Fractals and the Geometry of Matrix Operators

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Abstract
In this project, we are interested in how certain fractals can be generated using introductory linear algebra techniques. A fractal is an object or a structure that is self-similar in all length scales. Sierpinski Carpet and Sierpinski Triangle are two classic examples of fractals. Fractals have many applications in real life, for instance, in imaging, art and biological sciences. In this project, we apply classes of linear transformation to describe and generate fractals in the Euclidean plane, and we generate different examples of fractal using MATLAB.

Introduction
There are twelve 2x2 matrices that perform specific transformations like reflection, rotation, compression, expansion, and shear. We introduce the effects of such matrix operators in \( \mathbb{R}^2 \) by giving a few examples:

Hausdorff dimension for self-similar objects
Hausdorff dimension (dH(X)) is concerned with different aspects of self-similar objects as opposed to other kinds of dimensions (topological dimension dT(X)).

\[ X \text{ is a unit cube. Let } K \text{ represent the number of smaller cubes inside } X \text{ of side } \frac{1}{p}. K = p^3 \]

(1) Hausdorff dimension

Notice how dH(X) = dT(X) = 3 (integer)

More generally:

\[ \frac{1}{K} = \frac{1}{S^{dh(X)}} \]

\[ K = \frac{1}{S^{dh(X)}} \]

\[ dH(X) = \ln(K) \ln(\frac{1}{3}) \]

\[ dH(S^f) = \frac{\ln(K)}{\ln(\frac{1}{3})} = \frac{\ln(4)}{\ln(3)} \approx 1.261859 \]

One can think of a fractal as an object with non-integer Hausdorff dimension, but, a more accurate definition is: X is a fractal if its Hausdorff dimension is strictly greater than its topological dimension.

\[ dH(X) > dT(X) \]

Snowflake is a fractal as seen from dT(S^f) = 1 < dH(S^f) = 1.26

Generate fractals in MATLAB

We used similitudes to generate examples of fractals in \( \mathbb{R}^2 \).

Definition: A similitude with scale factor s is a mapping of \( \mathbb{R}^2 \) into \( \mathbb{R}^2 \) of the form

\[ T_{\theta} \left( \frac{e}{f} \right) = s \left[ \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right] \left( \begin{array}{c} x \\ y \end{array} \right) + \left( \begin{array}{c} e \\ f \end{array} \right) \]

where s, e, and f are scalars. \( \theta \) is the degree of rotation for the transformation. Vector \( \left[ \begin{array}{c} e \\ f \end{array} \right] \) contains one initial point’s position and the positions of that initial point after each transformation.

The figure below is an example of fractal. We generated this Sierpinski Carpet in MATLAB.

![Sierpinski Carpet](image)

In MATLAB, the similitudes would be written as:

\[ x_n = \frac{1}{3} x + \frac{2}{3} ; \text{ } y_n = \frac{1}{3} y \]

A part of the MATLAB code used:

```
for i = 1:iterations
    if p < 1/8
        % One out of every eight iterations this
        % transformation is applied
        xn = 1/3*x; yn = 1/3*y;
        x = xn; y = yn;
    end
    if i = 1:iterations
        % & p is a random number in (0-1)
        x = rand(1);
        y = rand(1);
    end
    % The x, y values used in iteration i=2
    xn = 1/3*x; yn = 1/3*y;
    x = x[n]; y = y[n];
end
```