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Fractals and the Geometry of Matrix Operators

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Abstract

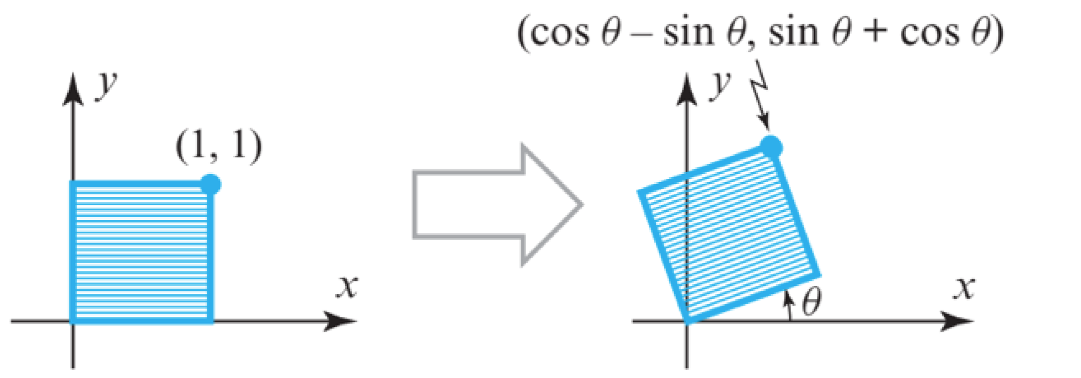
In this project, we are interested in how certain fractals can be generated by using introductory linear algebra techniques. A fractal is an object or a structure that is self-similar in all length scales. Sierpinski Carpet and Sierpinski Triangle are two classic examples of fractals. Fractals have many applications in real life, for instance, in imaging, art and biological sciences. In this project, we apply classes of linear transformation to describe and generate fractals in the Euclidean plane, and we generate different examples of fractal using MATLAB.

Introduction

There are twelve 2x2 matrices that perform specific transformations like reflection, rotation, compression, expansion, and shear. We introduce the effects of such matrix operators in $\mathbb{R}^2$ by giving a few examples:

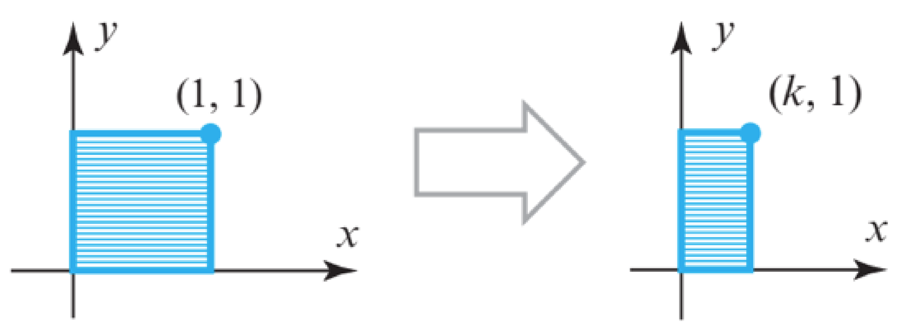
Rotation about the origin by angle θ: multiply by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



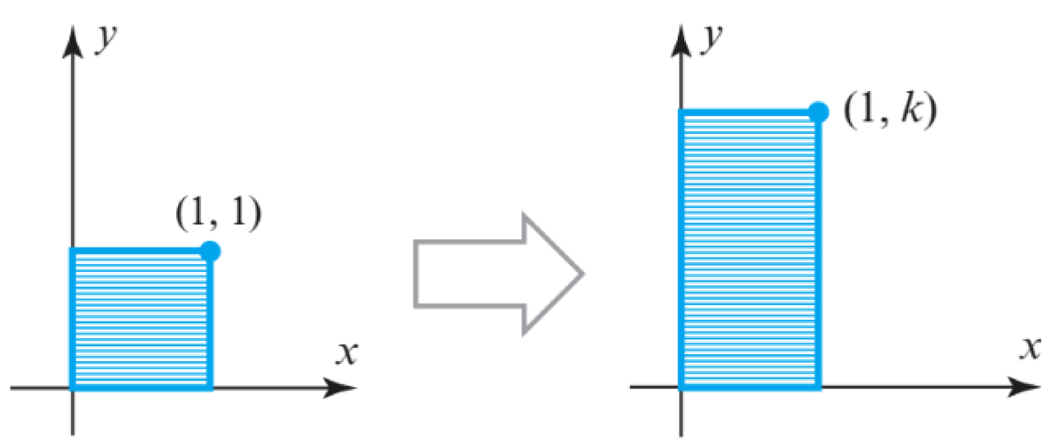
Compression in the x-direction by factor k: multiply by the matrix

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$



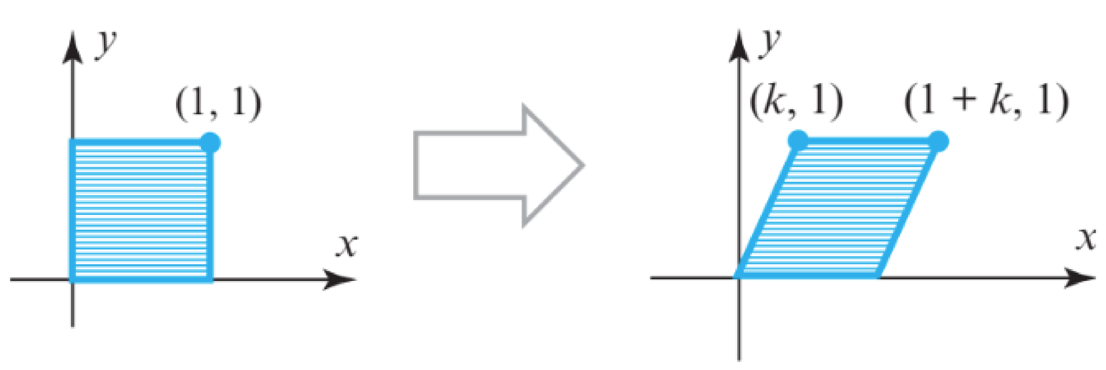
Expansion in the y-direction by factor k: multiply by the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$



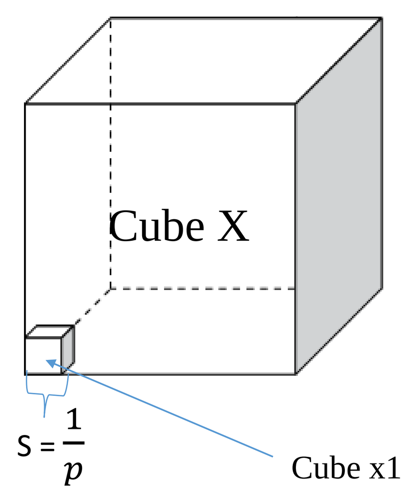
Shear in the x-direction by a factor k: multiply by the matrix

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$



Hausdorff dimension for self-similar objects

Hausdorff dimension (dH(X)) is concerned with different aspects of self-similar objects as opposed to other kinds of dimensions (topological dimension dT(X)).



X is a unit cube. Let K represent the number of smaller cubes inside X of side $S = \frac{1}{p}$. K= p^3

$$K = p^3 = \frac{1}{S^3}$$

$$\frac{1}{K} = S^3$$

(The exponent is Hausdorff dimension)

Notice how dH(X) = dT(X) = 3 (integer)

More generally:

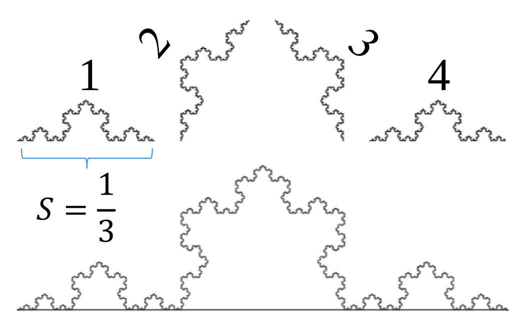
$$\frac{1}{K} = S^{dH(X)}$$

$$K = \frac{1}{S^{dH(X)}} = \left(\frac{1}{S}\right)^{dH(X)}$$

$$\ln(K) = \ln\left(\frac{1}{S}\right)^{dH(X)} = \mathrm{dH(X)}\ln\left(\frac{1}{S}\right)$$

$$\mathrm{dH(X)} = \frac{\ln(K)}{\ln\left(\frac{1}{S}\right)}$$

Use the formula above to find the dimension of snow flake:



Scaling factor S = $\frac{1}{3}$, numbers of self-similar snow flakes $K = 4$

$$\mathrm{dH}(Sf) = \frac{\ln(K)}{\ln\left(\frac{1}{S}\right)} = \frac{\ln(4)}{\ln(3)} \approx 1.261859$$

One can think of a fractal as an object with non-integer Hausdorff dimension, but, a more accurate definition is: X is a fractal if its Hausdorff dimension is strictly greater than its topological dimension.

$$dH(X) > dT(X)$$

Snowflake is a fractal as seen from $dT(Sf) = 1 < dH(Sf) \approx 1.26$

Generate fractals in MATLAB

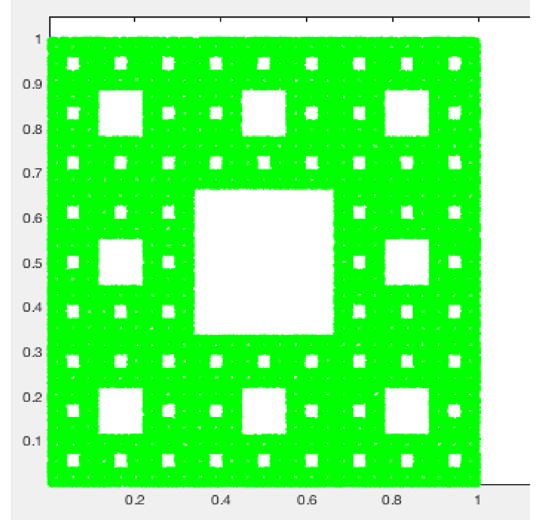
We used similitudes to generate examples of fractals in $\mathbb{R}^2$.

Definition: A similitude with scale factor s is a mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ of the form

$$\mathrm{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = s\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

where s, e, and f are scalars. θ is the degree of rotation for the transformation. Vector $\begin{bmatrix} e \\ f \end{bmatrix}$ contains one initial point's position and the positions of that initial point after each transformations.

The figure below is an example of fractal. We generated this Sierpinski Carpet in MATLAB.



$dH(X) \approx 1.892789$

$$\mathrm{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

Scalar factor

0° of rotation

8 sets of (e , f) are(0,0), $\left(\frac{1}{3}, 0\right)$, $\left(\frac{2}{3}, 0\right)$, $\left(0, \frac{1}{3}\right)$, $\left(\frac{2}{3}, \frac{1}{3}\right)$, $\left(0, \frac{2}{3}\right)$ $\left(\frac{1}{3}, \frac{2}{3}\right)$, $\left(\frac{2}{3}, \frac{2}{3}\right)$.

For this fractal, K=8, S=$\frac{1}{3}$, then

$$dH(X) = \frac{\ln(8)}{\ln(3)} \approx 1.892789$$

In MATLAB, the similitudes would be written as:

```
xn = 1/3*x+1/3;  % an example as (e,f) =(1/3,0)
yn = 1/3*y;
 x = xn;
 y = yn;
```

A part of the MATLAB code used:

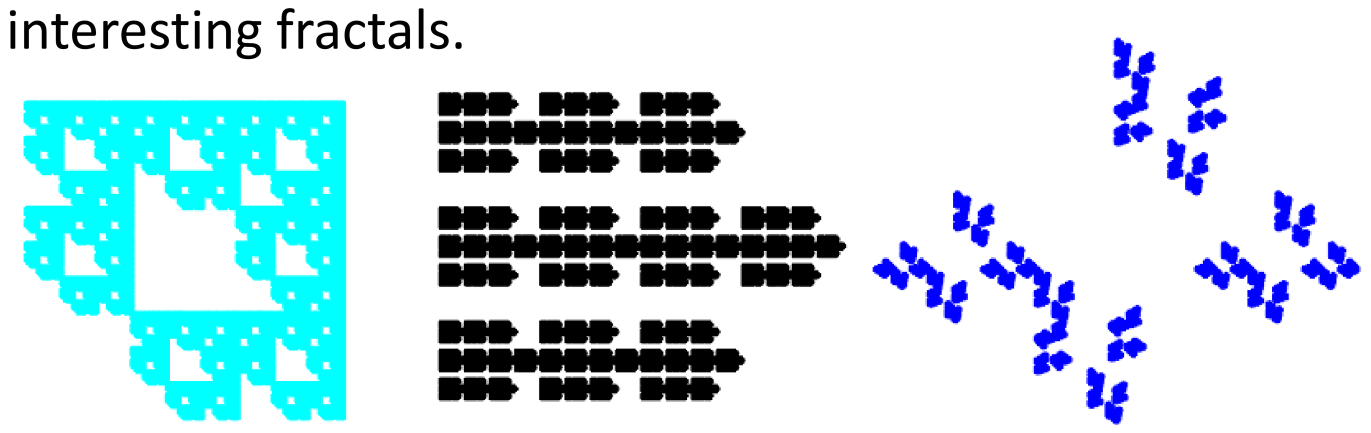
```
NumOfPts = 1000000;  % Numbers of points in figure
iterations = 5;          % Run the algorithm 5 times before a point is defined
pts = zeros(NumOfPts,2);    % pts is a 1000000x2 matrix with only 0s
for j = 1:NumOfPts           % j=1,2,3...1000000

   x = rand(1);       % x is a random number in (0-1)
   y = rand(1);       % y is a random number in (0-1)

   for i = 1:iterations   % i = 1,2,3,4,5
      p = rand(1);        % p is a random number in (0-1)

      if p < 1/8          % One out of every eight iterations this
                          % transformation is applied
        xn = 1/3*x;       % The x, y values used in iteration i=2
        yn = 1/3*y;       % come from  the x, y values generated in iteration
         x = xn;          % i=1, and this process repeats
         y = yn;
```

By using similar techniques in MATLAB, we generated other interesting fractals.



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