2010

TR-2010008: Knowability from a Logical Point of View

Sergei Artemov

Tudor Protopopescu

Knowability from a Logical Point of View

Sergei Artemov* & Tudor Protopopescu†

The CUNY Graduate Center

365 Fifth Avenue, rm. 4329

New York City, NY 10016, USA

October 18, 2010

Abstract

The well-known Church-Fitch paradox shows that the verificationist knowability principle *all truths are knowable*, yields an unacceptable omniscience property. Our semantic analysis establishes that the knowability principle fails because it misses the *stability* assumption 'the proposition in question does not change from true to false in the process of discovery,' hidden in the verificationist approach. Once stability is made explicit, the resulting *stable knowability principle* accurately represents verificationist knowability, does not yield the omniscience property, and can be offered as a resolution of the knowability paradox.

Two more principles are considered: *total knowability* stating that it is possible to know whether a proposition holds or not, and *monotonic knowability* stemming from the intrinsically intuitionistic reading of knowability. The study of these four principles yields a "knowability diamond" describing their logical strength. These results are obtained within a logical framework which opens the door to the systematic study of knowability from a logical point of view.

1 Introduction

Knowability is analyzed in a logical framework with the alethic modalities $\Box$ (necessary), $\Diamond$ (possible), and the epistemic modality $\mathbf{K}$. Modalities $\Box/\Diamond$ represent, in an abstract way, the process of discovery. The logical principles are considered as schemes of axioms hence subjects of the usual logical rules, e.g., necessitation.[1]

*The research is supported in part by NSF grant 0830450.

†Supported in part by a Doctoral Student Research Grant from the CUNY Graduate Center.

[1]This means, along with each principle P we adopt $\Box P$, $\mathbf{K}P$, etc. On the semantical level, once P holds in a model, P holds at each node of this model, hence $\Box P$, $\mathbf{K}P$, etc. also hold.

The knowability paradox purports to show that the verificationist knowability principle, *all truths are knowable*,

$$F \to \Diamond \mathbf{K} F,$$

implies that *all truths are known*, $F \to \mathbf{K}F$. The common reaction to the Church-Fitch proof is that it is not really valid, it is valid (so to speak) only on a technicality, it convicts an innocent principle according to the letter of the law but unjustly, and the task the paradox poses is to uncover where the mistake occured. The paradox is also usually considered as an objection to verificationist[2] views since the principle is often taken as definitive of their position, and an ancillary task taken up by commentators has been to show that such a view, whatever faults it may have, is not refuted by an argument as swift and simple as the Church-Fitch construction.

Our goal in this paper is not to defend verificationism but rather to analyse and clarify the concept of knowability expressed in the verificationist principle.

First (Section 2) we note that the formulation of verificationist knowability in classical logic, as $F \to \Diamond \mathbf{K} F$ is not intuitively valid once F is allowed to change from true to false during the verification process. For example, *even if it was raining when Holmes asked Watson to check for the rain, but the rain had stopped by the time Watson looked outside, Watson's answer will be "no rain"*; though true, that it is raining does not get verified. We then provide a new semantical proof that the classical verificationist knowability is indeed equivalent to the omniscience principle. This analysis indicates that the classical understanding of verificationist knowability is not intrinsically valid and should be augmented by some features representing the constructive view of truth and knowability it is supposed to express. One solution (Section 4) naturally stems from our semantical analysis: the verificationist knowability principle is valid for *stable truths*, those that remain true in the process of discovery, yielding the *stable knowability principle*

$$\Box F \to \Diamond \mathbf{K} F$$

which is equivalent to the version of verificationist knowability restricted to necessary truths $\Box F \to \Diamond \mathbf{K} \Box F$. We show that with this principle of stable knowability, the Church-Fitch paradox disappears. **Stable knowability can be offered as a correct classical version of verificationist knowability and a resolution of the knowability paradox**.

Yet another confirmation of the correctness of the stable knowability principle comes from the intuitionistic analysis of knowability (Section 5). The principle that *all truths are knowable* is supposed to represent the core position of the constructively inclined verificationist, a position which should also be naturally expressible in intuitionistic logic. As is well-known, the Church-Fitch construction carried out in intuitionistic logic proves only that

$$p \to \neg\neg \mathbf{K} p,$$

[2]We use the term 'verificationist' synonymously with 'anti-realist.'

which some, notably Dummett, consider a superior expression of the verificationist's position. We invoke Gödel's translation of intuitionsitic to modal logic to find out what sort of classical principles correspond to intuitionistic knowability, and show that it is equivalent to the classical principle of *monotonic knowability*

$$\Box F \to \Diamond\Box\mathbf{K}F$$

which implies stable knowability but differs from it by assuming that once F becomes known, it stays known, $\Box\mathbf{K}F$. This additional assumption appears because of the limited expressive power of the intuitionistic language and semantics, where it is implicit in the understanding of the intuitionistic truth relation $\Vdash$.

Furthermore, we offer a logical analysis of the total knowability principle: *it is possible to know whether a proposition holds or not*,

$$\Diamond\mathbf{K}F \vee \Diamond\mathbf{K}\neg F,$$

which provides a more general view of knowability (Section 6). We show that total knowability is strictly stronger than stable knowability and yet does not succumb to the knowability paradox. This indicates that total knowability is another option to consider for expressing the idea that all truths are knowable.

These observations open the door to a bi-modal framework for the systematic study of knowability.[3]

2 Semantical Analysis of Verificationist Knowability

By the well-known Church-Fitch construction [7, 21], assuming only that knowledge is factive and distributes across conjunction, along with a minimal amount of modal and classical logic[4]:

$$F \to \Diamond\mathbf{K}F \qquad (VK)$$

generates the knowability paradox since it implies the *omniscience principle* that 'all truths are known':

$$F \to \mathbf{K}F. \qquad (OMN)$$

Theorem 1 [Church-Fitch] *VK yields OMN.*

Proof. The idea is to consider the instance of *VK* with F being the well-known Moore sentence $p \wedge \neg\mathbf{K}p$ (which we will call *Moore*):

$$\mathit{Moore} \to \Diamond\mathbf{K}(\mathit{Moore}). \qquad (VK(Moore))$$

[3]See [3, 40, 42, 44] for other bi-modal approaches to the knowability paradox.

[4]The logic we use is no stronger than that used in the Church-Fitch proof; in most cases, T suffices.

We first establish that *Moore* cannot be known.

1. $\mathbf{K}(\mathit{Moore}) \to \mathbf{K}p$ - since $\mathbf{K}(X \wedge Y) \to \mathbf{K}X$;
2. $\mathbf{K}(\mathit{Moore}) \to (p \wedge \neg\mathbf{K}p) \to \neg\mathbf{K}p$ - factivity of knowledge.

Therefore, $\mathbf{K}(\mathit{Moore})$ yields a contradiction $\mathbf{K}p$ and $\neg\mathbf{K}p$, hence

3. $\mathbf{K}(\mathit{Moore}) \to \bot$ where $\bot$ is the the propositional constant *false*.

Moreover, the Moore sentence cannot possibly be known:

4. $\Diamond\mathbf{K}(\mathit{Moore}) \to \Diamond\bot$ - from 3, by modal reasoning;
5. $\neg\Diamond\mathbf{K}(\mathit{Moore})$ - from 4, since $\Diamond\bot \to \bot$.

Now we apply these findings to *VK(Moore)*:

6. $\neg\mathit{Moore}$ - from 5 and *VK(Moore)*;
7. $p \to \mathbf{K}p$ - from 6, since $\neg(X \wedge \neg Y)$ yields $X \to Y$ in classical logic.

The last line is nothing but the omniscience principle *OMN*. □

Given this, a stronger result is provable.

Corollary 1 *Schemas VK and OMN are equivalent.*

Proof. It remains to show that *OMN* yields *VK*. By *OMN*, $F \to \mathbf{K}F$. Since $\mathbf{K}F \to \Diamond\mathbf{K}F$ for reflexive modality $\Box$, $F \to \Diamond\mathbf{K}F$. □

We first present the intuitive semantics of *VK* (Section 2.1) and then produce a semantic proof that *VK* is equivalent to *OMN* (Section 2.2).

2.1 Verificationist Knowability is not intuitively valid

We begin by first showing that *VK* is invalid even under circumstances acceptable to the verificationist. The reason for *VK*'s invalidity is that even if a correct verification procedure has been applied to a true proposition F, we can expect a positive result only with the assumption that F stays true in the process of discovery.

However, this stability assumption is missing in *VK* and our logical analysis reveals that this missing assumption is the source of the knowability paradox.[5]

In a more formal setting, a proposition F is *stable* in a given model, if it satisfies

$$F \to \Box F.$$

Note that for a reflexive modality $\Box$, F is stable if and only if $F \leftrightarrow \Box F$. A stable sentence can be false at some (or even all) states of a given model, but once it is true at a state, it

[5] Indeed, it has been noted that non-stability, in particular empirical contingency, is a thorn in the side of verificationists, see [39, 43, 59, 66]. Verificationist theories of meaning, replacing truth with warranted assertibility, seem not to be able to handle satisfactorily the semantics of sentences whose warranted assertibilty changes between states, i.e. which are not stably warrantedly assertible. As we will see, stability is necessary for the validity of *VK*, suggesting an explanation of these problems.

remains true at all $\Box$-accessible states. There are sentences, e.g., propositional constants $\top$ (*true*) and $\bot$ (*false*) which are stable in any model. Sentences $\Box F$ are stable in any transitive model.

There are no reasons to believe that without the assumption that F is stable, principle *VK* is valid in all situations acceptable for the verificationist.

> *Consider a correct, i.e., knowledge- and knowability-producing verification procedure, $\mathfrak{Ver}$, applied to a true proposition F. However, during (or, perhaps, due to) verification, F changes its truth value and $\mathfrak{Ver}$ eventually certifies that F is false. Then VK fails despite the fact that a correct verification procedure has been applied to a true proposition and terminates with a definitive answer.*[6]

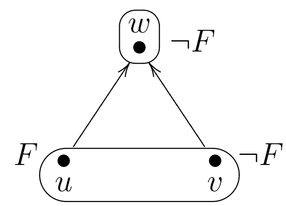

Figure 1: Model $\mathcal{M}_1$ where *VK* fails.

To make this precise, consider the following bi-modal model, $\mathcal{M}_1 = (W, R_\Box, R_\mathbf{K}, \Vdash)$. $\mathcal{M}_1$ has three states $W = \{u, v, w\}$ and two accessibility relations, an alethic accessibility relation $R_\Box$ (represented by arrows) which is reflexive and according to which w is accessible from u and v, and an epistemic accessibility relation $R_\mathbf{K}$ (represented by ovals) which is an equivalence relation on W such that u and v are equivalent (epistemically indistinguishable) and w is equivalent only to itself. A requirement for the truth of $\mathbf{K}F$ is just that F be true at all epistemically, $R_\mathbf{K}$, indistinguishable states. Suppose also that F holds at u but not at v or w. We can think of $\mathcal{M}_1$ as modeling the process of verification, moving from ignorance to knowledge by shrinking one's epistemic possibilities.[7]

As model $\mathcal{M}_1$ shows, F is not stable: though it is true at u, it is not true at w which is accessible from u. At u and v, $\mathfrak{Ver}$ cannot come to a definitive conclusion concerning F: u and v are epistemically indistinguishable so F is not known at either of them. At w, $\mathfrak{Ver}$ can only conclude that F is false, since $w \Vdash \neg F$. In $\mathcal{M}_1$, $\neg \mathbf{K}F$ holds at all words and hence $u \Vdash \neg \Diamond \mathbf{K}F$, accordingly

$$u \nVdash F \to \Diamond \mathbf{K}F.$$

[6] The process of discovery can be viewed as consisting of (a) choosing the verification procedure $\mathfrak{Ver}$ out of the ones available; (b) running $\mathfrak{Ver}$ on a given input. We argue that there is no generality lost, if we consider one universal verification meta-procedure $\mathfrak{Ver}$ and count the choosing of a specific procedure into the time balance of this universal $\mathfrak{Ver}$. Indeed, the process of looking for a proper specific procedure can be regarded as a run of $\mathfrak{Ver}$. Since for the modal language, the meta-procedure is no different from any other procedure, we can view the whole process of discovery as an application of one universal verification procedure $\mathfrak{Ver}$. So, (a) is subsumed under (b).

[7] We can also consider $\mathcal{M}_1'$, which is $\mathsf{S5}$ with respect to $\Box$, by setting $R_\Box = W^2$ in $\mathcal{M}_1$.

The non-stability of F prevents F from being knowable in the sense of *VK*, despite the assumption that the agent possesses a correct verification procedure in all of the states. This suggests that *VK* does not properly express the idea that all truths are knowable.

2.2 The Knowability Paradox, semantically

The counter-model $\mathcal{M}_1$ for *VK* leads us to consider what the natural frame conditions for *VK* are. We find that the answer to this question yields a semantic proof that verificationist knowability, *VK*, is equivalent to the omniscience principle, *OMN*.

To keep our analysis concise, we consider models in which states can be specified: for each state u there is a *specifying proposition* F_u such that

$$u \Vdash F_u \ \text{ and for all } v \neq u, \ \ v \nVdash F_u.$$

We call such models *specifiable states models* or *specifiable models*, for short. In a sense, in epistemology, specifiable models are the natural ones since it is natural to expect an epistemic state to have some combination of features that makes it different from all other states, i.e., specifies this state. For example, in model $\mathcal{M}_1$ from Figure 1, the following formulas can be regarded as specifying propositions:

$F_u = F$ (*u is the state at which F holds*);
$F_v = \neg F \wedge \neg \mathbf{K} \neg F$ (*v is the state at which both F and $\mathbf{K}\neg F$ are false*);
$F_w = \neg F \wedge \mathbf{K} \neg F$ (*w is the state in which $\neg F$ holds and is known*).

In the theory of models for modal logics, specifying propositions can be represented by special atoms (called *nominals*, cf. [4]), but, to reduce bookkeeping, we allow specifying propositions to be regular compound formulas as well. Each model can be made specifiable by adding fresh atomic specifying propositions: the old formulas all retain their truth value. Standard soundness/completeness theorems of modal logic extend to specifiable models automatically.[8]

A state u is called *omniscient* if it forms a singleton with respect to $R_{\mathbf{K}}$:

$$uR_{\mathbf{K}}v \quad \text{yields} \quad u = v.$$

At an omniscient state, any true proposition is known, hence also knowable. The following theorem shows that at a non-omniscient state, specifying propositions, though true, are not knowable.

Theorem 2 *At a non-omniscient state, no specifying proposition is knowable: $u \nVdash \Diamond \mathbf{K} F_u$ for all non-onmiscient u's.*

[8] To make the picture complete, we give an example of a non-specifiable model: $W = \{a, a'\}$, $R_\Box = R_{\mathbf{K}} = W \times W$, and $a, a' \Vdash p$ for all atoms p. In this model, the same formulas hold at a and a' so none of these states can be specified in the given bi-modal language. From the epistemic point of view, states a and a' are rather the same state, since there is no property which separates these states from each other.

Proof. Consider an arbitrary non-omniscient state u. There is a state v such that $uR_{\mathbf{K}}v$ and $v \neq u$. Let F_u be a nominal for u. Then $u \nVdash \Diamond\mathbf{K}F_u$. Indeed, $w \nVdash \mathbf{K}F_u$ for each w: for $w = u$ - since $v \nVdash F_u$ and $uR_{\mathbf{K}}v$, for $w \neq u$ - since $w \nVdash F_u$. □

A model is *omniscient* if each of its states is omniscient. A model is a *model for a principle (schema) P* if all instances of P hold in this model.

The following theorem describes specifiable models in which the verificationist knowability principle *VK* holds: they are exactly the omniscient models.

Theorem 3 *A specifiable model $\mathcal{M}$ is a model of VK if and only if $\mathcal{M}$ is omniscient.*

Proof. Any omniscient model is a model of *VK*: for an omniscient model, $u \Vdash F$ yields $u \Vdash \mathbf{K}F$. Due to reflexivity of $\Box$, $u \Vdash \mathbf{K}F \to \Diamond\mathbf{K}F$, hence $u \Vdash \Diamond\mathbf{K}F$.

Consider a specifiable model $\mathcal{M}$ with a non-omniscient state u. The following instance of *VK*:

$$F_u \to \Diamond\mathbf{K}F_u$$

fails at u. Indeed, $u \Vdash F_u$, by the definition of F_u, and $u \nVdash \Diamond\mathbf{K}F_u$ by Theorem 2. □

Theorem 3 may be regarded as a semantical version of the Church-Fitch theorem. Note that Moore sentences play no role in this proof which shows that the knowability paradox is not intrinsically related to the Church-Fitch proof, and in particular the instance of *VK* that is its premise, but rather to the structure of the verificationist knowability principle *VK* itself.

We see that assuming the principle of verificationist knowability, *VK*, trivializes the epistemic picture: all states are epistemically distinguishable and everything which is true is known. Our analysis confirms that the Church-Fitch construction is indeed valid: *VK* really is equivalent to *OMN*, and that, strictly speaking, there is no 'paradox' just an unexpected result. Since assuming *VK* is equivalent to assuming *OMN*, if the latter is not acceptable, the former should also be rejected.

Accordingly the analysis affords us a deeper understanding of the verificationist conception of knowledge, and not just knowability. We see that *VK*-endorsing verificationism really has no room for ignorance, and more to the point, no room for the idea of investigation or verification as a process that reduces one's ignorance. *VK* embodies a picture of knowledge where the knower moves from one state of knowledge, in which they know all there is to know, to another (presumably larger) state of knowledge in which they know all there is to know.[9] Such a picture of knowledge seems not to make sense of any kind of inquiry; it cannot model a scenario where one asks whether F is true or not, engages

[9] The intuitionist's ideal mathematician constructing the mathematical universe would be a paradigm example of such a knower; indeed such a picture expresses a very hardcore constructivism - the constructed truths are all the truths there are. On such a view it is a truism that one knows all truths, and supposing there is an unknown truth contradicts this notion of truth, of which Theorem 1 is just a straightforward justification.

in research, and with evidence settles the question. Our analysis suggests that *VK* is not consistent with even this most schematic description of scientific inquiry.

3 Diagnosis

So, what has gone wrong? From the Church-Fitch proof, it appears that *Moore* is, if not the main culprit of the "mystery of the disappearing diamond" [31], then at least a willing accomplice; that it is integral to the "modal collapse" which [34, 35] argues is the heart of the paradox. We showed in Section 2.2 that this is not so, and that the problem stems from *VK* itself.

Though not guilty of generating the paradox, the presence of *Moore* is instructive, indeed an unknowable sentence, like *Moore*, for knowability principles is like a crash test for vehicles: it is not a test under normal conditions, but a test of its behavior during a disaster. It takes quite a different set of specifications and tests to check whether a vehicle also does its normal job. The semantical analysis of 2.1 and 2.2 was the test under normal conditions, and we saw that the result of the knowability paradox bears no intrinsic connection to *Moore*.

Let us, then, examine the results of the crash test to see how exactly *Moore* reveals the structural weakness in *VK*. In the Church-Fitch proof, *VK* has been applied to the Moore sentence which **cannot be known**. Indeed, $\mathbf{K}(\mathit{Moore})$ is inconsistent in any logic of knowledge[10]: $\mathbf{K}(\mathit{Moore})$ yields both $\mathbf{K}p$ and $\neg\mathbf{K}p$,[11] so $\neg\mathbf{K}(\mathit{Moore})$.

This observation allows us to see where the collapse occurs. We have seen (Theorem 1, line 4) that $\mathbf{K}(\mathit{Moore}) \to \bot$. Since $\bot$ implies anything,

$$\mathbf{K}(\mathit{Moore}) \leftrightarrow \bot.$$

This is the moment when "the diamond disappears." In any normal modal logic, "the diamond disappears" from the constant *false*:

$$\bot \leftrightarrow \Diamond\bot,$$

and this is exactly what happens here:

$$\mathbf{K}(\mathit{Moore}) \leftrightarrow \Diamond\mathbf{K}(\mathit{Moore})$$

as well. Therefore, *VK(Moore)* of the form

$$\mathit{Moore} \to \Diamond\mathbf{K}(\mathit{Moore})$$

[10] In this respect our analysis agrees with [49, 51], see Section 7.2 below. See also [59] for a similar argument.

[11] See Theorem 1.

is logically equivalent to

$$Moore \rightarrow \mathbf{K}(Moore),$$

which is equivalent to

$$Moore \rightarrow \bot,$$

i.e., $\neg Moore$.

It is mentioned frequently that Church-Fitch results hold not just for **K** but for any factive and conjunction-distributive operator (as Fitch himself makes clear), e.g., [20, 34, 35, 49, 60]. The "modal collapse" on $\bot$ is a well-known phenomenon and hence this explanation holds for any other factive and distributive operator, **O**, applied to its un-**O**-able and un-**O**-ed Moore sentence $p \wedge \neg \mathbf{O}p$.

4 Stable Knowability

Our semantic analysis showed that the non-stability of F invalidates the principle of verificationist knowability. Without the stability assumption stated explicitly, the verificationist knowability principle *VK* does not represent verificationist approaches fairly. In this section, we show that with the stability assumption made explicit, no paradox results.

Clearly, not all truths are stable, so to distinguish between the knowability of stable and non-stable truths, we need to explicitly incorporate stability into a knowability principle. Since the possibility of knowing stable versus non-stable truths captures some of the distinction between the possibility of knowing necessary versus contingent truths, this would go some way to distinguishing between, e.g., mathematical and empirical knowability.

How might one represent the knowability of stable truths? Consider the principle **all stable truths are knowable**, e.g., if F is a stable truth ($F \rightarrow \Box F$ and F), then F is knowable ($\Diamond \mathbf{K} F$):

$$[(F \rightarrow \Box F) \wedge F] \rightarrow \Diamond \mathbf{K} F. \tag{1}$$

Since for reflexive modality $\Box$, the formula $(F \rightarrow \Box F) \wedge F$ is logically equivalent to $\Box F$, the principle (1) can be equivalently presented as $\Box F \rightarrow \Diamond \mathbf{K} F$. These considerations prompt the following definition of the principle of *stable knowability*

$$\Box F \rightarrow \Diamond \mathbf{K} F. \tag{SK}$$

Let *SK(F)*, *VK(F)*, etc., denote principles *SK*, *VK*, etc., for a given proposition F. Note that one could envision a seemingly stronger version of stable knowability, e.g., the principle SK':

$$\Box F \rightarrow \Diamond \mathbf{K} \Box F.$$

However, SK' as a schema has the same strength as *SK*. Indeed, $SK(\Box F) = \Box\Box F \rightarrow \Diamond \mathbf{K} \Box F$ which is logically equivalent to $SK'(F)$.[12]

[12] for transitive $\Box$'s.

We can also consider the *principle of eventual stable knowability*, *ESK*,

$$\Diamond\Box F \rightarrow \Diamond\mathbf{K}F.$$

Informally *ESK* expresses that the knowability of F can be concluded from the assumption that there is a state after which F holds everywhere. It can be shown that $\Box(SK)$ implies *ESK* which implies *SK*. Hence principle *ESK* is equivalent to *SK* as well.[13]

A stronger principle of *monotonic knowability*, *MK*, appears as the result of the Gödel translation of intuitionisitic knowability into the classical bi-modal language, cf. Section 5:

$$\Box F \rightarrow \Diamond\Box\mathbf{K}F.$$

The justification of *MK* requires some additional assumptions concerning the relation between the alethic modality $\Box$ and the epistemic modality $\mathbf{K}$ and this restricts the domain of situations in which such knowability holds. We postpone the discussion of *MK* and corresponding assumptions to Section 5 and concentrate on analyzing the basic and well-justified stable knowability principle *SK*.

The semantical analysis in Section 2.2 sheds some light on how stable knowability *SK* escapes the knowability paradox: it allows non-omniscient (and meaningful) specifiable models and hence does not yield the omniscience principle *OMN*. Due to Theorem 3, to make this point it suffices to provide a non-omniscient frame in which *SK* holds in all models.

Theorem 4 *Any model with the frame in Figure 2*[14] *is an SK-model:*

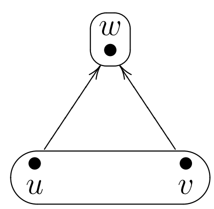

Figure 2: Non-omniscient frame for *SK*.

Proof. Indeed, if $\Box X$ holds at some state, then $w \Vdash X$ as well, hence $w \Vdash \mathbf{K}X$. Since w is $\Box$-accessible from each state, $\Diamond\mathbf{K}X$ holds at each state.[15] $\Box$

We can now show that stable knowability does not suffer from the knowability paradox, in particular, *SK* does not yield omniscience.

[13] *ESK* might be considered a rendering of specifically constructive knowability. In a constructive setting, a state after which a proposition holds everywhere can be regarded as a state where the proposition has a proof, in which case *ESK* can be read as saying that if it is possible that F has a proof, then F is knowable.

[14] e.g., model $\mathcal{M}_1$.

[15] This argument prompts a general characterization of specifiable *SK*-models: every state has a $\Box$-accessible omniscient state.

Corollary 2 *Schema SK does not yield OMN.*

Proof. By Theorem 4, all instances of *SK* hold in $\mathcal{M}_1$, but *OMN(F)* does not. Hence no combination of instances of *SK* can yield omniscience with respect to F. □

5 Intuitionistic vs. Stable Knowability

Given its intuitionistic inspiration, the natural logical perspective to have on *VK* is intuitionistic. Indeed one of the first reactions to the knowability paradox, [56], showed that in intuitionistic logic the Church-Fitch construction yielded only $p \to \neg\neg\mathbf{K}p$, which, when read from an intuitionistic view point, is not absurd. Indeed some, [9, 16, 43],[16] most notably Dummett, argue that for the verificationist it is superior to *VK* as a represention of their sense of knowability. Accordingly, let us try to read verificationist knowability *VK* as an intuitionistic principle.

Even a brief perusal of the literature on intuitionistic modal logic shows there is no widely accepted interpretation of intuitionistic modality, and many logics and families of logics have been constructed (see for instance [25, 63, 64, 65] for some overviews). Some of these systems consider independent intuitionistic 'possibility' modalities $\Diamond$, and some do not. Kripke semantics shows that intuitionistic logic itself can be regarded as a fragment of the classical modal logic $\mathsf{S4}$ with the modality $\Box$, hence introducing a new modality $\Diamond$ adds more modality than seems to be needed to express what the verificationist wants to say (Rasmussen in [43] makes this a central point of his Mapping Objection). For a discussion of intuitionistic modality bearing on the knowability paradox, and verificationism more generally, see [58]; see also [8, 19, 41]. To be precise, by intuitionistic modal logic in this paper we understand standard modal systems with reflexive $\Box$ and intuitionistic logic. The 'possibility' modality $\Diamond$ is viewed as the dual of $\Box$: $\Diamond X$ is an abbreviation for $\neg\Box\neg X$.

Theorem 5 *With intuitionsitic logic, principles VK and* $p \to \neg\neg\mathbf{K}p$ *are equivalent.*

Proof. We follow the first six steps of the proof of Theorem 1 and verify that each of them is made according to intuitionsitic logical rules (we repeat these steps here for the reader's convenience):

1. $\mathbf{K}(\mathit{Moore}) \to \mathbf{K}p$;
2. $\mathbf{K}(\mathit{Moore}) \to (p \wedge \neg\mathbf{K}p) \to \neg\mathbf{K}p$;
3. $\mathbf{K}(\mathit{Moore}) \to \bot$;

[16] Both Dummett and Rasmussen (via Dummett), attribute this view to Bernhard Weiss, who appears not to have published this. Incidentally, Dummett's informal reading of *IK* is "the possibility that F will come to be known always remains open," which fits nicely with the frame conditions characterizing *SK*. See [39] for objections to *IK* on the grounds that it yields that no truths are undecided (see also [5, 66] for related discussions), and see [6, 9, 57] for responses on behalf of the intuitionistically inclined. As with *VK* it is not our purpose to defend *IK*, but to find its proper relation to other knowability principles.

4. $\Diamond\mathbf{K}(Moore) \to \Diamond\bot \to \bot$;
5. $\neg\Diamond\mathbf{K}(Moore)$;
6. $\neg Moore$.

Since intuitionsitically $\neg(X \wedge Y)$ yields $X \to \neg Y$, we conclude

7. $p \to \neg\neg\mathbf{K}p$.

Here is the proof of the other direction.

1. $p \to \neg\neg\mathbf{K}p$;
2. $\Box\neg\mathbf{K}p \to \neg\mathbf{K}p$ - by reflexivity of $\Box$;
3. $\neg\neg\mathbf{K}p \to \neg\Box\neg\mathbf{K}p$ - since $X \to Y$ yields $\neg Y \to \neg X$;
4. $p \to \neg\Box\neg\mathbf{K}p$ - from 1 and 3, by syllogism;
5. $p \to \Diamond\mathbf{K}p$ - since here $\Diamond$ is $\neg\Box\neg$. $\Box$

The aforementioned arguments and endorsements justify the following definition. By *intuitionistic knowability* we mean the following principle:

$$F \to \neg\neg\mathbf{K}F. \qquad (IK)$$

In the 1930s Gödel found a faithful embedding of intuitionsitic logic into modal logic with the S4-style modality: one translates intuitionistic formulas by means of the rule *box every sub-formula* [26].[17] By $g(F)$ we denote the Gödel translation of formula F. The Gödel translation provides a complete characterization of intuitionistic validity: a formula F is intuitionistically valid if and only if its translation $g(F)$ is valid in S4.

Gödel's motivation resulted from the provability reading of the $\Box$ modality, hence boxing a formula G forces a constructive reading of it as *G is provable* rather than classically as *G is true*. The Kripke semantics for intuitionistic and modal logics revealed that on the semantic level, the Gödel translation specifies intuitionistic logic as a fragment of the classical modal logic S4 satisfying the stability condition: *what is true remains true* (see Section 5.1 for more discussion on stability and constructive semantics).

Via the Gödel embedding we see that stability is a faithful modal reincarnation of provability. Indeed, once a proof of F becomes available at a state u, in all further states F holds true. Conversely, if there is no proof of F at u, then a state at which F fails is consistent with u and hence, in principle, possible at u. Stability is provability expressed in a modal language.[18]

Note that the Gödel translation of *IK(p)* is

$$g(IK(p)) = \Box(\Box p \to \Box\neg\Box\neg\Box\mathbf{K}\Box p) = \Box(\Box p \to \Box\Diamond\Box\mathbf{K}\Box p).$$

As a schema, *g(IK)* is equivalent to the following principle of *monotonic knowability*:

$$\Box F \to \Diamond\Box\mathbf{K}F. \qquad (MK)$$

[17] Gödel in [26] offered two translations, each of which is essentially equivalent in S4 to the rule *box every sub-formula*.

[18] In the Logic of Proofs which combines the relational and provability readings of intuitionistic logic, this argument is captured by the Realization Theorem and the notion of fully explanatory models ([1, 22]).

Theorem 6 *Principles g(IK) and MK are equivalent.*

Proof. It is an easy exercise in modal logic to show that for a reflexive $\Box$,

$$g(IK)(F) \rightarrow MK(F)$$

which yields that as a schema, *g(IK)* implies *MK*.

To show the converse, assume *MK* and consider $MK(\Box F) = \Box\Box F \rightarrow \Diamond\Box\mathbf{K}\Box F$. Then

1. $\Box F \rightarrow \Diamond\Box\mathbf{K}\Box F$ - by transitivity;
2. $\Box(\Box F \rightarrow \Diamond\Box\mathbf{K}\Box F)$ - by necessitation;
3. $\Box\Box F \rightarrow \Box\Diamond\Box\mathbf{K}\Box F$ - by distribution;
4. $\Box F \rightarrow \Box\Diamond\Box\mathbf{K}\Box F$ - by transitivity;
5. $\Box(\Box F \rightarrow \Box\Diamond\Box\mathbf{K}\Box F)$ which is nothing but *g(IK)(F)* - by necessitation. $\Box$

We will now show that *MK* is stronger than *SK* and reveal the additional assumptions which, given *SK*, should be made to justify *MK*.

Theorem 7 *Principle MK is strictly stronger than SK.*

Proof. It is easy to see that *MK* logically implies *SK* for a reflexive $\Box$.

Let us show in a model that *SK* as a schema does not yield *MK*. Consider model $\mathcal{M}_2$ in Figure 3.

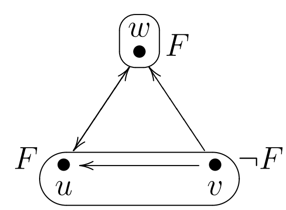

Figure 3: Model $\mathcal{M}_2$ where *MK* fails.

It is immediate that any instance of *SK* holds in $\mathcal{M}_2$. Indeed, if $\Box X$ holds at some node, then $w \Vdash X$, hence $w \Vdash \mathbf{K}X$ and $\Diamond\mathbf{K}X$ holds at each node.

Let us show that *MK* fails at u. Indeed, $u \Vdash \Box F$. On the other hand, $u \nVdash \mathbf{K}F$, hence $\Box\mathbf{K}F$ fails at both u and w. Therefore, $\Diamond\Box\mathbf{K}F$ fails at both u and w since these nodes are only $\Box$-accessible from each other. Hence $\Box F \rightarrow \Diamond\Box\mathbf{K}F$ fails at u. $\Box$

Apparently, schema *g(IK)* (i.e., *MK*) incorporates, along with stability, some other specifically intuitionistic assumptions concerning $\Box$ and $\mathbf{K}$. Stable knowability *SK* states that given a stable truth F, there is a possible state (moment of the discovery process) in which the verification procedure confirms F. Intuitionistic knowability *g(IK)/MK* seemingly states the same: there is a possible state in which the verification procedure confirms

F. However, since in the intuitionistic universe, once $\mathbf{K}F$ becomes true, it stays true, i.e., $\Box\mathbf{K}F$ holds. In this respect, model $\mathcal{M}_2$ is not an intuitionistic universe: $\mathbf{K}F$ holds at w, but does not stay true there, i.e., $w \nVdash \Box\mathbf{K}F$.

In terms of verification, *g(IK)/MK* states that there is a possible world at which the verification procedure confirms F, i.e., F becomes known, $\mathbf{K}F$, and **hence stays known**, $\Box\mathbf{K}F$. This additional assumption appears because of the limited expressive power of the intuitionistic language and semantics, where this assumption is implicit in the semantics of $\Vdash$. The classical modal language is more flexible and is able to express both *SK* and *IK* (via the Gödel translation).

Analogous to the Gödel embedding, the explicit assumption of stability has the effect of making explicit some of the constructive meaning of *VK* in the classical language via *SK*. In [43] Rasmussen calls *VK* "an amphibious hybrid between the points of view of realism and of anti-realism" (in logical terms, between classical and intuitionistic logic); we see just how apt a description this is.

What we observe here is the remarkably robust character of stable knowability and its ability to represent the constructive intent of verificationist knowability and its compatibility with intuitionsitic knowability.

5.1 Stability vs. constructive semantics

Now we want to address the issue of whether our analysis of the role of the stability requirement gives us a deeper understanding of the verificationist conception of knowability.

The history of studies of constructive, e.g., intuitionistic, logic has two distinguished traditions. First, there is the constructive 'witness' semantics, which originated in Brouwer's works and manifested itself in the Brouwer-Heyting-Kolmogorov semantics in which witnesses are viewed as proofs. Within this tradition, *F is true* is understood as

there is a proof of F,

i.e., via the informal **existential** quantifier over proofs.

The second tradition can be traced back to intuitionistic Kripke semantics, according to which *F is true* means

F holds in all possible situations,

i.e., via the informal **universal** quantifier over possible worlds. Intuitionisitic truth in this second 'universal' setting is **stable**: if F is true at u, F stays true at all other worlds accessible from u.

Reconciling these two traditions in a comprehensive formal model has been a longtime challenge in the area of constructive semantics. The first steps were made by Gödel's embedding discussed above. Later Gödel sketched a way to assign proof-like objects to each occurrence of modality in S4 [27]; this project of Gödel's was completed in the Logic of Proofs [1] which connected the 'existential' and 'universal' intuitionistic semantics: a formula F is true in the monotone 'universal' semantics if and only if F is true in the 'existential' semantics of proofs.

Our study suggests that similar developments can occur in the study of verificationist knowability. The standard verificationist justification of *VK* is based on the 'existential' witness semantics of constructive truth. What we offer in our analysis is a Kripke-style 'universal' semantics of knowability with the core stability condition leading to *SK* and rely on the intrinsic connection of stability and provability which informally connects it to the 'existential' semantics.

We wish to think that the principle of stable knowability *SK* provides a plausible modal resolution of the Church-Fitch paradox: *SK* is the correct modal expression of the verificationist view of knowability. In this respect, the problem of finding a bi-modal principle to adequately express verificationist knowability finds a reasonable solution in *SK*. The natural next step would be to capture the 'existential' witness side of knowability and we hope that *SK* could play a role there too.

Despite the seemingly different character of our approach to the traditional verificationist justification of the knowability principle, the aforemantioned history of reconciling the two semantic traditions for intuitionsitic logic gives us a certain hope that a similar reconciliation over stable knowability principles *SK* is possible as well. We hope that this will become the subject of future studies.[19]

6 A Bigger Picture of Knowability

Generally speaking, knowability can be thought of as a generalization of decidability.[20] To say 'F is knowable' is to say that one knows of a procedure which, if carried out in an appropriate situation, would yield a decision on the truth of F. So the knowability of F amounts to the possibility of knowing F or the possibility of knowing $\neg F$. Accordingly, the proper formalisation of the claim that F is knowable is the principle of *total knowability*:

$$\Diamond \mathbf{K} F \vee \Diamond \mathbf{K} \neg F. \qquad (TK)$$

The principle of total knowability asserts that, for any proposition, there is a verification procedure, runs of which are represented by the alethic modality $\Diamond$, which yields a definitive answer as to whether F or $\neg F$. From this point of view, *TK* is a meaningful principle of possible knowledge. We show that *TK* is strictly stronger than stable knowability, but *TK* also escapes the Church-Fitch paradox.

[19] One may try some other format of representing this notion of knowability. Such a possibility is investigated by Dean and Kurokawa, [11], who consider the principle 'if F is true, then there exists a proof of F' instead of *VK* and use the framework of the Quantifed Logic of Proofs (see [1, 2] and [23]).

[20] A classic statement "a closed formula A is *(formally) decidable*, if A is either provable or refutable, i.e. if either $\vdash A$ or $\vdash \neg A$" [32].

How should we understand *TK*? *TK* represents a conception of knowability distinct from the verificationist's favoured version. Hence, unlike Melia [37], we do not argue that *TK* should be viewed as a re-interpretation of *VK*, which preserves the verificationist's constructive motivations but does not succumb to the knowability paradox. We will show that the latter point is indeed correct, but the former should be resisted.

It has been argued, [45, 46, 60, 68], that *TK* is in fact not compatible with the verificationist's motivations. Williamson and Rückert point out that *TK* is consistent with there being truths which are unknowable (we prove this in Theorem 8), which means they are inconsistent with the verificationist's manifestation requirement; the requirement that the understanding of a proposition be manifested in an (in principle) ability to verify it if true. Indeed, according to Dummett, denying *TK* is a necessary condition for even stating the verificationist's position[21] vis a vis the realist. The tension between verificationism (*VK*) and *TK* is further brought out by Wright and Salerno; according to them it is evident that we do not know ourselves to posses the means for deciding every proposition. Hence the verificationist is committed to the possibility that *TK* is false alongside also holding that *VK* is true. Accordingly the verificationist cannot accept *TK* as a better expression of the intent behind *VK*.[22]

We have no reason to disagree with any of this: *TK* represents a broadly constructive position that applies to all types of propositions, and which is distinct from *VK*. We prove formally that *TK* does not succumb to the knowability paradox.

What is the relation between *TK* and *VK*?

Theorem 8 *Principle TK does not yield VK.*

Proof. Consider again model $\mathcal{M}_1$. First, we note that since w forms a singleton with respect to $R_{\mathbf{K}}$, for each formula X, either $w \Vdash \mathbf{K}X$ or $w \Vdash \mathbf{K}\neg X$. Since w is accessible from each node, $\Diamond\mathbf{K}X \vee \Diamond\mathbf{K}\neg X$ holds at each node. Therefore all instances of TK hold in $\mathcal{M}_1$. However, $\mathcal{M}_1 \nVdash VK$. Indeed, $u \Vdash F$ but obviously, $\mathbf{K}F$ does not hold at any node, hence $u \nVdash \Diamond\mathbf{K}F$, and so

$$u \nVdash VK.$$

□

On the other hand the converse does hold.

Theorem 9 *Principle VK yields TK.*[23]

Proof.

1. $F \to \Diamond\mathbf{K}F$ - an instance of *VK*;

[21] [46] makes this point.

[22] In fact *VK* implies *TK*, given excluded middle, (see Theorem 9), hence the possibility that *TK* is false yields the possibility that verificationism, *VK*, is false. The resolution of this problem, according to the verificationist, is to reject excluded middle, see [46] for a more careful statement of this line of thought.

[23] Substantially the same proof can be found in [68].

2. $\neg F \rightarrow \Diamond \mathbf{K}\neg F$ - an instance of *VK*;
3. $F \vee \neg F$ - excluded middle;
4. $\Diamond \mathbf{K}F \vee \Diamond \mathbf{K}\neg F$ - from 1–3, by propositional reasoning. □

What is the relation between knowability and stable knowability?

Theorem 10 *Principle TK yields SK.*

Proof. More specifically, we establish that *TK(F)* yields *SK(F)*.
1. $\mathbf{K}(\neg F) \rightarrow \neg F$ - factivity of knowledge;
2. $F \rightarrow \neg \mathbf{K}\neg F$ - contrapositive of 1 and double negation principle;
3. $\Box F \rightarrow \Box \neg \mathbf{K}\neg F$ - from 2, by $\Box$-necessitation, distribution;
4. $\Box F \rightarrow \neg \Diamond \mathbf{K}\neg F$ - from 3, converting $\Box\neg$ into $\neg\Diamond$;
5. $\neg \Diamond \mathbf{K}\neg F \rightarrow \Diamond \mathbf{K}F$ - conversion of $X \vee Y$ into $\neg Y \rightarrow X$ in *TK(F)*;
6. $\Box F \rightarrow \Diamond \mathbf{K}F$ - from 4, 5, by propositional reasoning. The latter is *SK(F)*. □

The converse however does not hold.

Theorem 11 *Principle SK does not yield TK.*

Proof. Consider model $\mathcal{M}_3$ in Figure 4.

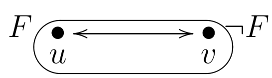

Figure 4: Model $\mathcal{M}_3$ where *SK* holds but *TK* fails.

It is easy to see that in $\mathcal{M}_3$ all instances of *SK* hold. Indeed, for any proposition X, if $\Box X$ holds at some node, then X holds in the whole of $\mathcal{M}_3$, hence both $\mathbf{K}X$ and $\Diamond\mathbf{K}X$ hold in $\mathcal{M}_3$. Therefore $\Box X \rightarrow \Diamond \mathbf{K}X$ holds in $\mathcal{M}_3$. It now suffices to show that *TK(F)* fails in $\mathcal{M}_3$. Indeed, $u \not\Vdash \mathbf{K}F$ and $u \not\Vdash \mathbf{K}\neg F$ and hence $u \not\Vdash \Diamond\mathbf{K}F$ and $u \not\Vdash \Diamond\mathbf{K}\neg F$, so $u \not\Vdash$ *TK(F)*. □

There is an informal explanation of the scenario represented by model $\mathcal{M}_3$. We have a thorough verification procedure, i.e., it can observe all epistemically possible states, which does not confirm F because it sees that F does not hold in all epistemic states, hence *TK* fails. But because it is thorough it sees that if F held in all states it would be known, hence *SK* is vacuously true at each node. Unlike *TK*, *SK* does not take any knowability obligations with respect to non-stable truths.

Theorem 12 *Principle VK yields MK.*

Proof. By Theorem 1, *VK* yields $F \to \mathbf{K}F$. By necessitation and distributivity, $\Box F \to \Box\mathbf{K}F$. By reflexivity, $\Box\mathbf{K}F \to \Diamond\Box\mathbf{K}F$, hence $\Box F \to \Diamond\Box\mathbf{K}F$. $\Box$

Theorem 13 *Principle MK does not yield TK.*

Proof. All instances of *MK* hold in model $\mathcal{M}_3$. Indeed, for any proposition X, if $\Box X$ holds at some node, then X holds in the whole of $\mathcal{M}_3$, hence, $\mathbf{K}X$, $\Box\mathbf{K}X$, and $\Diamond\Box\mathbf{K}X$ hold in $\mathcal{M}_3$. As shown in Theorem 11, *TK* fails in $\mathcal{M}_3$. $\Box$

Theorem 14 *Principle TK does not yield MK.*

Proof. Consider model $\mathcal{M}_2$ in Figure 3. According to Theorem 7, *MK* fails in $\mathcal{M}_2$. On the other hand, each instance of *TK* holds in $\mathcal{M}_2$. Indeed, w is $\Box$-accessible from each node and w is omniscient hence, for each X, either $\mathbf{K}X$ or $\mathbf{K}\neg X$ hold at w. $\Box$

Theorem 15 *Principle MK does not yield VK.*

Proof. Each instance of *MK* holds in model $\mathcal{M}_1$. Indeed, if $\Box X$ holds anywhere in $\mathcal{M}_1$ then $w \Vdash X$, hence $w \Vdash \mathbf{K}X$, and $w \Vdash \Box\mathbf{K}X$. Therefore, $\Diamond\Box\mathbf{K}X$ holds at each node of $\mathcal{M}_1$. As before, *VK(F)* fails in $\mathcal{M}_1$, in particular, $u \Vdash F$ and $u \nVdash \Diamond\mathbf{K}F$. $\Box$

6.1 Knowability Diamond

Figure 5 provides the diagram of relationships between the knowability principles.

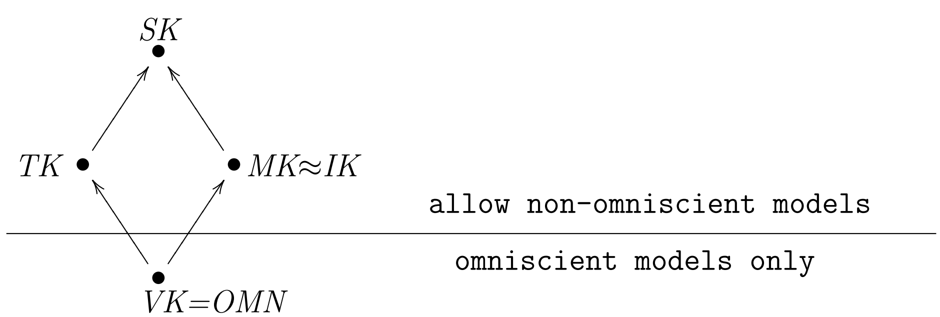

Figure 5: "Knowability Diamond"

Arrows represent generality/strength of the corresponding principles as schemes: $X \longrightarrow Y$ reads as *each model of* X *is a model for* Y, or, equivalently, *principle* X *yields principle*

Y. A more general principle is less strong logically. None of the converse arrows holds. So, *VK* is the most restrictive of the four - it holds only in omniscient models - and *SK* is the most general, has the most models of the four.

Does the "Knowability Diamond" offer a resolution to the Church-Fitch paradox? We argue that it does.

Principles *TK* and *IK* have been discussed before and the resolutions they provide have been criticized.[24] Stable knowability *SK* seems to be a better candidate for a resolution of the paradox since (a) *SK* provides a definitive answer to the core question of what went wrong with *VK* - the stability assumption was missing; (b) *SK* is a well-principled restriction of *VK* since it eliminates from consideration only non-stable truths which has been definitively diagnosed as the reason for the failure of *VK*; (c) *SK* has the same format as *VK* hence does not invoke any new scenarios, does not change the problem.

6.2 Safe knowability principles

It is immediate from Figure 5 that neither *TK*, nor *SK*, nor *MK* fall into the scope of the Church-Fitch paradox, i.e., none of them yield OMN.

Let us take a closer look at **how** with *stable knowability SK* instead of *verificationist knowability VK* the knowability paradox disappears. A simple repetition of the Church-Fitch argument with *SK(Moore)* only proves that $\neg\Box(p \wedge \neg\mathbf{K}p)$, i.e., $\Diamond OMN$, which is strictly weaker than OMN.[25] One can equivalently rewrite $\neg\Box(p \wedge \neg\mathbf{K}p)$ as

$$\Box p \to \Diamond\mathbf{K}p$$

which incidentally is *SK(p)* stating informally that

if p holds at all states, then it is possible it becomes known at one of them.

This conclusion does not appear paradoxical.[26]

[24] One could argue that both *TK* and *IK* change the problem rather than solve it. The format of *TK* is quite different from the original *VK*; it has been noted that *TK* is implied by *VK*, but is at the same time incompatible with what *VK* is supposed to express (see footnotes 16, 22 and 23 and the beginning of Section 6). Reading verificationist knowability as *SK* resolves this. The same holds for *IK* which is expressed in a different logic, with a quite different semantics; its translation, *MK*, into the classical modal language also does not match the original format of *VK*. It is not clear, in what sense a paradox associated with *VK* can be solved without providing a satisfactory explanation of the problems with *VK* and by considering other principles instead.

[25] This result is adumbrated in [45].

[26] Moreover, consider what happens if F is not stable. Let us assume that all non-stable truths are knowable, i.e., $(F\wedge\neg\Box F) \to \Diamond\mathbf{K}F$. A repetition of the Church-Fitch proof yields $(p\wedge\neg\mathbf{K}p) \to \Box(p\wedge\neg\mathbf{K}p)$. If all non-stable truths are knowable, then ignorance cannot disappear in the process of verification. This is rather counter-intuitive and provides further confirmation that the knowabilty paradox is due to the unstated assumption that known propositions are stable.

7 Comparisons

What we offer is a logical framework for studying different kinds of knowability. The above results suggest some plausible alternatives, *TK*, *MK*, or *SK*, to *VK* as an understanding of the notion that all truths are knowable. These alternatives in turn yield a principled restriction on *VK*. We have seen that *VK* implicitly assumes that truths are stable. Accordingly, if one wants to endorse *VK*, then its scope should be restricted to stable truths and when this is made explicit in *SK*, *OMN* does not result. How does this stability restriction compare to other prominent restriction strategies?

7.1 Edgington

In [17, 18], Edgington argues that only actual truths are knowable. She proposes that the knowability principle should be read as

$$\mathbf{A}F \rightarrow \Diamond\mathbf{K}\mathbf{A}F,$$

where '**A**' is the modal operator 'actually.' Methodologically, this approach looks compatible with our restriction of the knowability principle to 'stable truth' only. However, within the straightforward formalization of 'actuality' in [17] as

$$u \Vdash \mathbf{A}_w F \text{ if and only if } w \Vdash F,$$

Edgington's proposal turns out to be trivial. In any model, $\mathbf{A}F$ is equivalent to a propositional constant $\top$ (*true*) or $\bot$ (*false*). Therefore, assuming the 'actual knowability' schema above is equivalent to assuming *VK* for the propositional constants $\top$ and $\bot$, which trivially holds in each model regardless of any knowability assumptions.[27]

7.2 Tennant and Dummett

Tennant in [49, 50, 51] argues that only Cartesian propositions are knowable where "a Cartesian proposition is a proposition F such that $\mathbf{K}F$ is consistent" [50]. The Cartesian restriction provides a correct negative test for the knowability of a proposition - if a proposition is not Cartesian, then it is not knowable. Indeed, if F is non-Cartesian, then $\mathbf{K}F$ is formally inconsistent, hence in an appropriate logic L, $\mathsf{L} \vdash \neg\mathbf{K}F$. Then by necessitation, $\mathsf{L} \vdash \Box\neg\mathbf{K}F$ and hence $\mathsf{L} \vdash \neg\Diamond\mathbf{K}F$.

However, being Cartesian does not necessarily imply being knowable. In particular, a true Cartesian proposition may still not be knowable, and hence Cartesian-ness does not give the positive conditions under which a proposition is *knowable, if true*.[28] Consider

[27] An interesting development of Edgington's proposal can be found in [42], which considers a non-Kripkian semantics for actuality and knowledge.

[28] [28, 29, 35, 61] argue that the Cartesian restriction is *ad hoc*, our analysis (in 3) shows that while the restriction does not provide a positive test for knowability, it may not be as *ad hoc* as they argue.

VK(p) with a propositional letter p for F; this instance of F is clearly Cartesian. Consider model $\mathcal{M}$-p in Figure 6.

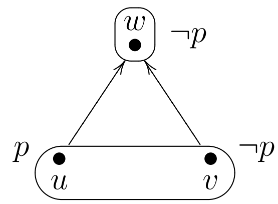

Figure 6: Model $\mathcal{M}$-p where *VK(p)* fails.

The same argument as accompanied $\mathcal{M}_1$ shows that $\mathcal{M}$-p is a scenario which is acceptable from a verificationist point of view in which *VK(p)* fails. Model $\mathcal{M}$-p shows that being Cartesian and true, without also being stable, is insufficient to guarantee the truth of *VK*. The validity of *VK(F)* depends on the stability of F, as we have seen, and Cartesian propositions are not necessarily stable.[29]

Note that if p is atomic or basic, then the above argument shows that the restriction on *VK* proposed in [15] is also too weak.

8 Conclusion

In [53] van Benthem argues that "what one really wants is a *new systematic viewpoint*" from which to approach the knowability paradox rather than just attempting to avoid it by weakening either the logic in the Church-Fitch proof or *VK* itself.[30] We argue that the knowability framework and our semantic analysis achieves this. The contribution of this work can be summarized as follows.

- An alternative, semantic proof that *VK* = *OMN*. This could be a serious argument that the paradox is due to the principle *VK* itself rather than the Church-Fitch proof.
- A case has been made that the key requirement of the verificationist argument in favor of *VK*, stability of the truth in question, is missing in *VK* and its lack alone is a sufficient explanation for the paradox.
- A corrected and justified version of verificationist knowability, the stable knowability principle *SK*, was offered and shown to be paradox-free. This is our suggested resolution of the Church-Fitch paradox.

[29] In the same vein, [61, 62] argues that a Cartesian proposition can still yield *OMN* via a more complex proof than the one by Church and Fitch (Theorem 1). But see [52] for a reply.

[30] One might argue that we do the latter, but our framework justifies why this is legitimate.

- The monotonic knowability principle *MK* was introduced and shown to be a faithful classical counterpart of intuitionistic knowability *IK*.
- A formal logical framework was offered. The resulting "Knowability Diamond" answers the question of the relative strength/generality of all four knowability principles considered.

We see that there is no need to adopt a non-classical logic[31] or to reject any of the epistemic principles used in the Church-Fitch proof. Indeed, the simplicity and plausibility of the principles appealed to in the proof is what makes the derivation of *OMN* so 'paradoxical.' Our approach preserves them all.

Our framework helps to clarify debates about the nature of verificationist knowability and its tenability. Due to the knowability paradox and the fact that *VK* yields the unacceptable *OMN*, there is a need for new bi-modal principles reflecting the constructive content of the verificationist principle of knowability. Our framework offers three possibilities, none of which leads to the knowability paradox.

- Stable knowability *SK*. This approach preserves the format of *VK* by limiting it to its epistemically justified stable version *SK*.
- Monotonic knowability *MK* which reflects the specifically intuitionistic reading of knowability. *MK* is stronger than *SK*, but more restrictive; it stipulates that once a proposition becomes known is stays known at all further steps.
- Knowability in a general setting reflected by the principle of total knowability *TK*. This approach attempts to preserve the idea that *all* truths are knowable.

The value of the framework comes not just from the results pertaining to the knowability paradox but also from the bigger explanatory picture it provides. It becomes possible not only to establish logical dependencies between principles, but also to definitively certify the absence of such dependencies which was not possible outside a rigorous logical framework. Such a framework allows us to begin systematically studying a concept which pervades epistemology. The possibility or impossibiliy of knowledge is central to debates about skepticism. The existence of *a priori* knowledge turns on the possibility or impossibility of knowing independently of experience.[32] To make more palatable the closure properties of the epistemic operator $\mathbf{K}$, formal epistemological approaches sometimes gloss it as *knowable* [10, 48], or as *is entitled to know* [24], or as *potential knowledge* [22]. Verificationists, of course, put the possibilities of knowledge at their center. Given how central knowability is to so many core epistemological topics, a direct investigation of it and its properties is warranted. We think the above analyses and principles constitute a step in this direction.

[31] See for instance [3, 40, 55, 56, 57].
[32] See [33, 38].

9 Acknowledgments

The authors are grateful to Melvin Fitting, Hidenori Kurokawa, Elena Nogina, and Graham Priest, for inspiring discussions and useful suggestions.

References

[1] S.N. Artemov. Explicit provability and constructive semantics. *Bulletin of Symbolic Logic*, 7(1):1–36, 2001.

[2] S.N. Artemov. The logic of justification. *Review of Symbolic Logic*, 2008.

[3] J.C. Beall. Knowability and possible epistemic oddities. In J. Salerno, editor, *New Essays on the Knowability Paradox*, pages 105–127. Oxford University Press, 2009.

[4] P. Blackburn and J. van Benthem Modal Logic: A Semantic Perspective. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*, pages 1–84. Elsevier, 2007.

[5] B. Brogaard and J. Salerno. Clues to the Paradoxes of Knowability: Reply to Dummett and Tennant. In E. N. Zalta, editor, *Stanford Encyclopedia of Philosophy*, Fall 2009.

[6] B. Brogaard and J. Salerno. Fitch's Paradox of Knowability. In *Analysis*, 62 (2), 143–151, 2002.

[7] A. Church. Referee reports on Fitch's "a definition of value." In Salerno [47], pages 13–20.

[8] T. De Vidi and G. Solomon. $\Box$ in intuitionistic modal logic. *Australasian Journal of Philosophy*, 75:2, 201–213, 1997.

[9] T. De Vidi and G. Solomon. Knowability and Intuitionistic Logic. *Philosophia*, 28, 319–334, 2001.

[10] T. De Vidi and D. Kenyon. Introduction. In T. De Vidi, D Kenyon, editors, *A Logical Approach to Philosophy: Essays in honour of Graham Solomon*, pages 1–21. Springer, 2006.

[11] W. Dean and H. Kurokawa. From the knowability paradox to the existence of proofs. *Synthese*, 175:2,177–225, 2010.

[12] M. Dummett. Truth. In *Truth and Other Enigmas*, pages 1–24. Harvard University Press, Cambridge, MA, 1959.

[13] M. Dummett. The philosophical basis of intuitionistic logic. In *Truth and Other Enigmas*, pages 215–247. Harvard University Press, Cambridge, Mass., 1973.

[14] M. Dummett. *The Logical Basis of Metaphysics*. Harvard University Press, Cambridge, 1991. The William James Lectures, 1976.

[15] M. Dummett. Victor's error. *Analysis*, 61:1–2, 2001.

[16] M. Dummett. Fitch's Paradox of Knowability. In Salerno [47], Chapter 4, pages 51–52.

[17] D. Edgington. The paradox of knowability. *Mind*, 94(376):557–568, October 1985.

[18] D. Edgington. Possible knowledge of unknown truth. *Synthese*, 2009.

[19] W.B. Ewald. Intuitionistic tense and modal logic. *Journal of Symbolic Logic*, 51(1): 166–179, 1963.

[20] M. Fara. Knowability and the capacity to knowability. *Synthese*, 2009.

[21] F. Fitch. A logical analysis of some value concepts. *Journal of Symbolic Logic*, 28(2): 135–142, 1963.

[22] M. Fitting. The logic of proofs, semantically. *Annals of Pure and Applied Logic*, 132: 1–25, 2005.

[23] M. Fitting. A quantified logic of evidence. *Annals of Pure and Applied Logic*, 152: 67–83, 2008.

[24] M. Fitting and R. Mendelsohn. *First-Order Modal Logic*. Kluwer Academic Publishers, Dordrecht, 1998.

[25] D.M. Gabbay, A. Kurucz, F. Wolter and M. Zakharyaschev. *Multi-Dimensional Modal Logic: Theory and Application*. Studies in Logic and the Foundations of Mathematics, 148. Elsevier, 1998.

[26] K. Gödel. Eine Interpretation des intuitionistischen Aussagenkalkuls. *Ergebnisse Math. Kolloq.*, 4:39–40, 1933. English translation in: S. Feferman et al., editors, *Kurt Gödel Collected Works, Vol. 1*, pages 301–303. Oxford University Press, Oxford, Clarendon Press, New York, 1986.

[27] K. Gödel. Vortrag bei Zilsel/Lecture at Zilsel's (*1938a). In Solomon Feferman, John W. Dawson, Jr., Warren Goldfarb, Charles Parsons, and Robert M. Solovay, editors, *Unpublished essays and lectures*, volume III of *Kurt Gödel Collected Works*, pages 86–113. Oxford University Press, 1995.

[28] M. Hand. Performance and paradox. In Salerno [47], chapter 17, pages 283–301.

[29] M. Hand and J.L. Kvanvig. Tennant on knowability. *Australasian journal of philosophy*, 77(4):422–428, 1999.

[30] Jaakko Hintikka. *Knowledge and belief; an introduction to the logic of the two notions*. Cornell University Press, Ithaca, N.Y., 1962. 23 cm. Contemporary philosophy.

[31] C. Jenkins. The mystery of the disappearing diamond. In Salerno [47].

[32] S. Kleene. *Introduction to Metamathematics*. Ishi Press, 1952.

[33] S. Kripke. *Naming and Necessity*. Basil Blackwell, Oxford, 1980.

[34] J. Kvanvig. *The Knowability Paradox*. Oxford University Press, 2006.

[35] J. Kvanvig. Restriction strategies for knowability. In Joe Salerno, editor, *New Essays on the Knowability Paradox*, pages 205–222. Oxford University Press, New York, 2009.

[36] J. L. Mackie. Truth and knowability. *Analysis*, 40(2):90–92, 1980.

[37] J. Melia. Anti-realism untouched. *Mind*, 100(3):341–342, July 1991.

[38] P. Peacocke, C. Boghossian. Introduction. In *New Essays on the A Priori*, pages 1–10. Oxford University Press, 2000.

[39] P. Percival. Fitch and Intuitionistic Knowability. *Analysis*, 50(3):182–187, 1990.

[40] G. Priest. Beyond the limits of knowledge. In J. Salerno, editor, *New Essays on the Knowability Paradox*, pages 93–104. Oxford University Press, 2009.

[41] A.N. Prior. *Time and Modality*. Oxford University Press, 1957.

[42] W. Rabinowicz and C. Segerberg. Actual truth, possible knowledge. *Topoi*, 13(2): 101–116, 1994.

[43] S.A. Rasmussen. The Paradox of Knowability and the Mapping Objection. In J. Salerno, editor, *New Essays on the Knowability Paradox*, pages 53–75. Oxford University Press, 2009.

[44] G. Restall. Not every truth is knowable: At least not all at once. In J. Salerno, editor, *New Essays on the Knowability Paradox*, pages 339–354. Oxford University Press, 2009.

[45] H. Ruckert. A solution to Fitch's paradox of knowability. In S. Rahman and J. Symons, editors, *Logic, Epistemology and the Unity of Science*, pages 351–380. Kluwer Academic Publishers, 2004.

[46] J. Salerno. Revising the logic of logical revision. *Philosophical Studies*, 99:2211–227, 1999.

[47] J. Salerno. *New Essays on the Knowability Paradox*. Oxford University Press, Oxford; New York, 2009.

[48] S. Shapiro. Externalism, anti-realism, and the kk-thesis. In T. De Vidi, D. Kenyon, editor, *A Logical Approach to Philosophy*, pages 22–35. Springer, 2006.

[49] N. Tennant. *The Taming of the True*. Oxford University Press, 1997.

[50] N. Tennant. Is every truth knowable: Reply to Williamson. *Ratio*, 14:263–280, 2001.

[51] N. Tennant. Revamping the restriction strategy. In Joe Salerno, editor, *New Essays on the Knowability Paradox*, pages 223–238. Oxford University Press, New York, 2009.

[52] N. Tennant. Williamson's Woes. *Synthese*, 173(1):2010.

[53] J. van Benthem. Actions that make us know. In Salerno. J., editor, *New Essays on the Knowability Paradox*, pages 129–146. Oxford University Press, 2009.

[54] W. Kooi B. van Ditmarsch, H. van der Hoek. *Dynamic Epistemic Logic*. Springer, 2008.

[55] H. Wansing. Diamonds are a philosopher's best friend: The knowability paradox and modal epistemic relevance logic. *Journal of Philosophical Logic*, 36(6):591–612, 2002.

[56] T. Williamson. Intuitionism disproved? *Analysis*, 42(4):203–207, 1982.

[57] T. Williamson. Knowability and constructivism. *Philosophical Quarterly*, 38(153): 422–432, 1988.

[58] T. Williamson. On intuitionistic epistemic modal logic. *Journal of Philosophical Logic*, 21:63–89, 1992.

[59] T. Williamson. Never say never. *Topoi*, 13:135–145, 1994.

[60] T. Williamson. *Knowledge and Its Limits*. Oxford University Press, Oxford, 2000.

[61] T. Williamson. Tennant on knowable truth. *Ratio*, XIII:99–114, 2000.

[62] T. Williamson. Tennant's Troubles. In Salerno. J., editor, *New Essays on the Knowability Paradox*, pages 183–205. Oxford University Press, 2009.

[63] F. Wolter and M. Zakharyaschev. On the relation between intuitionistic and classical modal logics. *Algebra and Logic*, 36:121–155, 1997.

[64] F. Wolter and M. Zakharyaschev. Intuitionistic modal logics as fragments of classical bimodal logics. In Orlowska. E., editor, *Logic at Work*, pages 168–186. Springer, 1999.

[65] F. Wolter and M. Zakharyaschev. Intuitionistic modal logic. In A. Cantini, E. Casari, and P. Minari, editors, *Logic and Foundations of Mathematics*, pages 227–238. Kluwer Academic Publishers, 1999.

[66] C. Wright. Can a Davidsonian meaning-theory be construed in terms of warranted assertibility? In C. Wright, *Realism, Meaning and Truth*, 286–315, 1983.

[67] C. Wright. Truth as sort of epistemic: Putnam's peregrinations. *Journal of Philosophy*, 97(6):335–364, 2000.

[68] C. Wright. On being in a Quandry: Relativism, Vagueness and Logical Revisionism. *Mind*, 110(437):45–98, 2001.