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# A Simulation Based Optimal Control System For Water Resources

Asier Lacasta

Mario Morales-Hernández

Pilar Brufau

Pilar García-Navarro

# A SIMULATION BASED OPTIMAL CONTROL SYSTEM FOR WATER RESOURCES

ASIER LACASTA (1), MARIO MORALES-HERNÁNDEZ (1), PILAR BRUFAU (1), PILAR GARCÍA-NAVARRO (1)

*(1): LIFTEC-CSIC, University Zaragoza, C/ Maria de Luna 3, Zaragoza, Spain*

Nowadays, the increasing demand of water resources has become one of the most relevant issues when water supply must be guaranteed as in the case of irrigation comunities. It becomes difficult to provide the predictive optimal behavior of regulation structures when the regime of the flow is changing in time and, moreover, when this regulation modifies the hydraulic conditions of the structure. One of the most widely used solution for control is PID. The disadvantages of PID control are the difficulties in the presence of non-linearities as well as the PID parameters tuning.
Taking into account previous works about sluice gate formulation and its hydraulics, an equivalent formulation is going to be applied to obtain the control signal for the gate. This will make possible to build a system that allows the optimal control of the gate in order to obtain, for instance, a desired volume of water in the least time possible with the minimum water loss.
In this work, a reformulation of the problem is analyzed and then, an equivalent the control is applied to sluice gate flow to obtain a simulation based control system and a near real-time solution is explored using adjoint variable formulation.

## INTRODUCTION

The idea of regulation applied to PDE's is the use of the mathematical model as the system which delivers the results of the physical variables which are used to apply the regulation. In other words, the mathematical model is used as the representation of the reality when the control is applied. Finally, the main question is the inverse, that is how to determine the control to optimize a given functional.
PDE-constrained optimization and the adjoint method for solving the optimal control problem appears to be an interesting option when the complexity of the problem does not allow to obtain the easy way to know the optimal solution.
Following the Algorithm (1) and using the technique provided in this work, it is possible to find the optimal control. In the future this control will be transformed in a gate opening signal.
The main ideaas has been obtained from the work [1] and [2] using the numerical resolution for the physical model described in [3].

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**Algorithm 1:** Gradient descent method

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Result: A new set of values for f closer to the solution that minimizes J
input  : f^0, ε, tolX_max, tolX_max, iter_max
output: f^n
nIter=0;
while n_i ≤ iter_max do
    f^{n+1} ← f^n − ε∇J;
    if ||∇J(f^{n+1})|| ≤ tol_max then
        /* Converged on critical point                              */
        return;
    else
        if ||f^{n+1} − f^n|| ≤ tolX_max then
            /* Converged on an f value                              */
            return;
        else
            if J(f^{n+1}) > J(f^{n+1}) then
                /* It has diverged                                  */
                return;
            else
                n_i++;
```

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**ONE DIMENSIONAL SCALAR PROBLEM WITH SOURCE TERMS**

In order to establish the basis in a simple example, the technique is introduced by its application to the 1D scalar equation with constant velocity and source term identified such as a subtraction or injection of some quantity. It is modelled as

$$\frac{\partial C(x,t)}{\partial t} + A\frac{\partial C(x,t)}{\partial x} = f(x,t) \qquad C(0,t) = C(x,0) = 0, \tag{1}$$

being $A$ a constant propagation velocity. The role of $f(x,t)$ for our control will be the punctual location of the injection in order to produce some profile in \$t\$ of the controllable quantity. This means that, $f(x,t)$ will be placed at $x = x_s \in (0,L)$ being 0 in any other location in order to satisfy some condition in another location $x_t$ the functional that covers the quadratic error may be written, such as

$$\mathbf{J} = \int_0^T \int_0^L \mathcal{E}(x,t;h,q)dxdt = \int_0^T (C(x_t,t) - C_{obj}(t))^2 dt \tag{2}$$

The control applied in (2) considers a physical location over the entire time domain.

**Adoint Formulation**

In order to establish some relation between the error and the controllable source $f(x_S,t)$ it is necessary to multiply (1) by $\sigma(x,t)$ and then, integrate over $x \in (0,L)$ and $t \in (0,T)$, taking into account that it satisfies

$$Q = \int_0^T \int_0^L \sigma \left( \frac{\partial C}{\partial t} + A \frac{\partial C}{\partial x} - f \right) dxdt = 0 \tag{3}$$

The new variable $\sigma(x,t)$ is also known as a Lagrange Multiplier and it is continuous and differentiable (once at least). Integrating (3) by parts and taking the first variation with respect to *C* and *f* it follows that

$$\begin{aligned} \delta Q = \int_0^L [C\sigma]_0^T dx - \int_0^T \int_0^L \delta C \frac{\partial \sigma}{\partial t} dxdt + \\ \int_0^T [\sigma A \delta C]_0^L dt - \int_0^T \int_0^L A \delta C \frac{\partial \sigma}{\partial x} dxdt - \int_0^T \int_0^L \sigma \delta f dxdt = 0 \end{aligned} \tag{4}$$

It is also possible to take the first variation of $\mathbf{J}$ in order to obtain $\delta\mathbf{J}$ as

$$\delta \mathbf{J} = \int_0^T \int_0^L 2(C - C_{obj}) \delta C dxdt \tag{5}$$

Adding both (4) and (5) to obtain $\delta\mathbf{J} = \delta\mathbf{J} + \delta Q$ applying boundary and initial conditions for *C* and placing the following contraints on $\delta C$ and $\sigma$

$$\begin{aligned} &\delta C(0,t) = 0, \delta C(L,t) = 0, \delta C(x,0) = 0, \\ &\sigma(0,t) = 0, \sigma(L,t) = 0, \sigma(x,T) = 0 \end{aligned} \tag{6}$$

The variation of the objective function can be rewritten as

$$\delta \mathbf{J} = \int_0^T \int_0^L \delta C \Big( -\frac{\partial \sigma}{\partial t} - A \frac{\partial \sigma}{\partial x} + 2(C - C_{obj}) \Big) dxdt - \int_0^T \int_0^L \sigma \delta f dxdt \tag{7}$$

Then, the gradient of $\mathbf{J}$ can be expressed as

$$\nabla \mathbf{J} = \frac{\delta \mathbf{J}}{\delta f(x,t)} = \sigma(x,t) \tag{8}$$

If

$$-\frac{\partial \sigma}{\partial t} - A \frac{\partial \sigma}{\partial x} = -2(C - C_{obj}) \tag{9}$$

For our purposes and taking into account the punctual source location in $\mathrm{x}_s$ the regulation will be applied by means of the perturbation in the value of $f(x_s,t)$ using the discrete version of (10) for every times $t_i \in (0,T)$ such

$$\nabla \mathbf{J}(\mathbf{x_s}, \mathbf{t_i}) = \frac{\delta \mathbf{J}}{\delta f(x_s, t_i)} = \sigma(x_s, t_i) \tag{10}$$

**Discretization**

The numerical resolution of equation (1) is performed using an upwind scheme for the spatial integration and Forward Euler for the time integration, making possible to formulate the updated value at cell *i* and time step $n+1, C_i^{n+1}$ as

$$C_i^{n+1}(x) = \begin{cases} C^n - A\Delta t\Big(\dfrac{(C_i^n - C_{i-1}^n)}{\Delta x} + f_i\Big) & \text{if } A > 0 \\ C^n - A\Delta t\Big(\dfrac{(C_{i+1}^n - C_i^n)}{\Delta x} + f_i\Big) & \text{if } A < 0 \end{cases} \tag{11}$$

The same scheme can be applied in order to solve the adjoint variable $\sigma_i^n$ but taking care with the forward integration by means of choosing negative time lengths and updating from the time $n+1$ to $n$

$$\sigma_i^{n+1}(x) = \begin{cases} \sigma^{n+1} - A\Delta t\Big(\dfrac{(\sigma_i^{n+1} - \sigma_{i-1}^{n+1})}{\Delta x} + f_i\Big) & \text{if } A > 0 \\ \sigma^{n+1} - A\Delta t\Big(\dfrac{(\sigma_{i+1}^{n+1} - \sigma_i^{n+1})}{\Delta x} + f_i\Big) & \text{if } A < 0 \end{cases} \tag{12}$$

Being necessary to satisfy the CFL condition for the election of the time step

$$c = \left| \frac{A\Delta t}{\Delta x} \right| \leq 1 \tag{13}$$

**Application**

In order to show the applicability, the using of (10) inside Algorithm (1) with the previous schemes are going to applied to the next case. The conditions for the physical problem are

$$L = 1, T = 1, \Delta x = 0.01, CFL = 1, x_s = 0.45, x_t = 0.65 \tag{14}$$

Where $C_{obj}(x_t, t)$ is defined as

$$C_{obj}(x_t, t) = \begin{cases} 0 & \text{if } t < 0.5 \\ 20(t - 0.5) & \text{if } 0.5 \leq t < 0.6 \\ 2 & \text{if } 0.6 \leq t < 0.7 \\ 2 + 20(0.7 - t) & \text{if } 0.7 \leq t < 0.8 \\ 0 & \text{if } t \geq 0.8 \end{cases} \tag{15}$$

Using the next parameters for the gradient method

$$tol_{max} = 10^{-}7, Iter_{max} = 2000, \epsilon = 60.0 \tag{16}$$

It is possible to obtain exponential convergence to the solution, reaching the tolerance in 1307 iterations. It is displayed in Figure (1)

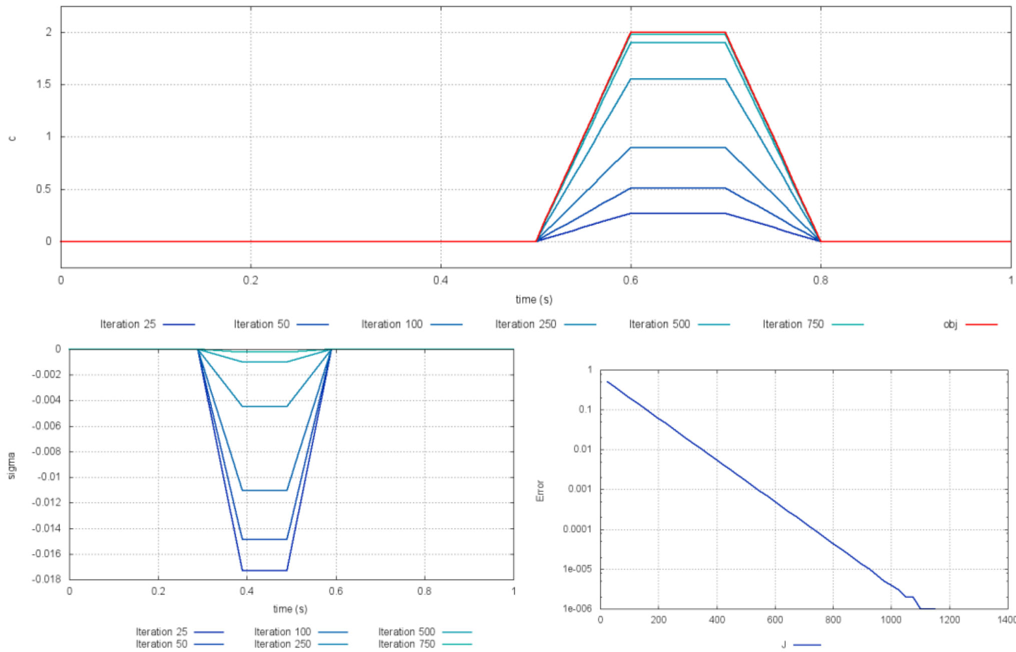


Figure 1 Convergence to the objective function. Evolution of $C(x_t, t)$ (top), $\sigma(x_s, t)$ used in the gradient and convergence (bottom, left) and $\mathbf{J}$ (bottom, right)

**ONE DIMENSIONAL SHALLOW WATER EQUATIONS**

The 1D Shallow Water Equations derive from the depth-averaged equations of mass conservation and of momentum. They form a 2x2 hyperbolic system of equations:

$$\frac{\partial \mathbf{U}(x,t)}{\partial t} + \frac{d\mathbf{F}(x,\mathbf{U})}{dx} = \mathbf{H}(x,\mathbf{U}) \tag{17}$$

Where

$$\mathbf{U} = \begin{pmatrix} h \\ q \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} q \\ \frac{q^2}{h} + gI_1 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} 0 \\ g\left[I_2 + h\left(S_0 - S_f\right)\right] \end{pmatrix} \tag{18}$$

In this study, we will consider constant section of width $B=1$ in order to make easier the analysis. This allow to rewrite the terms $I_1 = (1/2)Bh^2$ and $I_2 = 0$

Moreover, the objective function is introduced in a general form as

$$\mathbf{J} = \int_0^T \int_0^L \mathcal{E}(x,t;h,q)\,dx\,dt \tag{19}$$

This function includes the aspect of the flow to be characterized or regulated.

### Adjoint Formulation

Next, the formulation of the adjoint equation is obtained by means of multiplying the adjoint variable $\sigma_1(x,t)$ by the continuity equation and, the momentum equation by the adjoint variable $\sigma_2(x,t)$ .The sum of these two products is integrated in time and space.

$$Q = \int_0^T \int_0^L \sigma_1 \left( \frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} \right) + \sigma_2 \left( \frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{q^2}{h} + \frac{gh^2}{2} \right) - gh(S_0 - S_f) \right) dxdt = 0 \quad (20)$$

This expression can be integrated by parts as before. Then, variations are estimated by taking increments respect to *h* and *q*. The first variation of the functional becomes

$$\delta Q = Q(\delta h, \delta q) = 0 \quad (21)$$

Increments are also taken in the functional $\mathbf{J}$ , leading to

$$\delta J = \int_0^T \int_0^L \left( \frac{\partial \mathcal{E}}{\partial h} \delta h + \frac{\partial \mathcal{E}}{\partial q} \delta q \right) dxdt \quad (22)$$

Adding both terms and grouping (21) and (22), it follows that whenever the following is satisfied

$$\begin{aligned} &-\frac{\partial \sigma_1}{\partial t} + \left( \frac{q^2}{h^2} - gh \right) \frac{\partial \sigma_2}{\partial x} - g\sigma_2 (S_0 - S_f) + g\sigma_2 h \frac{\partial S_f}{\partial h} + \frac{\partial \mathcal{E}}{\partial h} = 0 \\ &-\frac{\partial \sigma_2}{\partial t} - \frac{\partial \sigma_1}{\partial x} - 2\frac{q}{h} \frac{\partial \sigma_2}{\partial x} + \sigma_2 gh \frac{\partial S_f}{\partial q} + \frac{\partial \mathcal{E}}{\partial q} = 0 \end{aligned} \quad (23)$$

Then $\delta \mathbf{J}$ can be rewritten as

$$\delta J = \underbrace{\int_0^L [\sigma_1 \delta h + \sigma_2 \delta q] \Big|_0^T dx}_{(a)} + \underbrace{\int_0^T \left[ \sigma_1 \delta q + \sigma_2 \left( \frac{2q}{h} \delta q - \left( \frac{q^2}{h^2} - gh \right) \right) \right] \Big|_0^L dt}_{(b)} \quad (24)$$

The expression of (24) establishes the relation between the error and the quantities that may be regulated (*h.q*). This will allow us to evaluate the gradient which will be introduced in the optimization method (Algorithm 1).

### Discretization

Both the shallow water system and (23) are hyperbolic. They include a Jacobian matrix in the formulation and share the same set of eigenvalues and eigenvectors. The upwind finite volume method applied to both systems relies on the sign of the eigenvalues. An upwind Riemann solver [4] for the locally linearized problem is formulated for both problems. For a given cell *i* with edges *i-1/2* and *i+1/2* differences the updating follows the scheme [5]:

$$\mathbf{U}_i^n = \mathbf{U}_i^{n+1} + \frac{\Delta t}{\Delta x}\left(\sum_{m=1}^{2}(\tilde{\lambda}^{+}\gamma\tilde{\mathbf{e}})_{i-1/2}^{m} + \sum_{m=1}^{2}(\tilde{\lambda}^{-}\gamma\tilde{\mathbf{e}})_{i+1/2}^{m}\right)^{n+1} \quad (25)$$

Where $\lambda^+$ and $\lambda^-$ are the eigenvalues and $\tilde{\mathbf{e}}$ are the eigenvectors. The coefficients, defined as:

$$\tilde{\gamma}_{i+1/2}^{m} = \left(\tilde{\alpha} - \frac{\tilde{\beta}}{\tilde{\lambda}}\right)_{i+1/2}^{m} \quad (26)$$

Are different for every system of equations. The common definition of (25) usually includes the evaluation of the next value *n+1* as a function of the previous one *n*. In this case, the integration is being backward in time and then, the expression is reversed. Obviously, in the Adjoint problem the conditions are usually established in $t = T$ forming a Final Value Problem.

**Application**

The adjoint variables are very useful to find the gradient of $\mathbf{J}$ to changes in inflow or outflow. When controlling one of these parameters, the value of $\nabla J$ may be introduced in a Iterative method to find the solution, (the optimal controlling parameters). In this case, the development has been made to control the inflow.
The first step is to choose initial conditions for both SWE and Adjoint Equations,

$$\begin{aligned} &\sigma_1(x,T) = 0, \sigma_2(x,T) = 0, \qquad 0 < x < L \\ &\delta h(x,0) = 0, \delta q(x,0) = 0, \qquad 0 < x < L \end{aligned} \quad (27)$$

Taking into account (27), $\delta\mathbf{J}$ of (24) can be simplified as

$$\delta J = -\int_0^T \sigma_1(0,t)\delta q(0,t)dt \quad (28)$$

Considering (28) sensitivity of $\mathbf{J}$, perturbations in *q* can be applied be means of the discrete version of the gradient

$$\frac{\delta\mathbf{J}}{\delta q(0,t_i)} = -\sigma_1(0,t_i) \quad (29)$$

The case was proposed in [6] where there is a channel without friction nor bed slope with constant width B=10m L=1000 m and initial depth $h_0 = 2.0m$ and initial flow $q_1 = 2.2627m^2/s$. The time domain is defined as $t \in (0,T)$ with T=400 s. and the inlet boundary condition is $q_o(t) = q_1 + \sec h^2(0.03(t-120))$ . The functional is oriented to regulate the water depth, considering $h_{obj}(x_t,t) = 2.05m$ for $t \in (0,T)$ with $x_t = 2.5$ . The evolution of the water depth as well as $\sigma_1(x,t)$ is displayed in figure (2)

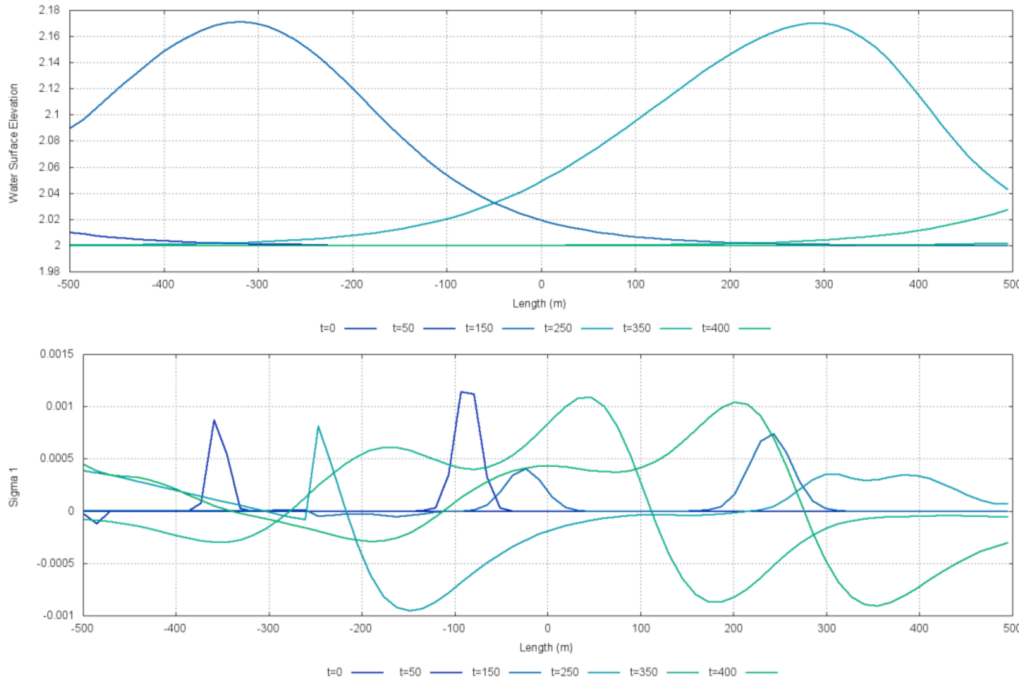


Figure 2 Evolution of water depth (top) and the adjoint variable $\sigma_1(x,t)$ for different times

In this case, $\sigma_1(x,t)$ evaluated in x=0 only provides information relative to the sensitivity of $\mathbf{J}$ respect to $q_0$ but the evaluation of $\sigma_1(0,t)$ can be used to regulate the inlet condition forbidding such discharges that may produce overflow in the channel.


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