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Abstract

An algorithmic approach to the differential Galois theory of second-order linear differential equations with differential parameters

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Advisor: Alexey Ovchinnikov

We present algorithms to compute the differential Galois group G associated via the parameterized Picard-Vessiot theory to a parameterized second-order linear differential equation

$$\frac{\partial^2}{\partial x^2}Y + r_1\frac{\partial}{\partial x}Y + r_0Y = 0,$$

where the coefficients r_1, r_0 belong to the field of rational functions $F(x)$ over a Π-field F, and the finite set of commuting derivations Π is thought of as consisting of derivations with respect to parameters. We build on an earlier procedure, developed by Dreyfus, that computes G when $r_1 = 0$, assuming either that G is reductive with unipotent radical $R_u(G) = 0$, or else that the maximal reductive quotient $G/R_u(G)$ of G is Π-constant. We

first show how to modify the space of parametric derivations to reduce the computation of the unipotent radical $R_u(G)$ to the case when the reductive quotient $G/R_u(G)$ is Π-constant, provided that the unimodularity condition $r_1 = 0$ holds. When $r_1 \neq 0$, we reinterpret a classical change-of-variables procedure in Galois-theoretic terms in order to reduce the computation of G to the computation of an associated unimodular differential Galois group H. We establish a parameterized version of the Kolchin-Ostrowski theorem and apply it to give more direct proofs than those found in the literature of the fact that the required computations can be performed effectively. We then extract from these algorithms a complete set of criteria to decide whether any of the solutions to a parameterized second-order linear differential equation is Π-transcendental over the underlying Π-field of $F(x)$. We give various examples of computation and some applications to differential transcendence.

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Contents

Introduction

Consider a linear differential equation

$$\delta_x^n Y + \sum_{i=0}^{n-1} r_i \delta_x^i Y = 0 \tag{0.1}$$

whose coefficients $r_i \in K = F(x)$ are rational functions in x with coefficients in a Π-field F, δ_x denotes the derivation with respect to x, and $\Pi = \{\partial_1, \ldots, \partial_m\}$ is a finite set of pairwise commuting derivations, which we think of as derivations with respect to parameters. Letting $\Delta := \{\delta_x\} \cup \Pi$, we consider K as a Δ-field by setting $\partial_j x = 0$ for each $1 \leq j \leq m$.

The parameterized Picard-Vessiot (PPV) theory developed by Cassidy and Singer in [9] associates a differential Galois group G (or PPV group) to (0.1), in analogy with the classical (or non-parameterized) Picard-Vessiot (PV) theory hinted at by Picard and Vessiot towards the end of the nineteenth century, and put on a firm modern footing by Kolchin [31] in the middle of the twentieth. The theory of [9] is a special case of the generalization of Kolchin's strongly normal differential Galois theory [32] to the parameterized setting, which was initiated by Landesman in [38].

This PPV group G is defined as the group of differential field automorphisms over K of the PPV extension M generated over K by the solutions to (0.1), together with all their derivatives with respect to Δ. It is shown in [9] that G admits a structure of linear differential algebraic group (LDAG). These groups, whose study was pioneered by Cassidy in [7], are the differential-algebraic analogues of linear algebraic groups: they are defined as subgroups of $\mathrm{GL}_n(F)$ by the vanishing of a system of Π-algebraic differential equations over F in the matrix entries. The PPV group G in this Galois theory encodes in its differential-algebraic structure the differential-algebraic relations amongst the solutions to (0.1).

In retrospect, the classical PV theory corresponds to the special case of the parameterized theory where the set or parameters $\Pi = \emptyset$ is empty. The first general algorithm to compute the PV group of (0.1) for equations of order $n = 2$ is due to Kovacic [37]. The first complete algorithm to compute the PV group of (0.1) for arbitrary order n was developed by Hrushovski [28]. Relying on the classification of the differential algebraic subgroups of $\mathrm{SL}_2(F)$ obtained by Sit in [54], and on Kovacic's algorithm [37] to compute the Liouvillian solutions of (0.2) (when they exist), Dreyfus has recently developed algorithms [19] to compute the PPV group H associated to

$$\delta_x^2 Y - qY = 0, \tag{0.2}$$

where $q \in K$. The effectivity of the procedures of [19] depends on the assumption that either H is reductive, or else the maximal reductive quotient $H/R_u(H)$ is Π-constant. These restrictions recur in the recent algorithms developed by Minchenko, Ovchinnikov, and Singer [41, 42] to compute the PPV group G associated to (0.1) whenever G is reductive or has Π-constant maximal reductive quotient $G/R_u(G)$. The algorithms of [41, 42] rely on Hrushovski's algorithm [28] to compute PV groups, together with the method of prolongations studied by Ovchinnikov in his approach to the PPV theory and the representation theory of LDAGs via differential Tannakian categories, initiated in [46–48], which has also been applied in [22] in the study of isomonodromy, and in [21] to recast and generalize the PPV theory.

In the case of a single parametric derivation $\Pi = \{\partial\}$, the computation of the PPV group of

$$\delta_x^2 Y + r_1 \delta_x Y + r_0 Y = 0 \tag{0.3}$$

for arbitrary $r_1, r_0 \in K$ is carried out in [1]. As a consequence of these algorithms, one can find simple and effective criteria [2, Thm. 3.2] to decide the ∂-transcendence over K of the solutions to

$$\delta_x^2 Y - p\delta_x Y = 0, \tag{0.4}$$

for $p \in K$. These criteria were applied in [2, Thm. 1.1] to give a new algebraic proof of the $\frac{\partial}{\partial t}$-transcendence of the incomplete Gamma function $\gamma(x,t)$, which satisfies (0.4) with $p = \frac{1-t-x}{x}$. The first proof of the $\frac{\partial}{\partial t}$-transcendence of $\gamma(x,t)$ was obtained by Johnson, Reinhart, and Rubel in [29]. This result was necessary in [9, Ex. 7.2] to compute the unipotent radical of the PPV group associated to $\gamma(x,t)$.

A new method to compute the unipotent radical $R_u(H)$ of the PPV group H associated to (0.2) is presented in [3], generalizing the methods of [2] to the setting of several parametric derivations. When the maximal reductive quotient $H/R_u(H)$ fails to be Π-constant, one computes effectively a new set of parametric derivations Π' such that the resulting PPV group H', obtained after replacing Π with Π' in the foregoing discussion, has the properties that $H'/R_u(H')$ is Π'-constant, and that $R_u(H)$ is defined by the same differential equations as $R_u(H')$. Thus the effective computation of $R_u(H)$, and therefore that of H, is reduced to the computation of H' carried out in [19].

A classical change-of-variables procedure relates the solutions of (0.3) to the solutions to an associated equation of the form (0.2). We reinterpret this change of variables in Galois-theoretic terms, to reconstruct the PPV group G corresponding to (0.3) from a lattice (2.22) of PPV fields and PPV groups. That the required computations can be performed effectively is proved in [1,4] through repeated, ad-hoc application of the classical Kolchin-Ostrowski Theorem [33]. The parameterized version of the Kolchin-Ostrowski Theorem 1.3 proved in Chapter 1 allows us to improve on some of the proofs in [1, 4].

We apply the results of [1–4, 19] to prove a complete set of criteria to decide whether any of the solutions to (0.3) is Π-transcendental over K, generalizing the criteria proved in [2, Thm. 3.2] to the several-parameter setting and general equations (0.3), not just those of the special form (0.4). In Chapter 3 we apply the algorithms of Chapter 2 to compute the PPV groups of several concrete second-order linear differential equations with differential parameters.

Chapter 1

Theory

In this chapter we summarize the main theoretical ingredients required in the sequel. In §1.1 we summarize some basic notions from differential algebra and set up some basic notation. In §1.2 we prove a parameterized version of the classical Kolchin-Ostrowski Theorem, which will allow us to give alternative proofs of some of the results presented in Chapter 2 to those found in the literature [1,2,4,19]. In §1.3 and §1.4 we present those aspects of the theory of linear differential algebraic groups and the parameterized Picard-Vessiot theory that are necessary to develop the algorithms presented in Chapter 2.

1.1 Differential-algebraic preliminaries

Let us begin by recalling some standard notions from differential algebra. See [35,50] for more details concerning the following definitions. A Δ-*ring* is a ring A equipped with a finite set $\Delta := \{\delta_1, \ldots, \delta_m\}$ of commuting derivations.

Concretely,

$$\delta_i(a+b) = \delta_i(a) + \delta_i(b); \qquad \delta_i(ab) = a\delta_i(b) + \delta_i(a)b;$$

$$\delta_i\delta_j = \delta_j\delta_i,$$

for each $a, b \in A$ and $1 \leq i, j \leq m$. We often omit the parentheses, and write δa for $\delta(a)$. An ideal $\mathfrak{I} \subseteq A$ is a *differential ideal* if $\delta(\mathfrak{I}) \subset \mathfrak{I}$ for each $\delta \in \Delta$.

For any subset $\Pi \subseteq \Delta$, we denote the subring of Π-*constants* of A by

$$A^{\Pi} := \{a \in K \mid \delta a = 0,\ \delta \in \Pi\}.$$

When $\Pi = \{\delta\}$ is a singleton, we write A^{δ} instead of A^{Π}. If $A = K$ happens to be a field, we say that (K, Δ) is a Δ-field. For any $a \in K$, we denote

$$\Delta a := (\delta_1 a, \ldots, \delta_m a),$$

and if $a \neq 0$ we also define

$$\tfrac{\Delta a}{a} := (\tfrac{\delta_1 a}{a}, \ldots, \tfrac{\delta_m a}{a}).$$

Every field is assumed throughout to be of characteristic zero.

The *ring of differential polynomials* over K (in m differential indeterminates) is denoted by

$$K\{Y_1, \ldots, Y_m\}_{\Delta}.$$

Algebraically, it is the free K-algebra in the countably infinite set of variables

$$\{\theta Y_i \mid 1 \leq i \leq m,\ \theta \in \Theta\}, \quad \text{where}$$

$$\Theta := \{\delta_1^{\ell_1} \ldots \delta_n^{\ell_n} \mid \ell_i \in \mathbb{Z}_{\geq 0} \ \text{ for } \ 1 \leq i \leq n\}$$

is the free commutative monoid on the set Δ. For $\theta := \delta_1^{\ell_1} \ldots \delta_n^{\ell_n}$, we let $\operatorname{ord}(\theta) := \sum_{i=1}^n \ell_i$. The ring $K\{Y_1, \ldots, Y_m\}_\Delta$ carries a natural structure of Δ-ring, given by $\delta_i(\theta Y_j) := (\delta_i \cdot \theta)Y_j$. The *field of differential rational functions* is the field of fractions

$$\operatorname{Frac}(K\{Y_1, \ldots, Y_m\}_\Delta) =: K\langle Y_1, \ldots, Y_m\rangle_\Delta.$$

We say $\mathbf{p} \in K\{Y_1, \ldots, Y_m\}_\Delta$ is a *linear differential polynomial* if it belongs to the K-linear span of the θY_j, for $\theta \in \Theta$ and $1 \leq j \leq m$. The K-vector space of linear differential polynomials will be denoted by $K\{Y_1, \ldots, Y_m\}_\Delta^1$.

The *ring of linear differential operators* $K[\Delta]$ is the K-linear span of Θ. Its (non-commutative) ring structure is defined by composition of additive endomorphisms of K, and is determined by the rule

$$\delta \circ a = a \circ \delta + \delta(a),$$

for $a \in K$ and $\delta \in \Delta$. The canonical identification of (left) K-vector spaces $K[\Delta] \simeq K\{Y\}_\Delta^1$ given by $\sum_\theta a_\theta \theta \leftrightarrow \sum_\theta a_\theta \theta Y$ will be assumed implicitly in what follows.

See [34, 56] and the references cited therein for more details concerning the following technical but important notion:

Definition 1.1. We say that a Δ-field K is Δ-*closed* if, for every prime differential ideal $\mathfrak{P} \subset K\{Y_1, \ldots, Y_m\}_\Delta$ and for any differential polynomial $\mathbf{q} \in K\{Y_1, \ldots, Y_m\}_\Delta$ such that $\mathbf{q} \notin \mathfrak{P}$, there exists an m-tuple $\underline{a} \in K^m$ such that $\mathbf{q}(\underline{a}) \neq 0$ and $\mathbf{p}(\underline{a}) = 0$ for every $\mathbf{p} \in \mathfrak{P}$.

If M is a Δ-field and $K \subseteq M$ is a subfield such that $\delta(K) \subset K$ for each $\delta \in \Delta$, we say K is a Δ-*subfield* of M and M is a Δ-*field extension* of K. If $y_1, \ldots, y_n \in M$, we denote the Δ-subfield of M generated over K by all the derivatives of the y_i by

$$K\langle y_1, \ldots, y_n\rangle_\Delta \subseteq M.$$

Let $K\langle y_1, \ldots, y_m\rangle_\Delta =: M$ be a Δ-field extension differentially generated by the m-tuple $\underline{y} := (y_1, \ldots, y_m)$. For any non-negative integer $s \in \mathbb{N}$, let M_s denote the (non-differential) field extension algebraically generated over K by the set

$$\{\theta y_i \mid 1 \leq i \leq m,\ \mathrm{ord}(\theta) \leq s\}.$$

It is shown in [35, §Thm. II.12.6] that there is a numerical polynomial $\omega_{\underline{y}/K}(T) \in \mathbb{Q}[T]$ such that

$$\omega_{\underline{y}/K}(s) = \mathrm{tr.deg}_K(M_s)$$

for large enough $s \in \mathbb{N}$, where $\text{tr.deg}_K(M_s)$ denotes the (algebraic) transcendence degree of M_s over K. The *differential type* of $\underline{y}$ over K, denoted by $\tau(\underline{y}/K)$ (see [35, II.13]), is defined as the degree of $\omega_{\underline{y}/K}(T)$ whenever this Kolchin polynomial is different from zero; otherwise, we set $\tau(\underline{y}/K) := -\infty$. By [35, Prop. II.12.15], $\tau(\underline{y}/K) =: \tau(M)$ depends only on M, and not on the choice of Δ-generators $\underline{y}$ for M over K. We observe that $\tau(M) \leq 0$ if and only if the algebraic transcendence degree $\text{tr.deg}_K(M) < \infty$.

We assume that K is Δ-closed for the remainder of §1.1. In analogy with the definition of the Zariski topology on affine m-space, we say $V \subseteq K^m$ is *Kolchin-closed* (cf. [35, §IV.1 and §IV.3] and [42, §2.1]) if there exist finitely many elements

$$\mathbf{p}_1, \ldots, \mathbf{p}_k \in K\{Y_1, \ldots, Y_m\}_\Delta \qquad \text{such that}$$

$$V = \{\underline{a} := (a_1, \ldots, a_m) \in K^m \mid \mathbf{p}_1(\underline{a}) = \cdots = \mathbf{p}_k(\underline{a}) = 0\},$$

and we let $\mathfrak{I}_V := \sqrt{\langle \mathbf{p}_1, \ldots, \mathbf{p}_k \rangle}$ denote the radical differential ideal generated by the $\mathbf{p}_i$. We define the *ring of differential regular functions* on V:

$$K\{V\} := K\{Y_1, \ldots, Y_m\}_\Delta / \mathfrak{I}_V.$$

We say that V is irreducible if and only if $K\{V\}$ is an integral domain; in this case we define the *field of differential rational functions* on V by $K\langle V \rangle := \text{Frac}(K\{V\})$, and the *differential type* $\tau(V) := \tau(K\langle V \rangle)$ (cf. [35,

§IV.3]). If $W \subseteq K^m$ is only Kolchin-closed, $\tau(W)$ denotes the maximum of $\tau(V)$ over the finitely many [35, Cor. IV.3.2] irreducible components V of W.

1.2 A parameterized Kolchin-Ostrowski Theorem

In this section we denote by $V \subset W$ an extension of Δ-fields (the choice of V and W prevents needless conflict with the notation adopted in Chapter 2), where $\Delta := \{\delta\} \cup \Pi$ is a set of pairwise commuting derivations. We think of the complement Π of δ in Δ as a set of derivations with respect to parameters. Assuming $V^\delta = W^\delta =: F$, let $e_1, \ldots, e_r \in W$ be such that $\frac{\delta e_i}{e_i} \in V$ for each $1 \leq i \leq r$, and let $f_1, \ldots, f_s \in W$ be such that $\delta f_j \in V$ for each $1 \leq j \leq s$.

Theorem 1.2 (Kolchin-Ostrowski)**.** *If there exists a polynomial*

$$0 \neq P \in V[Y_1, \ldots, Y_{r+s}] \qquad \textit{such that} \qquad P(e_1, \ldots, e_r, f_1, \ldots, f_s) = 0,$$

then at least one of the following possibilities holds:

1. *There exist integers $n_i \in \mathbb{Z}$ for $1 \leq i \leq r$, not all zero, such that*
$$\prod_{i=1}^{r} e_i^{n_i} \in V.$$

2. *There exist elements $p_j \in F$ for $1 \leq j \leq s$, not all zero, such that*
$$\sum_{j=1}^{s} p_j f_j \in V.$$

The second part of the previous result was proved by Ostrowski in [45], in the case where there are only *primitives* f_j as above, but no *exponentials* e_i, under the inessential hypothesis that V and W are fields of meromorphic functions. In [33], Kolchin proved the Kolchin-Ostrowski Theorem 1.2 as an application of the Picard-Vessiot theory. In Theorem 1.3, we generalize Theorem 1.2 to the parameterized setting as a corollary of the non-parameterized Theorem 1.2, applied to the underlying δ-fields of V and W.

Theorem 1.3 (Parameterized Kolchin-Ostrowski Theorem)**.** *If there exists a differential polynomial*

$$0 \neq P \in V\{Y_1, \ldots, Y_{r+s}\}_\Pi \qquad \textit{such that} \qquad P(e_1, \ldots, e_r, f_1, \ldots, f_s) = 0,$$

then at least one of the following possibilities holds:

1. *There exist integers $n_i \in \mathbb{Z}$ for $1 \leq i \leq r$, not all zero, such that*

$$\prod_{i=1}^{r} e_i^{n_i} \in V.$$

2. *There exist linear differential operators $\mathbf{p}_j \in F[\Pi]$ for $1 \leq j \leq s$, and (in case $\Pi \neq \emptyset$) linear differential polynomials $\mathbf{q}_i \in F\{Y_1, \ldots, Y_m\}^1_\Pi$ for $1 \leq i \leq r$, where at least one of the $\mathbf{p}_j$ or the $\mathbf{q}_i$ is non-zero, such that*

$$\sum_{j=1}^{s} \mathbf{p}_j(f_j) + \sum_{i=1}^{r} \mathbf{q}_i\left(\tfrac{\Pi e_i}{e_i}\right) \in V.$$

Proof of Thm. 1.3. The case $\Pi = \emptyset$ is Theorem 1.2, so we assume from now on that $\Pi \neq \emptyset$. Since the statement does not concern elements of W other than the e_i and the f_j, together with their Π-derivatives, we may assume that

$$W = V\langle e_1, \ldots, e_r, f_1, \ldots, f_s\rangle_\Pi$$

is differentially generated over V by these elements. Let

$$M := V(e_1, \ldots, e_r)\langle f_1, \ldots, f_s\rangle_\Pi \subseteq W;$$

in other words, M is algebraically generated over V by the e_i and the f_j, together with all the Π-derivatives of the f_j. Consider the Δ-subfield

$$\tilde{W} := M\left\langle \tfrac{\Pi e_1}{e_1}, \ldots, \tfrac{\Pi e_r}{e_r}\right\rangle_\Pi \subseteq W.$$

We claim that $\tilde{W} = W$. It is clear that $\tilde{W} \subseteq W$. To prove the opposite inclusion, it suffices to show that $\theta e_i \in \tilde{W}$ for each $1 \leq i \leq r$ and each $\theta \in \Theta$. We prove this by contradiction. It is clear that $e_i \in \tilde{W}$ and that $\partial_k e_i \in \tilde{W}$ for each $1 \leq i \leq r$ and $1 \leq k \leq m$. Assume that $\theta \in \Theta$ is of smallest order such that $\theta e_i \notin \tilde{W}$, and assume that $\partial_k \in \Pi$ appears effectively in θ. Letting θ' denote the element of Θ obtained from θ by decreasing the exponent $1 \leq \ell_k$ of ∂_k by 1, so that $\theta'\partial_k = \theta$, we observe that the element

$$\theta e_i - e_i\theta'\left(\tfrac{\partial_k e_i}{e_i}\right) \in \tilde{W},$$

since it is an algebraic expression over M that only involves elements $\theta'' e_i$ with

$$\operatorname{ord}(\theta'') \leq \operatorname{ord}(\theta') < \operatorname{ord}(\theta).$$

This contradiction concludes the proof that $\tilde{W} = W$.

It follows from the definition of $V\{Y_1, \ldots, Y_{r+s}\}_\Pi$ that any differential polynomial P as above gives rise to an algebraic polynomial

$$0 \neq \tilde{P} \in V[\theta Y_1, \ldots, \theta Y_{r+s}]_{\theta \in \Theta}.$$

Since $\tilde{W} = W$, every element of the form θe_i such that $\theta \neq 1$ may be rewritten as a rational expression in $\theta' \frac{\partial_k e_i}{e_i}$ with coefficients in M and with θ' ranging over Θ. Moreover, every element of M may be expressed as a rational expression in the e_i and θf_j, with coefficients in V, again with θ ranging over Θ. Thus we may rewrite $\tilde{P}$ as a rational expression in the elements e_i, $\theta \frac{\partial_k e_i}{e_i}$, and θf_j, and after clearing denominators we obtain a new polynomial

$$0 \neq Q \in V[Y_1, \ldots, Y_r, \theta Y_{r+1}, \ldots, \theta Y_{r+s+rm}]_{\theta \in \Theta}$$

such that

$$Q\Big(e_1, \ldots, e_r, \theta f_1, \ldots, \theta f_s, \theta\big(\tfrac{\partial_1 e_1}{e_1}\big), \ldots, \theta\big(\tfrac{\partial_m e_1}{e_1}\big), \ldots, \theta\big(\tfrac{\partial_1 e_r}{e_r}\big), \ldots, \theta\big(\tfrac{\partial_m e_r}{e_r}\big)\Big) = 0.$$

Since

$$\delta(\theta f_j) = \theta(\delta f_j) \in V \qquad \text{and} \qquad \delta\big(\theta \tfrac{\partial_k e_i}{e_i}\big) = \theta \partial_k \big(\tfrac{\delta e_i}{e_i}\big) \in V$$

for each $1 \leq i \leq r$; $1 \leq j \leq s$; $1 \leq k \leq m$; and $\theta \in \Theta$, an application of Theorem 1.2 to the underlying δ-fields of V and W implies that either there exist integers $n_i \in \mathbb{Z}$ for $1 \leq i \leq r$, not all zero, such that

$$\prod_{i=1}^{r} e_i^{n_i} \in V,$$

or else there exist elements $a_{j,\theta}, b_{i,k,\theta} \in F$, not all zero, such that

$$\sum_{j,\theta} a_{j,\theta}\theta f_j + \sum_{i,k,\theta} b_{i,k,\theta}\theta\big(\tfrac{\partial_k e_i}{e_i}\big) \in V.$$

Letting

$$\mathbf{p}_j := \sum_{\theta} a_{j,\theta}\theta \in F[\Pi] \qquad \text{and} \qquad \mathbf{q}_i := \sum_{k,\theta} b_{i,k,\theta}\theta Y_k \in F\{Y_1, \dots Y_m\}^1_\Pi$$

concludes the proof of the theorem. □

Remark 1.4. If $\Pi = \emptyset$, then $\Theta = \{1\}$ and $F[\Pi] = F$. Therefore, Theorem 1.2 is a special case of Theorem 1.3, corresponding to $\Pi = \emptyset$.

1.3 Linear differential algebraic groups

In this section we briefly recall some facts from the theory of linear differential algebraic groups, which was initiated in [7] (see also [36]).

Definition 1.5. Let F be a Π-closed field. A Kolchin-closed subgroup G of $\mathrm{GL}_n(F)$ is a *linear differential algebraic group* (or LDAG). We say that an LDAG G is Π-*constant* if it is conjugate to a subgroup of $\mathrm{GL}_n(F^\Pi)$.

The differential algebraic subgroups of the additive and multiplicative groups of F, which we denote respectively by $\mathbb{G}_a(F)$ and $\mathbb{G}_m(F)$, were classified by Cassidy in [7, Prop. 11, Prop. 31 and its Corollary]. The connectedness statements in the following result are in [7, p. 938 and p. 942].

Proposition 1.6 (Cassidy)**.** *If $B \leq \mathbb{G}_a(F)$ is a differential algebraic group, then B is connected, and there exist finitely many linear differential polynomials $\mathbf{p}_1, \ldots, \mathbf{p}_s \in F\{Y\}^1_\Pi$ such that*

$$B = \{b \in \mathbb{G}_a(F) \mid \mathbf{p}_i(b) = 0 \textit{ for each } 1 \leq i \leq s\}.$$

If $A \leq \mathbb{G}_m(F)$ is a differential algebraic group, either $A = \mu_\ell$, the group of ℓ^{th} roots of unity, or else $\mathbb{G}_m(F^\Pi) \subseteq A$ is connected, and there exist finitely many linear differential polynomials $\mathbf{q}_1, \ldots, \mathbf{q}_s \in F\{Y_1, \ldots, Y_m\}^1_\Pi$ such that

$$A = \left\{a \in \mathbb{G}_m(F) \;\middle|\; \mathbf{q}_i\big(\tfrac{\partial_1 a}{a}, \ldots, \tfrac{\partial_m a}{a}\big) = 0 \textit{ for } 1 \leq i \leq s\right\}.$$

We recall that an element $g \in \mathrm{GL}_n(F)$ is *unipotent* if it is conjugate to an upper-triangular matrix whose main diagonal consists entirely of 1's. Equivalently, if 1_n denotes the $n \times n$ identity matrix, g is unipotent if and only if $(g - 1_n)$ is a nilpotent matrix.

Definition 1.7. A LDAG G is *unipotent* if one of the following equivalent conditions is satisfied (cf. [8, Thm. 2] and [42, Defn. 2.1]):

1. G is conjugate to a subgroup of the group of upper triangular unipotent matrices.

2. G contains no elements of finite order other than the identity matrix.

3. G has a subnormal sequence of differential algebraic subgroups

$$G = G_0 \supset G_1 \supset \cdots \supset G_N = \{1_n\}$$

such that each intermediate quotient G_i/G_{i+1} is isomorphic to a differential algebraic subgroup of the additive group $\mathbb{G}_a(F)$.

Any LDAG G admits a maximal normal unipotent differential-algebraic subgroup, which is called its *unipotent radical* and denoted by $R_u(G)$. We say that G is *reductive* if its unipotent radical $R_u(G) = \{0\}$ is trivial.

Definition 1.8. A LDAG G is *differentially finitely generated* (or DFG) if it contains a Kolchin-dense finitely generated subgroup.

The following theorem, which is proved in [41, Thm. 2.8], is a key result in the algorithms to compute non-reductive parameterized Picard-Vessiot groups.

Theorem 1.9 (Minchenko-Ovchinnikov-Singer)**.** *Let G be a LDAG such that the reductive quotient $G/R_u(G)$ is differentially constant. Then G is differentially finitely generated if and only if the differential type $\tau(G) \leq 0$.*

1.4 Parameterized Picard-Vessiot theory

We now briefly recall the main facts that we will need from the parameterized Picard-Vessiot theory [9]. Let F be a Π-field, where $\Pi := \{\partial_1, \dots, \partial_m\}$, and let $K := F(x)$ be the field of rational functions in x with coefficients in F, equipped with the structure of $(\{\delta_x\} \cup \Pi)$-field determined by setting $\delta_x x = 1$, $K^{\delta_x} = F$, and $\partial_i x = 0$ for each i. We will sometimes refer to δ_x as the *main* derivation, and to Π as the set of *parametric* derivations. From now on, we will let $\Delta := \{\delta_x\} \cup \Pi$. Consider the following linear differential equation with respect to the main derivation, where $r_i \in K$ for each $0 \leq i \leq n-1$:

$$\delta_x^n Y + \sum_{i=0}^{n-1} r_i \delta_x^i Y = 0. \tag{1.1}$$

Definition 1.10. We say that a Δ-field extension $M \supseteq K$ is a *parameterized Picard-Vessiot extension* (or PPV extension) of K for (1.1) if:

(i) There exist n distinct, F-linearly independent elements $y_1, \dots, y_n \in M$ such that $\delta_x^n y_j + \sum_i r_i \delta_x^i y_j = 0$ for each $1 \leq j \leq n$.

(ii) $M = K\langle y_1, \dots y_n \rangle_\Delta$.

(iii) $M^{\delta_x} = K^{\delta_x}$.

The *parameterized Picard-Vessiot group* (or PPV group) is the group of field automorphisms of M that commute with all the derivations $\delta \in \Delta$ and fix every element of K, and we denote it by $\mathrm{Gal}_\Delta(M/K)$:

$$\mathrm{Gal}_\Delta(M/K) := \{\sigma \in \mathrm{Aut}_K(M) \mid \sigma\circ\delta = \delta\circ\sigma, \sigma(a) = a \text{ for } \delta \in \Delta \text{ and } a \in K\}.$$

The F-linear span of all the y_j is the *solution space* $\mathcal{S}$.

If F is Π-closed, it is shown in [9, Thm. 3.5] that a PPV extension of K for (1.1) exists and is unique up to K-Δ-isomorphism. Although this assumption allows for a simpler exposition of the theory, several authors [21,58] have shown that, in many cases of practical interest, the PPV theory can be developed without assuming that F is Π-closed. In any case, we may always embed F in a Π-closed field [34,56]. The action of $\mathrm{Gal}_\Delta(M/K)$ is determined by its restriction to $\mathcal{S}$, which defines an embedding $\mathrm{Gal}_\Delta(M/K) \hookrightarrow \mathrm{GL}_n(F)$ after choosing an F-basis for $\mathcal{S}$. It is shown in [9, Thm. 3.5] that this embedding identifies the PPV group with a linear differential algebraic group (Definition 1.5), and from now on we will make this identification implicitly.

There is a parameterized Galois correspondence [9, Thm. 3.5] between the linear differential algebraic subgroups H of $\mathrm{Gal}_\Delta(M/K)$ and the intermediate Δ-fields $K \subseteq L \subseteq M$, given by

$$H \mapsto M^H \qquad \text{and} \qquad L \mapsto \mathrm{Gal}_\Delta(M/L).$$

Under this correspondence, an intermediate Δ-field L is a PPV extension of K (for some linear differential equation with respect to δ_x) if and only if $\mathrm{Gal}_\Delta(M/L)$ is normal in $\mathrm{Gal}_\Delta(M/K)$. In this case, the restriction homomorphism $\mathrm{Gal}_\Delta(M/K) \to \mathrm{Gal}_\Delta(L/K)$ defined by $\sigma \mapsto \sigma|_L$ is surjective, with kernel $\mathrm{Gal}_\Delta(M/L)$.

The following result follows from the parameterized version of the Ramis Density Theorem proved by Dreyfus as part of his Ph.D. thesis [17, Thm. 1.2.12] and published in [18, Thm. 2.20]:

Proposition 1.11 (Dreyfus)**.** *If M is a PPV extension of K, then $\mathrm{Gal}_\Delta(M/K)$ is differentially finitely generated.*

The following result is proved in [42, Prop. 2.14 and Prop. 3.2]:

Proposition 1.12 (Minchenko-Ovchinnikov-Singer)**.** *Suppose that M is a PPV extension of K with PPV group $G = \mathrm{Gal}_\Delta(M/K)$. If the reductive quotient $G/R_u(G)$ is differentially constant, then G has differential type $\tau(G) \leq 0$ and M has finite algebraic transcendence degree over K.*

The Tannakian approach to LDAGs was initiated by Ovchinnikov in [46, 47], with the development of the notion of *differential Tannakian category*, which is applied to the PPV theory of [9] in [48], and towards a generalization of this theory developed in [21]. This generalization of the usual formalism of Tannakian categories [15,16] is essential to the concrete algorithms developed in [41,42] to compute PPV groups for higher order equations.

Chapter 2

Algorithms

In this chapter, we present a series of procedures that altogether amount to a complete algorithm to compute the PPV group G associated by the PPV theory to a second-order linear differential equation

$$\delta_x^2 Y + r_1 \delta_x Y + r_0 Y = 0, \tag{2.1}$$

where $r_0, r_1 \in K := F(x)$ is the Δ-field defined as follows: $F = K^{\delta_x}$ is a Π-closed field, $\Delta := \{\delta_x\} \cup \Pi$, $\delta_x x = 1$, and $\partial x = 0$ for each $\partial \in \Pi$.

In §2.1, we summarize Dreyfus' algorithms [19] to compute G when $r_1 = 0$ and either G is reductive or its maximal reductive quotient is Π-constant. In §2.2, still in the case where $r_1 = 0$, we then show how to modify the set of parametric derivations to remove the restriction that G is either reductive or has Π-constant maximal reductive quotient $G/R_u(G)$, following [3]. Then in §2.3 we show how to remove the assumption that $r_1 = 0$ by reinterpreting a classical change-of-variables procedure in Galois-theoretic terms, following [4]. We will apply the parameterized Kolchin-Ostroski Theorem 1.3 proved in §1.2 to give different arguments for some of the results of [1,4,19]. Finally, in §2.4 we apply the algorithms presented in this chapter to prove a substantial generalization of [2, Thm. 3.2], in the form of simple and effective criteria to decide whether any of the solutions to (2.1) is Π-transcendental over K.

2.1 Dreyfus' algorithm

In this section, we summarize the results of [19]. Consider a second-order parameterized linear differential equation

$$\delta_x^2 Y - qY = 0, \tag{2.2}$$

where $q \in K$. In [19], Dreyfus develops the following procedure to compute the PPV group H corresponding to (2.2) (see also [1, 3]). As in Kovacic's algorithm [37], one first decides whether there exists $u \in \bar{K}$ such that

$$(\delta_x + u) \circ (\delta_x - u) = \delta_x^2 - q, \tag{2.3}$$

where $\bar{K}$ is an algebraic closure of K. Expanding the left-hand side of (2.3) shows that such a factorization exists precisely when one can find a solution in $\bar{K}$ to the *Riccati equation*

$$P_q(u) = \delta_x u + u^2 - q = 0. \tag{2.4}$$

One can deduce structural properties of H from the algebraic degree of such a u over K [37, §1]. By [19, Thm. 2.10], precisely one of the following possibilities occurs.

I. If there exists $u \in K$ such that $P_q(u) = 0$, then there exist differential algebraic subgroups $A \leq \mathbb{G}_m(F)$ and $B \leq \mathbb{G}_a(F)$ such that H is conjugate to
$$\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in A,\ b \in B \right\}. \tag{2.5}$$

II. If there exists $u \in \bar{K}$, of degree 2 over K, such that $P_q(u) = 0$, then there exists a differential algebraic subgroup $A \leq \mathbb{G}_m(F)$ such that H is conjugate to
$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in A \right\} \cup \left\{ \begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix} \;\middle|\; a \in A \right\}.$$

III. If there exists $u \in \bar{K}$ of degree either 4, 6, or 12 over K such that $P_q(u) = 0$, then H is one of the finite primitive groups $A_4^{\mathrm{SL}_2}$, $S_4^{\mathrm{SL}_2}$, or $A_5^{\mathrm{SL}_2}$, respectively (see [53, §4]).

IV. If there is no u in $\bar{K}$ such that $P_q(u) = 0$, then there exists a subset $\Pi' \subset F \cdot \Pi$ (the F-vector space spanned by Π) consisting of F-linearly independent, pairwise commuting derivations ∂' such that H is conjugate to $\mathrm{SL}_2(F^{\Pi'})$.

In cases I and II the computation of A is obtained in [19, Lem. 2.2] and [1, Algo. 4 and Algo. 5] with different arguments than those presented here. We suppose that $\eta \neq 0$ satisfies

$$\delta_x Y - uY = 0, \tag{2.6}$$

and therefore $L := K(u)\langle\eta\rangle_\Delta$ is the corresponding PPV field over $K(u)$. The differential algebraic group A in cases I and II coincides with the PPV group $\mathrm{Gal}_\Delta(L/K(u)) =: \Sigma$, where u is the solution to the Riccati equation (2.4).

By Theorem 1.3, either

(1) the element η is Π-transcendental over $K(u)$; or

(2) there exists a non-zero integer $k \in \mathbb{Z}$ such that $\eta^k \in K(u)$; or

(3) there is a non-zero linear differential polynomial $\mathbf{p} \in F\{Y_1, \ldots, Y_m\}_\Pi$ such that $\mathbf{p}(\frac{\Pi\eta}{\eta}) \in K(u)$.

Case (1) holds if and only if $A = \mathbb{G}_m(F)$, and it follows from [52, Thm. 1.1 and Prop. 1.2] that no finite algebraic extension of K admits a PPV extension whose PPV group is $\mathbb{G}_m(F)$. Case (2) occurs precisely when there exists an element $0 \neq f \in K$ such that

$$ku = \tfrac{\delta_x f}{f}.$$

If k is the smallest such positive integer, then $L = K(\eta)$ is a cyclic algebraic extension of K of order k, and therefore $A = \mu_k$ [9, Ex. 3.1(bis), p. 121]. In case (3), the determination of the finite set of linear differential polynomials

$$\mathbf{p}_1, \ldots, \mathbf{p}_s \in F\{Y_1, \ldots, Y_m\}^1_\Pi \qquad \text{such that}$$
$$A = \{a \in \mathbb{G}_m(F) \mid \mathbf{p}_i(\tfrac{\Pi a}{a}) = 0 \text{ for } 1 \leq i \leq s\}$$

is reduced to the solution of the following *creative telescoping problem* (see [6, 10–12, 59]): let $a_\sigma := \frac{\sigma(\eta)}{\eta} \in A$ for each $\sigma \in \Sigma$, and note that

$$\delta_x\Big(\mathbf{p}\big(\tfrac{\Pi\eta}{\eta}\big)\Big) = \mathbf{p}(\Pi u); \qquad \text{and}$$
$$\sigma\Big(\mathbf{p}\big(\tfrac{\Pi\eta}{\eta}\big)\Big) = \mathbf{p}\big(\tfrac{\Pi\eta}{\eta}\big) + \mathbf{p}\big(\tfrac{\Pi a_\sigma}{a_\sigma}\big)$$

for each $\sigma \in \Sigma$ and each $\mathbf{p} \in F\{Y_1, \ldots, Y_m\}^1_\Pi$. By the parameterized Galois correspondence [9, Thm. 3.5], $\mathbf{p}(\frac{\Pi a}{a}) = 0$ for every $a \in A$ if and only if $\mathbf{p}(\Pi u) = \delta_x g$ for some $g \in K(u)$.

To compute the unipotent radical $R_u(H) = B$ in case I, recall [37, §1.1, p. 5] that there exists an element $\xi \in M$, where M is a PPV extension of K for (2.2), such that $\delta_x(\frac{\xi}{\eta}) = \eta^{-2} \in L$ and $\{\eta, \xi\}$ is an F-basis for the solution space for (2.2). Since $M = L\langle\frac{\xi}{\eta}\rangle_\Delta$, we may apply Theorem 1.3 to conclude that either ξ is Π-transcendental over L, or else there exists a linear differential polynomial $\mathbf{p}$ such that $\mathbf{p}(\frac{\xi}{\eta}) \in L$. It follows from Proposition 1.6

that we only need to determine finitely many such differential polynomials

$$\mathbf{p}_1, \ldots, \mathbf{p}_s \in F\{Y\}^1_\Pi.$$

When $A \subseteq \mathbb{G}_m(F^\Pi)$, it follows from Proposition 1.12 that M is of finite algebraic transcendence degree over K [42, Prop. 3.2], which allows for the effective computation of the $\mathbf{p}_i$ obtained in [42, §3.2.1]. But if $A \nsubseteq \mathbb{G}_m(F^\Pi)$, then M may be of infinite algebraic transcendence degree over L (see Theorem 2.21) and the methods of [19,42] no longer apply. We are not aware of *a priori* bounds on the orders of the $\mathbf{p}_i$ in this case, which raises the problem of deciding whether all the $\mathbf{p}_i$ have already been found, or whether it is still necessary to do more prolongations (see [42, §3.2.1]).

In the setting of one parametric derivation $\Pi = \{\partial\}$, the computation of $R_u(H)$ when $H/R_u(H)$ fails to be ∂-constant is carried out in [1,2]. In this case, it follows from [24, proof of Lem. 3.6(2)] that either $R_u(H) = \mathbb{G}_a(F)$, or else $R_u(H) = 0$. In light of [42], this has the counterintuitive consequence that the computation of $R_u(H)$ is actually easier when $H/R_u(H)$ is not ∂-constant (see [2, Thm. 3.2]), since in this case the parametric derivation ∂ is barred from appearing in the defining equations for $R_u(H)$. In Theorem 2.2, which was originally proved in [3, Thm. 3.2], we describe how this phenomenon generalizes to the setting of several parametric derivations. One can find a finite set of F-linearly independent, pairwise commuting derivations $\Pi' \subset F \cdot \Pi$, the F-vector space spanned by Π, with the property that if H' denotes the PPV group of (2.2) obtained by replacing Π with Π' in the foregoing discussion, and A' and B' are defined accordingly, then A' is Π'-constant and $B' = B$. Thus the computation of B is reduced to the computation of B' carried out in [19, §2.1] (see also [42, Prop. 3.2 and Algo. 1]).

In case III, [19, Thm. 2.10(3)] states that H coincides with the PV group for (2.2), and in this case the computation of H is carried out in [27, §2].

In case IV, [19, Thm. 2.10(4)] states that a parametric derivation $\partial \in F \cdot \Pi$ in the F-vector space spanned by Π belongs to the subspace $F \cdot \Pi'$ if and only if the following linear differential equation admits a solution in K:

$$-\tfrac{1}{2}\delta_x^3 Y + 2q\delta_x Y + (\delta_x q)Y = \partial q. \tag{2.7}$$

One can now write a parametric derivation $\partial = \sum_j c_j \partial_j$ with undetermined coefficients $c_j \in F$ and ∂_j ranging over Π, and the methods of [51, Lem. 3.1] show that the solvability of (2.7) in K is an F-linear condition on the coefficients c_j. Thus one can find a (possibly non-commuting) basis Π'' for the F-vector space of parametric derivations ∂ such that (2.7) admits a solution in K. The proof of [36, Prop. 0.6] gives a recipe to produce the commuting basis Π' of case IV. Cf. Remark 2.1 below and [22, Thm. 6.3].

2.2 Computation of the unipotent radical

In this section we address the computation of the unipotent radical $R_u(H)$ of the PPV group H for (2.2) in case I of Kovacic's algorithm, following [3]. We keep the notation from the previous section: there is a basis of solutions $\{\eta, \xi\}$ such that $\delta_x \eta = u\eta$, $\delta_x(\frac{\xi}{\eta}) = \eta^{-2}$, $u \in K$ satisfies the Riccati equation (2.4), and there exist differential algebraic subgroups $A \subseteq \mathbb{G}_m(F)$ and $B \subseteq \mathbb{G}_a(F)$ such that H is given by (2.5).

Let $\mathcal{D} := F \cdot \Pi$, the F-linear span of Π, and define

$$\mathcal{L} := \{\partial \in \mathcal{D} \mid \partial a = 0,\ \forall a \in A\}. \tag{2.8}$$

Note that $\mathcal{L}$ is a *Lie subspace* of $\mathcal{D}$, i.e., an F-subspace that is closed under the Lie bracket on derivations, because $[\partial, \partial'](a) = \partial(\partial' a) - \partial'(\partial a) = 0$ for any $a \in A$ and $\partial, \partial' \in \mathcal{L}$. By [7, Prop. 39] and [36, Prop. 0.6], there exists a commuting F-basis $\Pi' := \{\partial'_1, \ldots, \partial'_k\}$ for $\mathcal{L}$.

We let $\Delta' := \{\delta_x\} \cup \Pi'$, and consider K as a Δ'-field. Then, the Δ'-field $M' := K\langle \eta, \xi \rangle_{\Delta'}$ is a PPV extension of K for (2.2) and a Δ'-subfield of M. We identify the PPV group $H' := \mathrm{Gal}_{\Delta'}(M'/K)$ with a Π'-subgroup of $\mathrm{SL}_2(F)$ by means of the same basis $\{\eta, \xi\}$, and define A' and B' as in (I).

Remark 2.1. Let us describe how to compute Π'. For every $\partial \in \mathcal{D}$ and $\sigma \in H$,

$$\sigma\left(\tfrac{\partial\eta}{\eta}\right) = \tfrac{\partial\eta}{\eta} + \tfrac{\partial a_\sigma}{a_\sigma} \qquad \text{and} \qquad \delta_x\left(\tfrac{\partial\eta}{\eta}\right) = \partial u \tag{2.9}$$

(cf. §2.1). Hence, the parameterized Galois correspondence implies that

$$\mathcal{L} = \left\{\partial \in \mathcal{D} \;\middle|\; \tfrac{\partial\eta}{\eta} \in K\right\} = \{\partial \in \mathcal{D} \mid \partial u \in \delta_x(K)\}. \tag{2.10}$$

This condition can be tested in practice as follows: the methods of [20, Chapter 11] allow us to find an element

$$v = w + \sum_i e_i \log(z_i),$$

such that $\delta_x v = u$, where $e_i \in F$, $w \in K$, $z_i \in F[x]$ is a squarefree polynomial for each i, and $\log(z_i)$ satisfies $\delta_x(\log(z_i)) = \frac{\delta_x z_i}{z_i}$ for each i. Now write a derivation $\partial \in \mathcal{D}$ with undetermined F-coeficients: $\partial = \sum_j c_j \partial_j$, and note that the condition $\partial u \in \delta_x(K)$ is equivalent to $\partial e_i = 0$ for each i, which is an F-linear condition on the coefficients c_j of ∂ (cf. [9, Ex. 7.1] and [22, §5.2]). Thus the computation of a (possibly non-commuting) basis Π'' for $\mathcal{L}$ is reduced to linear algebra. The proof of [36, Prop. 0.6] gives an algebraic recipe to produce a commuting basis Π' for $\mathcal{L}$ from the (possibly non-commuting) basis Π''. This recipe was generalized and applied to the study of isomonodromy in [22, Thm. 6.3].

Theorem 2.2. *The reductive quotient $H'/R_u(H')$ is Π'-constant, and the linear differential operators $\{\mathbf{p}_i\}_{i=1}^s \subset F[\Pi']$ defining $R_u(H') \subseteq \mathbb{G}_a(F)$ are also the defining operators for $R_u(H) \subseteq \mathbb{G}_a(F)$, under the natural inclusion $F[\Pi'] \subseteq F[\Pi]$.*

Proof. That A' is Π'-constant follows from Remark 2.1: since $\frac{\partial \eta}{\eta} \in K$ for each $\partial \in \Pi'$, we have that $\frac{\partial a}{a} = 0$ for each $a \in A'$. We will prove that $B = B'$ in a series of lemmas. By Lemma 2.3, we have that $B \subseteq B'$. By Lemma 2.4, there is a finite set $\{\mathbf{p}_i\}_{i=1}^s \subset F[\Pi']$ such that B coincides with the set of those $b \in \mathbb{G}_a(F)$ such that $\mathbf{p}_i(b) = 0$ for each $1 \leq i \leq s$. By Lemma 2.6, $\mathbf{p}_i(b') = 0$ for each $b' \in B'$ and $1 \leq i \leq s$, whence $B' \subseteq B$. $\square$

The following three lemmas were used in the proof of Theorem 2.2.

Lemma 2.3. *The restriction homomorphism $H \hookrightarrow H' : \sigma \mapsto \sigma|_{M'}$ induces an inclusion $R_u(H) \hookrightarrow R_u(H')$.*

Proof. The actions of H and H' on M and M' are completely determined by their restrictions to the same solution space $\mathcal{S} = F \cdot \eta \oplus F \cdot \xi$, whose definition is independent of the chosen set of parametric derivations. Hence, the restriction homomorphism $H \hookrightarrow H'$ is injective, and it is clear from the definitions that $R_u(H)$ is then mapped (injectively) into $R_u(H')$ (cf. Remark 2.8). $\square$

The fact that B is the unipotent radical of (2.5), and not just any differential algebraic subgroup of $\mathbb{G}_a(F)$, allows to sharpen the classification result of [7, Prop. 11] in this very particular case, by producing a set of defining operators for B from $F[\Pi'] \subseteq F[\Pi]$. The following structural result, which was inspired by the results of [54] cited in its proof, holds true for any linear differential algebraic group G of the form (2.5), whether or not it happens to be a PPV group over K.

Lemma 2.4 (cf. [24, Lem. 3.6(2)], [54, Thm. II.1.3 and Thm. II.1.4])**.** *There exist finitely many linear differential operators* $\mathbf{p}_1, \ldots, \mathbf{p}_s \in F[\Pi'] \subseteq F[\Pi]$ *such that*

$$B = \{b \in F \mid \mathbf{p}_i(b) = 0,\ 1 \leq i \leq s\}.$$

Proof. By [36, Prop. 0.7] the F-basis Π' for $\mathcal{L}$ can be extended to a commuting F-basis $\tilde{\Pi} := \{\partial'_1, \ldots, \partial'_m\}$ for all of $\mathcal{D}$. We denote by $\tilde{\Theta}$ (resp., Θ') the free commutative monoid generated by $\tilde{\Pi}$ (resp., Π'). Consider the orderly ranking on $F\{Y\}_{\tilde{\Pi}}$ determined by the lexicographic order on $\tilde{\Theta}$ defined by setting $\delta'_i \leq \delta'_j$ if $j \leq i$; i.e., to compare two elements $\theta, \theta' \in \tilde{\Theta}$, first compare their total orders, and then the exponents of $\partial'_1, \ldots, \partial'_m$, in that order.

By [54, Thm. II.1.3(b) and Thm. II.1.4], there is a characteristic set $\{\mathbf{p}_1, \ldots, \mathbf{p}_s\}$ for the defining ideal of B (with respect to this ranking) such that $\mathbf{p}_i(aY) = a\mathbf{p}_i(Y)$ for each $a \in A$ and $1 \leq i \leq s$. Therefore, to show that $\{\mathbf{p}_i\}_{i=1}^s \subset F[\Pi']$, it suffices to prove that if $\mathbf{p} \in F[\tilde{\Pi}]$ does not belong to the image of $F[\Pi']$ under the natural inclusion $F[\Pi'] \subseteq F[\Pi]$, then there exists an element $a \in A$ such that $\mathbf{p}(aY) - a\mathbf{p}(Y) \neq 0$.

So suppose that $\mathbf{p} \in F[\tilde{\Pi}]$ and $\mathbf{p} \notin F[\Pi']$, and let $c_\theta \theta Y$ be the monomial in $\mathbf{p}$ of highest rank such that $c_\theta \neq 0$ and $\theta \notin \Theta'$. Assume that $\partial'_\ell \in \tilde{\Pi}$ is the derivation of lowest rank appearing effectively in θ with $\partial'_\ell \notin \Pi'$, and let $\tilde{\theta}$ denote the element of $\tilde{\Theta}$ obtained from θ by decreasing the order of ∂'_ℓ by 1. Since $\theta'(aY) = a\theta' Y$ for every $a \in A$ and $\theta' \in \Theta'$, the leader of $\mathbf{p}(aY) - a\mathbf{p}(Y)$ is $c_\theta \partial'_\ell(a)\tilde{\theta} Y$ whenever $a \in A$ and $\partial'_\ell(a) \neq 0$. Since $\partial'_\ell \notin \mathcal{L}$, there exists $a \in A$ such that $\partial'_\ell(a) \neq 0$, whence $\mathbf{p}(aY) - a\mathbf{p}(Y) \neq 0$. $\square$

Remark 2.5. When A is Π-constant, we may take $\Pi' = \Pi$, and Lemma 2.4 coincides with [7, Prop. 11]. In case that $\Pi = \{\partial\}$ is a singleton and $\Pi' = \emptyset$, Lemma 2.4 is equivalent to [24, Lem. 3.6(2)].

The previous result shows that B is definable as a subset of $\mathbb{G}_a(F)$ using derivations from Π' only. The following lemma rules out the possibility that B could somehow be defined by more Π'-differential equations than B' is.

Lemma 2.6. *If $\mathbf{p} \in F[\Pi']$ is such that $\mathbf{p}(b) = 0$ for every $b \in B$, then $\mathbf{p}(b') = 0$ for every $b' \in B'$. In other words, $B \subseteq B'$ is Π'-dense.*

Proof. Suppose that $\mathbf{p} \in F[\Pi']$ is such that $\mathbf{p}(b) = 0$ for each $b \in B$. Then by [19, §2.1, p. 7], we have $\mathbf{p}(\eta^{-2}) \in \delta_x(L)$. Moreover, since $\mathbf{p} \in F[\Pi']$,

$$\mathbf{p}(\eta^{-2}) \in K\langle\eta\rangle_{\Delta'} =: L',$$

the fixed field of $R_u(H')$. We will show that in fact $\mathbf{p}(\eta^{-2}) \in \delta_x(L')$. Again by [19, §2.1], this will imply that $\mathbf{p}(b') = 0$ for each $b' \in B'$, concluding the proof of the Lemma. We assume throughout the proof that η is algebraically transcendental over K, since otherwise $A \simeq \mu_k$, the group of k^{th} roots of unity (see §2.1), in which case $H = H'$ and in particular $B = B'$.

By [9, Prop. 3.9] (cf. Remark 2.1), the fact that A' is Π'-constant implies

$$v_j := \tfrac{\partial'_j \eta}{\eta} \in K \tag{2.11}$$

for each $\partial'_j \in \Pi'$, and therefore $L' = K(\eta)$ consists solely of rational expressions in η with coefficients in K. From (2.11) we obtain

$$-2v_j = \eta^2 \partial'_j(\eta^{-2}) \in K. \tag{2.12}$$

Let us prove by induction that $\eta^2\theta'(\eta^{-2}) \in K$ for each $\theta' \in \Theta'$, the free commutative monoid on the set Π'. The base case is (2.12). Assuming that $\eta^2\theta'(\eta^{-2}) =: v_{\theta'} \in K$, then

$$\eta^2\partial'_j\theta'(\eta^{-2}) = \eta^2\partial'_j(v_{\theta'}\eta^{-2}) = \partial'_j v_{\theta'} - 2v_j v_{\theta'} \in K$$

proves the induction step, and our claim. Hence, $\eta^2\mathbf{p}(\eta^{-2}) \in K$.

Since

$$L := K\langle\eta\rangle_\Delta = K(\eta)\langle\Pi\eta\rangle_\Delta = K(\eta)\left\langle\tfrac{\Pi\eta}{\eta}\right\rangle_\Delta,$$

L is algebraically generated as a field extension of $L' = K(\eta)$ by

$$\left\{\theta \tfrac{\partial_j \eta}{\eta} \mid \theta \in \Theta,\ 1 \leq j \leq m\right\}, \tag{2.13}$$

where Θ is the free commutative monoid on Π. By Proposition 1.12 and [24, Prop. 6.21], if we consider L and K as δ_x-fields, then L is a (non-parameterized) PV extension of K, and the algebraic transcendence degree of L over K is finite. Hence, we may choose a finite set $\beta_1, \ldots, \beta_s$ of field generators for L over L' from the set (2.13) such that the β_i are F-linearly independent modulo L'. It follows from (2.9) that $\delta_x \beta_i \in K$ for each $1 \leq i \leq s$. By Theorem 1.2, the elements $\beta_1, \ldots, \beta_s \in L$ are then algebraically independent over L'. We define

$$N := K(\beta_1, \ldots, \beta_s),$$

and observe that $L = N(\eta)$. Since, for each $1 \leq i \leq s$,

$$\tfrac{\delta_x \eta}{\eta} = u \in K \qquad \text{and} \qquad \delta_x \beta_i \in K, \tag{2.14}$$

Theorem 1.2 implies that η is algebraically transcendental over N.

Since A is abelian, the subgroup $\mathrm{Gal}_\Delta(L/N) \leq A$ is normal and consequently N is a PPV extension of K, by the parameterized Galois correspondence [9, Thm. 3.5]. Let $f \in L$ be such that $\delta_x(f) = \mathbf{p}(\eta^{-2})$. We claim that there exist elements $g \in N$ and $c \in F$ such that $f = g\eta^{-2} + c$. To see this, let $h \in K$ be such that

$$\delta_x f = \mathbf{p}(\eta^{-2}) = h\eta^{-2},$$

and write the partial fraction decomposition of f considered as a rational function in η, where the coefficients c_i, e_k, and $g_{j,k}$ belong to $\bar{N}$, a fixed algebraic closure of N:

$$\sum_i c_i \eta^i + \sum_{j,k} \frac{g_{j,k}}{(\eta - e_k)^j} = f. \tag{2.15}$$

Let $e_0 = 0$, and apply δ_x on both sides of (2.15) to obtain (cf. [1, Lem. 2.1]):

$$\begin{aligned} &\sum_i ((\delta_x c_i)\eta^i + c_i \delta_x(\eta^i)) + \sum_{j,k} \frac{\delta_x g_{j,k}}{(\eta - e_k)^j} - \frac{j g_{j,k}(\delta_x \eta - \delta_x e_k)}{(\eta - e_k)^{j+1}} \\ &= \sum_i (\delta_x c_i + iuc_i)\eta^i + \sum_{j,k} \frac{\delta_x g_{j,k}}{(\eta - e_k)^j} - \frac{j g_{j,k}(u\eta - ue_k + ue_k - \delta_x e_k)}{(\eta - e_k)^{j+1}} \\ &= \sum_i (\delta_x c_i + iuc_i)\eta^i + \sum_{j,k} \frac{\delta_x g_{j,k} - jug_{j,k}}{(\eta - e_k)^j} + \frac{j g_{j,k}(\delta_x e_k - ue_k)}{(\eta - e_k)^{j+1}} = \delta_x f = \frac{h}{\eta^2}. \end{aligned} \tag{2.16}$$

Comparing the coefficients of η^i in (2.16) shows that $\delta_x c_0 = 0$ and that $\delta_x c_i = -iuc_i$, which implies that $c_i = a\eta^{-i}$ for some $a \in F$, and therefore $c_0 \in F$ and $c_i = 0$ for $i > 0$. Now fix $k > 0$, so that $e_k \neq 0$, and let $j > 0$ be the smallest integer such that $g_{j,k} \neq 0$. Comparing the coefficients of $(\eta - e_k)^{-j}$ in (2.16), we obtain that $\delta_x g_{j,k} = jug_{j,k}$, which implies that $g_{j,k} = a\eta^j$ for some $0 \neq a \in F$. This is impossible, and therefore there is no such j, and only $k = 0$ appears in the sum (2.15). We obtain that

$\delta_x g_{j,0} = jug_{j,0}$ for $j \neq 2$ by comparing the coefficients of η^{-j} in (2.16), which again implies that $g_{j,0} = 0$ whenever $j \neq 2$. Therefore,

$$f = g_{2,0}\eta^{-2} + c_0,$$

where $c_0 \in F$ and $g_{2,0} \in \bar{N}$ is *algebraic* over N. Since

$$g_{2,0} = \eta^2(f - c_0) \in L = N(\eta),$$

the fact that η is algebraically transcendental over N implies that $g_{2,0} \in N$.

Having shown that $f = g\eta^{-2} + c$ for some $g \in N$ and $c \in F$, let us now show that the element g actually belongs to K. This will imply that $f \in L'$, concluding the proof of the Lemma by [19, §2.1]. Since

$$\big(\delta_x g - 2ug\big)\eta^{-2} = \delta_x f = h\eta^{-2}$$

for some $h \in K$, it follows that

$$\delta_x g - 2ug = h. \tag{2.17}$$

We begin by showing that $g \in K[\beta_1, \ldots, \beta_S]$ must be a polynomial expression in the β_i (recall that the β_i are algebraically independent over K). By (2.14), the underlying δ_x-field of N is a PV extension of K (see [24, Prop. 6.21]), with PV ring

$$P := K[\beta_1, \ldots, \beta_s] \subset N. \tag{2.18}$$

By (2.17), the K-vector space $\sum_j K \cdot \delta_x^j g \subset N$ is finite-dimensional over K. By [50, Cor. 1.38], the finite-dimensionality of $\sum_j K \cdot \delta_x^j g$ over K is a necessary and sufficient condition for $g \in N$ to belong to the PV ring P.

To show that $g \in K$, we proceed by contradiction. Suppose that $r_I \underline{\beta}^I$ is a monomial in g, considered as a polynomial in the β_i, with $0 \neq |I|$ maximal and $0 \neq r_I \in K$. Since the coefficient of $\underline{\beta}^I$ in the right-hand side of (2.17) is 0, we see that $\delta_x r_I = 2ur_I$, which implies that $r_I = a\eta^2$ for some $0 \neq a \in F$, a contradiction. Hence, no such monomial $r_I \underline{\beta}^I$ appears in g, which means that $g \in K$ and $g\eta^{-2} + c = \mathbf{p}(\eta^{-2}) \in L'$. Hence, $\mathbf{p}(b') = 0$ for every $b' \in B'$ (see §2.1), as we wanted to show. □

Remark 2.7. If $\Pi' = \emptyset$, then H' is the (non-parameterized) PV-group for (2.2), and Lemma 2.6 reduces to a special case of [9, Prop. 3.6(2)].

2.3 Computing the effect of the determinant

In this section, we follow [4] in computing the PPV group G for

$$\delta_x^2 Y - 2r_1\delta_x Y + r_0 Y = 0, \tag{2.19}$$

where $r_1, r_2 \in K$, and r_1 is not necessarily zero. The harmless normalization $-2r_1$ in lieu of r_1 will spare us the encumbrance of an ubiquitous factor of $-\frac{1}{2}$ in what follows.

We will now introduce the notation that will be used in the remainder of §2.3. The relationship between the different PPV fields and PPV groups to which we will refer in §2.3.1 and §2.3.2 is summarized in the diagram (2.22). The solutions for (2.19) are related to the solutions for an associated unimodular equation by a classical change of variables. Set

$$q := r_1^2 - \delta_x r_1 - r_0,$$

let M denote a PPV extension of K for (2.2), and denote by $H := \mathrm{Gal}_\Delta(M/K)$ the corresponding PPV group, which from now on we assume to be already known by the results presented in §2.1 and §2.2.

Let $\{\eta, \xi\}$ denote a basis for the solution space of

$$\delta_x^2 Y - qY = 0, \tag{2.20}$$

and let U denote a PPV extension of M for

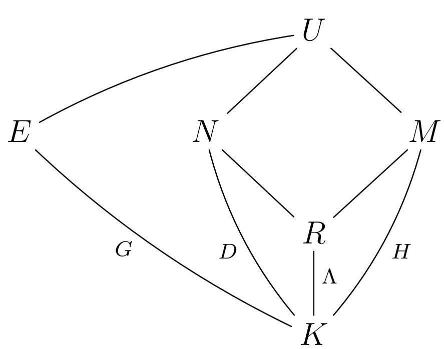
$$\delta_x Y - r_1 Y = 0. \tag{2.21}$$

Choose $0 \neq \zeta \in U$ such that $\delta_x \zeta = r_1 \zeta$, and let $\psi \in M$ be any solution to (2.20). Now expand $\delta_x^2(\zeta\psi) - 2r_1\delta_x(\zeta\psi) + r_0\zeta\psi$, to obtain:

$$\begin{aligned}
&\psi\delta_x^2\zeta + 2(\delta_x\zeta)(\delta_x\psi) + \zeta\delta_x^2\psi - 2r_1(\psi\delta_x\zeta + \zeta\delta_x\psi) + r_0\zeta\psi \\
=&(\delta_x r_1 + r_1^2)\zeta\psi + 2r_1\zeta\delta_x\psi + q\zeta\psi - 2r_1^2\zeta\psi - 2r_1\zeta\delta_x\psi + r_0\zeta\psi \\
=&(q - r_1^2 + \delta_x r_1 + r_0)\zeta\psi = 0,
\end{aligned}$$

and therefore $\{\zeta\eta, \zeta\xi\}$ is an F-basis for the solution space of (2.19), whence $E := K\langle\zeta\eta, \zeta\xi\rangle_\Delta \subseteq U$ is a PPV extension of K for (2.19), and its PPV group is $G := \mathrm{Gal}_\Delta(E/K)$.

Letting $N := K\langle\zeta\rangle_\Delta \subseteq U$, note that N is a PPV extension of K for (2.21), and denote its PPV group by D (the mnemonic is "determinant"). The computation of D is analogous to that of A in case I (see §2.1). Finally, define the Δ-field $R := M \cap N \subseteq U$. Since $D \subseteq \mathbb{G}_m(F)$ is abelian, $\mathrm{Gal}_\Delta(N/R) \subseteq D$ is normal, and therefore R is a PPV extension of K, with PPV group denoted by Λ. We obtain a lattice (2.22) of PPV extensions and PPV groups:



- E is a PPV extension of K for (2.19);
- N is a PPV extension of K for (2.21);
- M is a PPV extension of K for (2.20);
- U is a PPV extension of M for (2.21);
- $R = M \cap N$ is a PPV extension of K.

(2.22)

Remark 2.8. In fact, $U = K\langle\zeta\eta, \zeta\xi, \zeta\rangle_\Delta$ is a PPV extension of K for

$$\delta_x^3 Y - \left(3r_1 + \tfrac{\delta_x q}{q}\right)\delta_x^2 Y + \left(2r_1^2 - 2\delta_x r_1 + r_0 + 2r_1\tfrac{\delta_x q}{q}\right)\delta_x Y + \left(\delta_x r_0 - r_1 r_0 - r_0\tfrac{\delta_x q}{q}\right)Y = 0. \quad (2.23)$$

To verify that each of $\zeta\eta$, $\zeta\xi$, and ζ satisfies (2.23), note that

$$\delta_x^2\zeta - 2r_1\delta_x\zeta + r_0\zeta = -q\zeta,$$

and expand the following product in $K[\delta_x]$ to obtain the linear differential operator implicit in (2.23):

$$\left(\delta_x - r_1 - \tfrac{\delta_x q}{q}\right) \circ (\delta_x^2 - 2r_1\delta_x + r_0).$$

Denote by $\Gamma := \mathrm{Gal}_\Delta(U/K)$ the PPV group of U over K. The choice of Δ-field generators $\{\eta, \xi, \zeta\}$ for U over K produces the following embedding of Γ in $\mathrm{GL}_3(F)$:

$$\gamma \mapsto \begin{pmatrix} a_\gamma & b_\gamma & 0 \\ c_\gamma & d_\gamma & 0 \\ 0 & 0 & e_\gamma \end{pmatrix}, \qquad \text{where} \qquad \begin{aligned} \gamma(\eta) &= a_\gamma\eta + c_\gamma\xi; \\ \gamma(\xi) &= b_\gamma\eta + d_\gamma\xi; \\ \gamma(\zeta) &= e_\gamma\zeta. \end{aligned} \tag{2.24}$$

Embedding G in $\mathrm{GL}_2(F)$ by means of the basis $\{\zeta\eta, \zeta\xi\}$, the surjection $\Gamma \twoheadrightarrow G$ is then given by $\gamma \mapsto e_\gamma \cdot \left(\begin{smallmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{smallmatrix}\right)$, and therefore

$$G \simeq \left\{ \begin{pmatrix} e_\gamma a_\gamma & e_\gamma b_\gamma \\ e_\gamma c_\gamma & e_\gamma d_\gamma \end{pmatrix} \;\middle|\; \gamma \in \Gamma \right\}. \tag{2.25}$$

Our next task is to apply the parameterized Galois correspondence [9, Thm. 3.5] to the lattice (2.22) to compute Γ, and therefore G, in terms of H and D. The arguments are formal, and familiar from classical Galois theory.

Lemma 2.9. *The restriction homomorphisms*

$$\mathrm{Gal}_\Delta(U/M) \to \mathrm{Gal}_\Delta(N/R); \qquad \textit{and} \qquad \mathrm{Gal}_\Delta(U/N) \to \mathrm{Gal}_\Delta(M/R),$$

defined respectively by $\gamma \mapsto \gamma|_N$ *and* $\gamma \mapsto \gamma|_M$*, are isomorphisms.*

Proof. Since $U = M \cdot N$, these homomorphisms are injective. Since the image of $\mathrm{Gal}_\Delta(U/M)$ in $\mathrm{Gal}_\Delta(N/R)$ under the restriction homomorphism $\gamma \mapsto \gamma|_N$ is Kolchin-closed (by the parameterized Galois correspondence [9, Thm. 3.5]), its fixed field R' is an intermediate Δ-field extension $R \subseteq R' \subseteq N$. Since every $f \in R'$ is fixed by every $\gamma \in \mathrm{Gal}_\Delta(U/M)$, it follows that $f \in M$, whence $f \in R$. By [9, Thm. 3.5], the image of $\mathrm{Gal}_\Delta(U/M)$ must be all of $\mathrm{Gal}_\Delta(N/R)$. The surjectivity of $\mathrm{Gal}_\Delta(U/N) \to \mathrm{Gal}_\Delta(M/R)$ is proved analogously. $\square$

Proposition 2.10. *The canonical homomorphism*

$$\Gamma \to H \times_\Lambda D := \{(\sigma, \tau) \in H \times D \mid \sigma|_R = \tau|_R\},$$

given by $\gamma \mapsto (\gamma|_M, \gamma|_N)$*, is an isomorphism.*

Proof. Injectivity follows from the fact that $U = M \cdot N$. To establish surjectivity, let $\sigma \in H$ and $\tau \in D$ be such that $\sigma|_R = \tau|_R =: \lambda \in \Lambda$. Now choose $\tilde{\lambda} \in \Gamma$ such that $\tilde{\lambda}|_R = \lambda$, and define elements

$$\sigma' := \sigma \circ \tilde{\lambda}|_M^{-1} \in \mathrm{Gal}_\Delta(M/R); \quad \text{and}$$

$$\tau' := \tau \circ \tilde{\lambda}|_N^{-1} \in \mathrm{Gal}_\Delta(N/R).$$

There exist $\tilde{\sigma} \in \mathrm{Gal}_\Delta(U/N)$ and $\tilde{\tau} \in \mathrm{Gal}_\Delta(U/M)$ such that $\tilde{\sigma}|_M = \sigma'$ and $\tilde{\tau}|_N = \tau'$, by Lemma 2.9. Now a computation shows $\gamma := \tilde{\sigma} \circ \tilde{\tau} \circ \tilde{\lambda} \in \Gamma$ satisfies $\gamma|_M = \sigma$ and $\gamma|_N = \tau$. $\square$

Corollary 2.11. *The PPV group $\Lambda = \{1\}$ if and only if $\Gamma \simeq H \times D$. In this case,*

$$G \simeq \left\{ \begin{pmatrix} ea & eb \\ ec & ed \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H,\ e \in D \right\}.$$

Assuming H has already been computed, we now apply Proposition 2.10 to compute G, beginning with the comparatively simpler cases II, III, and IV, in which H (and hence G) is guaranteed to be reductive. See [41, 42] for a general discussion of unipotent radicals and reductivity in the context of LDAGs, and for algorithms to compute PPV groups for higher-order equations which either are reductive, or whose quotient by the unipotent radical is differentially constant.

2.3.1 Reductive cases: II, III, and IV

We will now apply Proposition 2.10 to compute G in case II. Recall that there exists a solution u to the Riccati equation (2.4) such that u is quadratic over K. We denote by $\bar{u}$ the unique Galois conjugate of u, and set

$$w := u - \bar{u}.$$

Then $w^2 \in K$, so $\frac{\delta_x w}{w} =: v \in K$. There is a differential algebraic subgroup $A \subseteq \mathbb{G}_m(F)$ such that $H \simeq A \rtimes \{\pm 1\}$.

Proposition 2.12. *In case II, with notation as above, exactly one of the following possibilities holds:*

(i) *D is finite of even order $2k$, and $v - kr_1 = \frac{\delta_x f}{f}$ for some $f \in K$.*

(ii) $\Lambda = \{1\}$.

Consequently, in each of these cases G coincides with the subset of matrices in

$$\left\{ \begin{pmatrix} e_1 a & 0 \\ 0 & e_1 a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -e_2 a \\ e_2 a^{-1} & 0 \end{pmatrix} \,\middle|\, a \in A;\ e_1, e_2 \in D \right\}$$

that satisfy the corresponding set of conditions below:

(1) In case (i), $e_1^k = 1$ and $e_2^k = -1$.

(2) In case (ii), there are no further conditions.

Proof. We recall the diagram of PPV groups and PPV fields (2.22) introduced at the beginning of §2.3 (see also Remark 2.8): $R := M \cap N$, $M := K\langle \eta, \xi \rangle_\Delta$, $N := K\langle \zeta \rangle$, and $\Lambda := \mathrm{Gal}_\Delta(R/K)$. Since the commutator subgroup $[H, H]$ of H coincides with the set of matrices $\left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \mid a \in A \right\}$, and Λ is abelian, the surjection $H \twoheadrightarrow \Lambda$ factors through $H/[H, H] \simeq \{\pm 1\}$, the PPV group of the quadratic subextension $K(u) \subset M$. Hence, $R \subseteq K(u)$ and Λ is finite of order at most 2.

If $D \subseteq \mathbb{G}_m(F)$ is infinite, then it is also connected [7, Corollary, p. 938] (cf. Proposition 1.6), so its only finite quotient is $\Lambda = \{1\}$. If D is finite of odd order, then $\Lambda = \{1\}$ is the only common quotient of $\{\pm 1\}$ and D. Hence, if $\Lambda \neq \{1\}$, D must be finite of even order $2k$. Since $D = \mu_{2k}$ is cyclic, the field $K(\zeta^k)$ is the only quadratic subextension of $K(\zeta)$. Therefore, either $R = K$ or $K(w) = R = K(\zeta^k)$. If $R = K$, or equivalently if $\Lambda = \{1\}$, we obtain G from Corollary 2.11.

We claim that $K(w) = K(\zeta^k)$ if and only if $w\zeta^{-k} \in K$. Indeed, since $w^2 \in K$, the set $\{1, w\}$ is a K-basis for $K(w)$ considered as a K-vector space. If $K(w) = K(\zeta^k)$, then there exist elements $f_1, f_2 \in K$ such that $\zeta^k = f_1 + f_2 w$. Since

$$\zeta^{2k} = f_1^2 + f_2^2 w^2 + 2 f_1 f_2 w \in K$$

and $\zeta^k, w \notin K$, this implies that $f_1 = 0$. Therefore

$$w\zeta^{-k} = f_2^{-1} =: f \in K, \qquad \text{whence}$$

$$v - kr_1 = \tfrac{\delta_x w}{w} - k\tfrac{\delta_x \zeta}{\zeta} = \tfrac{\delta_x (w\zeta^{-k})}{w\zeta^{-k}} = \tfrac{\delta_x f}{f}.$$

Letting $a_\gamma := \frac{\gamma w}{w}$ and $e_\gamma := \frac{\gamma\zeta}{\zeta}$ for $\gamma \in \Gamma$ (cf. Remark 2.8), then $\gamma(f) = a_\gamma e_\gamma^{-k} f = f$, and therefore $a_\gamma e_\gamma^{-k} = 1$. In other words, $\gamma w = w$ if and only if $e_\gamma^k = 1$, and $\gamma w = -w$ if and only if $e_\gamma^k = -1$. To conclude the proof of (1), note that the elements of H that fix w are precisely those of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ for $a \in A$ [37, §4.3]. $\square$

Remark 2.13. In case III, H is a finite subgroup of $\mathrm{SL}_2(F)$. If $D \subseteq \mathbb{G}_m(F)$ is infinite, then $\Lambda = \{1\}$, since it is finite and connected, and we obtain G from Corollary 2.11. If $D = \mu_s$ is finite, then U is algebraic over K, and therefore so is E. By [9, Prop. 3.6(2)], G coincides with the (non-parameterized) PV group of (2.19). The computation of G in this case is only sketched, as it is probably well-known but I remain unaware of a reference to this computation in the literature.

For each factor ℓ of s and each irreducible character $\chi : H \to \mu_\ell$, there is an element $w_\chi \in M$ such that $K(w_\chi) \subset M$ is cyclic of order ℓ and $\chi(\sigma) = \frac{\sigma w_\chi}{w_\chi}$ for each $\sigma \in H$. Thus, $K(w_\chi)$ is the fixed field of $\ker(\chi)$. Such an element w_χ can be computed effectively (cf. the *semi-invariants* discussed in [53, §4.3.1]). Let

$$v_\chi := \tfrac{\delta_x w_\chi}{w_\chi} \in K.$$

If there exist integers $0 < k_1 < \ell$ and $0 < k_2 < \frac{s}{\ell}$ such that

$$k_1 v_\chi - k_2 r_1 = \tfrac{\delta_x f}{f}$$

for some $f \in K$, then

$$G \simeq \{(\sigma, e) \in H \times D \mid \chi(\sigma)^{k_1} = e^{k_2}\}.$$

If no such k_1 and k_2 can be found for any $\chi \in H^*$, the character group of H, then $\Lambda = \{1\}$.

When H is finite in cases I and II (i.e., A is finite and $B = 0$), the computation of G performed in Theorem 2.15 and Proposition 2.12 coincides with the one just described.

Remark 2.14. In case IV, we recall from §2.1 that there is a finite subset Π' of the F-linear span of Π consisting of F-linearly independent, pairwise commuting derivations such that H is isomorphic to the simple group $\mathrm{SL}_2(F^{\Pi'})$ [7, Prop. 42]. Therefore, the only abelian quotient of H in this case is $\Lambda = \{1\}$, and by Corollary 2.11

$$G \simeq \left\{ \begin{pmatrix} ea & eb \\ ec & ed \end{pmatrix} \,\middle|\, e \in D \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F^{\Pi'}) \right\}.$$

2.3.2 Non-reductive case: I

In case I, there exists a solution $u \in K$ for the Riccati equation (2.4). We may choose the basis $\{\eta, \xi\}$ for the solution space of (2.20) so that $\delta_x \eta = u\eta$ and $\delta_x\left(\frac{\xi}{\eta}\right) = \eta^{-2}$ (cf. §2.1). The embedding $H \hookrightarrow \mathrm{SL}_2(F)$ is then given by the formulae $\sigma(\eta) = a_\sigma \eta$ and $\sigma(\xi) = a_\sigma^{-1}\xi + b_\sigma \eta$ (cf. Remark 2.8), and there are differential algebraic subgroups $A \subseteq \mathbb{G}_m(F)$ and $B \subseteq \mathbb{G}_a(F)$ such that H is as in (2.5). The Δ-field $L := K\langle\eta\rangle_\Delta$ is a PPV extension of K for (2.6), $A \simeq \mathrm{Gal}_\Delta(L/K)$ and $B \simeq \mathrm{Gal}_\Delta(M/L)$.

The following result consists of two parts. The second part describes the defining equations for G as a subset of

$$\left\{ \begin{pmatrix} ea & eb \\ 0 & ea^{-1} \end{pmatrix} \;\middle|\; a \in A,\ b \in B,\ e \in D \right\},$$

as per the conclusion of Proposition 2.10. There are four cases (1)–(4) that can occur, and the first part of the theorem describes a test to decide which case applies, depending on whether or not one can verify the conditions contained in (i)–(iv).

Theorem 2.15. *In case I, with notation as above, exactly one of the following possibilities holds:*

(i) Consider the set C_1 consisting of pairs of integers (n_1, n_2), both of them non-zero, such that:

- *there exists $a \in A$ such that $a^{n_1} \neq 1$; and*
- *there exists $e \in D$ such that $e^{n_2} \neq 1$; and*
- *there exists $f \in K^\times$ such that $n_1 u - n_2 r_1 = \frac{\delta_x f}{f}$.*

The set C_1 is nonempty. Moreover, in this case there exists a pair of integers $(k_1, k_2) \in C_1$ such that

$$K\langle \eta^{k_1} \rangle_\Delta = R = K\langle \zeta^{k_2} \rangle_\Delta. \tag{2.26}$$

(ii) Consider the set C_2 consisting of pairs $(\mathbf{p}, \mathbf{q})$ of linear differential polynomials $\mathbf{p}, \mathbf{q} \in F\{Y_1, \ldots, Y_m\}^1_\Pi$, both of them non-zero, such that:

- *there exists $a \in A$ such that $\mathbf{p}(\frac{\Pi a}{a}) \neq 0$; and*
- *there exists $e \in D$ such that $\mathbf{q}(\frac{\Pi e}{e}) \neq 0$; and*
- *there exists $f \in K$ such that $\mathbf{p}(\Pi u) - \mathbf{q}(\Pi r_1) = \delta_x f$.*

The set C_1 from case (i) is empty, whereas C_2 is non-empty. Moreover, in this case there exist finitely many pairs $(\mathbf{p_1}, \mathbf{q_1}), \ldots, (\mathbf{p}_s, \mathbf{q}_s) \in C_2$ such that

$$K\left\langle \mathbf{p}_1\left(\tfrac{\Pi\eta}{\eta}\right), \ldots, \mathbf{p}_s\left(\tfrac{\Pi\eta}{\eta}\right) \right\rangle_\Delta = R = K\left\langle \mathbf{q}_1\left(\tfrac{\Pi\zeta}{\zeta}\right), \ldots, \mathbf{q}_s\left(\tfrac{\Pi\zeta}{\zeta}\right) \right\rangle_\Delta. \tag{2.27}$$

(iii) Consider the set C_3 consisting of pairs $(\mathbf{p}, \mathbf{q})$ of linear differential polynomials $\mathbf{p} \in F\{Y\}^1_\Pi$ and $\mathbf{q} \in F\{Y_1, \ldots, Y_m\}^1_\Pi$, both of them non-zero, such that

- *there exists $b \in B$ such that $\mathbf{p}(b) \neq 0$; and*

- *there exists* $e \in D$ *such that* $\mathbf{q}(\frac{\Pi e}{e}) \neq 0$; *and*

- *there exists* $f \in K$ *such that* $\mathbf{p}(\eta^{-2}) - \mathbf{q}(\Pi r_1) = \delta_x f$.

The set C_3 *is non-empty. Moreover, in this case there exist finitely many pairs* $(\mathbf{p_1}, \mathbf{q_1}), \ldots, (\mathbf{p}_s, \mathbf{q}_s) \in C_3$ *such that*

$$K\Big\langle \mathbf{p}_1\big(\tfrac{\xi}{\eta}\big), \ldots, \mathbf{p}_s\big(\tfrac{\xi}{\eta}\big)\Big\rangle_\Delta = R = K\Big\langle \mathbf{q}_1\big(\tfrac{\Pi\zeta}{\zeta}\big), \ldots, \mathbf{q}_s\big(\tfrac{\Pi\zeta}{\zeta}\big)\Big\rangle_\Delta. \tag{2.28}$$

(iv) $\Lambda = \{1\}$.

Consequently, in each of these cases G *coincides with the subset of matrices in*

$$\left\{ \begin{pmatrix} ea & eb \\ 0 & ea^{-1} \end{pmatrix} \;\middle|\; a \in A,\ b \in B,\ e \in D \right\} \tag{2.29}$$

that satisfy the corresponding set of conditions below:

(1) In case (i), $a^{k_1} = e^{k_2}$.

(2) In case (ii), $\mathbf{p}_i(\frac{\Pi a}{a}) = \mathbf{q}_i(\frac{\Pi e}{e})$ *for* $1 \leq i \leq s$.

(3) In case (iii), $\mathbf{p}_i(b) = \mathbf{q}_i(\frac{\Pi e}{e})$ *for* $1 \leq i \leq s$.

(4) In case (iv), *there are no further conditions.*

We will show slightly more in the course of the proof. The following additional facts mediate criteria that, based solely on the data of A and D, eliminate from consideration certain possibilities from Theorem 2.15 without testing them directly.

Corollary 2.16. *The following cases refer to the list of possibilities in the first part of Theorem 2.15.*

(i) In case (i), either A and D are both finite, or else they coincide as subgroups of $\mathbb{G}_m(F)$.

(iii) In case (iii), $A \subseteq \{\pm 1\}$, whence $\eta^2 \in K$. Therefore, the test comprised in (iii) concerns elements of K only.

Proof of Theorem 2.15. That the possibilities (i)–(iv) are exhaustive and mutually exclusive will be proved in Propositions 2.17 and 2.20 below. Let us prove that each of these possibilities implies that G is defined as a subset of (2.29) by the corresponding equations contained in (1)–(4), and that no more equations are required. In case (iv), the fact that G coincides with (2.29) is Corollary 2.11.

In case (i), let $(n_1, n_2) \in C_1$, let $f \in K$ be such that $n_1 u - n_2 r_1 = \frac{\delta_x f}{f}$, and observe that

$$\tfrac{\delta_x(\eta^{n_1}\zeta^{-n_2}f^{-1})}{\eta^{n_1}\zeta^{-n_2}f^{-1}} = n_1\tfrac{\delta_x\eta}{\eta} - n_2\tfrac{\delta_x\zeta}{\zeta} - \tfrac{\delta_x f}{f} = n_1 u - n_2 r_1 - \tfrac{\delta_x f}{f} = 0.$$

Therefore, $\eta^{n_1}\zeta^{-n_2} = cf$ for some $c \in F$. Letting $a_\gamma := \frac{\gamma(\eta)}{\eta}$ and $e_\gamma := \frac{\gamma(\zeta)}{\zeta}$ for $\gamma \in \Gamma$ (recall that Γ is the PPV group of $U := K\langle\eta, \xi, \zeta\rangle_\Delta$ over K; these actions of Γ on $F \cdot \eta$ and $F \cdot \zeta$ result from the equations $\delta_x \eta = u\eta$ and $\delta_x \zeta = r_1 \zeta$), we have that

$$\gamma(cf) = a_\gamma^{n_1} e_\gamma^{-n_2} cf = cf,$$

and therefore $a_\gamma^{n_1} = e_\gamma^{n_2}$ for every $\gamma \in \Gamma$. Since Γ surjects both onto A and onto D, this implies that A is finite if and only if D is finite. If A and D are infinite, then A and D are defined by the same linear differential polynomials (see Proposition 1.6), because

$$n_1 \mathbf{r}(\tfrac{\Pi a_\gamma}{a_\gamma}) = \mathbf{r}(\tfrac{\Pi(a_\gamma^{n_1})}{a_\gamma^{n_1}}) = \mathbf{r}(\tfrac{\Pi(e_\gamma^{n_2})}{e_\gamma^{n_2}}) = n_2 \mathbf{r}(\tfrac{\Pi e_\gamma}{e_\gamma}) \tag{2.30}$$

for every $\gamma \in \Gamma$ and every $\mathbf{r} \in F\{Y_1, \dots, Y_m\}_\Pi^1$, and the integers n_1 and n_2 are both different from zero. It follows from (2.26) that $R \subseteq L$, and therefore $B \subseteq \ker(H \twoheadrightarrow \Lambda)$. It also follows from (2.26) that the surjection $A \twoheadrightarrow \Lambda$ (resp., $D \twoheadrightarrow \Lambda$) is given by $a \mapsto a^{k_1}$ (resp., $e \mapsto e^{k_2}$). By Proposition 2.10, (1) describes G as a subset of (2.29).

In case (ii), let $(\mathbf{p}, \mathbf{q}) \in C_2$, and let $f \in K$ be such that

$$\mathbf{p}(\Pi u) - \mathbf{q}(\Pi r_1) = \delta_x f.$$

Since $\delta_x(\frac{\partial \eta}{\eta}) = \partial u$ and $\delta_x(\frac{\partial \zeta}{\zeta}) = \partial r_1$ for each $\partial \in \Pi$, it follows that

$$\delta_x\big(\mathbf{p}(\tfrac{\Pi\eta}{\eta}) - \mathbf{q}(\tfrac{\Pi\zeta}{\zeta})\big) = \mathbf{p}(\Pi u) - \mathbf{q}(\Pi r_1) = \delta_x f,$$

and therefore there exists $c \in F$ such that $\mathbf{p}(\frac{\Pi\eta}{\eta}) - \mathbf{q}(\frac{\Pi\zeta}{\zeta}) = f + c$. Since, for each $\gamma \in \Gamma$,

$$\gamma\big(\mathbf{p}(\tfrac{\Pi\eta}{\eta})\big) = \mathbf{p}(\tfrac{\Pi\eta}{\eta}) + \mathbf{p}(\tfrac{\Pi a_\gamma}{a_\gamma}); \quad \text{and} \tag{2.31}$$

$$\gamma\big(\mathbf{q}(\tfrac{\Pi\zeta}{\zeta})\big) = \mathbf{q}(\tfrac{\Pi\zeta}{\zeta}) + \mathbf{q}(\tfrac{\Pi e_\gamma}{e_\gamma}), \tag{2.32}$$

the computation

$$\gamma(f + c) = f + c + \mathbf{p}(\tfrac{\Pi a_\gamma}{a_\gamma}) - \mathbf{q}(\tfrac{\Pi e_\gamma}{e_\gamma}) = f + c$$

implies that

$$\mathbf{p}(\tfrac{\Pi a_\gamma}{a_\gamma}) = \mathbf{q}(\tfrac{\Pi e_\gamma}{e_\gamma})$$

for each $\gamma \in \Gamma$. Since A is not Π-constant, it is infinite, and it follows from Lemma 2.18 that B is in the kernel of $H \twoheadrightarrow \Lambda$, and therefore the equations defining G as a subset of (2.29) do not involve B, by Proposition 2.10. It follows from (2.27) that the surjection $A \twoheadrightarrow \Lambda$ (resp., $D \twoheadrightarrow \Lambda$) is given by $a \mapsto (\mathbf{p}_1(\frac{\Pi a}{a}), \ldots, \mathbf{p}_s(\frac{\Pi a}{a}))$ (resp., $e \mapsto (\mathbf{q}_1(\frac{\Pi e}{e}), \ldots, \mathbf{q}_s(\frac{\Pi e}{e})))$. By Proposition 2.10, (2) describes G as a subset of (2.29).

In case (iii), let $(\mathbf{p}, \mathbf{q}) \in C_3$, let $f \in K$ be such that $\mathbf{p}(\eta^{-2}) - \mathbf{q}(\Pi r_1) = \delta_x f$, and observe that

$$\delta_x\big(\mathbf{p}(\tfrac{\xi}{\eta}) - \mathbf{q}(\tfrac{\Pi\zeta}{\zeta})\big) = \mathbf{p}(\eta^{-2}) - \mathbf{q}(\Pi r_1) = \delta_x f.$$

Therefore, there exists $c \in F$ such that $\mathbf{p}(\frac{\xi}{\eta}) - \mathbf{q}(\frac{\Pi\zeta}{\zeta}) = f + c$. It follows from Proposition 2.20 that case (iii) can occur only if $A \subseteq \{\pm 1\}$, which implies that $\eta^2 \in K$. Let $b_\gamma := \gamma\big(\frac{\xi}{\eta}\big) - \frac{\xi}{\eta}$ for each $\gamma \in \Gamma$ (this action is deduced from the fact that $\delta_x(\frac{\xi}{\eta}) = \eta^{-2} \in K$; cf. Remark 2.8). Since

$$\gamma\big(\mathbf{p}\big(\tfrac{\xi}{\eta}\big)\big) = \mathbf{p}\big(\tfrac{\xi}{\eta}\big) + \mathbf{p}(b_\gamma), \tag{2.33}$$

the computation

$$\gamma(f + c) = f + c + \mathbf{p}(b_\gamma) - \mathbf{q}(\tfrac{\Pi e_\gamma}{e_\gamma}) = f + c$$

implies that

$$\mathbf{p}(b_\gamma) = \mathbf{q}(\tfrac{\Pi e_\gamma}{e_\gamma})$$

for each $\gamma \in \Gamma$. By Proposition 2.17, $L \cap R = K$. Since $A \simeq \mathrm{Gal}_\Delta(L/K)$, the equations defining G in (2.29) do not involve A. It follows from (2.28) that the surjection $H \twoheadrightarrow \Lambda$ (resp., $D \twoheadrightarrow \Lambda$) is given by $\left(\begin{smallmatrix} \pm 1 & b \\ 0 & \pm 1 \end{smallmatrix}\right) \mapsto (\mathbf{p}_1(b), \ldots, \mathbf{p}_s(b))$ (resp., $e \mapsto (\mathbf{q}_1(\frac{\Pi e}{e}), \ldots, \mathbf{q}_s(\frac{\Pi e}{e}))$). By Proposition 2.10, (3) describes G as a subset of (2.29). $\square$

In proving the first part of Theorem 2.15, it is convenient to treat separately the cases where $A \subseteq \{\pm 1\}$ and $A \nsubseteq \{\pm 1\}$. This is done in Proposition 2.17 and Propostion 2.20, respectively. These results are obtained as consequences of Theorem 1.3.

When $A \subseteq \{\pm 1\}$, the first part of Theorem 2.15 follows from the following result.

Proposition 2.17. *If $A \subseteq \{\pm 1\}$, then $\eta^2 \in K$ and exactly one of the following possibilities holds:*

(i) *$A = \{\pm 1\}$, D is finite of even order $2k$, and*

$$u - kr_1 = \tfrac{\delta_x f}{f}$$

for some $f \in K$. Moreover, in this case

$$L = K(\eta) = R = K(\zeta^k).$$

(iii) *Consider the set C_3 consisting of pairs $(\mathbf{p}, \mathbf{q})$ of linear differential polynomials $\mathbf{p} \in F\{Y\}^1_\Pi$ and $\mathbf{q} \in F\{Y_1, \dots, Y_m\}^1_\Pi$, both of them non-zero, such that*

- *there exists $b \in B$ such that $\mathbf{p}(b) \neq 0$; and*
- *there exists $e \in D$ such that $\mathbf{q}(\frac{\Pi e}{e}) \neq 0$; and*
- *there exists $f \in K$ such that $\mathbf{p}(\eta^{-2}) - \mathbf{q}(\Pi r_1) = \delta_x f$.*

The set C_3 is non-empty. Moreover, in this case there exist finitely many pairs $(\mathbf{p_1}, \mathbf{q_1}), \dots, (\mathbf{p}_s, \mathbf{q}_s) \in C_3$ such that

$$K\Big\langle \mathbf{p}_1\big(\tfrac{\xi}{\eta}\big), \dots, \mathbf{p}_s\big(\tfrac{\xi}{\eta}\big)\Big\rangle_\Delta = R = K\Big\langle \mathbf{q}_1\big(\tfrac{\Pi\zeta}{\zeta}\big), \dots, \mathbf{q}_s\big(\tfrac{\Pi\zeta}{\zeta}\big)\Big\rangle_\Delta. \tag{2.34}$$

(iv) $\Lambda = \{1\}$.

Proof. If $A \subseteq \{\pm 1\}$, then $\sigma\eta = \pm\eta$ for every $\sigma \in H$, whence $\eta^2 \in K$ and $L = K(\eta)$. If $\Lambda \neq \{1\}$, there exists an element $g \in R$ such that $g \notin K$, and there exist non-constant differential rational functions

$$Q_1 \in K\langle Y\rangle_\Pi \qquad \text{and} \qquad Q_2 \in K\langle Y_1, Y_2\rangle_\Pi$$

with coefficients in K, such that

$$Q_1(\zeta) = g = Q_2\left(\eta, \tfrac{\xi}{\eta}\right). \tag{2.35}$$

Clearing denominators in (2.35) shows that the elements ζ, η, and $\frac{\xi}{\eta}$ are Π-algebraically dependent over K. By Theorem 1.3, either there exist integers $k_1, k_2 \in \mathbb{Z}$, none of them zero, such that $\eta^{k_1}\zeta^{k_2} \in K$, or else there exist non-zero linear differential polynomials $\mathbf{q} \in F\{Y_1, \dots, Y_m\}^1_\Pi$ and $\mathbf{p} \in F\{Y\}^1_\Pi$ such that

$$\mathbf{p}\big(\tfrac{\xi}{\eta}\big) - \mathbf{q}\big(\tfrac{\Pi\zeta}{\zeta}\big) =: f \in K. \tag{2.36}$$

First, suppose that $\eta^{k_1}\zeta^{k_2} \in K$ as above. Since $\Lambda \neq \{1\}$, we may assume that $k_1 = 1$, $\eta \notin K$, and therefore $\zeta^{k_2} \notin K$. Now $\eta\zeta^{k_2} \in K$ implies that $\zeta^{2k_2} \in K$, whence D is finite. If we denote by $k > 0$ the smallest such integer k_2, then $K(\zeta^k) = L = R$, the order of D is $2k$, and

$$u - kr_1 = \tfrac{\delta_x\eta}{\eta} - k\tfrac{\delta_x\zeta}{\zeta} = \tfrac{\delta_x(\eta\zeta^{-k})}{\eta\zeta^{-k}} = \tfrac{\delta_x f}{f}.$$

Supposing instead that there are linear differential polynomials $\mathbf{p}$ and $\mathbf{q}$ as in (2.36), we may assume that $\mathbf{p}\big(\frac{\xi}{\eta}\big) \notin K$, and that $\mathbf{q}(\frac{\Pi\zeta}{\zeta}) \notin K$, since $\Lambda \neq \{1\}$. This implies that D is infinite, and hence connected [7, Corollary, p. 938] (cf. Proposition 1.6), because whenever D is finite we have that $\frac{\partial\zeta}{\zeta} \in K$ for each $\partial \in \Pi$. Therefore, Λ is also connected (since it is a quotient of D). This implies that $L \cap R = K$, because otherwise we would have that $\eta \in R$ and $\eta \notin K$, but since $\eta^2 \in K$, the parameterized Galois correspondence would imply that $A = \{\pm 1\}$ is a quotient of Λ, contradicting the connectedness of Λ. Let $b_\gamma := \gamma(\frac{\xi}{\eta}) - \frac{\xi}{\eta}$ and $e_\gamma := \frac{\gamma(\zeta)}{\zeta}$ for each $\gamma \in \Gamma$ (cf. Remark 2.8), and note that

$$\gamma\big(\mathbf{p}\big(\tfrac{\xi}{\eta}\big)\big) = \mathbf{p}\big(\tfrac{\xi}{\eta}\big) + \mathbf{p}(b_\gamma) \qquad \text{and} \qquad \gamma\big(\mathbf{q}(\tfrac{\Pi\zeta}{\zeta})\big) = \mathbf{q}(\tfrac{\Pi\zeta}{\zeta}) + \mathbf{q}(\tfrac{\Pi e_\gamma}{e_\gamma}).$$

Hence, the parameterized Galois correspondence implies that there exists $b \in B$ such that $\mathbf{p}(b) \neq 0$ and there exists $e \in D$ such that $\mathbf{q}(\frac{\Pi e}{e}) \neq 0$. Observe that $\delta_x \mathbf{p}\big(\frac{\xi}{\eta}\big) = \mathbf{p}(\eta^{-2})$ and $\delta_x\big(\mathbf{q}(\frac{\Pi\zeta}{\zeta})\big) = \mathbf{q}(\Pi r_1)$, whence

$$\mathbf{p}(\eta^{-2}) - \mathbf{q}(\Pi r_1) = \delta_x(\mathbf{p}(\tfrac{\xi}{\eta}) - \mathbf{q}(\tfrac{\Pi\zeta}{\zeta})) = \delta_x f.$$

Therefore, $(\mathbf{p}, \mathbf{q}) \in C_3$.

We claim that there exists a finite collection of linear differential polynomials $\mathbf{p}_1, \dots, \mathbf{p}_s \in F\{Y\}^1_\Pi$ such that

$$R = K\langle \mathbf{p}_1(\tfrac{\xi}{\eta}), \dots, \mathbf{p}_s(\tfrac{\xi}{\eta})\rangle_\Delta.$$

To see this, note that the PPV group of $M' := K\langle\frac{\xi}{\eta}\rangle_\Delta$ is given by $B \subseteq \mathbb{G}_a(F)$ (since $\delta_x(\frac{\xi}{\eta}) = \eta^{-2} \in K$; cf. Remark 2.8), and that by the parameterized Galois correspondence [9, Thm. 3.5] R is determined by $\mathrm{Gal}_\Delta(M'/R) \subseteq B$. By Proposition 1.6, there exist finitely many linear differential polynomials $\mathbf{p}_1, \ldots, \mathbf{p}_s \in F\{Y\}^1_\Pi$ such that

$$\mathrm{Gal}_\Delta(M'/R) = \{b \in B \mid \mathbf{p}_i(b) = 0 \text{ for } 1 \leq i \leq s\},$$

and the fixed field of this differential algebraic subgroup of B is precisely $K\langle \mathbf{p}_1(\frac{\xi}{\eta}), \ldots, \mathbf{p}_s(\frac{\xi}{\eta})\rangle_\Delta$. To prove (2.34), it suffices to show that there exist $\mathbf{q}_1, \ldots, \mathbf{q}_s \in F\{Y_1, \ldots, Y_m\}^1_\Pi$ such that $(\mathbf{p}_i, \mathbf{q}_i) \in C_3$ for each $1 \leq i \leq s$, because then $\mathbf{p}_i(\frac{\xi}{\eta}) - \mathbf{q}_i(\frac{\Pi\zeta}{\zeta}) \in K$ implies that

$$R = K\langle \mathbf{p}_1(\tfrac{\xi}{\eta}), \ldots, \mathbf{p}_s(\tfrac{P\xi}{\eta})\rangle_\Delta = K\langle \mathbf{q}_1(\tfrac{\Pi\zeta}{\zeta}), \ldots, \mathbf{q}_s(\tfrac{\Pi\zeta}{\zeta})\rangle_\Delta.$$

By [41, Cor. 5.2], the differential type of $D = \mathrm{Gal}_\Delta(N/K)$ is 0. By [42, Prop. 3.2], this implies that N is of finite algebraic transcendence degree over K, so there exist finite subsets $\Theta_1, \ldots, \Theta_m \subset \Theta$, where Θ denotes the free commutative monoid generated by Π (cf. §1.1), such that

$$N' := K\langle\tfrac{\Pi\zeta}{\zeta}\rangle_\Delta = K\big(\theta_j(\tfrac{\partial_j\zeta}{\zeta}) \mid \theta_j \in \Theta_j, 1 \leq j \leq m\big).$$

We may assume without loss of generality that the $\theta_j(\frac{\partial_j\zeta}{\zeta})$, for θ_j ranging over Θ_j and j ranging over $\{1, \ldots, m\}$, are F-linearly independent modulo K. By

Theorem 1.2, the $\theta_j(\frac{\partial_j\zeta}{\zeta})$ must then also be K-algebraically independent, since $\delta_x(\theta\frac{\partial\zeta}{\zeta}) = \theta\partial r_1 \in K$ for each $\theta \in \Theta$ and $\partial \in \Pi$. This also implies that the underlying δ_x-field of N' is a (non-parameterized) PV extension of the δ_x-field K, whose corresponding PV ring is given by

$$P := K\big[\theta_j(\tfrac{\partial_j\zeta}{\zeta}) \mid \theta_j \in \Theta_j,\ 1 \leq j \leq m\big].$$

To conclude the proof of Proposition 2.17, we will show that if $\mathbf{p}(\frac{\xi}{\eta}) \in R$ for some $\mathbf{p} \in F\{Y\}^1_\Pi$, then there exists $f \in K$ and $\mathbf{q} \in F\{Y_1, \ldots, Y_m\}^1_\Pi$ such that

$$\mathbf{p}(\tfrac{\xi}{\eta}) = f + \mathbf{q}(\tfrac{\Pi\zeta}{\zeta}).$$

First, note that since $\delta_x\mathbf{p}(\frac{\xi}{\eta}) = \mathbf{p}(\eta^{-2}) \in K$, then $\sum_i K \cdot \delta_x^i(\mathbf{p}(\frac{\xi}{\eta}))$ is finite dimensional as a K-vector space, and therefore $\mathbf{p}(\frac{\xi}{\eta}) \in P$ by [50, Cor. 1.38] (cf. the proof of Lemma 2.6). For notational convenience, let us relabel:

$$\bigcup_{1\leq j\leq m} \{\theta(\tfrac{\partial_j\zeta}{\zeta}) \mid \theta \in \Theta_j\} = \{\beta_1, \ldots, \beta_S\}. \tag{2.37}$$

We recall that the β_i are K-algebraically independent and $\delta_x\beta_i \in K$ for each $1 \leq i \leq S$. Since $\mathbf{p}(\frac{\xi}{\eta}) \in P$, there exists an algebraic polynomial

$$Q \in K[Y_1, \ldots, Y_S]$$

such that $\mathbf{p}(\frac{\xi}{\eta}) = Q(\beta_1, \ldots, \beta_S)$, and therefore

$$\delta_x(Q(\underline{\beta})) = \delta_x\mathbf{p}(\tfrac{\xi}{\eta}) \in K.$$

Suppose $r_I\underline{\beta}^I$ is a term in $Q(\underline{\beta})$ (where $I = (i_1, \ldots, i_S)$ is a multiindex) with $r_I \neq 0$ and $|I| := \sum_j i_j > 0$ maximal. Letting $\mathbf{1}_j$ denote the multiindex with j-th entry 1 and all other entries 0, we obtain that

$$\delta_x(r_I\underline{\beta}^I) = (\delta_x r_I)\underline{\beta}^I + r_I \sum_{j=1}^{S} i_j(\delta_x\beta_j)\underline{\beta}^{I-\mathbf{1}_j},$$

whence it follows that $r_I \in F$, because the coefficient of $\underline{\beta}^I$ in $\delta_x(Q(\underline{\beta}))$ is precicely $\delta_x r_I = 0$.

Now fix $j \in \{1, \ldots, m\}$ such that i_j (the j-th entry in I) is non-zero, let $r_j \in K$ denote the coefficient of $\underline{\beta}^{I-\mathbf{1}_j}$ in $Q(\underline{\beta})$, and let c_k denote the coefficient of $\underline{\beta}^{I-\mathbf{1}_j+\mathbf{1}_k}$ in $Q(\underline{\beta})$ for each $k \neq j$. Since $|I - \mathbf{1}_j + \mathbf{1}_k| = |I|$ is also maximal, the argument above shows that $\delta_x c_k = 0$ for each $k \neq j$, so $c_k \in F$. If $I \neq \mathbf{1}_j$, then $\underline{\beta}^{I-\mathbf{1}_j} \notin K$, and the coefficient of $\underline{\beta}^{I-\mathbf{1}_j}$ in $\delta_x(Q(\underline{\beta}))$ is

$$i_j r_I \delta_x \beta_j + \delta_x r_j + \sum_{k\neq j} i_k c_k \delta_x \beta_k = \delta_x(i_j r_I \beta_j + r_j + \sum_{k\neq j} i_k c_k \beta_k) = 0,$$

which implies that

$$i_j r_I \beta_j + \sum_{k\neq j} i_k c_k \beta_k = -r_j + c \in K$$

for some $c \in F$, contradicting the F-linear independence of the β_i modulo K. Therefore, $I = \mathbf{1}_j$, and since we assumed that $|I| = 1$ was maximal, we have just shown that $Q(\underline{\beta}) = \sum_{j=1}^{S} r_j\beta_j + f$, for some $f \in K$ and $r_j \in F$. Recalling the relabelling (2.37), we see that there is a unique $\mathbf{q} \in F\{Y_1, \ldots, Y_m\}^1_\Pi$ such that $\sum_{j=1}^{S} r_j \beta_j = \mathbf{q}(\frac{\Pi\zeta}{\zeta})$, whence

$$\mathbf{p}(\tfrac{\xi}{\eta}) = Q(\underline{\beta}) = \mathbf{q}(\tfrac{\Pi\zeta}{\zeta}) + f,$$

as claimed. $\square$

To treat the case when $A \nsubseteq \{\pm 1\}$, we begin with two preliminary results.

Lemma 2.18. *If $A \nsubseteq \{\pm 1\}$, then B lies in the kernel of the restriction homomorphism $H \twoheadrightarrow \Lambda$.*

Proof. To show that $R \subseteq L$, we proceed by contradiction: suppose that $f \in R$ and $f \notin L$. There exist non-constant differential rational functions

$$Q_1, Q_2 \in L\langle Y \rangle_\Pi \qquad \text{such that}$$

$$Q_1\big(\zeta\big) = f = Q_2\big(\tfrac{\xi}{\eta}\big). \tag{2.38}$$

Clearing denominators in (2.38) yields a non-trivial differential-algebraic relation over L amongst the elements ζ and $\frac{\xi}{\eta}$. Since $\frac{\delta_x \zeta}{\zeta} = r_1 \in L$ and $\delta_x(\frac{\xi}{\eta}) = \eta^{-2} \in L$, by Theorem 1.3 there exist non-zero linear differential operators $\mathbf{q} \in F\{Y_1, \ldots, Y_m\}$ and $\mathbf{p} \in F\{Y\}^1_\Pi$ such that

$$\mathbf{p}\big(\tfrac{\xi}{\eta}\big) + \mathbf{q}\big(\tfrac{\Pi\zeta}{\zeta}\big) =: g \in L. \tag{2.39}$$

Since $f \notin L$, we may assume that $\mathbf{p}(\frac{\xi}{\eta}) \notin L$. Applying δ_x on both sides of (2.39), we obtain

$$\mathbf{p}(\eta^{-2}) + \mathbf{q}(\Pi r_1) = \delta_x g. \tag{2.40}$$

Now choose $\sigma \in \mathrm{Gal}_\Delta(L/K)$ such that $a_\sigma := \frac{\sigma\eta}{\eta} \in F^\Pi$ and $a_\sigma^2 \neq 1$, and apply $(\sigma - 1)$ to both sides of (2.40), to obtain

$$(a_\sigma^{-2} - 1)\mathbf{p}(\eta^{-2}) = \delta_x(\sigma g - g).$$

But this implies that $\mathbf{p}(\frac{\xi}{\eta}) \in L$, a contradiction. Therefore, $R \subseteq L$. $\square$

Lemma 2.19. *Suppose that $A \nsubseteq \{\pm 1\}$, and that $k \in \mathbb{Z}$ is such that $\eta^k \in R$ and $\eta^k \notin K$, then there exists an integer $n \in \mathbb{Z}$ such that $(k, n) \in C_1$.*

Proof. First assume that A is finite of order greater than 2. By Lemma 2.18, $B \subseteq \ker(H \twoheadrightarrow \Lambda)$, and therefore Λ must also be finite because it is a quotient of $H/B \simeq A$. Moreover, D must also be finite, because otherwise D would be connected [7, Corollary, p. 938], which would imply that its finite quotient Λ is also connected and therefore trivial, which is impossible because $R \neq K$. Let $\ell \in \mathbb{N}$ denote the order of D. Then $\eta^k \in N = K(\zeta)$ implies that there exist unique elements $r_0, \ldots, r_{\ell-1} \in K$ such that

$$\eta^k = \sum_{i=0}^{\ell-1} r_i \zeta^i, \tag{2.41}$$

Let $\gamma \in \Gamma$ be such that $e_\gamma := \frac{\gamma(\zeta)}{\zeta}$ is a primitive ℓ-th root of unity, and let $a_\gamma := \frac{\gamma(\eta)}{\eta}$ (these actions of Γ on η and ζ are deduced from the relations $\delta_x \eta = u\eta$ and $\delta_x \zeta = r_1 \zeta$, cf. Remark 2.8). Applying $a_\gamma^{-k}\gamma$ on both sides of

(2.41), we obtain

$$\sum_{i=0}^{\ell-1} r_i \zeta^i = \eta^k = a_\gamma^{-k}\gamma(\eta^k) = \sum_{i=0}^{\ell-1} r_i a_\gamma^{-k} e_\gamma^i \zeta^i,$$

which implies that $e_\gamma^i = a_\gamma^k$ whenever $r_i \neq 0$. Thus, there exists a unique $n \in \{0, \ldots, \ell-1\}$ such that $r_n \neq 0$. In fact, $n \neq 0$ because $\eta^k \notin K$. Thus, $\eta^k \zeta^{-n} = r_n \in K$ for some $1 \leq n \leq \ell - 1$, and

$$\tfrac{\delta_x r_n}{r_n} = \tfrac{\delta_x(\eta^k\zeta^{-n})}{\eta^k\zeta^{-n}} = k\tfrac{\delta_x\eta}{\eta} - n\tfrac{\delta_x\zeta}{\zeta} = ku - nr_1$$

implies that $(k, n) \in C_1$.

Now assume that A is infinite. Since A is connected by [7, Corollary, p. 938], its quotient $\Lambda \neq \{1\}$ is also connected, and therefore infinite, whence D must also be infinite because D surjects onto Λ by (2.22). Let

$$N' := K\langle \tfrac{\Pi\zeta}{\zeta}\rangle_\Delta,$$

so that $N = N'(\zeta)$. Since D is infinite, $2 \in \mathbb{G}_m(F^\Pi) \subset D$ (see Proposition 1.6), and therefore there exists $\gamma \in \Gamma$ such that $\gamma(\zeta) = 2\zeta$. Therefore, this γ fixes $\frac{\partial\zeta}{\zeta}$ for each $\partial \in \Pi$, so γ fixes all of N'. Since $\frac{\delta_x\zeta}{\zeta} = r_1 \in K$ and

$$\delta_x(\theta\tfrac{\partial\zeta}{\zeta}) = \theta\partial(\tfrac{\delta_x\zeta}{\zeta}) = \theta\partial r_1 \in K$$

for each $\theta \in \Theta$, Theorem 1.2 implies that ζ is algebraically transcendental over N'. Now let $\tilde{\gamma}$ denote the automorphism of

$$\tilde{N} := \overline{N'}(\zeta) = N \otimes_{N'} \overline{N'}$$

given by $\tilde{\gamma}(\zeta) = 2\zeta$ and $\tilde{\gamma}(r) = r$ for every $r \in \overline{N'}$, the algebraic closure of N'. Then if $\eta^k \in N \subset \tilde{N}$ there exists a unique partial fraction decomposition

$$\eta^k = \sum_{i \in \mathbb{Z}} c_i \zeta^i + \sum_{j=1}^{S} \sum_{\ell \geq 1} \frac{g_{j,\ell}}{(\zeta - d_j)^\ell}, \tag{2.42}$$

where $c_i \in N'$ and $c_i = 0$ for all but finitely many values of $i \in \mathbb{Z}$ (in other words, $\sum_i c_i \zeta^i$ is a Laurent polynomial in ζ), and $g_{j,\ell}, p_j \in \overline{N'}$ are such that $p_j \neq 0$ for every $j \in \{1, \ldots S\}$ and $g_{j,\ell} = 0$ for all but finitely many values of $\ell \in \mathbb{N}$. We claim that in fact $g_{j,k} = 0$ for every j and ℓ. To see this, let again $a_\gamma := \frac{\gamma(\eta)}{\eta}$, and apply $a_\gamma^{-k}\tilde{\gamma}$ on both sides of (2.42) to obtain

$$\sum_i c_i \zeta^i + \sum_{j,\ell} \frac{g_{j,\ell}}{(\zeta - p_j)^\ell} = \eta^k = a^{-k}\gamma(\eta^k) = \sum_i 2^i a_\gamma^{-k} c_i \zeta^i + \sum_{j,\ell} \frac{2^{-\ell} a_\gamma^{-k} g_{j,\ell}}{(\zeta - \frac{p_j}{2})^\ell}, \tag{2.43}$$

which implies that if $g_{j,\ell} \neq 0$ for some j, then there exists j' such that $p_{j'} = \frac{p_j}{2}$ and $g_{j',\ell} = 2^{-\ell} a_\gamma^{-k} g_{j,\ell} \neq 0$, a contradiction because there are only finitely many indices $j \in \{1, \ldots, S\}$ and $p_j \neq 0$ for each j. It also follows from (2.43) that $2^i = a^k$ for any $i \in \mathbb{Z}$ such that $c_i \neq 0$. Therefore, there exists a unique $n \in \mathbb{Z}$ such that $c_n \neq 0$, and therefore

$$ku - nr_1 = \tfrac{\delta_x(\eta^k \zeta^{-n})}{\eta^k \zeta^{-n}} = \tfrac{\delta_x c_n}{c_n}.$$

We claim moreover that $c_n \in K$ and that $n \neq 0$. If $c_n \notin K$, then c_n is a nontrivial rational expression over K in the elements $\theta(\frac{\partial \zeta}{\zeta})$, each one of whose δ_x-derivatives belongs to K. Since $\frac{\delta_x(\eta^k\zeta^{-n})}{\eta^k\zeta^{-n}} \in K$, Theorem 1.2 implies that the exponential element $\eta^k\zeta^{-n}$ is K-algebraically independent from the primitives $\theta(\frac{\partial\zeta}{\zeta})$, for θ ranging over Θ and ∂ ranging over Π. Having shown that $c_n \in K$, the claim that $n \neq 0$ follows from the fact that if $\eta^k = c_0 \in K$ then $a^k = 1$ for each $a \in A$, contradicting the assumtion that A is infinite. Therefore $(k, n) \in C_1$. □

The following result implies Theorem 2.15 when $A \nsubseteq \{\pm 1\}$.

Proposition 2.20. *If $A \nsubseteq \{\pm 1\}$, then $R \subseteq L$, and exactly one of the following possibilities holds:*

(i) *Consider the set C_1 consisting of pairs of integers (n_1, n_2), both of them non-zero, such that:*

- *there exists $a \in A$ such that $a^{n_1} \neq 1$; and*
- *there exists $e \in D$ such that $e^{n_2} \neq 1$; and*
- *there exists $f \in K^\times$ such that $n_1 u - n_2 r_1 = \frac{\delta_x f}{f}$.*

The set C_1 is nonempty. Moreover, in this case there exists a pair of integers $(k_1, k_2) \in C_1$ such that

$$K\langle \eta^{k_1} \rangle_\Delta = R = K\langle \zeta^{k_2} \rangle_\Delta. \tag{2.44}$$

(ii) *Consider the set C_2 consisting of pairs $(\mathbf{p}, \mathbf{q})$ of linear differential polynomials $\mathbf{p}, \mathbf{q} \in F\{Y_1, \ldots, Y_m\}^1_\Pi$, both of them non-zero, such that:*

- *there exists $a \in A$ such that $\mathbf{p}(\frac{\Pi a}{a}) \neq 0$; and*
- *there exists $e \in D$ such that $\mathbf{q}(\frac{\Pi e}{e}) \neq 0$; and*
- *there exists $f \in K$ such that $\mathbf{p}(\Pi u) - \mathbf{q}(\Pi r_1) = \delta_x f$.*

The set C_1 from case (i) is empty, whereas C_2 is non-empty. Moreover, in this case there exist finitely many pairs $(\mathbf{p_1}, \mathbf{q_1}), \ldots, (\mathbf{p}_s, \mathbf{q}_s) \in C_2$ such that

$$K\left\langle \mathbf{p}_1\left(\tfrac{\Pi\eta}{\eta}\right), \ldots, \mathbf{p}_s\left(\tfrac{\Pi\eta}{\eta}\right)\right\rangle_\Delta = R = K\left\langle \mathbf{q}_1\left(\tfrac{\Pi\zeta}{\zeta}\right), \ldots, \mathbf{q}_s\left(\tfrac{\Pi\zeta}{\zeta}\right)\right\rangle_\Delta. \tag{2.45}$$

(iv) $\Lambda = \{1\}$.

Proof. Since $A \nsubseteq \{\pm 1\}$, Lemma 2.18 says that $R \subseteq L$. Assume that $\Lambda \neq \{1\}$, and let $g \in R$ be such that $g \notin K$. Then there exist non-constant differential rational functions

$$Q_1, Q_2 \in K\langle Y\rangle_\Pi$$

such that

$$Q_1(\eta) = g = Q_2(\zeta). \tag{2.46}$$

Clearing denominators in (2.46) shows that the elements η and ζ are Π-algebraically dependent over K. Since $\frac{\delta_x \eta}{\eta} = u \in K$ and $\frac{\delta_x \zeta}{\zeta} = r_1 \in K$, Theorem 1.3 implies that either there are integers $n_1, n_2 \in \mathbb{Z}$, none of them zero, such that $\eta^{n_1}\zeta^{-n_2} \in K$, or else there exist $\mathbf{p}, \mathbf{q} \in F\{Y_1, \dots, Y_m\}^1_\Pi$ such that

$$\mathbf{p}(\tfrac{\Pi\eta}{\eta}) - \mathbf{q}(\tfrac{\Pi\zeta}{\zeta}) \in K. \tag{2.47}$$

If $\eta^{n_1}\zeta^{-n_2} =: f \in K$ for $n_1, n_2 \in \mathbb{Z}$ as above, since $\Lambda \neq \{1\}$ we may assume that $\eta^{n_1} \notin K$ and that $\zeta^{n_2} \notin K$. Hence, there exist $a \in A$ and $e \in D$ such that $a^{n_1} \neq 1$ and $e^{n_2} \neq 1$, and

$$n_1 u - n_2 r_1 = \tfrac{\delta_x(\eta^{n_1}\zeta^{-n_2})}{\eta^{n_1}\zeta^{-n_2}} = \tfrac{\delta_x f}{f}. \tag{2.48}$$

Therefore, $(n_1, n_2) \in C_1$. Now let

$$k_1 := \inf\{n_1 \mid (n_1, n_2) \in C_1, \text{ and } n_1 > 0\}$$

and let $k_2 \in \mathbb{Z}$ be such that $(k_1, k_2) \in C_1$. To prove (2.44), note that the Δ-subfield $R \subseteq L$ is completely determined by the differential-algebraic group $\mathrm{Gal}_\Delta(L/R) \subseteq A$, by the parameterized Galois correspondence [9, Thm. 3.5]. Since $\eta^{n_1}\zeta^{-n_2} \in K$ whenever $(n_1, n_2) \in C_1$ (cf. the proof of Theorem 2.15(1)), we have in particular that

$$K\langle\zeta^{k_2}\rangle_\Delta = K\langle\eta^{k_1}\rangle_\Delta \subseteq R.$$

Again by the parameterized Galois correspondence, $\mathrm{Gal}_\Delta(L/R) \subseteq \mu_{k_1}$, the group of k_1-th roots of unity. If the inclusion $K\langle\eta^{k_1}\rangle_\Delta$ were proper, then there would exist an integer $0 < k < k_1$ such that k divides k_1 and $\mathrm{Gal}_\Delta(L/R) = \mu_k$. Since the fixed field of $\mu_k \subset A$ in L is $K\langle \eta^k \rangle_\Delta$, this would imply that $\eta^k \in R \subseteq N$, but since k divides k_1 and $\eta^{k_1} \notin K$, Lemma 2.19 implies that $(k, n) \in C_1$ for some $n \in \mathbb{Z}$, contradicting the minimality of k_1. Therefore,

$$K\langle \eta^{k_1} \rangle_\Delta = R = K\langle \zeta^{k_2} \rangle_\Delta.$$

Now suppose there do not exist non-zero integers $n_1, n_2 \in \mathbb{Z}$ such that $\eta^{n_1}\zeta^{-n_2} \in K$. Then it follows from (2.48) that C_1 is empty. Since $\Lambda \neq \{1\}$, there must exist linear differential polynomials $\mathbf{p}$ and $\mathbf{q}$ as in (2.47), and we may assume that $\mathbf{p}(\frac{\Pi\eta}{\eta}) \notin K$ and that $\mathbf{q}(\frac{\Pi\zeta}{\zeta}) \notin K$. Since for each $\gamma \in \Gamma$

$$\gamma\big(\mathbf{p}(\tfrac{\Pi\eta}{\eta})\big) = \mathbf{p}(\tfrac{\Pi\eta}{\eta}) + \mathbf{p}(\tfrac{\Pi a_\gamma}{a_\gamma}) \qquad \text{and}$$
$$\gamma\big(\mathbf{q}(\tfrac{\Pi\zeta}{\zeta})\big) = \mathbf{q}(\tfrac{\Pi\zeta}{\zeta}) + \mathbf{q}(\tfrac{\Pi e_\gamma}{e_\gamma}),$$

there exists $a \in A$ such that $\mathbf{p}(\frac{\Pi a}{a}) \neq 0$ and there exists $e \in D$ such that $\mathbf{q}(\frac{\Pi e}{e}) \neq 0$, which implies that both A and D are infinite. Note that

$$\delta_x\big(\mathbf{p}(\tfrac{\Pi\eta}{\eta})\big) = \mathbf{p}(\Pi u); \qquad \text{and} \qquad \delta_x\big(\mathbf{q}(\tfrac{\Pi\zeta}{\zeta})\big) = \mathbf{q}\big(\Pi r_1\big).$$

We claim that there exists a finite collection of linear differential polynomials $\mathbf{p}_1, \dots, \mathbf{p}_s \in F\{Y_1, \dots, Y_m\}^1_\Pi$ such that

$$R = K\langle \mathbf{p}_1(\tfrac{\Pi\eta}{\eta}), \dots, \mathbf{p}_s(\tfrac{\Pi\eta}{\eta})\rangle_\Delta.$$

By the parameterized Galois correspondence [9, Thm. 3.5], R is determined by $\mathrm{Gal}_\Delta(M'/R) \subseteq B$. Since C_1 is empty, Lemma 2.19 says that there is no $k \in \mathbb{N}$ such that $\eta^k \in R$ (as A being infinite guarantees that $\eta^k \notin K$ for every $k \in \mathbb{N}$). The parameterized Galois correspondence then implies that

$$\mathrm{Gal}_\Delta(L/R) \nsubseteq \mathrm{Gal}_\Delta(L/K(\eta^k)) = \mu_k,$$

the group of k-th roots of unity, for each $k \in \mathbb{N}$. By Proposition 1.6, there exist finitely many linear differential polynomials $\mathbf{p}_1, \dots, \mathbf{p}_s \in F\{Y\}^1_\Pi$ such that

$$\mathrm{Gal}_\Delta(L/R) = \{a \in A \mid \mathbf{p}_i(\tfrac{\Pi a}{a}) = 0 \text{ for } 1 \leq i \leq s\},$$

and the fixed field of this differential algebraic subgroup of A is precisely $K\langle \mathbf{p}_1(\frac{\Pi\eta}{\eta}), \dots, \mathbf{p}_s(\frac{\Pi\eta}{\eta})\rangle_\Delta = R$. To prove (2.45), it suffices to show that there exist $\mathbf{q}_1, \dots, \mathbf{q}_s \in F\{Y_1, \dots, Y_m\}^1_\Pi$ such that $(\mathbf{p}_i, \mathbf{q}_i) \in C_2$ for each $1 \leq i \leq s$, because then $\mathbf{p}_i(\frac{\Pi\eta}{\eta}) - \mathbf{q}_i(\frac{\Pi\zeta}{\zeta}) \in K$ implies that

$$R = K\langle \mathbf{p}_1(\tfrac{\Pi\eta}{\eta}), \dots, \mathbf{p}_s(\tfrac{\Pi\eta}{\eta})\rangle_\Delta = K\langle \mathbf{q}_1(\tfrac{\Pi\zeta}{\zeta}), \dots, \mathbf{q}_s(\tfrac{\Pi\zeta}{\zeta})\rangle_\Delta.$$

By [41, Cor. 5.2], the differential type of $D = \mathrm{Gal}_\Delta(N/K)$ is 0. By [42, Prop. 3.2], this implies that N is of finite algebraic transcendence degree over K, so there exist finite subsets $\Theta_1, \dots, \Theta_m \subset \Theta$, where Θ denotes the free commutative monoid generated by Π (cf. §1.1), such that

$$N' := K\langle \tfrac{\Pi\zeta}{\zeta}\rangle_\Delta = K\big(\theta_j(\tfrac{\partial_j\zeta}{\zeta}) \mid \theta_j \in \Theta_j, 1 \leq j \leq m\big).$$

We may assume without loss of generality that the $\theta_j(\frac{\partial_j\zeta}{\zeta})$, for θ_j ranging over Θ_j and j ranging over $\{1,\ldots,m\}$, are F-linearly independent modulo K. By Theorem 1.2, the $\theta_j(\frac{\partial_j\zeta}{\zeta})$ must then also be K-algebraically independent, since $\delta_x(\theta\frac{\partial\zeta}{\zeta}) = \theta\partial r_1 \in K$ for each $\theta \in \Theta$ and $\partial \in \Pi$. This also implies that the underlying δ_x-field of N' is a (non-parameterized) PV extension of the δ_x-field K, whose corresponding PV ring is given by

$$P := K\big[\theta_j(\tfrac{\partial_j\zeta}{\zeta}) \mid \theta_j \in \Theta_j,\ 1 \leq j \leq m\big].$$

To conclude the proof of Proposition 2.20, we will show that if $\mathbf{p}(\frac{\Pi\eta}{\eta}) \in R\backslash K$ for some $\mathbf{p} \in F\{Y\}^1_\Pi$, then there exists $f \in K$ and $\mathbf{q} \in F\{Y_1,\ldots,Y_m\}^1_\Pi$ such that

$$\mathbf{p}(\tfrac{\Pi\eta}{\eta}) = f + \mathbf{q}(\tfrac{\Pi\zeta}{\zeta}).$$

First, note that since $\delta_x\mathbf{p}(\frac{\Pi\eta}{\eta}) = \mathbf{p}(\Pi u) \in K$, then $\sum_i K \cdot \delta_x^i(\mathbf{p}(\frac{\Pi\eta}{\eta}))$ is finite dimensional as a K-vector space, and therefore $\mathbf{p}(\frac{\Pi\eta}{\eta}) \in P$ by [50, Cor. 1.38] (cf. the proofs of Lemma 2.6 and Proposition 2.17). For notational convenience, let us relabel:

$$\bigcup_{1\leq j\leq m} \{\theta(\tfrac{\partial_j\zeta}{\zeta}) \mid \theta \in \Theta_j\} = \{\beta_1,\ldots,\beta_S\}. \tag{2.49}$$

We recall that the β_i are K-algebraically independent and $\delta_x\beta_i \in K$ for each $1 \leq i \leq S$. Since $\mathbf{p}(\frac{\Pi\eta}{\eta}) \in P$, there exists an algebraic polynomial

$$Q \in K[Y_1,\ldots,Y_S]$$

such that $\mathbf{p}(\frac{\Pi\eta}{\eta}) = Q(\beta_1, \ldots, \beta_S)$, and therefore

$$\delta_x(Q(\underline{\beta})) = \delta_x \mathbf{p}(\tfrac{\Pi\eta}{\eta}) \in K.$$

Suppose $r_I \underline{\beta}^I$ is a term in $Q(\underline{\beta})$ (where $I = (i_1, \ldots, i_S)$ is a multiindex) with $r_I \neq 0$ and $|I| := \sum_j i_j > 0$ maximal. Letting $\mathbf{1}_j$ denote the multiindex with j-th entry 1 and all other entries 0, we obtain that

$$\delta_x(r_I \underline{\beta}^I) = (\delta_x r_I)\underline{\beta}^I + r_I \sum_{j=1}^{S} i_j (\delta_x \beta_j) \underline{\beta}^{I-\mathbf{1}_j},$$

whence it follows that $r_I \in F$, because the coefficient of $\underline{\beta}^I$ in $\delta_x(Q(\underline{\beta}))$ is precicely $\delta_x r_I = 0$.

Now fix $j \in \{1, \ldots, S\}$ such that i_j (the j-th entry in I) is non-zero, let $r_j \in K$ denote the coefficient of $\underline{\beta}^{I-\mathbf{1}_j}$ in $Q(\underline{\beta})$, and let c_k denote the coefficient of $\underline{\beta}^{I-\mathbf{1}_j+\mathbf{1}_k}$ in $Q(\underline{\beta})$ for each $k \neq j$. Since $|I - \mathbf{1}_j + \mathbf{1}_k| = |I|$ is also maximal, the argument above shows that $\delta_x c_k = 0$ for each $k \neq j$, so $c_k \in F$. If $I \neq \mathbf{1}_j$, then $\underline{\beta}^{I-\mathbf{1}_j} \notin K$, and the coefficient of $\underline{\beta}^{I-\mathbf{1}_j}$ in $\delta_x(Q(\underline{\beta}))$ is

$$i_j r_I \delta_x \beta_j + \delta_x r_j + \sum_{k \neq j} i_k c_k \delta_x \beta_k = \delta_x(i_j r_I \beta_j + r_j + \sum_{k \neq j} i_k c_k \beta_k) = 0,$$

which implies that

$$i_j r_I \beta_j + \sum_{k \neq j} i_k c_k \beta_k = -r_j + c \in K$$

for some $c \in F$, contradicting the F-linear independence of the β_i modulo K. Therefore, $I = \mathbf{1}_j$, and since we assumed that $|I| = 1$ was maximal, we have just shown that $Q(\underline{\beta}) = \sum_{j=1}^{S} r_j\beta_j + f$, for some $f \in K$ and $r_j \in F$. Recalling the relabelling (2.49), we see that there is a unique $\mathbf{q} \in F\{Y_1, \dots, Y_m\}_\Pi^1$ such that $\sum_{j=1}^{S} r_j\beta_j = \mathbf{q}(\frac{\Pi\zeta}{\zeta})$, whence

$$\mathbf{p}(\tfrac{\Pi\eta}{\eta}) = Q(\underline{\beta}) = \mathbf{q}(\tfrac{\Pi\zeta}{\zeta}) + f,$$

as claimed. □

2.4 Criteria for differential transcendence

In the applications of the PPV theory, it is often useful to know whether any of the solutions to (2.19) are Π-transcendental (see e.g. [2, 23, 24, 49]). As in §2.2, we denote by $\mathcal{D} := F \cdot \Pi$ the Lie vector space of parametric derivations obtained by taking the F-linear span of Π. The PPV field E for (2.19) defined in §2.3 has infinite algebraic transcendence degree over K if and only if there exists a ∂-transcendental solution to (1.1) for some $\partial \in \mathcal{D}$. By [9, §9] and the definition of $\tau(G)$ (see §1.1), this holds if and only if $\tau(G) > 0$, which can be decided effectively [42, Algo. 2]. For second-order equations, the algorithms of the previous sections allow us to give a set of criteria to decide whether any of the solutions to (2.19) is Π-transcendental over K.

Theorem 2.21. *There exists a solution ω for (2.19) that is Π-transcendental over K if and only if one of the following possibilities holds:*

(1) There exists a solution $u \in K$ to the Riccati equation (2.4), and none of the equations

$$\delta_x Y - 2uY = 1 \qquad \textit{and} \qquad \delta_x Y = (\partial u)Y$$

admits a solution in K, for ∂ ranging over $\mathcal{D}$.

(4) There is no solution $u \in \bar{K}$, with algebraic degree at most 12 over K, to the Riccati equation (2.4), and none of the equations

$$\delta_x^3 Y = 4q\delta_x Y + 2(\delta_x q)Y - 2\partial q \tag{2.50}$$

admits a solution in K, for ∂ ranging over $\mathcal{D}$.

Proof. Since the solution $0 \neq \zeta \in U$ to (2.21) is ∂-algebraic over K for each $\partial \in \mathcal{D}$, it suffices to prove the theorem for equations of the form (2.20). We will consider separately each of the cases I, II, III, and IV of Kovacic's algorithm [19, 37], outlined in §2.1.

In case I, there exists a solution $u \in K$ to the Riccati equation (2.4), and we may choose a basis $\{\eta, \xi\}$ for the solution space of (2.20) such that $\delta_x \eta = u\eta$ and $\delta_x(\frac{\xi}{\eta}) = \eta^{-2}$. It is proved in [52, Lem. 3.2] that, for any $\partial \in \mathcal{D}$, the group $\mathbb{G}_m(F)$, considered as ∂-algebraic group, is not the PPV group of any PPV extension of K, considered as $\{\delta_x, \partial\}$-field. Let $a_\sigma := \frac{\sigma\eta}{\eta}$ for each

$\sigma \in H$. Since for every $\mathbf{p} \in K[\partial]$ we have that

$$\sigma\big(\mathbf{p}(\tfrac{\partial\eta}{\eta})\big) = \mathbf{p}\big(\tfrac{\partial\eta}{\eta}\big) + \mathbf{p}\big(\tfrac{a_\sigma}{a_\sigma}\big),$$

It follows that $L := K\langle\eta\rangle_\Delta$ is ∂-algebraic for each $\partial \in \mathcal{D}$.

Therefore, (2.20) admits a Π-transcendental solution in case I if and only if the element $\frac{\xi}{\eta}$ is Π-transcendental over K. Now let $b_\sigma := \sigma(\frac{\xi}{\eta}) - \frac{\xi}{\eta}$ for each $\sigma \in \mathrm{Gal}_\Delta(M/L) \simeq B$. For each $\mathbf{p} \in F[\Pi]$, the relation

$$\sigma\big(\mathbf{p}(\tfrac{\xi}{\eta})\big) = \mathbf{p}\big(\tfrac{\xi}{\eta}\big) + \mathbf{p}(b_\sigma)$$

shows that $\frac{\xi}{\eta}$ is Π-transcendental if and only if there is no $0 \neq \mathbf{p} \in F[\Pi]$ such that $\mathbf{p}(b) = 0$ for every $b \in B$, which holds if and only if $B = \mathbb{G}_a(F)$. The following result is proved in [2, Lem. 4.3].

Lemma 2.22. *The equation*

$$\delta_x Y - 2uY = 1 \tag{2.51}$$

admits a solution in K if and only if $\mathrm{Gal}_\Delta(M/L) = 0$.

Proof of Lemma 2.22. If $f \in K$ satisfies $\delta_x f - 2uf = 1$, we have that

$$\delta_x(f\eta^{-2}) = (\delta_x f - 2uf)\eta^{-2} = \eta^{-2}.$$

Therefore $\delta_x(\frac{\xi}{\eta} - f\eta^{-2}) = 0$, which implies that $\frac{\xi}{\eta} \in L$ because $M^\delta = L^\delta = F$.

For the opposite implication, suppose that $\mathrm{Gal}_\Delta(M/L) = 0$. Then it follows from Remark 2.8 that $\sigma(\eta\xi) = \eta\xi$ for every $\sigma \in \mathrm{Gal}_\Delta(M/K)$. By the parameterized Galois correspondence [9, Thm. 3.5], there exists $f \in K$ such that $\eta\xi = f$, and therefore

$$f\eta^{-2} = \tfrac{\xi}{\eta}. \tag{2.52}$$

Applying δ_x on both sides of (2.52), we obtain

$$(\delta_x f - 2uf)\eta^{-2} = \eta^{-2}, \tag{2.53}$$

and therefore f satisfies (2.51). This concludes the proof of Lemma 2.22. $\square$

We assume from now on that there is no solution $f \in K$ for (2.51). Denote by $\mathcal{L} \subseteq \mathcal{D}$ the Lie subspace of $\mathcal{D}$ consisting of parametric derivations ∂ such that $A \subset \mathbb{G}_m(F^\partial)$ (cf. §2.2). It follows from Remark 2.1 that $\mathcal{L} = 0$ if and only if the equation $\delta_x Y = \partial u$ does not admit a solution in K for any $\partial \in \mathcal{D}$. We claim that $\mathcal{L} = \{0\}$ if and only if $\frac{\xi}{\eta}$ is Π-transcendental over K. If $\mathcal{L} \neq \{0\}$, then there is a non-empty set $\Pi' \subset \mathcal{D}$ of pairwise commuting derivations such that $\mathcal{L} = F \cdot \Pi'$, $B = B' \simeq \mathrm{Gal}_{\Delta'}(M'/L')$, and the reductive quotient $H/R_u(H) \simeq A'$ is Π'-constant (cf. §2.2). It follows from Proposition 1.12 that $B' \neq \mathbb{G}_a(F)$. By Proposition 1.6, there exists a linear differential polynomial $0 \neq \mathbf{p} \in F\{Y\}^1_{\Pi'}$ such that $\mathbf{p}(b') = 0$ for each

$b' \in B'$. Finally, if $\mathcal{L} = \{0\}$ then Theorem 2.2 implies that either $B = \{0\}$ or $B = \mathbb{G}_a(F)$. This concludes the proof of Theorem 2.21 in case I.

In case II, it is proved in [37, §4.3] that the PV-extension (resp., PPV extension) for (2.20) is given by $M' := K(u, \eta)$, where $u \in \bar{K}$ is quadratic over K and satisfies the Riccati equation (2.4), and $0 \neq \eta$ satisfies $\delta_x \eta = u\eta$. Therefore, $M = K(u)\langle\eta\rangle_\Delta$ (cf. [19, §2.2]). It is proved in [52, Lem. 3.2] that, for any $\partial \in \mathcal{D}$, the PPV group $\mathbb{G}_m(F)$, considered as ∂-algebraic group, is not the PPV group of any PPV extension of $L := K(u)$, considered as $\{\delta_x, \partial\}$-field. By Proposition 1.6, there exists $0 \neq \mathbf{p} \in F[\partial]$ such that

$$\mathbf{p}\left(\tfrac{\partial a}{a}\right) = 0$$

for every $a \in A \simeq \mathrm{Gal}_{\{\delta_x, \partial\}}(M/L)$. Letting $\frac{\sigma\eta}{\eta} =: a_\sigma \in A$ for each σ in this PPV group, the fact that

$$\sigma\left(\mathbf{p}\left(\tfrac{\partial \eta}{\eta}\right)\right) = \mathbf{p}\left(\tfrac{\partial \eta}{\eta}\right) + \mathbf{p}\left(\tfrac{\partial a_\sigma}{a_\sigma}\right)$$

implies that $\mathbf{p}(\frac{\partial\eta}{\eta}) \in L$. Hence, there are no ∂-transcendental elements $\omega \in M$ (whence there are no ∂-transcendental elements $\omega \in E$, either), for any $\partial \in \mathcal{D}$, concluding the proof of Theorem 2.21 in case II.

The proof of Theorem 2.21 is obvious in case III, since the solutions to (2.20) are algebraic over K.

In case IV, it is shown in [19, §2.4] that there exists a set of F-linearly independent, pairwise commuting derivations $\Pi' \subseteq \mathcal{D}$ such that H is conjugate to $\simeq \mathrm{SL}_2(F^{\Pi'})$. In fact, Π' is the F-basis of the Lie subspace $\mathcal{L} \subseteq \mathcal{D}$ consisting of derivations ∂ such that (2.20) is isomonodromic with respect to ∂, i.e., the fundamental matrix of solutions corresponding to (2.20) satisfies matrix differential equations with respect to each $\partial' \in \Pi'$ (see [22, Defn. 6.1 and Thm. 6.3]). Therefore the solutions to (2.20) are Π-transcendental over K if and only if $\mathcal{L} = \{0\}$. By [19, Thm. 2.10(4)], this is equivalent to (2.50) not admitting a solution in K. This concludes the proof of Theorem 2.21 in the last case IV. $\square$

Chapter 3

Examples

In this section we apply the algorithms developed in Chapter 2 to compute the PPV groups of some concrete second-order linear differential equations.

3.1 Example 1

In this example we compute the PPV group G corresponding to the incomplete Gamma function $\gamma(t, x)$, which satisfies

$$\delta_x^2 Y - \frac{t-1-x}{x}\delta_x Y = 0. \tag{3.1}$$

We will now compute G using the methods presented in Chapter 2 (see also [9, Ex. 7.2] and [2, Thm. 1.1]), by first computing the PPV group H of the associated unimodular equation (2.20) with coefficient q given by

$$q = r_1^2 - \delta_x r_1,$$

where $r_1 = \frac{t-1-x}{2x}$. We see that

$$u := -r_1 \tag{3.2}$$

satisfies the Riccati equation (2.4). Since $\partial_t u = -\frac{1}{2x}$, we have that $\partial_t^2 u = 0$ and therefore the PPV group A for $\delta_x Y - uY = 0$ is

$$A \simeq \left\{ a \in \mathbb{G}_m(F) \;\middle|\; \partial_t \left(\frac{\partial_t a}{a} \right) = 0 \right\}.$$

Since the only derivation $\partial \in F \cdot \partial_t$ such that $\partial a = 0$ for every $a \in A$ is the trivial one $\partial = 0$, Theorem 2.2 shows that the unipotent radical B coincides with the unipotent radical B' of the PV group H' for (2.20), obtained by forgetting the parametric derivation ∂_t and regarding F as a (non-differential) algebraically closed field.

We follow [2, proof of Thm. 1.1] in proving that $B' = \mathbb{G}_a(F)$. By Lemma 2.22, we have to show that there is no element $f \in K$ such that

$$\delta_x f + \frac{t-1-x}{x} f = 1. \tag{3.3}$$

We proceed by contradiction: assume that $f \in K$ does satisfy (3.3). Then f cannot be δ_x-constant, so it must have a pole somewhere on $\mathbb{P}^1(F)$. But f cannot have a pole outside of $\{0, \infty\}$, for otherwise the left-hand side of (3.3) would have a pole. If f had a pole at 0, then the residue of $\frac{t-1-x}{x}$ at 0 would have to be an integer, which is clearly false. Therefore, f can only have a pole at $\infty \in \mathbb{P}^1(F)$, i.e., $f \in F[x]$ is a polynomial in x. Now f must be divisible by x, for otherwise the left-hand side of (3.3) would have a pole at 0. But then the left-hand side of (3.3) is a polynomial of degree equal to that of f, which is at least 1. This contradiction concludes the proof that $B' = \mathbb{G}_a(F)$, and therefore

$$H \simeq \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a, b \in F;\; a \neq 0;\; \partial_t \left(\frac{\partial_t a}{a} \right) = 0 \right\}.$$

The PPV group D for $\delta_x Y - r_1 Y = 0$ coincides with A, since $u = -r_1$. It also follows from (3.2) that the integers k_1 and k_2 from Theorem 2.15(i) are both equal to 1, and therefore

$$G \simeq \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a, b \in F;\; a \neq 0;\; \partial_t \left(\frac{\partial_t a}{a} \right) = 0 \right\}.$$

3.2 Example 2

In this example we again let F denote a differential closure of $\mathbb{Q}(t)$, where there is only one parametric derivation $\Pi = \{\partial_t\}$, where ∂_t denotes the derivation with respect to t, and $K = F(x)$ as in Chapter 2. Consider

$$\delta_x^2 Y - \frac{2t}{x} \delta_x Y + \left(\frac{4t^2 + 4t + 1}{4x^2} \right) Y = 0. \tag{3.4}$$

The associated unimodular equation is given by

$$\delta_x^2 Y + \frac{1}{4x^2} Y = 0. \tag{3.5}$$

The Riccati equation (2.4) admits the solution $u = \frac{1}{2x}$. Since the coefficients of (3.5) are ∂_t-constant, [9, Prop. 3.9] implies that the PPV group of (3.5) is ∂_t-constant. We find the basis of solutions $\{\eta, \xi\} = \{\sqrt{x}, \sqrt{x}\log(x)\}$ with Kovacic's algorithm [37, §1.1, p. 5], and the PPV group H for (3.5) is

$$H \simeq \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a, b \in F^{\partial_t};\ a^2 = 1 \right\}.$$

From the coefficient $r_1 = \frac{t}{x}$ we obtain that the PPV group D corresponding to the chage of variables from (3.4) to (3.5) is (cf. [9, Ex. 3.1])

$$D \simeq \left\{ e \in \mathbb{G}_m(F) \;\middle|\; \partial_t\left(\frac{\partial_t e}{e}\right) = 0 \right\}.$$

Since $A = \{\pm 1\}$, and D is infinite, possibilites (i) and (ii) of Theorem 2.15 cannot occur (see Corollary 2.16). Since

$$\eta^{-2} - \partial_t r_1 = \tfrac{1}{x} - \tfrac{1}{x} = 0,$$

we have verified the hypotheses of Theorem 2.15(iii), and the linear differential polynomials $\mathbf{p}, \mathbf{q} \in F\{Y\}^1_\Pi$ are both equal to 1 in this case. By Theorem 2.15(3), the PPV group G for (3.4) is given by

$$G \simeq \left\{ \begin{pmatrix} e & \frac{\partial_t e}{e} \\ 0 & e \end{pmatrix} \;\middle|\; e \in F;\ e \neq 0;\ \partial_t\left(\frac{\partial_t e}{e}\right) = 0 \right\}.$$

3.3 Example 3

We again let $K = F(x)$ with F a differential closure of $\mathbb{Q}(t_1, t_2)$, as in the previous section. Following [3, §4], we will not apply Theorem 2.2 to compute the PPV-group H corresponding to the parameterized linear differential equation

$$\delta_x^2 Y - \left(\frac{x^2 + (2 - 2t_1t_2)x + t_1^2t_2^2 - 3t_1t_2 + 2}{x^2}\right) Y = 0. \tag{3.6}$$

The Riccati equation (2.4) admits the solution

$$u := \frac{t_1t_2 - 1 - x}{x}.$$

Therefore, by [37, §1.1, p. 5] there is a basis $\{\eta, \xi\}$ for the solution space of (3.6) such that $\delta_x \eta = u\eta$ and $\delta_x\left(\frac{\xi}{\eta}\right) = \eta^{-2}$, and by [19, §2.1] there exist differential algebraic subgroups $A \leq \mathbb{G}_m(F)$ and $B \leq \mathbb{G}_a(F)$ such that H is given by (2.5). Since $\partial_1 u = \frac{t_2}{x}$ and $\partial_2 u = \frac{t_1}{x}$, we have that

$$\partial_1^2 u = 0 = \partial_2^2 u \qquad \text{and} \qquad t_1\partial_1 u - t_2\partial_2 u = 0. \tag{3.7}$$

Therefore, the $F[\Pi]$-span of $\{\partial_1 u, \partial_2 u\}$ and the F-span of $\{\partial_1 u, \partial_2 u, \partial_1\partial_2 u\}$ coincide modulo $\delta_x(K)$. By [19, §2.1],

$$A = \left\{a \in \mathbb{G}_m(F) \;\middle|\; t_1\tfrac{\partial_1 a}{a} = t_2\tfrac{\partial_2 a}{a};\ \partial_1\left(\tfrac{\partial_1 a}{a}\right) = 0 = \partial_2\left(\tfrac{\partial_2 a}{a}\right)\right\}.$$

Since $\partial_1 u \notin \delta_x(K)$, the Lie subspace of derivations $\mathcal{L} \subset F \cdot \Pi$ defined in (2.8), or equivalently in (2.10), has dimension at most 1 over F. Hence, by

(3.7) $\mathcal{L}$ coincides with $F \cdot (t_1\partial_1 - t_2\partial_2)$, the F-span of $\partial_1' := t_1\partial_1 - t_2\partial_2$, and we may take $\Pi' := \{\partial_1'\}$.

By Theorem 2.2, to compute the unipotent radical $R_u(H) = B$ it suffices to compute $R_u(H') =: B'$, where H' denotes the PPV-group of (3.6) relative to the new set of parametric derivations $\Pi' = \{\partial_1'\}$. It follows from (3.7) that $\delta_x Y - uY = 0$ is *isomonodromic* [22, Defn. 6.1] (or *completely integrable*, in the terminology of [9, Defn. 3.8]). Therefore, by [9, Prop. 3.9] $L' = K(\eta)$ is a (non-parameterized) PV-extension of K for $\delta_x Y - uY = 0$, and (cf. Theorem 2.2)

$$\mathrm{Gal}_{\Delta'}(L'/K) \simeq H'/R_u(H') \simeq A' = \mathbb{G}_m(F^{\partial_1'}).$$

By Lemma 2.22, to see that $B' \neq 0$ it suffices to show that there is no $f \in K$ such that $\delta_x f - 2uf = 1$. We prove this by contradiction, along the lines of [2, proof of Thm. 1.1]. Assume that $f \in K$ satisfies

$$\delta_x f - 2\left(\frac{t_1t_2 - 1 - x}{x}\right) f = 1. \tag{3.8}$$

First, note that f cannot be δ_x-constant, whence it must a have a pole somewhere in $\mathbb{P}^1(F)$. But f cannot have a pole outside of $\{0, \infty\}$, for otherwise the left-hand side of (3.8) would have a pole. If f had a pole at 0, the residue of $2u$ at 0 would have to be an integer, which is clearly false. Therefore, f can only have a pole at ∞, i.e., f must be a polynomial in x. Moreover, f

must be divisible by x, because otherwise the left-hand side of (3.8) would have a pole at 0. But then the degree of the polynomial on the left-hand side of (3.8) is equal to the degree of f, which is at least 1, since f is not constant. This contradiction concludes the proof that there is no solution in K for (3.8), and therefore that $B' \neq 0$. Since

$$\delta_x\left(\partial_1' \tfrac{\xi}{\eta}\right) = \partial_1' \eta^{-2} = 0 \qquad \Longrightarrow \qquad \partial_1' \tfrac{\xi}{\eta} \in F = \delta_x(F \cdot x) \subset \delta_x(K),$$

it follows from [19, §2.1] that $B' = \mathbb{G}_a(F^{\partial_1'})$. Therefore, by Theorem 2.2,

$$H \simeq \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; \begin{array}{c} a, b \in F; \quad a \neq 0; \\[1ex] t_1 \frac{\partial_1 a}{a} = t_2 \frac{\partial_2 a}{a}; \\[1ex] \partial_1\left(\frac{\partial_1 a}{a}\right) = 0 = \partial_2\left(\frac{\partial_2 a}{a}\right); \\[1ex] t_1 \partial_1 b = t_2 \partial_2 b \end{array} \right\}.$$

3.4 Example 4

We let $K = F(x)$ denote the Δ-field of the previous sections, where now $\Pi := \{\partial_1, \partial_2\}$, $\partial_j := \frac{\partial}{\partial t_j}$ for $j = 1, 2$, and F denotes a Π-closed field containing $\mathbb{Q}(t_1, t_2)$. Following [4, §5], we will now compute the PPV group G for (2.19), with

$$r_1 := \frac{t_1 - t_2}{x} + \frac{t_2}{x - 1};$$

and

$$r_0 := \frac{(t_1-2t_2)(t_2-1)+2(t_1-t_2)^2x}{x^2} + \frac{t_1(2t_2-t_1+1)-2(t_1-t_2)^2(x-1)}{(x-1)^2}.$$

The coefficient q in the unimodular equation (2.20) associated to (2.19) is

$$q = \frac{t_1(t_1-1)(1-2x)}{x^2} + \frac{(t_1-t_2)(2t_1x-t_1-t_2-1)}{(x-1)^2},$$

and the Riccati equation $\delta_x u + u^2 = q$ admits the solution

$$u := \frac{t_1}{x} + \frac{t_1-t_2}{x-1}.$$

Hence, we are in case I, and the PPV-group H for (2.20) is defined by (2.5). By [19, §2.1] (see also §2.1 and Remark 2.1), for every $a \in A$ and every linear differential polynomial $\mathbf{q} \in F\{Y_1, Y_2\}^1_\Pi$,

$$\mathbf{q}\left(\tfrac{\partial_1 a}{a}, \tfrac{\partial_2 a}{a}\right) = 0 \quad \Longleftrightarrow \quad \mathbf{q}(\partial_1 u, \partial_2 u) \in \delta_x(K). \tag{3.9}$$

Since $\partial_1 u = \frac{1}{x} + \frac{1}{x-1}$ and $\partial_2 u = \frac{-1}{x-1}$, we see that

$$A = \left\{a \in \mathbb{G}_m(F) \;\middle|\; \partial_j\left(\tfrac{\partial_1 a}{a}\right) = 0 = \partial_j\left(\tfrac{\partial_2 a}{a}\right) \text{ for } j = 1, 2\right\}.$$

A similar computation shows that the PPV-group D for (2.21) is defined by the same linear differential polynomials as A.

Since the only derivation $\partial \in F \cdot \Pi$ such that $\partial u \in \delta_x(K)$ is $\partial = 0$, by Theorem 2.2 we have that $B = B'$, where B' is the unipotent radical of the (non-parameterized) PV group H' of (2.20). To show that the unipotent radical B' of H' is $\mathbb{G}_a(F)$, we proceed as in [2, proof of Thm. 1.1] (cf. the previous example): by Lemma 2.22 we have to show that there is no $f \in K$ such that

$$\delta_x f - 2\left(\frac{t_1}{x} + \frac{t_1 - t_2}{x-1}\right) f = 1. \tag{3.10}$$

We proceed by contradiction, and assume there does exist such an element $f \in K$ satisfying (3.10). Since f cannot be δ_x-constant, it must have a pole somewhere on $\mathbb{P}^1(F)$. But f cannot have a pole outside of $\{0, 1, \infty\}$, for otherwise the left hand side of (3.10) would have a pole. But it follows from (3.10) that if f had a pole at 0 or (resp., 1) then the residue of $2u$ at 0 (resp., 1), would have to be an integer, which it is not. Therefore, f can only have a pole at ∞, viz. f must be a polynomial in x. Moreover, f must be divisible by $x(x-1)$, since otherwise the left-hand side of (3.10) would have a pole. But then the degree of f as a polynomial in x is at least 2, and the left-hand side of (3.10) must be a polynomial of degree at least 1, which is the desired contradiction. Therefore, $B' = \mathbb{G}_a(F) = B$, and

$$H \simeq \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \begin{array}{c} a, b \in F;\ a \neq 0; \\ \partial_i \left(\frac{\partial_j a}{a}\right) = 0 \\ \text{for } 1 \leq i, j \leq 2 \end{array} \right\}.$$

We now apply Theorem 2.15 to compute G. For any integers $k_1, k_2 \in \mathbb{Z}$,

$$k_1u - k_2r_1 = \frac{t_1(k_1 - k_2) + t_2k_2}{x} + \frac{t_1k_1 - t_2(k_1 - k_2)}{x - 1}. \tag{3.11}$$

If $k_1u - k_2r_1 = \frac{\delta_x f}{f}$ for some $f \in K$, the residues of (3.11) would have to be integers, which is clearly impossible unless $k_1 = 0 = k_2$, and therefore we have verified that case (i) of Theorem 2.15 doesn't hold.

The relations

$$\partial_1 u + \partial_2 u = \partial_1 r_1 \qquad \text{and} \qquad \partial_1 r_1 + \partial_2 r_1 = -\partial_2 u$$

correspond to the linear differential polynomials

$$\mathbf{p}_1 := Y_1 + Y_2; \qquad \mathbf{p}_2 := -Y_2;$$

$$\mathbf{q}_1 := Y_1; \qquad \mathbf{q}_2 := Y_1 + Y_2.$$

Since F is Π-closed (Definition 1.1), there exist $a \in A$ and $e \in D$ such that $\mathbf{p}_i(\frac{\partial_1 a}{a}, \frac{\partial_2 a}{a}) \neq 0$ and $\mathbf{q}_i(\frac{\partial_1 e}{e}, \frac{\partial_2 e}{e}) \neq 0$ for each $i \in \{1, 2\}$, and we have verified the hypotheses of Theorem 2.15(ii).

Since $\theta\partial_j u = 0 = \theta\partial_j r_1$ for $\theta \in \Theta$ and $1 \leq j \leq 2$, the set

$$\big\{(\mathbf{p}_1, \mathbf{q}_1);\ (\mathbf{p}_2, \mathbf{q}_2)\big\}$$

forms a basis for the F-vector space of pairs $(\mathbf{p}, \mathbf{q})$, with $\mathbf{p}, \mathbf{q} \in F\{Y_1, Y_2\}^1_\Pi$, such that

$$\mathbf{p}(\partial_1 u, \partial_2 u) - \mathbf{q}(\partial_1 r_1, \partial_2 r_1) \in \delta_x(K).$$

Therefore, the PPV group G for (2.19) is given by

$$G \simeq \left\{ \begin{pmatrix} ea & eb \\ 0 & ea^{-1} \end{pmatrix} \;\middle|\; \begin{array}{c} a, b, e \in F;\ a, e \neq 0; \\[1em] \partial_i(\frac{\partial_j a}{a}) = 0 = \partial_i(\frac{\partial_j e}{e}) \\[1em] \text{for } 1 \leq i, j \leq 2; \\[1em] \frac{\partial_1 a}{a} + \frac{\partial_2 a}{a} = \frac{\partial_1 e}{e}; \\[1em] \frac{\partial_1 e}{e} + \frac{\partial_2 e}{e} = -\frac{\partial_2 a}{a} \end{array} \right\}.$$

Bibliography

[1] Arreche, C.E.: Computing the differential Galois group of a one-parameter family of second order linear differential equations (2012). Preprint: arXiv:1208.2226

[2] Arreche, C.E.: A Galois-theoretic proof of the differential transcendence of the incomplete Gamma function. J. Algebra **389**, 119–127 (2013). doi:10.1016/j.jalgebra.2013.04.037

[3] Arreche, C.E.: Computation of the unipotent radical of the differential Galois group for a parameterized second-order linear differential equation. Adv. Appl. Math. **57**, 44–59 (2014). doi:10.1016/j.aam.2014.03.001

[4] Arreche, C.E.: Computing the differential Galois group of a parameterized second-order linear differential equation. Proceedings of ISSAC 2014, pp. 43–50 (2014). doi:10.1145/2608628.2608680

[5] Baldassarri. F., Dwork, B.: On Second Order Linear Differential Equations with Algebraic Solutions. Amer. J. Math. **101**(1), 42–76 (1979). doi:10.2307/2373938

[6] Bostan, A., Chen, S., Chyzak, F., Li, Z., Xin, G.: Hermite Reduction and Creative Telescoping for Hyperexponential Functions. Proceedings of ISSAC 2013, pp. 77–84 (2013). doi:10.1145/2465506.2465946

[7] Cassidy, P.J.: Differential algebraic groups. Amer. J. Math. **94**(3), 891–954 (1972). doi:10.2307/2373764

[8] Cassidy, P.J.: Unipotent differential algebraic groups. In: H. Bass, P. Cassidy, J. Kovacic (eds.) Contributions to Algebra (A collection of pa-

pers dedicated to Ellis Kolchin), pp. 83–115. Academic Press [Harcourt Brace Jovanovich Publishers], New York (1977)

[9] Cassidy, P.J., Singer, M.F.: Galois theory of parameterized differential equations and linear differential algebraic groups. In: IRMA Lect. Math. Theor. Phys., vol. 9, pp. 113–155. Eur. Math. Soc. Publishing House (2007). doi:10.4171/020-1/7

[10] Chen, S., Feng, R., Li, Z., Singer, M.F.: Parallel Telescoping and Parameterized Picard-Vessiot Theory. Proceedings of ISSAC 2014, pp. 99–106 (2014). doi:10.1145/2608628.2608638

[11] Chen, S., Kauers, M., Singer, M.F.: Telescopers for Rational and Algebraic Functions via Residues. Proceedings of ISSAC 2012, pp. 130–137 (2012). doi:10.1145/2442829.2442851

[12] Chen, S., Singer, M.F.: Residues and Telescopers for Bivariate Rational Functions. Adv. Appl. Math. **49**, 111–133 (2012). doi:10.1016/j.aam.2012.04.003

[13] Chevalley, C.: Introduction to the Theory of Algebraic Functions of One Variable. No. VI in Math. Surveys, Amer. Math. Soc., New York (1951)

[14] Compoint, E., Singer, M.F.: Computing Galois Groups of Completely Reducible Differential Equations. J. Symbolic Comput. **28**(4–5), 473–494 (1999). doi:10.1006/jsco.1999.0311

[15] Deligne, P.: Catégories Tannakiennes. In: The Grothendieck Festschrift, Vol. II, pp. 111–195. Progr. Math. **87**, Birkhäuser Boston (1990)

[16] Deligne, P., Milne, J.S.: Tannakian Categories. In: Hodge cycles, motives, and Shimura varieties, *Lecture Notes in Math.*, vol. 900, pp. 101–228. Springer-Verlag, Berlin (1982). doi:10.1007/978-3-540-38955-2_4

[17] Dreyfus, T.: Phénomènes de Stokes et approche galoisienne des problèmes de confluence. Ph.D. thesis, Université Paris Diderot (Paris 7), Paris (2013)

[18] Dreyfus, T.: A density theorem for parameterized differential Galois theory (2014). To appear in Pacific J. Math. arXiv:1203.2904

[19] Dreyfus, T.: Computing the Galois group of some parameterized linear differential equation of order two. Proc. Amer. Math. Soc. **142**, 1193–1207 (2014). doi:10.1090/S0002-9939-2014-11826-0

[20] Geddes, K.O., Czapor, S.R., Labahn, G.: Algorithms for Computer Algebra. Kluwer Academic Publishers, Norwell, MA (1992)

[21] Gillet, H., Gorchinskiy, S., Ovchinnikov, A.: Parameterized Picard-Vessiot extensions and Atiyah extensions. Adv. in Math. **238**, 322–411 (2013). doi:10.1016/j.aim.2013.02.006

[22] Gorchinskiy, S., Ovchinnikov, A.: Isomonodromic differential equations and differential Tannakian categories. J. Math. Pures et Appliquées **102**(1), 48–78 (2014). doi:10.1016/j.matpur.2013.11.001

[23] Hardouin, C.: Hypertranscendance des systèmes aux différences diagonaux. Compos. Math. **144**(3), 565–581 (2008). doi:10.1112/S0010437X07003430z

[24] Hardouin, C., Singer, M.F.: Differential Galois theory of linear difference equations. Math. Ann. **342**(2), 333–377 (2008). doi:10.1007/s00208-008-0238-z

[25] van Hoeij, M.: Factorization of differential operators with rational function coefficients. J. Symbolic Comput. **24**(5), 537–561 (1997). doi:10.1006/jsco.1997.0151

[26] van Hoeij, M., Weil, J.-A.: An algorithm for computing invariants of differential Galois groups. J. Pure Appl. Algebra **117–118**, 353–379 (1997). doi:10.1016/S0022-4049(97)00018-2

[27] van Hoeij, M., Weil, J.-A.: Solving second-order linear differential equations with Klein's algorithm. Proceedings of ISSAC 2005, pp. 340–347 (2005). doi:10.1145/1073884.1073931

[28] Hrushovski, E.: Computing the Galois group of a linear differential equation. Banach Center Publ. **58**, 97–138 (2002). doi:10.4064/bc58-0-9

[29] Johnson, J., Reinhart, G., Rubel, L.: Some counterexamples to separation of variables. J. Differential Equations **121**(1), 42–66 (1995). doi:10.1006/jdeq.1995.1121

[30] Katz, N.M., Oda, T.: On the differentiation of De Rham cohomology classes. J. Math. Kyoto Univ. **8**(2), 199–213 (1968). http://projecteuclid.org/euclid.kjm/1250524135

[31] Kolchin, E.R.: Algebraic matric groups and the Picard-Vessiot theory of homogeneous linear ordinary differential equations. Annal. Math. **49**(1), 1–42 (1948). doi:10.2307/1969111

[32] Kolchin, E.R.: Galois Theory of Differential Fields. Amer. J. Math. **75**(4), 753–824 (1953). doi:10.2307/2372550

[33] Kolchin, E.R.: Algebraic groups and algebraic dependence. Amer. J. Math. **90**(4), 1151–1164 (1968). doi:10.2307/2373294

[34] Kolchin, E.R.: Constrained Extensions of Differential Fields. Adv. in Math. **12**(2), 141–170 (1974). doi:10.1016/S0001-8708(74)80001-0

[35] Kolchin, E.R.: Differential Algebra and Algebraic Groups. Academic Press, New York (1976)

[36] Kolchin, E.R.: Differential Algebraic Groups. Pure Appl. Math. 114. Academic Press, Orlando, FL (1985)

[37] Kovacic, J.J.: An algorithm for solving second order linear homogeneous differential equations. J. Symbolic Comput. **2**(1), 3–43 (1986). doi:10.1016/S0747-7171(86)80010-4

[38] Landesman, P.: Generalized differential Galois theory. Trans. Amer. Math. Soc. **360**(8), 4441–4495 (2008). doi:10.1090/S0002-9947-08-04586-8

[39] Manin, J.I.: Algebraic Curves over Fields with Differentiation (*English translation*). Amer. Math. Soc. Transl. Ser. 2 **37**, 59–78 (1964)

[40] Minchenko, A., Ovchinnikov, A.: Zariski closures of linear differential algebraic groups. Adv. in Math. **227**(3), 1195–1224 (2011). doi:10.1016/j.aim.2011.03.002

[41] Minchenko, A., Ovchinnikov, A., Singer, M.F.: Reductive linear differential algebraic groups and the Galois groups of parameterized linear differential equations. To appear in Int. Math. Res. Not. (2014). doi:10.1093/imrn/rnt344

[42] Minchenko, A., Ovchinnikov, A., Singer, M.F.: Unipotent differential algebraic groups as parameterized differential Galois groups. To appear in J. Inst. Math. Jussieu (2014). doi:10.1017/S1474748013000200

[43] Mitschi, C., Singer, M.F.: Monodromy groups of parameterized linear differential equations with regular singularities. Bull. Lond. Math. Soc. **44**(5), 913–930 (2012). doi:10.1112/blms/bds021

[44] Mitschi, C., Singer, M.F.: Projective isomonodromy and Galois groups. Proc. Amer. Math. Soc. **141**, 605–617 (2013). doi:10.1090/S0002-9939-2012-11499-6

[45] Ostrowski, A.: Sur les rélations algébriques entre les intégrales indéfinies. Acta Math. **78**(1), 315–318 (1946). doi:10.1007/BF02421605

[46] Ovchinnikov, A.: Tannakian Approach to Linear Differential Algebraic Groups. Transformation Groups **13**(2), 413–446 (2008). doi:10.1007/s00031-008-9010-4

[47] Ovchinnikov, A.: Differential Tannakian Categories. J. Algebra **321**(10), 3043–3062 (2009). doi:10.1016/j.jalgebra.2009.008

[48] Ovchinnikov, A.: Tannakian Categories, Linear Differential Algebraic Groups, and Parameterized Linear Differential Equations. Transformation Groups **14**(1), 195–223 (2009). doi:10.1007/s00031-008-9042-9

[49] Ovchinnikov, A., Wibmer, M.: σ-Galois theory of linear difference equations. To appear in Int. Math. Res. Not. (2014). doi:10.1093/imrn/rnu060

[50] van der Put, M., Singer, M.F.: Galois Theory of Linear Differential Equations. No. 328 in Grundlehren der mathematischen Wissenschaften. Springer-Verlag (2003)

[51] Singer, M.F.: Liouvillian solutions of n-th order homogeneous linear differential equations. Amer. J. Math. **103**(4), 661–682 (1981). doi:10.2307/2374045

[52] Singer, M.F.: Linear algebraic groups as parameterized Picard-Vessiot Galois groups. J. Algebra **373**, 151–161 (2013). doi:10.1016/j.jalgebra.2012.09.037

[53] Singer, M.F., Ulmer, F.: Liouvillian and Algebraic Solutions of Second and Third Order Linear Differential Equations. J. Symbolic Comput. **16**, 37–74 (1993). doi:10.1006/jsco.1993.1033

[54] Sit, W.Y.: Differential algebraic subgroups of SL(2) and strong normality in simple extensions. Amer. J. Math. **97**(3), 627–698 (1975). doi:10.2307/2373770

[55] Springer, T.A.: Linear Algebraic Groups, 2nd edn. Progr. Math. **9**, Birkhäuser Boston (1983)

[56] Trushin, D.: Splitting fields and general differential Galois theory. Sbornik: Math. **201**(9), 1323–1353 (2010). doi:10.1070/SM2010v201n09ABEH004114

[57] Ulmer, F., Weil, J.-A.: Note on Kovacic's algorithm. J. Symbolic Comput. **22**(2), 179–200 (1996). doi:10.1006/jsco.1996.0047

[58] Wibmer, M.: Existence of ∂-parameterized Picard-Vessiot extensions over fields with algebraically closed constants. J. Algebra **361**, 163–171 (2012). doi:10.1016/j.jalgebra.2012.03.035

[59] Zeilberger, D.: The method of creative telescoping. J. Symbolic Comput. **11**(3), 195–204 (1991). doi:10.1016/S0747-7171(08)80044-2