TR-2010002: Reasoning About Games

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Abstract

A mixture of propositional dynamic logic and epistemic logic is used to give a formalization of Artemov's knowledge-based reasoning approach to game theory (KBR), [4, 5, 6, 7]. We call the (family of) logics used here $\text{PDL} + \text{E}$. It is in the general family of Dynamic Epistemic Logics [21], was applied to games already in [20], and investigated further in [18, 19]. Epistemic states of players, usually treated informally in game-theoretic arguments, are here represented explicitly and reasoned about formally. The heart of the presentation is a detailed analysis of the Centipede game using both the proof theoretic and the semantic machinery of $\text{PDL} + \text{E}$. The present work can be seen partly as an argument for the thesis that $\text{PDL} + \text{E}$ should be the basis of the logical investigation of game theory.

1 Background

Game theory attempts to predict or explain the behavior of agents under tightly specified sequences of interactions. What agents do commonly depends on their reasoning abilities and their information about other agents. An agent may behave one way in an exchange with another agent if the agent knows the second agent is rational, yet behave differently if the agent doesn’t know that. Recently in a series of technical reports Sergei Artemov has developed a knowledge-based approach to games, [4, 5, 6, 7]. (It should be noted that the development is not strictly linear—some things in earlier reports are overridden in later ones.) Artemov’s work provides new and significant insights. At the heart of it is the idea that the game tree is only a partial specification of the game—the epistemic states of the players have a fundamental role, and while conditions on these are generally stated more informally, they are no less significant. Thus if one assumes common knowledge of player rationality, or merely mutual knowledge of player rationality, the same game tree may lead to different outcomes. In this report a formal treatment of game tree + epistemic state machinery is investigated. In particular there is a semantics in which epistemic states are explicitly present, and a proof theory is provided for reasoning about them.

Artemov’s presentation is a mixture of formal and informal reasoning. When one asserts that something is so, an argument using some formalism and some conventional English is appropriate and convincing. There is a certain amount of hand-waving involved, as there is in most mathematics. When one asserts that something is not so, one must generally be more careful because informally presented counter-examples might omit a crucial but overlooked detail. In this report I provide formal machinery, and apply it formally. This does not mean I advocate that all arguments in this
area must be formal—it would kill the subject. But it is a truism that a correct mathematical proof is one that can be formalized, and for this to be applicable, formal machinery must exist. It is the elaboration of such formal machinery that this report concentrates on, for knowledge based reasoning applied to game theory.

*Dynamic epistemic logic* is really an umbrella term covering various combinations of logics of knowledge and belief with logics of actions. The book [21] has become the standard reference in this area generally. Nonetheless, the specific variety of dynamic epistemic logic we use here does not appear in that book. What we need, quite simply, is a straightforward combination of two very, very traditional logics, propositional dynamic logic, [16, 10] and epistemic logic [12]. This combination has been considered before, and specifically applied to the analysis of games, in [20]. The approach here is rather different than this, however. In [20], roughly, game states are also considered to be epistemic states, which is appropriate for the questions investigated there. But here we are interested in examining Artemov’s knowledge based reasoning ideas, and this leads us to think of a game state as containing possibly several distinct epistemic states. This provides flexible machinery suitable for examining player uncertainty even in perfect information games. I call the (family of) logics used here PDL + E. It is really a family, rather than a single logic, because on the one hand assumptions of varying strengths can be made concerning player knowledge, and on the other hand, various ‘cross’ axioms can be assumed as to how the epistemic and the dynamic operators relate to each other. The interaction axioms were thoroughly investigated in [18, 19], where axiomatizations, semantics, and completeness proofs can be found (essentially for the one agent case).

As a case study I apply the machinery in detail to the well-known Centipede game previously analyzed by Aumann, [8], and discussed again in Artemov’s technical reports (and many other places, of course). In order to make this report relatively self-contained, I will explain what is needed from the combination of epistemic and dynamic logics. In particular, completeness results from [18, 19] will not be needed, only soundness results, and these are straightforward. The game theory side is a different thing, however, and it is recommended that [4, 5, 6, 7] be consulted for the motivation behind the knowledge-based approach.

I recognize that this report has a rather forbidding appearance, with many detailed formal proofs. I apologize for this. The intention is that informal discussions should be read, and the ontology of the semantics should be grasped, including how epistemic states figure in. Formal proofs should generally be skipped, or a few spot checked. The existence of formal proofs is significant, the details of them are not, but proofs are given in full so that a reader may check that they really do exist and do what is claimed for them.

## 2 Logics

I begin with a brief discussion of epistemic logic, and a longer one of propositional dynamic logic, then I discuss their combination, which is much less familiar. I give the combination a name, PDL + E. The name is slightly misleading since it is really a family of logics, depending on what epistemic assumptions are made, as well as what assumptions are made concerning the connections between the dynamic and the epistemic operators. It should be noted again that PDL + E is in the general family of dynamic epistemic logics [21].

### 2.1 Epistemic Logic

This is a well-known, standard subject, so I will be brief. There is a set (generally finite but not necessarily) of agents, $A, B, C, \ldots$. For each agent $A$ there is a modal operator $K_A$, with $K_A X$
read as: agent $A$ knows $X$. A dual operator $\hat{K}_A$ is sometimes introduced, with $\hat{K}_A X$ abbreviating $\neg K_A \neg X$. $\hat{K}_A X$ can be read as asserting that $X$ is compatible with the knowledge of agent $A$.

Axiomatically each $K_A$ is a normal modal operator, so there is a knowledge necessitation rule, from $X$ conclude $K_A X$. There are also modus ponens, and the following axiom schemes (note that these are schemes, not individual axioms).

E-1 All tautologies (or enough schemes to generate them)

E-2 $K_A(X \supset Y) \supset (K_A X \supset K_A Y)$

In addition there may be some or all of the following as axiom schemes.

E-3 $K_A X \supset X$, Factivity. Its presence or absence is what distinguishes knowledge from belief, in this approach.

E-4 $K_A X \supset K_A K_A X$, Positive Introspection.

E-5 $\neg K_A X \supset K_A \neg K_A X$, Negative Introspection.

I will explicitly note when any of these three axiom schemes are needed, and will note their absence if that is significant.

Semantics is the usual Kripke/Hintikka possible world version. A model, then, is a structure $\langle G, R_A, \ldots, \vdash \rangle$, where $G$ is a non-empty set of epistemic states or possible worlds, $R_A$ is a binary relation on $G$ for each agent $A$, and $\vdash$ is a relation between states and formulas, the “at this state the formula is true” relation. The $\vdash$ relation meets the usual condition, $\Gamma \vdash K_A X$ iff $\Delta \vdash X$ for every $\Delta \in G$ such that $\Gamma R_A \Delta$. Factivity for agent $A$ corresponds to requiring that $R_A$ be reflexive, Positive Introspection corresponds to transitivity, and Factivity, Positive Introspection and Negative Introspection together corresponds to an equivalence relation. Other conditions are often considered, but they will not be needed here.

2.2 Propositional Dynamic Logic

Propositional dynamic logic (PDL) is a logic of actions, which originally were thought of as corresponding to computer programs, though there is no need to restrict things to this. Informally $[\alpha]X$ is to be read: after action $\alpha$ is complete then $X$ will be true. Actions can be non-deterministic—the action ‘go to a store and buy milk’ could be executed in many ways since a choice of store is unspecified. So to be more precise, $[\alpha]X$ is intended to express that $X$ will be true after $\alpha$ is executed, no matter how this is done. If $\alpha$ cannot be executed, $[\alpha]X$ is automatically true. The dual, $(\alpha)X$, is to be read: there is at least one way of executing action $\alpha$ that leaves $X$ true. If $\alpha$ cannot be executed $(\alpha)X$ is false. For background and further information about propositional dynamic logic (PDL), the on-line treatments [1, 9] are recommended. Also strongly recommended is [11]. Here is a brief summary of PDL, beginning with the language.

There is a collection of actions. Actions are built up from atomic actions (given arbitrarily) using the following machinery.

- if $\alpha$ and $\beta$ are actions so is $(\alpha; \beta)$, representing action $\alpha$ followed by action $\beta$.
- if $\alpha$ and $\beta$ are actions so is $(\alpha \cup \beta)$, representing a non-deterministic choice of $\alpha$ or $\beta$. This is sometimes written $(\alpha + \beta)$.
- if $\alpha$ is an action so is $\alpha^*$, representing $\alpha$ repeated an arbitrary number of times, possibly 0.
• if $A$ is a formula then $A?$ is an action, representing a test for $A$.

Formulas are built up from atomic formulas in the usual way, with the additional condition: if $\alpha$ is an action and $X$ is a formula then $[\alpha]X$ and $\langle \alpha \rangle X$ are formulas. As noted above, $[\alpha]X$ can be read as: after $\alpha$ is finished, $X$ is true. $\langle \alpha \rangle X$ will be taken as an abbreviation for $\neg[\alpha]\neg X$.

Incidentally, note the following. One condition above says that if $A$ is a formula then $A?$ is an action. Another condition says that if $\alpha$ is an action and $X$ is a formula, then $[\alpha]X$ is a formula. These two together tell us that formulas and actions must be defined by a simultaneous recursion.

The standard axiom system for PDL is as follows. For rules there are modus ponens and action necessitation, from $X$ conclude $[\alpha]X$, for any action $\alpha$. Then there are the following axiom schemes (again, schemes, not individual axioms).

PDL-1 All tautologies (or enough of them)

PDL-2 $[\alpha](X \supset Y) \supset ([\alpha]X \supset [\alpha]Y)$

PDL-3 $[\alpha; \beta]X \equiv [\alpha][\beta]X$

PDL-4 $[\alpha \cup \beta]X \equiv ([\alpha]X \land [\beta]X)$

PDL-5 $[\alpha^*]X \equiv (X \land [\alpha][\alpha^*]X)$

PDL-6 $[A?]X \equiv (A \supset X)$

Finally, a version of induction can be captured by either an axiom scheme or a rule of inference—they are interderivable. The axiom and rule are as follows.

PDL-7 $[\alpha^*](X \supset [\alpha]X) \supset (X \supset [\alpha^*]X)$

PDL-8 From $X \supset [\alpha]X$ infer $X \supset [\alpha^*]X$

A semantics for PDL amounts to a multi-modal Kripke structure with some additional conditions. A model is a structure $\langle G, R_\alpha, \ldots, \models \rangle$ where $G$ is a non-empty set of states (originally, states in the execution of a program); $R_\alpha$ is a binary accessibility relation between states, for each action $\alpha$; and $\models$ is a relation between states and formulas. The $\models$ relation must satisfy the familiar Kripkean condition: $\Gamma \models [\alpha]X$ iff $\Delta \models X$ for every $\Delta \in G$ such that $\Gamma R_\alpha \Delta$.

The special PDL machinery imposes relationships between the accessibility relations. Specifically, one requires the following.

• $R_{\alpha;\beta}$ is the relation product $R_\alpha \circ R_\beta$, where $\Gamma(R_\alpha \circ R_\beta)\Delta$ if there is some $\Omega$ so that $\Gamma R_\alpha \Omega$ and $\Omega R_\beta \Delta$.

• $R_{\alpha \cup \beta}$ is the relation $R_\alpha \cup R_\beta$.

• $\Gamma R_{A?} \Gamma$ just in case $\Gamma \models A$.

• $R_{\alpha^*}$ is the reflexive and transitive closure of $R_\alpha$ (with respect to $\circ$).

Note that the condition on $A?$ means that $\models$ and the family of accessibility relations must be defined by a mutual recursion.

If there were no $*$ operation, completeness would be simple to prove. With it, the proof is more complex, and there were incorrect versions proposed. Completeness was first properly established by Parikh in [15], with another proof along somewhat different lines in [13]. The logic is decidable. Tableau systems are known.
The axiomatics and semantics described in this section come from [18, 19], where more detailed discussions can be found, for the mono-modal case. A summarization sufficient for present purposes is given here. In subsequent sections the resulting formal system will be extended by axioms special to particular games. Not surprisingly, these game axioms will not be axiom schemes, and so the resulting logics will not be closed under substitution—they will not be normal. This makes for quite a few difficulties where completeness and decidability are concerned, a primary consideration of [18, 19], but it is not a relevant issue here. Here logics will not be investigated as such, but rather their applicability to particular problems is central. Soundness, relative to a semantics having to do with the game in question, will be easy to establish, and that is all that is needed for present purposes. The discussion in this section is more general, however, and will narrow down when particular games are discussed.

To begin, we start with the fusion of the logics from Sections 2.1 and 2.2. Axiomatically one simply combines all the machinery of epistemic logic with that of PDL. Semantically one uses possible world models in which there are accessibility relations for each PDL action, and an accessibility relation for each agent. It is assumed that the PDL relations meet the conditions of Section 2.2 and each knowledge relation meets the general epistemic conditions of Section 2.1, and whichever of the reflexive, symmetric, transitive conditions are desired.

In addition, some interaction conditions between the PDL and the epistemic machinery may be imposed. In [18, 19] three are considered, and these are discussed now.

The first condition is a No Learning condition, given by the following axiom scheme, where $i$ is any agent.

$$\text{PDLE-1} \ [\alpha]K_iX \supset K_i[\alpha]X \ (\text{No Learning})$$

Informally this says that if an agent knows $X$ after an action $\alpha$ is performed, the agent already knew that $X$ would be true after $\alpha$, and so executing the action brought no new knowledge.

Semantically, the axiom corresponds to the diagram given in Figure 1. I will say what “correspondence” amounts to more precisely below. The diagram should be read as follows. Assume there are arbitrary states 1, 3, and 4, with 3 accessible from 1 under the epistemic accessibility relation associated with $K_i$, and 4 accessible from 3 under the PDL accessibility relation associated with action $\alpha$. Then there is a state 2, accessible from 1 under the PDL relation associated with $\alpha$, with 4 accessible from 2 under the epistemic accessibility relation associated with $K_i$. In short, given states 1, 3, and 4 meeting the accessibility conditions shown by the two solid arrows, there is a state 2 with accessibilities shown by the two dashed arrows, so that the diagram commutes.

The next connecting condition is that of Perfect Recall. Axiomatically it is given as follows.

$$\text{PDLE-2} \ K_i[\alpha]X \supset [\alpha]K_iX \ (\text{Perfect Recall})$$

The corresponding semantic condition is shown in Figure 2. The diagram is read similarly to the previous one. If the relations between states 1, 2, and 4 shown by the solid arrows obtain, then a state 3 must exist with the relations shown by the dashed arrows holding, with the diagram commuting.

The third condition is called Church-Rosser in [18, 19], but I prefer to call it the Reasoning Ability condition. Axiomatically it is the following.

$$\text{PDLE-3} \ \langle \alpha \rangle K_iX \supset K_i(\alpha)X \ (\text{Reasoning Ability})$$
Informally this says that if an agent could know $X$ after an action $\alpha$, the agent is able to figure that out and so knows now that $X$ could be the case after $\alpha$. The semantic condition corresponding to this is given in figure Figure 3.

The exact connections between the three axiomatic conditions given above, and the corresponding semantic conditions, is a bit tricky. Much of it comes down to whether one takes all conditions as axiom schemes or as particular axioms—equivalently, whether or not there is closure under substitution. Consider Perfect Recall, $\text{[PDLE-2]}$ as a representative example. I will assume it not as a general scheme, but for a particular action $\alpha$. As such, it is simple to verify that Perfect Recall for $\alpha$ evaluates to true at every world of any model meeting the Perfect Recall semantic condition for $\alpha$. As it happens, this is all that is needed for intended applications. If one takes the axioms as schemes and asks about completeness issues, things become complicated. It is shown in $[18, 19]$ that if the epistemic part is strong enough, $S5$, the $\text{PDL}+E$ combination collapses to the $\text{PDL}$ part. It is also shown that the heart of the problem is the $\text{PDL}$ notion of test. If one does not have tests, or if tests are restricted to atomic, this collapse does not happen.

In fact, tests in game formulations will not be used here, and all work will be with particular versions of the conditions considered above and so there will be no closure under substitution. Completeness considerations are not important, only soundness ones. And as was noted, these hold, with routine verifications.
2.4 Common Knowledge

A common knowledge operator, $C$, is introduced in one of the usual ways. There is one axiom scheme and one rule. As is standard, one first adds an ‘everybody knows’ operator: $EX$ abbreviates $K_AX \land K_BX \land K_CY \land \ldots$. Here are the common knowledge assumptions.

- **CK-1** (Axiom Scheme) $C_X \supset E(X \land CX)$
- **CK-2** (Rule)

\[
\frac{X \supset E(Y \land X)}{X \supset CY}
\]

This particular formulation serves for both common knowledge and common belief. One can prove $C_X \supset X$ if agents satisfy Factivity, $\text{E-3}$. Likewise one can prove $CCX \supset CX$ with Factivity.

Certain useful items are provable without assumptions of Factivity, Positive Introspection, or Negative Introspection, and hence apply even to very weak versions of belief. Here are some of them.

The following are provable from $\text{CK-1}$ and $\text{CK-2}$:

- **Com-1** $E(X \land CX) \supset CX$
- **Com-2** $CX \supset CCX$
- **Com-3** $CX \supset CEX$
- **Com-4** $CX \supset CK_iX$
- **Com-5** $C(X \supset Y) \supset (CX \supset CY)$
- **Com-6** Necessitation, from $X$ conclude $CX$

In addition there are connections with some of the special assumptions given earlier in this section. These are now discussed.

**Proposition 2.1** Assume each agent satisfies Reasoning Ability, $\text{PDLE-3}$, using action $\alpha$. Then analogous results hold for $E$ and $C$, specifically one has the following.

1. $(\alpha)EX \supset E(\alpha)X$
2. $(\alpha)CX \supset C(\alpha)X$

Proof We give the proof for two agents. The extension to more is obvious. For item 1,

$$\langle \alpha \rangle EX \supset \langle \alpha \rangle [K_A X \land K_B X]$$

$$\supset [\langle \alpha \rangle K_A X \land \langle \alpha \rangle K_B X]$$

$$\supset [K_A(\alpha) X \land K_B(\alpha) X]$$

$$\supset E \langle \alpha \rangle X$$

For item 2, using both the axiom and rule for common knowledge,

$$CX \supset E (X \land CX)$$

$$\langle \alpha \rangle CX \supset \langle \alpha \rangle E (X \land CX)$$

$$\supset E \langle \alpha \rangle (X \land CX)$$

$$\supset E [\langle \alpha \rangle X \land \langle \alpha \rangle CX]$$

$$\langle \alpha \rangle CX \supset C(\alpha)X$$

A similar result holds for No Learning. I state it and omit the proof.

**Proposition 2.2** Assume each agent satisfies No Learning, \textit{PDLE-1} for action $\alpha$. Then analogous results hold for $E$ and $C$, specifically one has the following.

1. $[\alpha]EX \supset E[\alpha]X$

2. $[\alpha]CX \supset C[\alpha]X$

Perfect Recall is more of a problem. There is the following.

**Proposition 2.3** Assume each agent satisfies Perfect Recall, \textit{PDLE-2} for action $\alpha$. Then we have $E^n[\alpha]X \supset [\alpha]E^n X$, and also $C[\alpha]X \supset [\alpha]E^n X$, for every $n$.

Proof Again the proof is given for two agents, to keep things simple. The argument is by induction, and only the first item will be proved.

The base case, $n = 0$, is trivial. The induction step is as follows.

$$E^{n+1}[\alpha]X \supset E^n \{K_A[\alpha]X \land K_B[\alpha]X\}$$

$$\supset E^n \{[\alpha]K_A X \land [\alpha]K_B X\}$$

$$\supset E^n[\alpha]\{K_A X \land K_B X\}$$

$$\supset E^n[\alpha]EX$$

$$\supset [\alpha]E^n EX$$

$$\supset [\alpha]E^{n+1}X$$

The analogous result for common knowledge does not seem to follow (though I have no proof of this). $E^n$ suffices for results about particular finite games, but for results about families of games one needs $C$ and so, when appropriate, I will assume the following as an additional condition.

CK-3 $C[\alpha]X \supset [\alpha]CX$ (Extended Perfect Recall)
3 PDL + E For Games

In this section I give axioms general enough to apply to many games. Semantics comes later, in Section 5. Following Artemov, [4, 5, 6, 7], Knowledge of the Game Tree includes possible moves, payoffs, etc., all of which should be common knowledge. I leave issues of payoffs until later, and concentrate initially on what might be called ‘general structure.’ To this end I adopt some specific axioms—these are not axiom schemes since they refer to particular players. It is assumed that our necessitation rules apply to these axioms, necessitation both for knowledge operators and for action operators.

3.1 General Game Tree Knowledge

Games can have any number of players. The ones of interest here have two, and so things are formulated for this situation only. Generalizations are obvious, but the assumption of two players keeps notation simpler. I designate two special propositional letters, $A$ and $B$, with the intended meaning that $A$ is true if it is the turn of agent $A$ to move, and $B$ is true if it is the turn of agent $B$ to move. I also assume there are knowledge operators (and their duals) for each agent, $K_A$ and $K_B$. In addition to general PDL + E axioms and rules there are the following axioms (again, not axiom schemes). They say that exactly one player is to move, and everybody knows whose move it is. They are in a family of axioms representing knowledge of the game; accordingly I number these axioms in a KG sequence.

$$\begin{align*}
\text{KG-1} & \quad A \lor B \\
\text{KG-2} & \quad \neg (A \land B) \\
\text{KG-3} & \quad A \supset K_A A \\
\text{KG-4} & \quad B \supset K_B B \\
\text{KG-5} & \quad A \supset K_B A \\
\text{KG-6} & \quad B \supset K_A B
\end{align*}$$

An aside: it is possible at this point to introduce a defined operator $K$ with the intended meaning of $KX$ being: the player whose turn it is to move knows $X$. More formally, $KX \equiv ((A \supset K_A X) \land (B \supset K_B X))$. I will not make use of this defined operator here, but it is interesting to note that $K$ has almost all the properties of a normal modal operator. There is no closure of theorems involving it under substitution, since I have given propositional letters $A$ and $B$ special roles, and this is obviously not preserved under substitution. Nonetheless there is the following, whose proof is omitted.

**Theorem 3.1** The following are provable using the axioms, rules, and definitions introduced so far.

1. $K(X \supset Y) \supset (KX \supset KY)$ is a theorem.
2. If $X$ is a theorem, so is $KX$.
3. If $E$ Factivity, holds for both $K_A$ and $K_B$ then it holds for $K; KX \supset X$.
4. If $E$ Positive Introspection, holds for both $K_A$ and $K_B$ then it holds for $K; KX \supset KXX$. 

5. If $E$ Negative Introspection, holds for both $K_A$ and $K_B$ then it holds for $K$; $\neg KX \supset K \neg KX$.

This ends the side remarks, and I now continue with general game tree considerations. In a game each player has a choice of moves, say these are represented by propositional letters $m_1, m_2, \ldots, m_k$. At each turn the appropriate player picks exactly one move, so there are the following minimal assumptions about moves.

KG-7 $m_1 \lor m_2 \lor \ldots \lor m_k$

KG-8 $\neg(m_1 \land m_2), \neg(m_1 \land m_3), \neg(m_2 \land m_3), \ldots$

Each $m_i$ represents a decision by a player. In addition there are transitions from one state of the game to another state. Some choices by players end the game, some choices trigger these transitions. In this report transitions are dynamic operators, distinct from any of the $m_i$, though there is certainly a connection between player choices and transitions—choices that do not end play trigger transitions. Let us assume $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the atomic transitions available, represented by atomic dynamic operators.

It is assumed players alternate, and so a transition to a new active state of the game switches the player whose turn it is to move. For each game transition $\alpha_i$ I assume the following.

KG-9 $A \supset [\alpha_i]B$
KG-10 $B \supset [\alpha_i]A$

Some game states are terminal in the sense that players can choose plays, but all plays end the game—no transitions to further states are possible. We will use this terminology throughout, referring to a game state as terminal, or not. It is assumed players know which states are the terminal ones. More generally, I assume that for each atomic transition $\alpha_i$, each player knows if transition $\alpha_i$ is possible or not. If no transition is possible at a state, the state is terminal. This is easily represented. The formula $\langle \alpha_i \rangle \top$ asserts that an $\alpha_i$ transition is possible ($\top$ is truth), while the formula $[\alpha_i] \bot$, equivalently $\neg \langle \alpha_i \rangle \top$, asserts that an $\alpha_i$ transition is impossible ($\bot$ is falsehood). Likewise $\langle \alpha_1 \cup \ldots \cup \alpha_n \rangle \top$ asserts that some atomic transition is possible, while $[\alpha_1 \cup \ldots \cup \alpha_n] \bot$ asserts that no atomic transition can be made—one is at a terminal game state. I assume the following, for each atomic game transition $\alpha_i$.

KG-11 $[\alpha_i] \bot \supset K_A[\alpha_i] \bot$
KG-12 $[\alpha_i] \bot \supset K_B[\alpha_i] \bot$
KG-13 $\langle \alpha_i \rangle \top \supset K_A \langle \alpha_i \rangle \top$
KG-14 $\langle \alpha_i \rangle \top \supset K_B \langle \alpha_i \rangle \top$

Incidentally, KG-11 and KG-12 follow easily from the No Learning condition, [PDLE-1], and KG-13 and KG-14 likewise follow from the Reasoning Ability condition, [PDLE-3]. Since one doesn’t always want to assume these powerful conditions, it is reasonable to make them separate assumptions.

**Proposition 3.2** The following are provable.

1. $[\alpha_1 \cup \ldots \cup \alpha_n] \bot \supset K_A[\alpha_1 \cup \ldots \cup \alpha_n] \bot$
2. $[\alpha_1 \cup \ldots \cup \alpha_n] \bot \supset K_B[\alpha_1 \cup \ldots \cup \alpha_n] \bot$
3. \( (\alpha_1 \cup \ldots \cup \alpha_n) \top \supset K_A(\alpha_1 \cup \ldots \cup \alpha_n) \top \)

4. \( (\alpha_1 \cup \ldots \cup \alpha_n) \top \supset K_B(\alpha_1 \cup \ldots \cup \alpha_n) \top \)

**Proof** For item 1 (item 2 is similar)

\[
[\alpha_1 \cup \ldots \cup \alpha_n] \bot \equiv [\alpha_1] \bot \land \ldots \land [\alpha_n] \bot \\
\supset K_A[\alpha_1] \bot \land \ldots \land K_A[\alpha_n] \bot \\
\equiv K_A([\alpha_1] \bot \land \ldots \land [\alpha_n] \bot) \\
\equiv K_A[\alpha_1 \cup \ldots \cup \alpha_n] \bot
\]

Next for item 3 (item 4 is similar)

\[
(\alpha_1 \cup \ldots \cup \alpha_n) \bot \equiv (\alpha_1) \bot \lor \ldots \lor (\alpha_n) \bot \\
\supset K_A(\alpha_1) \bot \lor \ldots \lor K_A(\alpha_n) \bot \\
\supset K_A((\alpha_1) \bot \lor \ldots \lor (\alpha_n) \bot) \\
\equiv K_A(\alpha_1 \cup \ldots \cup \alpha_n) \bot
\]

3.2 Rationality Considerations

The last of our general game principles is the only one that is essentially non-trivial. While it appears in one form or another in the works of a number of authors, it was given special emphasis in [4, 5, 6, 7]. Loosely it says that a player who is rational and who knows what his or her best move is, given the limitations imposed by the knowledge the player possesses, will play that best move. This, of course, presupposes that such best moves must exist. In fact Artemov has shown that, under very broad conditions, each player in a game, when it is his turn to move, must have a move that is best possible given what the player knows at that point of the game. This is called the *best known move*, and is represented formally by a propositional letter. Suggestively, this propositional letter is written in a special format, for example \( k_{\text{best}}^A(m) \) which is intended to express that move \( m \) is the best known move for player \( A \). Rationality for a player is also represented by a propositional letter, and again suggestive notation is used: \( r^A \) or \( r^B \), for rationality of \( A \) or \( B \) respectively. The fundamental *rationality conditions* assert that a player who is rational and is aware of what his or her best known move is, will play it. Here is the formal version of these *rationality conditions*, for each move \( m_i \).

- **RC-A** \( (A \land K_A k_{\text{best}}^A(m_i) \land r^A) \supset m_i \)
- **RC-B** \( (B \land K_B k_{\text{best}}^B(m_i) \land r^B) \supset m_i \)

3.3 Backward Induction

So-called *backward induction* plays a central role in the analysis of a number of games. In this section I give a schematic version of backward induction, using the machinery introduced so far. This version is quite straightforward, but will serve to simplify some of the arguments later on. Roughly, backward induction shows something, say \( X \), is true throughout a game by showing \( X \) is true at terminal game states, and also showing \( X \) is true at a state provided some transition from that state takes the game to another state at which \( X \) is true. Thus one works backward from terminal states to encompass the entire game tree.
Theorem 3.3 (Backward Induction Derived Rule) Let $\alpha_1, \ldots, \alpha_n$ be all the atomic game transitions, and let $X$ be some formula. Assume the following.

1. $\langle (\alpha_1 \cup \ldots \cup \alpha_n)^* \rangle [\alpha_1 \cup \ldots \cup \alpha_n] \perp$
2. $[\alpha_1 \cup \ldots \cup \alpha_n] \perp \supset X$
3. $\langle (\alpha_1 \cup \ldots \cup \alpha_n) \rangle X \supset X$

Then $X$ follows.

Before giving the simple proof some comments are in order since PDL formulas are not easy to read until some experience has been gained. I noted earlier that $[\alpha_1 \cup \ldots \cup \alpha_n] \perp$ asserts one is at a terminal game state—every atomic transition is impossible. Then condition 1 of the theorem asserts that one can always reach a terminal state through some sequence of atomic transitions. Likewise condition 2 asserts that the formula $X$ is true at terminal states. And finally condition 3 asserts that if some atomic transition takes us to a state at which $X$ is true, then $X$ is true at the original state. These are the conditions for backward induction, stated less formally earlier.

Proof By condition 3, $\neg X \supset [\alpha_1 \cup \ldots \cup \alpha_n] \neg X$. Then by the rule of inference PDLE-8 we have $\neg X \supset [(\alpha_1 \cup \ldots \cup \alpha_n)^*] \neg X$. By standard modal reasoning, and using condition 2, we have the following.

\[
[(\alpha_1 \cup \ldots \cup \alpha_n)^*] \neg X \land [(\alpha_1 \cup \ldots \cup \alpha_n)^*] [\alpha_1 \cup \ldots \cup \alpha_n] \perp \\
\supset [(\alpha_1 \cup \ldots \cup \alpha_n)^*] (\neg X \land [\alpha_1 \cup \ldots \cup \alpha_n] \perp) \\
\supset [(\alpha_1 \cup \ldots \cup \alpha_n)^*](\neg X \land X) \\
\supset \perp
\]

Then using condition 1 we have $[(\alpha_1 \cup \ldots \cup \alpha_n)^*] \neg X \supset \perp$ which combines with the result above to give us $\neg X \supset \perp$, or $X$. $\blacksquare$

4 Formal PDL + E Proofs

Payoffs have not entered into the discussion so far. These are not formalized directly in our formal logic. It can be complicated mixing a complex propositional modal logic with elementary arithmetic. Instead I assume payoffs induce general strategy principles which vary from game to game, and I attempt to formulate these strategy principles using the PDL + E machinery introduced so far. For this it is simplest to discuss specific games, and so I turn to the well-known Centipede game. I begin with a standard presentation, giving the extensive form diagram and an informal analysis. The usual conclusion is that if there is common knowledge of player rationality, the first move in the game will be down. Not surprisingly, this will come out of the present formalization too, but the analysis of Centipede is only part of the point here. I present a methodology that I believe will be applicable to other games as well; consequently I proceed in gradual stages to make the presentation as transparent as possible. Only axiomatic proofs are considered in this section—semantics is reserved for Section 5.

4.1 The Centipede Game Tree

Figure 4 displays the extensive form diagram for a five-move version of the Centipede game, a game which in its 100 move version first appeared in [17]. Play starts at the upper left, alternating between A and B. Each player can choose to move right or move down. Payoffs shown are for A
and \(B\) in that order. The payoffs are arranged so that if a player does not terminate the game (by moving down), and the other player ends the game on the next move, the first player receives a little less than if that player had simply terminated the game directly. There are many ways payoffs can be arranged to meet this condition. We have used a common version: at each move the payoffs are switched around, and 2 is added to second payoff number.

![Figure 4: Five-Move Centipede](image)

It is important to point out that the game tree displayed does not take knowledge into consideration. Nowhere is it represented that agent \(A\) knows, or does not know, that \(B\) is rational, for instance. It can be seen as a kind of PDL model in which transitions to the right are shown, taking players from one game state to another. But without any explicit representation of knowledge, it is not in any sense a model for PDL + \(E\). I postpone the semantic introduction of epistemic states, and work entirely proof-theoretically for this section.

It is clear that the game presented in Figure 4 is intended to be one of a family—the pattern can be continued to arbitrary length. As given in [4] the analysis is tailored to the specific five move game, though it is obvious that the ideas generalize. At first glance a direct generalization would seem to involve a formal logic of knowledge and an induction, applied from the outside of the logic. The machinery of PDL allows us to carry this out inside our formal system, and so our analysis will apply to Centipede of any length.

One can reason informally about the game, as follows. If the game were to start at the right-hand node, it is obvious that \(A\) is best off moving down, so if \(A\) is rational, the move will be down. If the game were to start at the node second from the right, \(B\) could reason as follows: at the next node, if \(A\) is rational the move will be down; so I am better off moving down right now since I will get 6 instead of 5. Therefore, at this node, if \(B\) is rational and if \(B\) knows \(A\) is rational, \(B\) will move down. This reasoning can be repeated, as a backward induction, leading to the conclusion that if the rationality of everybody is common knowledge (or at least sufficiently so), then \(A\) will move down at the start.

4.2 Centipede Game Tree Knowledge

There are two players, \(A\) and \(B\), and axioms \([\text{KG-1}] – \text{KG-6}\) from Section 3.1 are directly adopted, but are relabeled \(\text{KGcent-1} – \text{KGcent-6}\) to be uniform with axioms introduced below. At each node one of two moves can be selected by a player, right and down. Instead of \(m_1\) and \(m_2\), I represent these more suggestively by \(ri\) and \(do\). Then \([\text{KG-7}]\) and \([\text{KG-8}]\) specialize to the following.

\[
\text{KGcent-7} \quad ri \lor do
\]

\[
\text{KGcent-8} \quad \neg(ri \land do)
\]

Continuing, one needs to say what atomic transitions there are in the Centipede game. In fact there is only one, which I denote by \(R\), representing a transition to the next active node to the
right. $R$ is an atomic action in $\text{PDL} + \text{E}$. For convenience I repeat the Centipede game tree in Figure 5, with some of the labels changed. I have suppressed player information and inserted node labels (positive integers) to make it easier to discuss things. I have also added explicit transition labels, $R$.

![Centipede Game Tree](image)

Figure 5: Five-Move Centipede, Again

Now [KG-9] and [KG-10] specialize to the following.

$\text{KGcent-9 } A \supset [R]B$

$\text{KGcent-10 } B \supset [R]A$

In a similar way [KG-11] – [KG-14] specialize to the following.

$\text{KGcent-11 } [R] \bot \supset K_A[R] \bot$

$\text{KGcent-12 } [R] \bot \supset K_B[R] \bot$

$\text{KGcent-13 } \langle R \rangle \top \supset K_A[R] \top$

$\text{KGcent-14 } \langle R \rangle \top \supset K_B[R] \top$

There is one more axiom to be stated, whose importance turns out to be fundamental. In Figure 5 it is obvious that no matter at which of the five nodes one is, a sequence of transitions is possible that will take one to node 5, the terminal state, namely a sequence of moves to the right. Recall, the Centipede game displayed is one of many, since the length need not be 5 but could be anything. An axiom is needed to cover the general situation—a terminal state is always possible to reach.

$\text{KGcent-15 } \langle R^* \rangle [R] \bot$

4.3 Rationality and Strategy Considerations

General rationality conditions were stated in Section 3.2 as [RC-A] and [RC-B]. These now specialize as follows.

$\text{RCcent-A } (A \land K_A kbest_A (do) \land ra_A) \supset do$

$\quad (A \land K_A kbest_A (ri) \land ra_A) \supset ri$

$\text{RCcent-B } (B \land K_B kbest_B (do) \land ra_B) \supset do$

$\quad (B \land K_B kbest_B (ri) \land ra_B) \supset ri$

I also assume that players do not abruptly become irrational, rationality persists.
Next I turn to issues of strategy. The logical machinery that has been introduced is inadequate to represent numerical payoffs. Rather than expanding this machinery, I extract from the game formulation general statements that follow from the payoff information, but that can be formulated using the machinery that has been introduced.

For the Centipede game I have been displaying a five-move version, but in fact the game could be of any length and certain strategy assumptions would still apply. The first pair of conditions stated below say that if the game has reached the terminal node (node 5 in Figure 5), down is the best move for the player whose turn it is to play, and this is obvious so it is also the best known move. In the five-move game the last player to play is A, but could be either depending on the length of the game. Recall that \([R] \perp\) distinguishes the terminal node in the Centipede game. I call these endgame strategy axioms.

The final conditions I call midgame strategy axioms. These conditions apply to the cases where play is not at the terminal node, and so a transition to the right is possible, after which there is still at least one more move. A quick inspection of Figure 4 shows that at any non-terminal node, N, if the play is right and then down, the player whose turn it is to play at node N will receive less than would be the case if down were played directly. So if the player whose turn it is at node N somehow knows that a play of down might be made if he plays right, then the player’s known best move at N is down. Likewise if the play is right and then right, the player whose turn it is to play will receive more, no matter how the play goes afterward, than would be the case if he played down. So if the player whose turn it is at node N knows that a move of right must be made if he moves right, then right is his known best move at N. This is the content of the next two axioms.

A remark about the two axioms having to do with moves to the right. A ‘pre-condition’ of \(\langle R \rangle \top\) is included because \([R]ri\) is trivially true at a terminal node, but the conclusions of the two axioms conflict with those of \[E\text{Scent-A}\] and \[E\text{Scent-B}\]. The present axioms are meant to be applicable at midgame nodes only, and the presence of \(\langle R \rangle \top\) restricts things to these nodes. There isn’t a corresponding condition in the two axioms having to do with moves down because it isn’t needed. \(\langle R \rangle do\) implies \(\langle R \rangle \top\) and so, within a knowledge condition, terminal nodes are implicitly ruled out.

### 4.4 Proofs

Let \(ra\) abbreviate \(ra_A \land ra_B\), so that \(ra\) is a general rationality assertion. The entire of this section is a formal proof, for Centipede of any length, of \(Cra \supset Cdo\), which says that if it is common knowledge that both players are rational, then it is common knowledge that the move is down. The assumptions that will be used in this section are the following:
• The general PDL + E axiom schemes and rules from Section 2.3

• The Centipede game tree knowledge axioms KGcent-1 – KGcent-15 from Section 4.2

• The Centipede rationality and strategy axioms RCcent-A, RCcent-B, RPcent-A, RPcent-B, EScent-A, EScent-B, MScent-A, and MScent-B all from Section 4.3

In addition to these general assumptions, I will be explicit about whether and where \( E^{-3} \), \( E^{-4} \), Positive Introspection, and \( E^{-5} \), Negative Introspection are used epistemically—in fact, rather remarkably, they are not needed at all. I will be similarly explicit about the cross-conditions PDLE-1, No Learning, PDLE-2, Perfect Recall, PDLE-3, Reasoning Ability, and CK-3, Extended Perfect Recall, some of which do play a role.

The general plan in this section is to establish results that will allow us to apply the Backward Induction Derived Rule, Theorem 3.3. Our first result says that if the game is at a terminal node, and both players are rational, the move will be down. In fact, it is enough for the player whose turn it is to move to be rational, but the stronger hypothesis will certainly do no harm. It follows easily that this result is common knowledge. In giving proofs I assume standard modal reasoning is familiar, and omit steps involving it.

**Proposition 4.1** \([R] \perp \supset (ra \supset do)\), and hence also \([R] \perp \supset (Cra \supset Cdo)\).

**Proof**

\[
\begin{align*}
A \supset ([R] \perp \supset kbest_A(do)) & \quad \text{EScent-A} \\
K_A A \supset (K_A[R] \perp \supset K_A kbest_A(do)) & \quad \text{KG-3 KGcent-11} \\
A \supset ([R] \perp \supset K_A kbest_A(do)) & \\
A \supset ((ra \land [R] \perp) \supset (ra \land K_A kbest_A(do))) & \\
A \supset ((ra \land [R] \perp) \supset do) & \quad \text{RCcent-A}
\end{align*}
\]

Similarly we show \((B \supset ((ra \land [R] \perp) \supset do))\). Now \((ra \land [R] \perp) \supset do\), and hence \([R] \perp \supset (ra \supset do)\) follow using KGcent-1, identical to [KG-1].

For the second part, from what was just proved we have \(C[R] \perp \supset (Cra \supset Cdo)\), so it is enough to show \([R] \perp \supset C[R] \perp\). Now, by KGcent-11 we have \([R] \perp \supset K_A[R] \perp\) and from KGcent-12 we have \([R] \perp \supset K_B[R] \perp\), and so also \([R] \perp \supset E[R] \perp\). Then \([R] \perp \supset C[R] \perp\) follows by rule CK-2.

The next item says that if the game is at an intermediate node, everybody is rational, and it is common knowledge that after a transition right the next move might be down, then a down move will be made. Again it is enough for the player whose turn it is to play to be rational, and this is proved first.

**Lemma 4.2** The following are provable.

1. \((A \land ra_A \land C(R)do) \supset do\)
2. \((B \land ra_B \land C(R)do) \supset do\)
3. \((ra \land C(R)do) \supset do\)
Proof

\[(A \land K_A \langle R \rangle do) \implies \text{kbest}_A \langle do \rangle\]  
\[\text{MScent-A}\]

\[(C A \land CK_A \langle R \rangle do) \implies \text{ckbest}_A \langle do \rangle\]

\[\text{Com-4}\]

\[(A \land C \langle R \rangle do) \implies \text{Com-2}\]

\[\langle R \rangle (C \langle R \rangle do) \implies C \langle (R \rangle do \rangle \]

\[\text{Proposition 2.1}\]

In a similar way we prove \[(B \land ra_B \land C \langle R \rangle do) \implies \text{do}\]. Then \[(ra \land C \langle R \rangle do) \implies \text{do}\] follows using the definition of ra, and KGcent-1, identical to KG-1. 

Next, a few utility items.

Lemma 4.3

1. Assume \[CK-3\] Extended Perfect Recall. Then \[Cra \implies [R]Cra\].

2. Assume \[PDLE-3\] Reasoning Ability. Then \[Cra \implies \langle (R \rangle do \implies do\].

Proof For part 1,

\[ra \implies [R]ra\]  
\[Cra \implies C[R]ra\]

\[\text{RPcent-A  RPcent-B  CK-3}\]

And for part 2,

\[C \langle R \rangle do \implies (ra \implies do)\]  
\[\text{Lemma 4.2}\]

\[CC \langle R \rangle do \implies C(ra \implies do)\]

\[\text{Com-2}\]

\[\langle R \rangle do \implies C(ra \implies do)\]  
\[\text{Proposition 2.1}\]

\[\text{RKcent-A}\]

And now what amounts to the induction step of Backwards Induction.

Proposition 4.4 Assume \[CK-3\] Extended Perfect Recall, and \[PDLE-3\] Reasoning Ability. Then \[\langle R \rangle (Cra \implies do) \implies (Cra \implies do)\].

Proof

\[\langle R \rangle (Cra \implies do) \equiv \langle R \rangle (\neg Cra \lor do)\]

\[\equiv \langle R \rangle (\neg Cra \lor (R \rangle do)\]

\[\equiv \neg [R]Cra \lor (R \rangle do)\]

\[\equiv \neg Cra \lor (R \rangle do\]  
\[\text{Lemma 4.3  part 1}\]

\[\equiv Cra \lor (R \rangle do\]

\[\text{Lemma 4.3  part 2}\]

Finally, the result we have been aiming at.
Theorem 4.5 Assume $\text{CK-3}$, Extended Perfect Recall, and $\text{PDLE-3}$ Reasoning Ability. Then $C_{\text{ra}} \supset C_{\text{do}}$.

Proof Axiom $\text{KGcent-15}$ says $\langle R^* \rangle [R] \bot$. Proposition 4.1 says $[R] \bot \supset (C_{\text{ra}} \supset C_{\text{do}})$. And Proposition 4.4 says $\langle R \rangle (C_{\text{ra}} \supset C_{\text{do}}) \supset (C_{\text{ra}} \supset C_{\text{do}})$. Since $R$ is the only atomic transition, Theorem 3.3 immediately yields $C_{\text{ra}} \supset C_{\text{do}}$. ■

It is worth noting that $\text{E-3}$ Factivity, $\text{E-4}$ Positive Introspection, and $\text{E-5}$ Negative Introspection, were not used in any proof of this section. In particular, this means the result holds under the assumption that one is modeling player belief rather than player knowledge. This should not be surprising. After all, play is based on what one believes to be the case—what the full situation ‘actually’ is may be unobtainable. Further, no introspection of any kind is needed by players, which is somewhat curious given the standard assumption of $S5$ knowledge. Also $\text{PDLE-1}$ No Learning, was not used, but this is of lesser significance.

5 PDL + E Semantics

I have noted several times that extensive form game trees have no machinery to keep track of what players know or do not know at various states. It is time to bring this in. Whimsically expressed, the idea is that one can envision the nodes of a game tree not as featureless dots, but as containing (while concealing) the epistemic states of the players. These epistemic states will be represented by Hintikka-style possible world models—that is, we use the semantic machinery discussed in Section 2.3.

Throughout the rest of this report the following terminology will be systematically used. Game nodes or game states are the nodes seen in the usual extensive form game diagrams. Epistemic states are possible worlds in the usual Hintikka/Kripke epistemic sense. Each game node has associated with it an epistemic model, and we refer to the epistemic states of this model as being of or in the game node. A game tree with its epistemic states displayed will be referred to as an augmented game tree.

5.1 Augmented Game Tree Examples

Suppose that somewhere in a game tree one has the fragment shown in Figure 6. In it two game nodes are shown, though presumably there are others as well. On the left $A$ is to play, on the right $B$, and there is a transition $\alpha$, from left to right.

![Figure 6: Game Tree Fragment, GTF](image)

Suppose that at the left node $A$ has no uncertainty—for every formula $Z$ either $K_A Z$ or $K_A \neg Z$. Suppose also that at the right node $B$ is uncertain of the status of some proposition $P$. In Figure 7 I have expanded the ‘dots’ of Figure 6 to reflect these conditions. In this augmented game tree the left game node contains one epistemic state and the right two. On the left I have only displayed
things appropriate to \( A \), the player to move at this state, and on the right I have only displayed things appropriate to \( B \). More could be shown, but it makes things hard to read and is not relevant for this example. It is assumed that the epistemic state of the left game node is accessible from itself, with respect to \( A \)’s accessibility relation, and for the right game node the two epistemic states are mutually accessible, including from themselves, with respect to \( B \)’s accessibility relation. (That is, \( S5 \) knowledge is assumed in both cases.) In one of \( B \)’s epistemic states \( P \) is true, and in one \( P \) is false, reflecting the uncertainty \( B \) has concerning \( P \). The lack of uncertainty possessed by \( A \) is reflected by the single epistemic state in the left game node.

![Figure 7: GTF, First Epistemic Augmentation](image)

Note that there are two transition arrows labeled \( \alpha \) in Figure 7. This does not mean two different moves are available from the left game node, both called \( \alpha \). Rather, one move \( \alpha \) is available, but there is some question about what epistemic state we will find \( B \) in after the move is made.

I now follow the rules for evaluation of formulas at states, as discussed in Section 2. At both of the epistemic states for \( B \) in the right game node, \( K_B P \) is false, since there is an epistemic state in which \( P \) itself is false, and hence \( B \) does not know \( P \). Likewise \( K_B \neg P \) is also false in both states. Hence at both states of the right game node we have \( \neg K_B P \) and \( \neg K_B \neg P \). Then since all \( \alpha \) transitions from the single epistemic state in the left node lead to states in which both of these formulas are true, in the epistemic state of the left game node we have the truth of \( [\alpha] \neg K_B P \) and \( [\alpha] \neg K_B \neg P \). And since there is only one epistemic state for \( A \) in the left game node, we have \( K_A [\alpha] \neg K_B P \) and \( K_A [\alpha] \neg K_B \neg P \) both true at it. On the other hand, we also have \( \langle \alpha \rangle P \) and \( \langle \alpha \rangle \neg P \) true at the epistemic state in the left game node, and hence also \( K_A \langle \alpha \rangle P \) and \( K_A \langle \alpha \rangle \neg P \).

Stating things more colloquially, at the epistemic state in the left game node \( A \) knows that a transition of \( \alpha \) could leave \( P \) true or could leave it false, and so \( A \) is uncertain of the effect of the transition on \( P \). But also \( A \) knows that after a transition of \( \alpha \), \( B \) will likewise be uncertain about \( P \).

Figure 8 shows a modification of Figure 7, one of the transition arrows is missing. In this, at the epistemic state in the left game node, it is still the case that \( K_A [\alpha] \neg K_B P \) and \( K_A [\alpha] \neg K_B \neg P \), and it is the case that \( K_A \langle \alpha \rangle P \), but it is no longer true that \( K_A \langle \alpha \rangle \neg P \). Instead \( K_A [\alpha] P \).

![Figure 8: GTF, Second Epistemic Augmentation](image)

For the third and last of our simple examples, Figure 9 shows the game tree fragment of
Figure 6 but augmented with different epistemic states. This time the right game node contains three epistemic states for B, with the top two mutually accessible, and the bottom one not accessible from either of the top two. (I state these accessibility conditions in words to keep the diagrams from getting too complicated.) At both of the top two epistemic states for B, \( \neg KBP \) is true, while at the bottom one \( KBP \) is true. I leave it to you to check that at the epistemic state in the left game node, \( KA[\langle \alpha \rangle KBP \land \langle \alpha \rangle \neg KBP \land \neg \langle \alpha \rangle KB \neg P] \) is true.

![Figure 9: GTF, Third Epistemic Augmentation](image)

5.2 Centipede Augmented

The examples discussed in Section 5.1 are not related to any particular game of interest. They are based on the game tree fragment of Figure 6 which is quite generic. Now things become very specific indeed—I make use of semantic machinery to prove that, in Theorem 4.5, common knowledge of rationality is essential. To this end I give an augmented version of Centipede, but to keep things relatively simple I use a three-move version of the game, as shown in Figure 10. Of course similar models could be given with more moves involved.

![Figure 10: Three-Move Centipede](image)

An augmented version of this game is shown in Figure 11 with epistemic states shown for both players. Since the diagram is rather complicated, I explain how to read it. First, the three game nodes of Figure 10 are expanded, and labeled G1, G2, and G3. Each game node now has internal structure, each with three epistemic states labeled E1, E2, and E3. I will, in effect, use coordinates to refer to particular epistemic states, as in (G1, E3) for game node G1, epistemic state E3, for instance. The epistemic states for each game node have accessibility relations defined on them, for each player. These are represented by ellipses, and happen to be the same for each game node. Thus, epistemic state E1 is related only to itself for B. Epistemic states E1 and E2 are related to each other and to themselves for A. Similarly E2 and E3 are related to each other and to themselves for B. And finally, E3 is related only to itself for A. The accessibility relations for each player, at each game state, are equivalence relations, and hence S5 knowledge is assumed for each player.

Transitions to the right are shown by arrows. It is left to the reader to check in detail that the diagram satisfies all the conditions discussed in Section 2.3, No Learning, Perfect Recall, and
Reasoning Ability, for each player. For instance, for player A there is an epistemic arrow from (G1, E1) to (G1, E2), and a game arrow from (G1, E2) to (G2, E2). (Epistemic accessibility is an equivalence relation, so states that are related have accessibility arrows in both directions.) But there is also a game arrow from (G1, E1) to (G2, E1) and an epistemic arrow from (G2, E1) to (G2, E2), filling out the square shown in Figure [I] for the No Learning condition. Many cases must be checked, but verifying No Learning, Perfect Recall, and Reasoning Ability is straightforward.

At each epistemic state truth or falsity for various atomic propositions is shown. For instance, at epistemic state (G1, E1), \( ra_A \) and \( ra_B \) are both true, both players are rational at this state. Also \( do \) is false, that is, player A does not choose to move down. Finally \( kbest_A(do) \) is also false. It is easy to verify that at this state A knows the rationality of both A and B, as does B, but A does not know that B knows A is rational. This latter is the case because, at epistemic state (G1, E3) A is not rational; B cannot distinguish between (G1, E2) and (G1, E3) so at (G1, E2) B does not know A is rational, and A cannot distinguish between (G1, E1) and (G1, E2), so at (G1, E1) A does not know that B knows A is rational.

It is implicitly assumed that the propositional letter \( A \) is true at each of the three epistemic states of G1, that \( B \) is true at each epistemic state of G2, and that \( A \) is true at each epistemic state of G3. It follows easily that each of the axioms \( KG_{cent-1} \) through \( KG_{cent-6} \) is true at each of the nine epistemic states of the model, or as I will more simply say, these axioms are valid in the model.

It is also assumed that \( ri \) is true at exactly the epistemic nodes where \( \neg do \) is displayed. It
follows that both of the axioms $\text{KGcent-7}$ and $\text{KGcent-8}$ are valid in the model.

Each of the arrows shown is implicitly labeled $R$, and validity of axioms $\text{KGcent-9}$ through $\text{KGcent-15}$ is easily checked.

It is assumed in the diagram that $k_{bestA}(ri)$ is true at exactly the nodes at which $k_{bestA}(do)$ is false, and similarly for $k_{bestB}(ri)$. It follows that each of $\text{RCcent-A}$ and $\text{RCcent-B}$ is valid. Consider, for example, $(A \land K_Ak_{bestA}(do) \land ra_A) \supset do$. It is true at each of $(G3, E1)$ and $(G3, E2)$ because $do$ is true, and at $(G3, E3)$ because $ra_A$ is false. It is true at each of $(G2, E1)$, $(G2, E2)$ and $(G2, E3)$ because $A$ is false at each. And it is true at each of $(G1, E1)$, $(G1, E2)$ and $(G1, E3)$ because $K_Ak_{bestA}(do)$ is false. I leave it to the reader to check validity for the other three axioms.

Axioms $\text{RPcent-A}$ and $\text{RPcent-B}$ are easily seen to be valid, as are $\text{EScent-A}$ and $\text{EScent-B}$. Finally there are axioms $\text{MScent-A}$ and $\text{MScent-B}$. At all epistemic states of $G3$ $\text{MScent-A}$ is true because $\langle R \rangle do$ is false and hence so is $K_A\langle R \rangle do$. At all epistemic states of $G2$ $\text{MScent-A}$ is trivially true because $A$ is false. Finally we consider $G1$. At state $(G1, E2) \langle R \rangle do$ is false, because $do$ is false at $(G2, E2)$, and so $K_A\langle R \rangle do$ is false at both epistemic states $(G1, E1)$ and $(G1, E2)$, which are indistinguishable for $A$. Likewise $\langle R \rangle do$ is false at $(G1, E3)$ because $do$ is false at $(G2, E3)$, and so $K_A\langle R \rangle do$ is false at $(G1, E3)$. It follows that $\text{MScent-A}$ is true at the three epistemic states of $G1$, and hence is valid in the model. Validity of $\text{MScent-B}$ is checked similarly.

Thus every axiom for Centipede, given in Section 4.2, is valid in the model of Figure 11, as are the No Learning, Perfect Recall, and Reasoning Ability conditions from Section 2.3. In addition, since all epistemic accessibility relations are equivalence relations, Factivity, Positive Introspection, and Negative Introspection are valid for both players.

Now let us concentrate on epistemic node $(G1, E1)$. $Cra$ is not true at this epistemic state because if it were, $ra$ would have to be true at every epistemic state reachable from here, but $ra_A$ is not true at the reachable node $(G1, E3)$. On the other hand, $Era$ is true at $(G1, E1)$—this abbreviates a formula equivalent to $K_Ara_A \land K_Ara_B \land K_Bra_A \land K_Bra_B$ and expresses that everybody knows that all players are rational. At $(G1, E1)$ $ra_A$ and $ra_B$ are true, and hence so are $K_Bra_A$ and $K_Bra_B$. Likewise $ra_A$ and $ra_B$ are true at both $(G1, E1)$ and $(G1, E2)$, and hence $K_Ara_A$ and $K_Ara_B$ are true at $(G1, E1)$. Since $do$ is not true at $(G1, E1)$, this establishes that $Era \supset do$ is not derivable from our axioms for Centipede, together with additional strong knowledge assumptions.

6 One More Example

I conclude with one more Centipede-based example, in which the game tree is the same but the epistemic assumptions are substantially different. They are asymmetric—the two players are not interchangeable, so to speak.

6.1 Irrational Play

Theorem 4.5 concludes that, in Centipede, play will be down provided rationality of players is common knowledge. What does it mean to be rational? I have taken an essentially operational view here, though the wording sometimes obscures this. Axioms $\text{RC-A}$ and $\text{RC-B}$ are really necessary conditions for a play to be rational—a rational move must be in accordance with what is most advantageous to the player, within the limits of what the player knows at the moment of play. Rationality is not treated as a predisposition or psychological state; rather it is moves of the player that are rational. In effect, then, a player is rational if the player only makes rational moves. Then our game axioms $\text{RCcent-A}$ and $\text{RCcent-B}$ combined with axioms $\text{RPcent-A}$ and $\text{RPcent-B}$ amount to rationality assumptions about players—from them it follows that rational play at the
start yields rational play throughout, and one can identify rational play throughout with rationality of the player.

I now want to consider *irrational play*—moves that are not in the best interests of the player. I add two new axioms, counterparts of [RC-A] and [RC-B]. For each move $m_i$ of a game, I assume the following *irrationality conditions*.

$I C-A \ (A \land K_A k_{best}(m_i) \land \neg ra_A) \supset \neg m_i$

$I C-B \ (B \land K_B k_{best}(m_i) \land \neg ra_B) \supset \neg m_i$

Briefly, a player plays irrationally if the player does not choose the known best move. Note that these axioms, combined with [RC-A] and [RC-B], give us simple equivalence versions.

- $(A \land K_A k_{best}(m_i)) \supset (ra_A \equiv m_i)$
- $(B \land K_B k_{best}(m_i)) \supset (ra_B \equiv m_i)$

In investigating strategy, rationality or irrationality is not really what is important. Rather it is what players *know* about these things that matters. And for this there are three, not two, possibilities. A player might know another player is rational, or he might know another player is irrational, or he might not know either the rationality or the irrationality of the other player. We do not consider this third possibility here—the state of ignorance concerning rationality. We only examine the consequences of knowing a player will act irrationally.

To make the discussion concrete, I now examine the Centipede game under the assumption that one of the players is irrational—that is, always makes irrational moves.

### 6.2 Centipede Again

Figure 12 displayed a five move version of Centipede. A bigger game is more illustrative for present purposes, so a nine move version is shown in Figure 12. The numbering starts from the end, rather than from the beginning as is more usual. This simplifies the discussion somewhat. We write $A_0$ above the terminal state, for example, to indicate that $A$ is to play, and it is state number 0.

Given our epistemic assumptions, one can reason informally as follows. If the game were to start at state 0 the best move for $A$ would be down, but since $A$ plays irrationally, $A$ would play
right. At state 1, B should be able to duplicate our reasoning and so would know that the best move for it would be right, and being rational, B would move right at state 1. At state 2, A can also reason as we just did, and so would know that B would move right at state 1, and so would know that its best move is right. But playing irrationally, A would move down at state 2. And so on. This generates the following table, indicating the moves that would be made if the game were to start at various nodes, given our epistemic assumptions about rationality. The task is to formalize this informal discussion.

<table>
<thead>
<tr>
<th>Node</th>
<th>Player</th>
<th>move</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>A</td>
<td>right</td>
</tr>
<tr>
<td>1</td>
<td>B</td>
<td>right</td>
</tr>
<tr>
<td>2</td>
<td>A</td>
<td>down</td>
</tr>
<tr>
<td>3</td>
<td>B</td>
<td>down</td>
</tr>
<tr>
<td>4</td>
<td>A</td>
<td>right</td>
</tr>
<tr>
<td>5</td>
<td>B</td>
<td>right</td>
</tr>
<tr>
<td>6</td>
<td>A</td>
<td>down</td>
</tr>
<tr>
<td>7</td>
<td>B</td>
<td>down</td>
</tr>
<tr>
<td>8</td>
<td>A</td>
<td>right</td>
</tr>
</tbody>
</table>

Figure 13: Nine Move Centipede Play

We conclude this section by setting up our goal, which is to state and prove a formal embodiment of Table 13. In subsequent sections we will add appropriate axioms that will enable us to reach this goal.

An inspection of the informal reasoning above shows that things repeat in patterns of 4. For instance, the move will be right at node 0, 4, 8, . . . . We are at node 0, which is terminal, if we have \([R] \bot\). We are at node 4 if we have \(⟨R; R; R; R⟩[R] \bot\), or more compactly \(⟨R^4⟩[R] \bot\), and so on. Most generally, when we have \(⟨(R^4)^*⟩[R] \bot\) the move should be right. At nodes 2, 6, . . . the move should be down, and we can express this by asserting that when we have \(⟨R; R⟩⟨(R^4)^*⟩[R] \bot\) the move should be down. And similarly for the other two general cases. This leads us to the following convenient abbreviations. (We write \(R^{4\ast}\) in place of the more cluttered \((R^4)^\ast\).

\[
F_0 = ⟨R^{4\ast}⟩[R] \bot ⊃ ri
\]
\[
F_1 = ⟨R⟩⟨R^{4\ast}⟩[R] \bot ⊃ ri
\]
\[
F_2 = ⟨R; R⟩⟨R^{4\ast}⟩[R] \bot ⊃ do
\]
\[
F_3 = ⟨R; R; R⟩⟨R^{4\ast}⟩[R] \bot ⊃ do
\]
\[
F = F_0 \land F_1 \land F_2 \land F_3
\]

Also we adopt the following.

\[
ra' = \neg ra_A \land ra_B
\]

Our goal is to show, formally, that from appropriate assumptions concerning Centipede, and player rationality, there is a proof of the following, analogous to our earlier Theorem 4.5.

\[
Cra' ⊃ CF
\]

And just as we did earlier, we will use Backward Induction in the formulation of Theorem 3.3.
6.3 Beginning Formalization

Axioms \text{RPcent-A} and \text{RPcent-B} assert that rationality persists in the Centipede game. I now add axioms saying a similar thing about irrationality, in effect positing that it is players who are rational or irrational because they always make rational, or irrational, moves. Of course these assumptions could be modified for other examples. One could consider a player whose rationality varies, but the present example is complicated enough for now.

\[ \text{IPcent-A} \quad \neg \text{ra}_A \supset [R] \neg \text{ra}_A \]
\[ \text{IPcent-B} \quad \neg \text{ra}_B \supset [R] \neg \text{ra}_B \]

The irrationality conditions \text{IC-A} and \text{IC-B} specialize to Centipede as follows, taking \text{KGcent-1} and \text{KGcent-8} into account.

\[ \text{ICcent-A} \quad (A \land K_A \text{kbest}_A(\text{do}) \land \neg \text{ra}_A) \supset \text{ri} \]
\[ \quad \quad \quad (A \land K_A \text{kbest}_A(\text{ri}) \land \neg \text{ra}_A) \supset \text{do} \]
\[ \text{ICcent-B} \quad (B \land K_B \text{kbest}_B(\text{do}) \land \neg \text{ra}_B) \supset \text{ri} \]
\[ \quad \quad \quad (B \land K_B \text{kbest}_B(\text{ri}) \land \neg \text{ra}_B) \supset \text{do} \]

As noted earlier, \( A \) plays last, that is, has the turn at the terminal node if it is reached. This is not really necessary because one could formalize things in terms of the player who plays last, but it would make things unnecessarily complicated. I simply postulate the following special condition.

\[ \text{Last-A} \quad [R] \bot \supset A \]

Backwards Induction will be used to prove \( Cr'_A \supset CF \). Since we already have item 1 from Theorem 3.3 it is one of our axioms, we just need the other two, and of these, the first is obtainable now without any further assumptions.

\textbf{Proposition 6.1 (Backwards Induction, Base Case)} Given the basic assumptions thus far, \( [R] \bot \supset (Cr'_A \supset CF) \).

\textbf{Proof} It is enough to show \([R] \bot \supset (Cr'_A \supset CF_i)\) for \( i = 0, 1, 2, 3 \). To show the \( i = 0 \) case we will first show \([R] \bot \supset (\neg \text{ra}_A \supset \text{ri})\). To show the \( i = 1, 2, 3 \) cases we will first show \([R] \bot \supset \neg (R^i \langle R^i \rangle (R^{i'}) [R] \bot)\). The proofs are presented in some detail, though they are all essentially straightforward.

1. \([R] \bot \supset (\neg \text{ra}_A \supset \text{ri})\). Proof:

\[
\begin{align*}
([R] \bot \land A) & \supset \text{kbest}_A(\text{do}) & \text{EScent-A} \\
[R] \bot & \supset \text{kbest}_A(\text{do}) & \text{Last-A} \\
K_A[R] \bot & \supset K_A \text{kbest}_A(\text{do}) & \text{KGcent-11} \\
[R] \bot & \supset K_A \text{kbest}_A(\text{do}) & \text{Last-A} \\
[R] \bot & \supset (A \land K_A \text{kbest}_A(\text{do})) & \\
([R] \bot \land \neg \text{ra}_A) & \supset (A \land K_A \text{kbest}_A(\text{do}) \land \neg \text{ra}_A) & \text{ICcent-A} \\
& \supset \text{ri} &
\end{align*}
\]
2. \([R] \bot \supset (\text{Cra}' \supset CF_0)\). Proof:

\[
\begin{align*}
[R] \bot & \supset (\neg ra_A \supset ri) & \text{part 1} \\
[R] \bot & \supset (\neg ra_A \supset F_0) \\
C[R] \bot & \supset (C \neg ra_A \supset CF_0) \\
C[R] \bot & \supset (\text{Cra}' \supset CF_0) \\
[R] \bot & \supset (\text{Cra}' \supset CF_0) & \text{using KGcent-11 and KGcent-12}
\end{align*}
\]

3. \([R] \bot \supset \neg \langle R^i \rangle \langle R^{4*} \rangle [R] \bot\) for \(i = 1, 2, 3\). Proof: Assume \(i\) is 1, 2, or 3.

\[
\begin{align*}
([R] \bot \land \langle R^i \rangle \langle R^{4*} \rangle [R] \bot) & \supset \langle R \rangle (\bot \land \langle R^{i-1} \rangle \langle R^{4*} \rangle [R] \bot) \\
& \supset \langle R \rangle \bot \\
& \supset \bot \\
[R] \bot & \supset \neg \langle R^i \rangle \langle R^{4*} \rangle [R] \bot
\end{align*}
\]

4. \([R] \bot \supset F_i\) for \(i = 1, 2, 3\). Proof: Assume \(i\) is 1, 2, or 3.

\[
\begin{align*}
[R] \bot & \supset \neg \langle R^i \rangle \langle R^{4*} \rangle [R] \bot & \text{part 3} \\
& \supset F_i \\
& \supset (\text{ra}' \supset F_i) \\
C[R] \bot & \supset (\text{Cra}' \supset CF_i) \\
[R] \bot & \supset (\text{Cra}' \supset CF_i) & \text{using KGcent-11 and KGcent-12}
\end{align*}
\]

We now have part of what we need for an application of Backwards Induction. To get the rest we need more information about the structure of augmented game trees for Centipede. We take this up in the next section.

6.4 Additional Structural Assumptions

In Section 4.2 axioms were given embodying structural information about the Centipede game, KGcent-1 – KGcent-6, identical to KG-1 – KG-6 and KGcent-7 – KGcent-15 but they are not complete. In this section more assumptions are proposed. They are stated semantically since they seem difficult to properly capture in axioms. They do, however, lead to a formula we can use to complete our discussion of Centipede with an irrational player.

The game tree for Centipede is obviously linear, so each move to the right takes us from a game state to a unique “next” state. The game is one of perfect information, so there is no ambiguity as to which state we are in. This should be reflected from the game tree to augmented game trees for Centipede, where epistemic states within game states are shown.

**Linearity Assumption:** Suppose there is an \(R\) transition from an epistemic state \(e\) in Centipede game state \(g_1\) to some epistemic state in game state \(g_2\). Then every \(R\) transition from \(e\) must be to an epistemic state of \(g_2\).

The next assumption roughly says that different epistemic states in the same game state should be capable of having some affect on each other. There should be no isolated epistemic states. I propose this as a useful assumption, but admit it needs more thought and exploration.
Reachability Assumption: Suppose \( g \) is a Centipede game state and \( e_1 \) and \( e_2 \) are two different epistemic states of \( g \). Then \( e_2 \) is reachable from \( e_1 \) via a path of epistemic states in which each is accessible from its predecessor either via the accessibility relation for player \( A \) or the accessibility relation for player \( B \).

Notice that both of these assumptions hold in the diagram of Figure 11.

The Reachability Assumption provides a solution to a minor lacuna that we have so far avoided discussing. We have been making tacit assumptions about terminal nodes in our informal discussions, though this has had no affect on our axiomatic arguments. We have characterized terminal nodes by the truth of \([R] \bot\), but truth where? A game state might contain many epistemic states—could \([R] \bot\) be true at some while false at others. The Reachability Assumption rules this out.

Suppose some epistemic state \( e \) of a game state has \([R] \bot\) true at it. It follows from \([KGcent-11] \) and \([KGcent-12] \) that \([R] \bot\) is common knowledge at \( e \). The state \( e \) might not be accessible from itself, we might not have Factivity, but every other state will be reachable, and then \([CK-1] \) implies \([R] \bot\) will be true at every other state. Of course we began with the assumption that \([R] \bot\) was true at \( e \), and hence \([R] \bot\) is true at every epistemic state of the game node. In short, if the Reachability Assumption is assumed, then if \([R] \bot\) is true at any epistemic state of a game state, it is true at all of them, and so any ambiguity about being a terminal node is bypassed.

There is one more important consequence of these semantic assumptions, one that can be formulated succinctly, and which we will adopt as an additional Centipede axiom. Accept the Linearity Assumption and the Reachability Assumption. Suppose \( \langle R \rangle CX \) is true at some epistemic state, \( e \), of a game state \( g_1 \). Then there must be an \( R \) transition to an epistemic state of some game node \( g_2 \), at which \( CX \) is true. Using the Reachability Assumption and \([CK-1]\) \( CX \) must be true at every epistemic state of \( g_2 \). Using the Linearity Assumption, every \( R \) transition from \( e \) must be to an epistemic state of \( g_2 \), hence to an epistemic state at which \( CX \) is true. It follows that \([R]CX \) is true at \( e \). We have shown that the assumptions above imply the validity of \( \langle R \rangle CX \vdash [R]CX \), and we will make this our final Centipede axiom.

\[
\text{Lin} + \text{Reach} \quad \langle R \rangle CX \vdash [R]CX
\]

We note again, the Reachability Assumption and the Linearity Assumption together imply Lin + Reach.

6.5 The Irrational Player Concluded

By making use of the additional assumptions discussed in the previous section, or rather of their consequence, we can complete our proof of \( Cra' \supset CF \), where \( r \) and \( F \) were defined in Section 6.2. We do this by applying the Backwards Induction scheme, Theorem 3.3. The base case has already been shown, in Proposition 6.1. We have yet to show the induction step, but before doing this it will simplify things to get a few easy items out of the way first.

Our first preliminary item says that where we are in the game determines whose move it is. This is not surprising. The point is that it does follow from our axioms, hence is common knowledge, and can be used in our proofs. It says \( A \) moves at even positions, and \( B \) moves at odd ones.

Lemma 6.2

1. \( \langle R^{2*} \rangle [R] \bot \supset A \)
2. \( \langle R \rangle \langle R^{2*} \rangle [R] \bot \supset B \)
Proof By KG-9 and KG-10. \( A \supset [R]B \) and \( B \supset [R]A \), so \( B \supset [R^2]B \). Then by the induction rule PDL-8, \( B \supset [R^2\ast]B \). By Last-A, \( [R] \perp \supset A \), so using KG-1 and KG-2, \( B \supset [R]\perp \). Combining results, \( B \supset [R^2\ast][R] \perp \). Then by the induction rule PDL-8, \( B \supset [R^2\ast][R] \perp \). By Last-A, \( [R] \perp \supset A \), so using KG-1 and KG-2, \( B \supset [R] \perp \). Combining results, \( B \supset [R^2\ast]\perp \). Then part 2 follows immediately from part 1.

The second preliminary result says that common knowledge of the status of player rationality is preserved under a game move.

Lemma 6.3 Assume No Learning, PDLE-1. Then \( Cra' \supset [R]Cra' \).

Proof Using Pcent-A and RPcent-A
\[
ra' \equiv (\neg ra_A \land ra_B)
\supset ([R][\neg ra_A \land [R]ra_B])
\supset [R]ra'
\]
Then, using Proposition 2.2
\[
Cra' \supset [R]ra'
\supset [R]Cra'
\]

The final preliminary item allows us to simplify the formulas \( F_0, F_1, F_2, \) and \( F_3 \) that make up the formula \( F \).

Lemma 6.4 Assume No Learning, PDLE-1 and Reasoning Ability, PDLE-3. Then:

1. \( \langle R^n \rangle [R] \perp \supset C[\langle R^n \rangle [R] \perp \),
2. \( [R^n][R] \perp \supset C[R^n][R] \perp \),
3. \( C[\langle R^n \rangle [R] \perp \supset X] \equiv [\langle R^n \rangle [R] \perp \supset CX] \).

Proof
1. Using KGcent-11 and KGcent-12 \( [R] \perp \supset C[R] \perp \). Then \( \langle R^n \rangle [R] \perp \supset [R^n][R] \perp \), and the result follows using Proposition 2.1.
2. \( \perp \supset CT \), so \( \langle R^n \rangle \perp \supset C[\langle R^n \rangle C] \perp \). By Proposition 2.1 again, \( \langle R^n \rangle \perp \supset C[\langle R^n \rangle \perp \), and so \( [R^n][R] \perp \supset [R]C[R^n] \perp \). The result follows using Proposition 2.2.
3. The implication from left to right follows by distributing \( C \) across the implication, and then using part 1 of this Lemma. For the implication from right to left, we reason as follows.
\[
[\langle R^n \rangle [R] \perp \supset CX] \supset [\neg [\langle R^n \rangle [R] \perp \vee CX]
\supset [\neg [\langle R^n \rangle \perp \vee CX]
\supset [\neg [\langle R^n \rangle \perp \vee CX]
\supset [\neg [\langle R^n \rangle \perp \vee CX]
\supset [\neg [\langle R^n \rangle [R] \perp \supset X]
\]
Now we are ready for the remaining part of our backwards induction argument. The argument looks long, but the overall structure is rather simple.

**Proposition 6.5 (Backwards Induction, Induction Step)** Assume No Learning, PDLE-1, Reasoning Ability, Lin + Reach, and Lin + Reach. Then \( \langle R \rangle \lceil Cra' \supset CF \rceil \supset [Cra' \supset CF] \).

**Proof** Essentially, the argument divides into four cases, based on the following.

\[
\begin{align*}
\{ \langle R \rangle |Cra' \supset CF \rceil \supset Cra' \} & \supset \{ \langle R \rangle |Cra' \supset CF \rceil \supset |R|Cra' \} & \text{Lemma 6.3} \\
& \supset \langle R \rangle CF \\
& \supset \langle R \rangle C\{F_0 \land F_1 \land F_2 \land F_3\} \\
& \supset \langle R \rangle \{CF_0 \land CF_1 \land CF_2 \land CF_3\} \\
& \supset \{\langle R \rangle CF_0 \land \langle R \rangle CF_1 \land \langle R \rangle CF_2 \land \langle R \rangle CF_3\} \\
& \supset \{\langle R \rangle CF_0 \land \langle R \rangle CF_1 \land \langle R \rangle CF_2 \land \langle R \rangle CF_3\} & \text{Lin + Reach}
\end{align*}
\]

We will prove each of the following.

1. \( \{Cra' \land [R]CF_0 \} \supset CF_1 \)
2. \( \{Cra' \land [R]CF_1 \} \supset CF_2 \)
3. \( \{Cra' \land [R]CF_2 \} \supset CF_3 \)
4. \( \{Cra' \land [R]CF_3 \} \supset CF_0 \)

Once this is shown, by combining this with the result above we are finished, as follows.

\[
\begin{align*}
\{ \langle R \rangle |Cra' \supset CF \rceil \supset Cra' \} & \supset \{\langle R \rangle CF_0 \land \langle R \rangle CF_1 \land \langle R \rangle CF_2 \land \langle R \rangle CF_3\} \\
& \supset \{CF_1 \land CF_2 \land CF_3 \land CF_0\} \\
& \supset C\{F_1 \land F_2 \land F_3 \land F_0\} \\
& \supset CF
\end{align*}
\]

Of the four numbered items, we only show 1 and 4, which have proofs that differ in details. The other two have proofs similar to the ones we show. We begin with item 1. Using Lemma 6.4 part 3, item 1 is equivalent to

\[
\begin{align*}
\{Cra' \land [R] (\langle R^{4^*} \rangle [R] \supset Cri) \land \langle R \rangle (\langle R^{4^*} \rangle [R] \uparrow)\} & \supset Cri \\
& \supset \langle R \rangle Cri \\
& \supset [R] Cri \\
& \supset \{B \land [R] Cri\} & \text{Lemma 6.2} \\
& \supset \{B \land [R] Cri \land (\langle R \rangle \top)\} \\
& \supset \{B \land [R] K B ri \land (\langle R \rangle \top)\} \\
& \supset \{B \land K B [R] ri \land (\langle R \rangle \top)\} & \text{PDLE-1} \\
& \supset k b s t _ B (ri) & \text{MScent-B} \\
& \supset \{B \land k b s t _ B (ri)\} & \text{Lemma 6.2} \\
& \supset \{B \land k b s t _ B (ri)\} & \text{Com-2} \\
& \supset \{B \land k b s t _ B (ri)\} & \text{Lemma 6.4[1]} \\
& \supset \{B \land k b s t _ B (ri)\} & \text{Kccent-B}
\end{align*}
\]
This is almost but not quite \(1\). The following finishes our derivation of it.

\[
[R]\left(\langle R^{4*}\rangle[R]\downarrow \triangleright C\text{ri}\right) \supset [R]C\left(\langle R^{4*}\rangle[R]\downarrow \triangleright ri\right) \quad \text{Lemma 6.4(3)}
\]

\[
\supset [R]CC\left(\langle R^{4*}\rangle[R]\downarrow \triangleright ri\right) \quad \text{Com-2}
\]

\[
\supset C[R]C\left(\langle R^{4*}\rangle[R]\downarrow \triangleright ri\right) \quad \text{Proposition 2.2}
\]

\[
\supset C[R]\left(\langle R^{4*}\rangle[R]\downarrow \triangleright Cri\right) \quad \text{Lemma 6.4(3)}
\]

Finally we establish item 4, which has a complication unique to this case. As in the proof of item 1, we use Lemma 6.4 part 3 and convert item 4 into the following equivalent version, which we then proceed to prove.

\[
\{Cra' \land [R]\left(\langle R^{3}\rangle\langle R^{4*}\rangle[R] \downarrow \supset Cdo\right) \land \langle R^{4*}\rangle[R] \land \supset \right\} \supset Cri
\]  

(2)

Generally in PDL, \(\langle \alpha^*\rangle X \equiv [X \lor \langle \alpha^*\rangle X]\), so in particular, \(\langle R^{4*}\rangle[R] \land \equiv [\langle R\rangle \lor \langle R\rangle] \equiv [\langle R\rangle \lor \langle R\rangle] \equiv [\langle R\rangle \lor \langle R\rangle] \equiv [\langle R\rangle \lor \langle R\rangle] \equiv [\langle R\rangle \lor \langle R\rangle].\)

Then to prove (2) it is enough to prove both of the following.

\[
\{Cra' \land [R]\left(\langle R^{3}\rangle\langle R^{4*}\rangle[R] \downarrow \supset Cdo\right) \land \langle R^{4*}\rangle[R] \land \supset \right\} \supset Cri
\]

(3)

\[
\{Cra' \land [R]\left(\langle R^{3}\rangle\langle R^{4*}\rangle[R] \downarrow \supset Cdo\right) \land \langle R^{4*}\rangle\langle R^{4*}\rangle[R] \land \supset \right\} \supset Cri
\]

(4)

Of these, (3) follows easily from Proposition 6.1 and we omit details. Here is the argument for (4). As in the proof of item 1, we begin by using the modal tautology \(\square (X \supset Y) \supset (\diamond X \supset \diamond Y)\).

\[
\{Cra' \land [R]\left(\langle R^{3}\rangle\langle R^{4*}\rangle[R] \downarrow \supset Cdo\right) \land \langle R^{4*}\rangle\langle R^{4*}\rangle[R] \land \supset \right\} \supset (R)Cdo
\]

\[
\supset C(R)do \quad \text{Proposition 2.1}
\]

\[
\supset K_A\langle R\rangle do \quad \{A \land K_A\langle R\rangle do\} \quad \text{Lemma 6.2}
\]

\[
\supset kbest_A(do) \quad \text{MMScen-A}
\]

\[
\supset \{A \land kbest_A(do)\} \quad \text{Lemma 6.2}
\]

\[
C\left\{Cra' \land [R]\left(\langle R^{3}\rangle\langle R^{4*}\rangle[R] \downarrow \supset Cdo\right) \land \langle R^{4*}\rangle\langle R^{4*}\rangle[R] \land \supset \right\} \supset C\left\{A \land kbest_A(do)\right\}
\]

\[
CCra' \land C[R]\left(\langle R^{3}\rangle\langle R^{4*}\rangle[R] \downarrow \supset Cdo\right) \land C\langle R^{4*}\rangle\langle R^{4*}\rangle[R] \land \supset \]  

\[
\supset C\left\{A \land kbest_A(do)\right\}
\]

\[
\supset C[R]\left(\langle R^{3}\rangle\langle R^{4*}\rangle[R] \downarrow \supset Cdo\right) \land C\langle R^{4*}\rangle\langle R^{4*}\rangle[R] \land \supset \]  

\[
\supset C\left\{A \land kbest_A(do)\right\}
\]

\[
\text{Com-2}
\]

\[
\supset C\left\{A \land kbest_A(do)\right\} \land \neg ra_A \}
\]

\[
\supset Cri \quad \text{MCen-A}
\]

To finish the proof of item 4 we need

\[
[R]\left(\langle R^{3}\rangle\langle R^{4*}\rangle[R] \downarrow \supset Cdo\right) \supset C[R]\left(\langle R^{3}\rangle\langle R^{4*}\rangle[R] \downarrow \supset Cdo\right)
\]

and the proof of this is essentially the same as one given at the end of the argument for item 1.

This finishes the proof of item 4, and concludes the overall argument.
We have everything we need to get the promised result for Centipede with an irrational player.

**Theorem 6.6** Assume No Learning, PDLE-1, Reasoning Ability, PDLE-3 and Lin + Reach.

\[ Cra' \supset CF \]

**Proof** As we have said several times along the way, this is by an application of the Backwards Induction Scheme, Theorem 3.3. Of the three hypotheses in that, item 1 is an axiom, KGcent-15. Item 2 is Proposition 6.1. And item 3 is Proposition 6.3 above. ■

**7 Conclusion**

The Centipede analysis strongly suggests that PDL + E is a natural tool for reasoning about games in which an epistemic component is central. The paper [20] makes the same point. As has been seen, PDL + E machinery provides the flexibility to establish derivability, and non-derivability, of various statements—there is both a proof theory and a model theory. Nonetheless, much more remains to be done.

Completeness has not been essential here. This does not mean it is without interest, but since we have been concerned with specific games, it is not entirely clear what completeness should mean. Perhaps it is best to break the issue into smaller pieces. For example, can one characterize, semantically or syntactically, games of perfect information? After all, game history is not part of the formalism.

The structural assumptions introduced semantically in Section 6.4 need further thought. What is a/the natural way of capturing them proof-theoretically? Are they required in full strength, or will something weaker do for most purposes? In particular, what is the status of Lin + Reach? The Reachability Assumption, in particular, needs thought. As stated, it does not rule out reachability stretching beyond the epistemic states of a single game state. Indeed, in [20] this was done in order to model games that are not perfect information, in particular, where a player does not know exactly which game node he or she might be at. To what extent do we want to assume, and to what extent do we want to restrict reachability.

We need more examples involving other games, worked through using PDL + E machinery. Certainly the strategy assumptions MScent-A, MScent-B, EScent-A, EScent-B would need to be replaced with other conditions appropriate to other games, as would Lin + Reach, and this is partly why completeness issues remain a sideline. The general assumptions introduced in Section 3 are likely to survive for a range of games. Then there is the matter of the interaction conditions, No Learning, PDLE-1, Perfect Recall, PDLE-2 and Reasoning Ability, PDLE-3 To what extent do we want these? Reasoning Ability, \( \langle R \rangle KX \supset K \langle R \rangle X \), is a case in point. It seems improbable that we would want \( \langle \alpha \rangle KX \supset K \langle \alpha \rangle X \) to be a general logical principle for any action \( \alpha \) because, for instance, it might be true that after conducting a certain experiment we will know the truth of some scientific fact, but we might not know that this is the experiment we need to carry out.

On the other hand Perfect Recall, \( K[R]X \supset [R]KX \), seems a plausible candidate to generalize, \( K[\alpha]X \supset [\alpha]KX \), though it would not be so if the knowledge operator were replaced with a belief operator. The logic must be tailored to the situation.

There is also the issue of common knowledge in general. It is bothersome that Extended Perfect Recall, CK-3 needed to be assumed. What, exactly, is its status?

What about a practical proof procedure for PDL+E, an issue that partly depends on what can be done with tableau procedures for PDL itself, as well as tableau procedures for common knowledge. It may be that an interactive system would be helpful, in which straightforward tableau rules
take care of most things, while the user suggests appropriate induction formulas to guide it in the applications of the * or the C operator. At any rate, a start has already been made in [14].

Finally, Justification logics, [2, 3], have developed into an important field of research. These are epistemic logics in which, instead of simple knowledge or belief operators, there is machinery to express reasons for particular items of knowledge. There are operations on these reasons, but so far the operations investigated have been motivated by formal proofs in modal logic. The various cross axioms of Section 2.3 suggest there may be other operations on justifications, arising in a combined dynamic-epistemic context. Investigating this would be a long-term project, but one of considerable interest.

References


