal{L}$ of type B with no free variables, except possibly for the variable x. Two arrows $A \to B$ in $\mathbf{C}(\mathcal{L})$ satisfy $(x \in A, \varphi(x)) = (y \in A, \psi(y))$ if and only if the equation $\varphi(x) \underset{\{x\}}{=} \psi(x)$ holds in $\mathcal{L}$. Identity arrows, composition of arrows and the cartesian closed structure of $\mathbf{C}(\mathcal{L})$ is defined as in [LS86]. If $\mathcal{L}$ contains a natural number type $\mathbb{N}$, $\mathbf{C}(\mathcal{L})$ contains a weak natural numbers object, also denoted $\mathbb{N}$, as given in [LS86]. In addition to the cartesian closed structure, $\mathbf{C}(\mathcal{L})$ inherits the proof term algebra $\mathcal{P}$ of $\mathcal{L}$. The action of $\mathcal{P}$ on objects of $\mathbf{C}(\mathcal{L})$ is defined as it is in $\mathcal{L}$. LP arrow terms of $\mathbf{C}(\mathcal{L})$ are defined from the evidence terms of $\mathcal{L}$, as follows.

$$\begin{aligned}
\mathrm{N}^{s,t}_{A,B} &:= \Big(x \in \big(\big(t : B^A\big) \wedge (s : A)\big)\,,\, \mathrm{N}^{s,t}_{A,B}(x) \in (t \cdot s : B)\Big) \\
\mathrm{P}^{s,t}_{A,B} &:= \Big(x \in ((s : A) \wedge (t : B))\,,\, \mathrm{P}^{s,t}_{A,B}(x) \in (s \,\&\, t : A \wedge B)\Big) \\
\mathrm{L}^{v,t}_{A,B} &:= \Big(x \in \Big((t : B)^{(v:A)}\Big),\, \mathrm{L}^{v,t}_{A,B}(x) \in \big(\overline{\lambda} v.t : B^A\big)\Big) \\
\mathrm{V}^{!t}_{(t:A)} &:= \Big(x \in (!t : (t : A)),\, \mathrm{V}^{!t}_{(t:A)}(x) \in (t : A)\Big) \\
\mathrm{C}^{t}_{A} &:= \big(x \in (t : A),\, \mathrm{C}^{t}_{A}(x) \in A\big) \\
\bigcirc^{c}_{A} &:= (x \in A,\, \bigcirc^{c}_{A}(x) \in (c : \top))
\end{aligned}$$

To a translation $\Phi : \mathcal{L} \to \mathcal{L}'$ of LP-typed lambda calculi, the corresponding LP-functor $\mathbf{C}(\Phi) : \mathbf{C}(\mathcal{L}) \to \mathbf{C}(\mathcal{L}')$ follows the definition of [LS86] exactly. If A is an object of $\mathbf{C}(\mathcal{L})$, it is a type of $\mathcal{L}$, and we set $\mathbf{C}(\Phi)(A) = \Phi(A)$, which is a type of $\mathcal{L}'$, and hence an object of $\mathbf{C}(\mathcal{L}')$. If $f = (x \in A,\, \varphi(a)) : A \to B$ is an arrow in $\mathbf{C}(\mathcal{L})$, then we set $\mathbf{C}(\Phi)(f) = (x \in \Phi(A),\, \Phi(\varphi(x))) : \Phi(A) \to \Phi(B)$. Since Φ is a translation of LP-typed lambda calculi, $\mathbf{C}(\Phi)$ preserves the structure of an LP-category. This makes the $\mathbf{C}$ construction into a functor.

2 Results

An analog of the LP internalization theorem holds in an LP deductive calculus.

Proposition 7 (Internalization)**.** *If $\varphi \mid A_1 \wedge \cdots \wedge A_n \vdash B$ holds in an LP deductive calculus $\mathcal{D}$, then there is a proof φ', proof variables $x_1, \ldots, x_n$ and a proof polynomial $t\,(x_1, \ldots, x_n)$ depending on φ and $x_1, \ldots, x_n$ such that*

$$\varphi' \mid (x_1 : A_1) \wedge \cdots \wedge (x_n : A_n) \vdash (t\,(x_1, \ldots, x_n) : B)$$

holds in $\mathcal{D}$.

Theorem 8. *The functors $\mathbf{L} : \mathbf{CCC}_{\mathcal{P}} \to \lambda\text{-}\mathbf{Calc}_{\mathcal{P}}$ and $\mathbf{C} : \lambda\text{-}\mathbf{Calc}_{\mathcal{P}} \to \mathbf{CCC}_{\mathcal{P}}$ define an equivalence of categories.*

3 Proofs

3.1 Proof of internalization

Proof of Proposition 7. If φ is an axiom, there are several cases. If $\varphi = 1_A$ for some object A, take

$$\varphi' = 1_{(x:A)} \mid (x : A) \vdash (x : A)\,.$$

The cases of $\varphi = \pi^i_{A,B}$ for $i = 1, 2$ are handled similarly.

If $\varphi = \bigcirc_A \mid A \vdash \top$, take

$$\varphi' = \bigcirc^c_{(x:A)} \mid (x : A) \vdash (c : \top)$$

for some proof constant c and for some proof variable x.

If $\varphi = \varepsilon_{A,B} \mid (B \to A) \wedge B \vdash A$, take

$$\varphi' = \mathrm{N}^{x,y}_{B,A} \mid (y : (B \to A)) \wedge (x : B) \vdash (y \cdot x : A)$$

for proof variables x and y.

If $\varphi = \mathrm{V}^t_A \mid (t : A) \vdash A$, take

$$\varphi' = \mathrm{V}^x_{(t:A)} \mid (x : (t : A)) \vdash (t : A),$$

where x is a proof variable not occurring in the proof term t.

If $\varphi = \bigcirc^c_A \mid A \vdash (c : \top)$, take

$$\varphi' = \mathrm{C}^c_A \circ \bigcirc^c_A \circ \mathrm{V}^x_A \mid (x : A) \vdash (!c : (c : \top))\,.$$

If φ is an evidence arrow term of the form $\varphi \mid A_1 \wedge \cdots \wedge A_n \vdash (t : B)$, then proceed as follows. Repeated applications (in the correct order) of the following derived rule

$$\frac{\mathrm{V}^{x_i}_{A_i} \mid (x_i : A_i) \vdash A_i \qquad \mathrm{V}^{x_j}_{A_j} \mid (x_j : A_j) \vdash A_j}{\langle \mathrm{V}^{x_i}_{A_i} \pi^1_{(x_i:A_i),(x_j:A_j)}, \mathrm{V}^{x_j}_{A_j} \pi^2_{(x_i:A_i),(x_j:A_j)} \rangle \mid (x_i : A_i) \wedge (x_j : A_j) \vdash A_i \wedge A_j}$$

yields an arrow term $f \mid (x_1 : A_1) \wedge \cdots \wedge (x_n : A_n) \vdash A_1 \wedge \cdots \wedge A_n$ (additional parentheses in the correct order are not shown).

Take

$$\varphi' = \mathrm{C}^t_B \circ \varphi \circ f \mid (x_1 : A_1) \wedge \cdots \wedge (x_n : A_n) \vdash (!t : (t : B))\,.$$

For the cases where φ is obtained from other arrow terms by one of the operations on arrow terms, we will assume that in $\varphi \mid A_1 \wedge \cdots \wedge A_n \vdash B$ that $n = 1$ as the general case is similar.

If φ has the form $\langle f', g' \rangle \mid Z \vdash X \wedge Y$, then by induction there exist proofs f and g as in the proof tree below, the last line of which has the required form.

$$\dfrac{\dfrac{f \mid (x : Z) \vdash (s(x) : X) \qquad g \mid (x : Z) \vdash (t(x) : Y)}{\langle f, g \rangle \mid (x : Z) \vdash (s(x) : X) \wedge (t(x) : Y)} \qquad \mathrm{P}^{s(x),t(x)}_{X,Y}}{\mathrm{P}^{s(x),t(x)}_{X,Y} \circ \langle f, g \rangle \mid Z \vdash (s(x) \,\&\, t(x) : (X \wedge Y))}$$

where

$$\mathrm{P}^{s(x),t(x)}_{X,Y} \mid (s(x) : X) \wedge (t(x) : Y) \vdash ((s(x) \,\&\, t(x)) : (X \wedge Y))$$

If $\varphi = g^* \mid X \vdash Y \to Z$ for some proof g, by induction there exists a proof f and a proof terml t as in the proof tree below, the last line of which has the required form.

$$\dfrac{\dfrac{f \mid (x : X) \wedge (y : Y) \vdash (t(x,y) : Z)}{f^* \mid (x : X) \vdash (y : Y) \to (t(x,y) : Z)} \qquad \mathrm{L}^{y,t(x,y)}_{Y,Z}}{\mathrm{L}^{y,t(x,y)}_{Y,Z} \circ f^* \mid (x : X) \vdash \big(\overline{\lambda} y.t(x,y) : (Y \to Z)\big)}$$

where

$$\mathrm{L}^{y,t(x,y)}_{Y,Z} \mid (y : Y) \to (t(x,y) : Z) \vdash \big(\overline{\lambda} y.t(x,y) : (Y \to Z)\big)$$

If $\varphi = g' f'$ for proofs $f' \mid X \vdash Y$ and $g' \mid Y \vdash Z$, then by induction there exist proofs $f \mid (x : X) \vdash (s(x) : Y)$ and $g \mid (y : Y) \vdash (t(y) : Z)$. Adjoin the assumption $\zeta \mid \top \vdash (x : X)$ to $\mathcal{D}$; we reason in $\mathcal{D}(\zeta)$ and eliminate ζ, shown in square brackets, using the deduction theorem.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{g \mid (y : Y) \vdash (t(y) : Z)}{\ulcorner g \urcorner \mid \top \vdash (y : Y) \to (t(y) : Z)} \qquad \mathrm{L}^{y,t(y)}_{Y,Z}}{\mathrm{L}^{y,t(y)}_{Y,Z} \circ \ulcorner g \urcorner \mid \top \vdash \big(\overline{\lambda} y.t(y) : (Y \to Z)\big)} \qquad \dfrac{[\zeta] \qquad f}{f \circ \zeta \mid \top \vdash (s(x) : Y)}}{\langle \mathrm{L}^{y,t(y)}_{Y,Z} \circ \ulcorner g \urcorner, f \circ \zeta \rangle \mid \top \vdash \big(\overline{\lambda} y.t(y) : (Y \to Z)\big) \wedge (s(x) : Y)}}{\mathrm{N}^{s(x),\overline{\lambda} y.t(y)}_{Z,Y} \circ \langle \mathrm{L}^{y,t(y)}_{Y,Z} \circ \ulcorner g \urcorner, f \circ \zeta \rangle \mid \top \vdash \big(\overline{\lambda} y.t(y) \cdot s(x) : Z\big)}}{\kappa_{\zeta \in (x:X)} \Big(\mathrm{N}^{s(x),\overline{\lambda} y.t(y)}_{Z,Y} \langle \mathrm{L}^{y,t(y)}_{Y,Z} \ulcorner g \urcorner, f\zeta \rangle\Big) \mid (x : X) \wedge \top \vdash \big(\overline{\lambda} y.t(y) \cdot s(x) : Z\big)}}{\big(\kappa_{\zeta \in (x:X)} (\cdots)\big) \circ \langle 1_{(x:X)}, \bigcirc_{(x:X)} \rangle \mid (x : X) \vdash \big(\overline{\lambda} y.t(y) \cdot s(x) : Z\big)}$$

□

3.2 Proof of the equivalence $\mathbf{L} : \mathbf{CCC}_{\mathcal{P}} \leftrightarrow \lambda\text{-}\mathbf{Calc}_{\mathcal{P}} : \mathbf{C}$

Proof of Theorem 8. This follows [LS86] exactly. We define two natural isomorphisms of functors

$$\varepsilon : \mathbf{CL} \to 1_{\mathbf{CCC}_{\mathcal{P}}} \quad \text{and} \quad \eta : 1_{\lambda\text{-}\mathbf{Calc}_{\mathcal{P}}} \to \mathbf{LC}.$$

The component $\eta_{\mathcal{L}} : \mathcal{L} \to \mathbf{LC}(\mathcal{L})$ of η at the LP-typed lambda calculus $\mathcal{L}$ is the translation given as follows. For a type $A \in \mathcal{L}$, we set $\eta_{\mathcal{L}}(A) := A$. For a term $\varphi\,(x_1, \ldots, x_n)$ of type B, we set

$$\eta_{\mathcal{L}}(\varphi\,(x_1, \ldots, x_n)) := (z \in 1,\, \varphi\,(x_1, \ldots, x_n)) : 1 \to B$$

in $\mathbf{C}(\mathcal{L}\,(x_1, \ldots, x_n))$. The construction $\mathcal{L}\,(x_1, \ldots, x_n)$ requires some explanation: this is the LP-typed lambda calculus with parameters $x_1, \ldots, x_n$ adjoined. This LP-typed lambda calculus has the same types, terms and term forming operations as $\mathcal{L}$, but $x_1, \ldots, x_n$ are are treated as if they were no longer variables. Closed terms of $\mathcal{L}\,(x_1, \ldots, x_n)$ are terms of $\mathcal{L}$ in which there are no free variables other than $x_1, \ldots, x_n$. Equality $\underset{X}{=}$ in $\mathcal{L}\,(x_1, \ldots, x_n)$ means $\underset{X \cup Y}{=}$ in $\mathcal{L}$, where $Y := \{x_1, \ldots, x_n\}$ and $X \cap Y = \emptyset$. This construction is the analog the polynomial LP (cartesian closed) category $\mathbb{A}[x_1, \ldots, x_n]$; the universal property of the polynomial LP category yields the isomorphism $\mathbf{C}(\mathcal{L})[x_1, \ldots, x_n] \simeq \mathbf{C}(\mathcal{L}\,(x_1, \ldots, x_n)))$ [LS86, Theorem 11.2].

The arrow $(z \in 1,\, \varphi\,(x_1, \ldots, x_n)) : 1 \to B$ in $\mathbf{C}(\mathcal{L}\,(x_1, \ldots, x_n))$ comes from the closed term $\varphi\,(x_1, \ldots, x_n)$ in $\mathcal{L}\,(x_1, \ldots, x_n)$; the variable $z \in 1$ is taken to be different from $x_1, \ldots, x_n$. An arrow of the form $1 \to B$ in $\mathbf{C}(\mathcal{L}\,(x_1, \ldots, x_n))$ is by definition a term of type B in $\mathbf{LC}(\mathcal{L}\,(x_1, \ldots, x_n))$. Under the isomorphism

$$\mathbf{LC}(\mathcal{L}\,(x_1, \ldots, x_n)) \simeq \mathbf{L}\,(\mathbf{C}(\mathcal{L})[x_1, \ldots, x_n])$$

it is a term $\varphi\,(x_1, \ldots, x_n)$ of type B in the internal language $\mathbf{LC}(\mathcal{L})$, which is by definition a polynomial $(z \in 1,\, \varphi\,(x_1, \ldots, x_n)) : 1 \to B$ in $\mathbf{C}(\mathcal{L})[x_1, \ldots, x_n]$.

The inverse map is given on terms by $\eta_{\mathcal{L}}^{-1}((z \in 1,\, \varphi(z))) := \varphi(*)$.

Let $\Phi : \mathcal{L} \to \mathcal{L}'$ be a translation. The following diagram commutes.

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\eta_{\mathcal{L}}} & \mathbf{LC}(\mathcal{L}) \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \mathbf{LC}(\Phi)} \\ \mathcal{L}' & \xrightarrow[\eta_{\mathcal{L}'}]{} & \mathbf{LC}(\mathcal{L}') \end{array}$$

The component $\varepsilon_{\mathbb{A}} : \mathbf{CL}(\mathbb{A}) \to \mathbb{A}$ of ε at the LP category $\mathbb{A}$ is an LP functor, defined as follows. For an object A of $\mathbf{CL}(\mathbb{A})$, which is a type of $\mathbf{L}(\mathbb{A})$ and hence an object of A, we set $\varepsilon_{\mathbb{A}}(A) = A$.

An arrow in $\mathbf{CL}(\mathbb{A})$ has the form $f = (x \in B,\, x \in \varphi(x)) : B \to C$, where $\varphi(x)$ has type C in $\mathbf{L}(\mathbb{A})$, and where $\varphi(x)$ has at most the variable x of type B free. By definition, the term $\varphi(x)$ in $\mathbf{L}(\mathbb{A})$ is a polynomial $\varphi(x) \mid 1 \vdash C$ in the indeterminate $x \mid \top \vdash B$; i.e., t is an arrow in the polynomial category $\mathbb{A}[x]$. By functional completeness (Corollary 5, (i)), there is a unique arrow $g : B \to C$ in $\mathbb{A}$ such that the equation $gx = \varphi(x)$ holds in $\mathbb{A}[x]$. Set $\varepsilon_{\mathbb{A}}(f) := g$.

Let $F : \mathbb{A} \to \mathbb{A}'$ be an LP-functor. The following diagram commutes.

$$\begin{array}{ccc} \mathbf{CL}(\mathbb{A}) & \xrightarrow{\varepsilon_{\mathbb{A}}} & \mathbb{A} \\ {\scriptstyle \mathbf{CL}(F)}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbf{CL}(\mathbb{A}') & \xrightarrow[\varepsilon_{\mathbb{A}'}]{} & \mathbb{A}' \end{array}$$

Functional completeness implies that $\varepsilon_{\mathbb{A}}$ induces an isomorphism

$$\mathbf{CL}(\mathbb{A})(B, C) = \mathbf{CL}(\mathbb{A})(\mathbf{CL}(B), \mathbf{CL}(B)) \simeq \mathbb{A}(B, C).$$

Hence ε is a natural isomorphism.

□

3.3 Proof of the corollary to functional completeness

Proof of Corollary 5 (i). Functional completeness implies that given

$$\varphi(x) \mid \top \vdash C,$$

there is a unique $f \mid A \wedge \top \vdash C$ in $\mathbb{C}$ with $f\langle x\bigcirc_{\top}, 1_{\top}\rangle = \varphi(x)$. Define

$$g = f\langle 1_A, \bigcirc_A\rangle \mid A \vdash C.$$

For existence, calculate:

$$gx = f\langle x, \bigcirc_A x\rangle = f\langle x\bigcirc_{\top}, \bigcirc_{\top}\rangle = f\langle x\bigcirc_{\top}, 1_{\top}\rangle = \varphi(x).$$

For uniqueness, suppose that $h \mid A \vdash C$ satisifies $hx = \varphi(x)$. Then $h\pi^1_{A,\top} \mid A \wedge \top \vdash C$ would satisfy

$$h\pi^1_{A,\top}\langle x\bigcirc_{\top}, 1_{\top}\rangle = hx\bigcirc_{\top} = hx = \varphi(x).$$

Uniqueness implies that $h\pi^1_{A,\top} = f$. But then

$$g = f\langle 1_A, \bigcirc_A\rangle = h\pi^1_{A,\top}\langle 1_A, \bigcirc_A\rangle = h1_A = h.$$

□

Proof of Corollary 5 (ii). Define $h := \ulcorner g \urcorner \mid \top \vdash A \to C$. Then

$$\varepsilon_{C,A}\langle h, x\rangle = h_{\bullet}x = \ulcorner g \urcorner_{\bullet} x = gx = \varphi(x).$$

□

3.4 Proof of the deduction theorem

This proof is identical to that of Lambek and Scott, except for the inclusion of proof trees. It is included for the convenience of the reader.

The arrow $f \mid A \wedge B \vdash C$ will be defined inductively. We will write $f = \kappa_{x\in A}\phi(x)$ to suggest an abstraction operation from $\mathcal{D}(x)$ to $\mathcal{D}$, in which the "variable x of type A" in $\phi(x)$ becomes "bound."

Proof of Theorem 1. Case (i): $\varphi(x) = k \mid B \vdash C$ in $\mathcal{D}$. Set $\kappa_{x\in k} = k\pi^2_{A,B}$. This case subsumes the axiom arrow terms.

Case (ii): $\varphi(x) = x \mid \top \vdash A$. Set $\kappa_{x\in A}x = \pi^1_{A,\top} \mid A \wedge \top \vdash A$. (The point is that this is an arrow of $\mathcal{D}$.)

The remaining cases concern arrows arising from operations.

Case (iii): $\varphi(x) = \langle f(x), g(x)\rangle \mid B \vdash C \wedge D$, where $f(x) \mid B \vdash C$, $g(x) \mid B \vdash D$. Set $\kappa_{x\in A}\langle f(x), g(x)\rangle = \langle \kappa_{x\in A}f(x), \kappa_{x\in A}g(x)\rangle$.

Case (iv): $\varphi(x) = g(x)f(x)$, where $f(x) \mid B \vdash D$ and $g(x) \mid D \vdash C$. Set

$$\kappa_{x\in A}\left(g(x)f(x)\right) = \kappa_{x\in A}g(x)\langle \pi^1_{A,B}, \kappa_{x\in A}f(x)\rangle$$

Here the composition on the left is in $\mathcal{D}(x)$, whereas the composition on the right is in $\mathcal{D}$. Illustration: the proof

$$\frac{f(x) \mid B \vdash D \qquad g(x) \mid D \vdash C}{g(x)f(x) \mid B \vdash C}$$

in $\mathcal{D}(x)$ transforms to the following proof in $\mathcal{D}$.

$$\frac{\dfrac{\pi^1_{A,B} \mid A \wedge B \vdash A \qquad \kappa_{x\in A}f(x) \mid A \wedge B \vdash D}{\langle \pi^1_{A,B}, \kappa_{x\in A}f(x)\rangle \mid A \wedge B \vdash A \wedge D} \qquad \kappa_{x\in A}g(x) \mid A \wedge D \vdash C}{\kappa_{x\in A}g(x)\langle \pi^1_{A,B}, \kappa_{x\in A}f(x)\rangle \mid A \wedge B \vdash C}$$

Case (v): $\varphi(x) = f(x)^* \mid B \vdash D \to E$ (here the codomain C of $\varphi(x)$ is $D \to E$). Set $\kappa_{x\in A}\big(f(x)^*\big) = (\kappa_{x\in A}f(x)\alpha_{A,B,D})^*$, where

$$\alpha_{A,B,D} = \langle \pi^1_{A,B}\pi^1_{A\wedge B,D}, \langle \pi^2_{A,B}\pi^1_{A\wedge B,D}, \pi^2_{A\wedge B,D}\rangle\rangle \mid (A \wedge B) \wedge D \vdash A \wedge (B \wedge D)\,.$$

We illustrate how the proof in $\mathcal{D}(x)$ transforms to one in $\mathcal{D}$. In $\mathcal{D}(x)$, $\varphi(x)$ was obtained from an arrow $f(x) \mid B \wedge D \vdash E$ by curry.

$$\frac{f(x) \mid B \wedge D \vdash E}{f(x)^* \mid B \vdash D \to E}$$

This becomes a proof in $\mathcal{D}$ as follows.

$$\frac{\dfrac{\alpha_{A,B,D} \mid (A \wedge B) \wedge D \vdash A \wedge (B \wedge D) \qquad \kappa_{x\in A}f(x) \mid A \wedge (B \wedge D) \vdash E}{\kappa_{x\in A}\left(f(x)\right)\alpha_{A,B,D} \mid (A \wedge B) \wedge D \vdash E}}{\left(\kappa_{x\in A}\left(f(x)\right)\alpha_{A,B,D}\right)^* \mid A \wedge B \vdash D \to E}$$

□

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