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Possible World Semantics for First Order LP*

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Abstract

In [1] an elegant formulation of the first-order logic of proofs was given, FOLP. That report also proved an arithmetic completeness theorem, and a realization theorem for the logic. In this report we provide a possible-world semantics for FOLP, based on the propositional semantics of [2]. Motivation and intuition for the logic itself can be found in [1], and are not discussed here. We also give an Mkrtychev semantics for FOLP. This report was essentially completed June 10, 2011.

1 The Language and the Axioms

This is directly from a technical report of Artemov and Yavorskaya, [1]. The language definition is not repeated here, but the axiomatization is, for reference purposes.

A1 classical axioms of first order logic

A2 $t{:}_{Xy}A \rightarrow t{:}_X A, \quad y \notin FVar(A)$

A3 $t{:}_X \rightarrow t{:}_{Xy}A$

B1 $t{:}_X A \rightarrow A$

B2 $s{:}_X(A \rightarrow B) \rightarrow (t{:}_X \rightarrow (s \cdot t){:}_X A)$

B3 $t{:}_X A \rightarrow (t+s){:}_X A, \quad s{:}_X A \rightarrow (t+s){:}_X A$

B4 $t{:}_X A \rightarrow !t{:}_X t{:}_X A$

B5 $t{:}_X A \rightarrow \mathsf{gen}_x(t){:}_X \forall x A, \;\; x \notin X$

R1 $\vdash A, A \rightarrow B \;\Rightarrow\; \vdash B$

R2 $\vdash A \;\Rightarrow\; \vdash \forall x A$

R3 $\vdash c{:}A$, where A is an axiom, c is a proof constant

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2 Models

A possible world model for FOLP is a structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E} \rangle$, where the various items are as follows.

$\mathcal{G}$ A non-empty set, of *possible worlds* or *states*.

$\mathcal{R}$ An *accessibility relation* on $\mathcal{G}$—a binary relation that is reflexive and transitive.

$\mathcal{D}$ A *domain function* on $\mathcal{G}$—mapping $\mathcal{G}$ to non-empty domains, subject to the *monotonicity condition* that $\Gamma, \Delta \in \mathcal{G}$ and $\Gamma \mathcal{R} \Delta$ imply $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Delta)$.

1. We call $\cup\{\mathcal{D}(\Gamma) \mid \Gamma \in \mathcal{G}\}$ the *domain of the model* $\mathcal{M}$.
2. A *valuation* in $\mathcal{M}$ is a mapping v from individual variables to the domain of the model $\mathcal{M}$.
3. A valuation *lives in possible world* Γ provided, for each individual variable x, $v(x) \in \mathcal{D}(\Gamma)$.

$\mathcal{I}$ An *interpretation function*—for each n-place relation symbol R and each $\Gamma \in \mathcal{G}$, $\mathcal{I}(R, \Gamma)$ is an n-place relation on the domain of the model $\mathcal{M}$.

$\mathcal{E}$ An *evidence function*—for each justification term t, each formula A, each finite set X of individual variables, and each valuation v, $\mathcal{E}(t, A, X, v)$ is a set of possible worlds in which v lives.

The idea behind the evidence function is this. If $\Gamma \in \mathcal{E}(t, A, X, v)$, then informally Γ is a possible world in which t serves as relevant evidence for the formula A, where the members of X are treated as parameters whose values are supplied by v.

Special conditions are imposed on evidence functions. Most come from LP; three are new to FOLP.

$\cdot$ **Condition** $\mathcal{E}(s, A \to B, X, v) \cap \mathcal{E}(t, A, X, v) \subseteq \mathcal{E}((s \cdot t), B, X, v)$.

$+$ **Condition** $\mathcal{E}(s, A, X, v) \cup \mathcal{E}(t, A, X, v) \subseteq \mathcal{E}((s + t), A, X, v)$.

$\mathcal{R}$ **Closure Condition** $\Gamma \mathcal{R} \Delta$ and $\Gamma \in \mathcal{E}(t, A, X, v)$ imply $\Delta \in \mathcal{E}(t, A, X, v)$.

! **Condition** $\mathcal{E}(t, A, X, v) \subseteq \mathcal{E}(!t, t{:}_X A, X, v)$.

Contraction Condition $\mathcal{E}(t, A, Xy, v) \subseteq \mathcal{E}(t, A, X, v)$ provided $y \notin FVar(A)$.

Expansion Condition $\mathcal{E}(t, A, X, v) \subseteq \mathcal{E}(t, A, Xy, v)$

$\mathbf{gen}_x$ **Condition** $\mathcal{E}(t, A, X, v) \subseteq \mathcal{E}(\mathsf{gen}_x(t), \forall x A, X, v)$ provided $x \notin X$.

Truth of formulas at worlds of model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{A}, \mathcal{I} \rangle$, with respect to a valuation v, denoted $\mathcal{M}, \Gamma \Vdash_v X$, is evaluated as follows.

1. $\mathcal{M}, \Gamma \Vdash_v R(x_1, \ldots, x_n) \iff \langle v(x_1), \ldots, v(x_n) \rangle \in \mathcal{I}(\Gamma, R)$ for R an n-place relation symbol;
2. $\mathcal{M}, \Gamma \nVdash_v \bot$;
3. $\mathcal{M}, \Gamma \Vdash_v A \to B \iff \mathcal{M}, \Gamma \nVdash_v A$ or $\mathcal{M}, \Gamma \Vdash_v B$, and similarly for other propositional connectives;

4. $\mathcal{M}, \Gamma \Vdash_v \forall x A \iff \mathcal{M}, \Gamma \Vdash_w A$ for every x-variant w of v such that $w(x) \in \mathcal{D}(\Gamma)$.

5. $\mathcal{M}, \Gamma \Vdash_v t{:}_X A \iff$

   (a) $\Gamma \in \mathcal{E}(t, A, X, v)$ and

   (b) $\mathcal{M}, \Delta \Vdash_w A$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, and for every valuation w that lives in Δ and agrees with v on the variables in X.

We say A is *valid* in the model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{A}, \mathcal{I} \rangle$ if, for every $\Gamma \in \mathcal{G}$, and for every valuation v that lives in Γ, $\mathcal{M}, \Gamma \Vdash_v A$.

A model *meets constant specification* $\mathcal{C}$ provided, if $c{:}_\emptyset A \in \mathcal{C}$ then for each $\Gamma \in \mathcal{G}$ and for each valuation v that lives in Γ, $\Gamma \in \mathcal{E}(c, A, \emptyset, v)$.

3 Soundness

Each of the FOLP axioms is valid in all models, and the rules preserve validity, hence each theorem is valid. We show this for a few of the axioms, and omit details for the rest. In what follows, assume $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{A}, \mathcal{I} \rangle$ is an FOLP model.

A2 Validity Suppose $\Gamma \in \mathcal{G}$ and v is a valuation that lives in Γ. Suppose also that $\mathcal{M}, \Gamma \Vdash_v t{:}_{Xy} A$, where $y \notin FVar(A)$. We show $\mathcal{M}, \Gamma \Vdash_v t{:}_X A$.

Since $\mathcal{M}, \Gamma \Vdash_v t{:}_{Xy} A$ then $\Gamma \in \mathcal{E}(t, A, Xy, v)$, and so $\Gamma \in \mathcal{E}(t, A, X, v)$ by the Contraction Condition.

Let $\Delta \in \mathcal{G}$ be such that $\Gamma \mathcal{R} \Delta$, and let w be a valuation that lives in Δ and agrees with v on the variables in X; we show $\mathcal{M}, \Delta \Vdash_w A$, which is enough to conclude this case. Let w' be like w except that $w'(y) = v(y)$. Then v and w' agree on the variables in Xy. Also since the domain function is monotonic, it follows that w' lives in Δ. Since $\mathcal{M}, \Gamma \Vdash_v t{:}_{Xy} A$, then $\mathcal{M}, \Delta \Vdash_{w'} A$. Finally, since w and w' differ only on y and this does not occur free in A, $\mathcal{M}, \Delta \Vdash_w A$.

A3 Validity Suppose $\Gamma \in \mathcal{G}$, v is a valuation that lives in Γ, and $\mathcal{M}, \Gamma \Vdash_v t{:}_X A$. We show $\mathcal{M}, \Gamma \Vdash_v t{:}_{Xy} A$.

Since $\mathcal{M}, \Gamma \Vdash_v t{:}_X A$ then $\Gamma \in \mathcal{E}(t, A, X, v)$, and so $\Gamma \in \mathcal{E}(t, A, Xy, v)$ by the Expansion Condition.

Let $\Delta \in \mathcal{G}$ be such that $\Gamma \mathcal{R} \Delta$, and let w be a valuation that lives in Δ and agrees with v on the variables in Xy; we must show $\mathcal{M}, \Delta \Vdash_w A$. But since v and w agree on variables in Xy, in particular they agree on variables in X. Then since $\mathcal{M}, \Gamma \Vdash_v t{:}_X A$, it follows by definition that $\mathcal{M}, \Delta \Vdash_w A$.

B5 Validity Suppose $\Gamma \in \mathcal{G}$, v is a valuation that lives in Γ, $x \notin X$, and $\mathcal{M}, \Gamma \Vdash_v t{:}_X A$. We show $\mathcal{M}, \Gamma \Vdash_v \mathsf{gen}_x(t){:}_X \forall x A$.

Since $\mathcal{M}, \Gamma \Vdash_v t{:}_X A$ then $\Gamma \in \mathcal{E}(t, A, X, v)$, so $\Gamma \in \mathcal{E}(\mathsf{gen}_x(t), \forall x A, X, v)$ by the gen_x Condition.

Let $\Delta \in \mathcal{G}$ be such that $\Gamma \mathcal{R} \Delta$, and let w be a valuation that lives in Δ and agrees with v on the variables in X; we show $\mathcal{M}, \Delta \Vdash_w \forall x A$. For this, let w' be an arbitrary x-variant of w, with $w'(x) \in \mathcal{D}(\Delta)$; we will show that $\mathcal{M}, \Delta \Vdash_{w'} A$. But v and w' agree on the variables of X, since $x \notin X$. Also w' lives in Δ. Then since $\mathcal{M}, \Gamma \Vdash_v t{:}_X A$, it follows that $\mathcal{M}, \Delta \Vdash_{w'} A$, which is what was wanted.

The other axioms are valid, and the rules preserve validity—results left to the reader. It follows that the axiom system is sound with respect to the semantics. This allows for taking constant specifications into account.

4 Constant Specifications

We will be constructing a 'canonical' model, and showing it establishes a basic completeness result. Subsequently we discuss what else it does. While soundness holds with respect to any constant specification, completeness requires something more. The first is familiar from propositional LP.

Definition 4.1 A constant specification $\mathcal{C}$ is *axiomatically appropriate* if, for every axiom A there is a proof constant c such that $c{:}_{\emptyset}A \in \mathcal{C}$.

An *internalization theorem* can be proved for FOLP provided an axiomatically appropriate constant specification is assumed—see [1].

The completeness proof will use a Henkin construction, so the original language will need to be extended by the addition of 'witnesses'. Presumably a constant specification is for the original language, so we will need a way of extending it usefully to the larger language. The following requirement makes it straightforward to do that—other ways may be possible, but this will do for now.

Definition 4.2 Two formulas, A and B of FOLP are *variable variants* of each other if the following conditions are met:

1. the free variables of A are $x_1, \ldots, x_n$, and the free variables of B are $y_1, \ldots, y_n$,
2. the substitution of y_i for x_i is free in A, for all $i = 1, \ldots, n$, and turns A into B,
3. the substitution of x_i for y_i is free in B, for all $i = 1, \ldots, n$, and turns B into A.

Informally, A and B are variable variants if they are identical except for the choice of free variables. In other words, $A = \varphi(x_1, \ldots, x_n)$, and $B = \varphi(y_1, \ldots, y_n)$ for some φ.

Definition 4.3 A constant specification $\mathcal{C}$ is *variant closed* provided that whenever A and B are variable variants, $c{:}_{\emptyset}A \in \mathcal{C}$ if and only if $c{:}_{\emptyset}B \in \mathcal{C}$.

Our completeness proof will assume constant specifications are axiomatically appropriate and variant closed. Note that a schematic constant specification is trivially variant closed.

5 Language Extensions

Throughout the rest of this note L is the language of FOLP, with a fixed set of individual variables. It is for formulas of L that completeness will be proved, but we need to extend the language, Henkin-style, for purposes of proving completeness. Let $\mathcal{P}$ be some set of individual variables not used in L—following one tradition we call these variables *parameters*. By $L(\mathcal{P})$ we mean the language formulated like L, but also allowing variables from $\mathcal{P}$, parameters, but with the proviso that *parameters are never quantified.* Parameters are intended to serve as witnesses for existential statements. The condition that parameters are never bound means that if p is a parameter, $(\forall p)$ and gen_p may not appear in formulas of $L(\mathcal{P})$. In particular, the rule of universal generalization can only be applied to a formula $\varphi(x)$ if x is an individual variable that is not a parameter. This gives rise to some minor complications, which are addressed later in the section. The reason for requiring that parameters are never bound is simply that it allows us to avoid problems when substituting parameters for free variable occurrences—such substitutions are automatically free.

Axiom schemes of FOLP are still as specified in Section 1, but the language $L(\mathcal{P})$ may be used instead of L. We will say which we are using, as appropriate. Next we discuss extending constant specifications from L to $L(\mathcal{P})$.

Definition 5.1 Let $\mathcal{C}$ be a variant closed constant specification for the language L. We define a constant specification $\mathcal{C}(\mathcal{P})$ for $L(\mathcal{P})$ as follows. Let $\varphi(p_1, \ldots, p_n)$ be a formula of $L(\mathcal{P})$ with p_1, $\ldots$, p_n as parameters. Let x_1, $\ldots$, x_n be individual variables that are not parameters, and that do not occur in $\varphi(p_1, \ldots, p_n)$, free or bound. If $c{:}_{\emptyset}\varphi(x_1, \ldots, x_n) \in \mathcal{C}$, put $c{:}_{\emptyset}\varphi(p_1, \ldots, p_n) \in \mathcal{C}(\mathcal{P})$.

Some observations. First, since $\mathcal{C}$ is variant closed, the actual choice of free variables x_1, $\ldots$, x_n doesn't matter. Second, if φ has no parameter occurrences then substitution for parameters doesn't change anything, and it follows that $\mathcal{C}(\mathcal{P})$ extends $\mathcal{C}$. It is easy to see that it does so conservatively. Third, it is easy to check that $\mathcal{C}(\mathcal{P})$ is also variant closed. And finally, since FOLP is axiomatized using axiom schemes, it is easy to see that if $\mathcal{C}$ is axiomatically appropriate with respect to L, then $\mathcal{C}(\mathcal{P})$ will be axiomatically appropriate with respect to $L(\mathcal{P})$. It follows that the Internalization Theorem still applies, and with the same proof. We omit the verification. Note that since parameters are not quantified, if universal generalization is applied to $\psi(x)$ in the course of a proof of φ, a non-parameter x must be involved, and so gen_x is available to be used in a justification term.

Finally we consider the important issue of universal generalization and parameters. If $\varphi(x)$ is provable for an individual variable x that is not a parameter, the universal generalization rule tells us we can conclude $(\forall x)\varphi(x)$. But if $\varphi(p)$ is provable where p is a parameter, we cannot conclude that $(\forall p)\varphi(p)$ is provable, since parameters cannot occur bound. Proposition 5.3 provides a substitute for this.

Proposition 5.2 *Assume the following.*

1. *$\mathcal{C}$ is a constant specification that is variant closed,*
2. *$L(\mathcal{P})$ extends L with countably many parameters, and $\mathcal{C}(\mathcal{P})$ is the extension of $\mathcal{C}$ to $L(\mathcal{P})$,*
3. *Z_1, Z_2, $\ldots$, Z_n is an FOLP proof in the language $L(\mathcal{P})$, using constant specification $\mathcal{C}(\mathcal{P})$,*
4. *p is a parameter, and x is an individual variable that is not a parameter, but that does not occur in Z_1, Z_2, $\ldots$, Z_n free or bound,*
5. *Z_1', Z_2', $\ldots$, Z_n' is the sequence that results from Z_1, Z_2, $\ldots$, Z_n by replacing all occurrences of p with occurrences of x.*

Then Z_1', Z_2', $\ldots$, Z_n' is also an FOLP proof.

Proof The proof is by induction on proof length.

If Z is an axiom of FOLP, so is Z', since FOLP axiomatization is by schemes.

If Z follows from X and $X \supset Z$ by *modus ponens*, then Z' similarly follows from X' and $(X \supset Z)' = (X' \supset Z')$.

If $(\forall z)\varphi(z)$ follows from $\varphi(z)$ using universal generalization, since z occurs bound in the proof then z is not the parameter p. Then $[(\forall z)\varphi(z)]' = (\forall z)[\varphi(z)']$ follows from $\varphi(z)'$, also by universal generalization.

Finally, suppose $c{:}_{\emptyset}X$ is by Axiom Necessitation, and so $c{:}_{\emptyset}X \in \mathcal{C}(\mathcal{P})$. Then $[c{:}_{\emptyset}X]' = c{:}_{\emptyset}[X']$ also follows by Axiom Necessitation, using the fact that $\mathcal{C}(\mathcal{P})$ is variant closed. ∎

Now we can prove that we have the effect of universal generalization applied to parameters, even though parameters themselves can not be quantified.

Proposition 5.3 *Let $\mathcal{C}$ be a variant closed constant specification for the language L, and let $\mathcal{C}(\mathcal{P})$ be its extension to $L(\mathcal{P})$. Let $\varphi(p)$ be a formula of $L(\mathcal{P})$, where p is a parameter. If $\varphi(p)$ is provable using constant specification $\mathcal{C}(\mathcal{P})$, so is $(\forall x)\varphi(x)$, where x is any individual variable of L that is free for p in $\varphi(p)$.*

Proof Let z be an individual variable that is not a parameter, and that does not occur in the proof of $\varphi(p)$. Using Proposition 5.2, $\varphi(z)$ is also provable. Then $(\forall z)\varphi(z)$ is provable using universal generalization. Further, $(\forall z)\varphi(z) \supset (\forall x)\varphi(x)$ is classically valid, hence $(\forall x)\varphi(x)$ is provable. (The need for this last step is because the variable x might have occured in the proof of $\varphi(p)$.) ∎

6 Completeness

For this section $\mathcal{P}$ is a countable set of parameters, not occurring as individual variables in the language L. The usual Henkin/Lindenbaum construction can be applied, and one can prove the following (details omitted, though Proposition 5.3 comes in). Let $W_1 \subseteq \mathcal{P}$ be a set of parameters, possibly empty, and let F be a consistent set of formulas in the language $L(W_1)$. Assume $W_2 \subseteq \mathcal{P}$ extends W_1 with countably many additional parameters. Then F extends to a set F' that is maximally consistent and E-complete in the language $L(W_2)$, with parameters as witnesses. (that is, every existential or negated universal quantifier has a parameter witness.)

The following model construction assumes a constant specification $\mathcal{C}$ for L that is variant closed and axiomatically appropriate. Parameters play a fundamental role in creating the model, but finally it is only formulas from the original language L that we are concerned with—formulas without parameters. We now specify the *canonical model* $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E} \rangle$.

Special Notation Let $W \subseteq \mathcal{P}$ be a set of parameters.

1. $L(W)$ is the set of formulas whose individual variables are those of L and also members of W, with the proviso that parameters from W are not bound (essentially, this is as before).

2. $L^*(W)$ is the subset of $L(W)$ consisting of those formulas all of whose free variables are parameters.

3. Let S be a set of formulas of $L^*(W)$. Then $S^\sharp$ is the set of formulas A such that $t{:}_X A \in S$, for some justification term t and some set X of parameters.

Definition of $\mathcal{G}$ Call $\Gamma = \langle \mathsf{form}(\Gamma), \mathsf{parm}(\Gamma) \rangle$ an *HL-world* (standing for Henkin/Lindenbaum) if:

1. $\mathsf{parm}(\Gamma)$ is a countable set of parameters (that is, a countable subset of $\mathcal{P}$) that also omits countably many parameters.
2. $\mathsf{form}(\Gamma)$ is a set of formulas in the language $L^*(\mathsf{parm}(\Gamma))$.

3. $\mathsf{form}(\Gamma)$ is consistent, maximally so among sets of formulas in the language $L^*(\mathsf{parm}(\Gamma))$, and E-complete. Note that the definition of consistency brings constant specification $\mathcal{C}$ into things.

We let $\mathcal{G}$ be the collection of all *HL*-worlds.

Definition of $\mathcal{R}$ For $\Gamma = \langle \mathsf{form}(\Gamma), \mathsf{parm}(\Gamma)\rangle$ and $\Delta = \langle \mathsf{form}(\Delta), \mathsf{parm}(\Delta)\rangle$, set $\Gamma\mathcal{R}\Delta$ provided:

1. $\mathsf{parm}(\Gamma) \subseteq \mathsf{parm}(\Delta)$
2. $(\mathsf{form}(\Gamma))^\sharp \subseteq \mathsf{form}(\Delta)$

Definition of $\mathcal{D}$ For $\Gamma \in \mathcal{G}$ where $\Gamma = \langle \mathsf{form}(\Gamma), \mathsf{parm}(\Gamma)\rangle$, set $\mathcal{D}(\Gamma) = \mathsf{parm}(\Gamma)$.

Definition of $\mathcal{I}$ For an n-place relation symbol R and for $\Gamma \in \mathcal{G}$, let $\mathcal{I}(R, \Gamma)$ be the set of all $\langle p_1, \ldots, p_n\rangle$ where each p_i is a parameter, and $R(p_1, \ldots, p_n) \in \mathsf{form}(\Gamma)$.

Valuations A valuation v maps individual variables *of the language* L to the domain of the model (which happens to be the set of parameters). Then a valuation v induces a substitution, which we call σ_v, on formulas of L: replace each free occurrence of individual variable x with parameter $v(x)$. If A is a formula of L, $A\sigma_v$ is a formula whose free variables are all parameters. Note that since formulas of L cannot have quantifiers that bind parameters, σ_v is automatically a free substitution on formulas in L.

Definition of $\mathcal{E}$ Let A be a formula of L and let X be a set of variables of L. Set $\Gamma \in \mathcal{E}(t, A, X, v)$ provided v lives in Γ and $[t{:}_X A]\sigma_v \in \mathsf{form}(\Gamma)$.

A remark about the substitution σ_v associated with a valuation v. Suppose A is a formula of L that is provable in FOLP, and v is a valuation in the canonical model just defined. Then $A\sigma_v$ is a formula in the language $L(\mathcal{P})$, and it is also provable in FOLP. There is more than one argument for this—here is a simple one. Say $A = \varphi(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are the free variables of A, none of which are parameters. Since $\varphi(x_1, \ldots, x_n)$ is provable, by universal generalization so is $(\forall x_1)\cdots(\forall x_n)\varphi(x_1, \ldots, x_n)$. The formula $(\forall x_1)\cdots(\forall x_n)\varphi(x_1, \ldots, x_n) \supset \varphi(x_1\sigma_v, \ldots, x_n\sigma_v)$ of $L(\mathcal{P})$ is provable, hence so is $\varphi(x_1\sigma_v, \ldots, x_n\sigma_v) = A\sigma_v$.

We have finished the definition of the canonical model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}\rangle$. It must be checked that it is a model with respect to the language L. Most of this is straightforward—we consider only the $\mathcal{R}$ Closure Condition as an example. Suppose $\Gamma, \Delta \in \mathcal{G}$ and $\Gamma\mathcal{R}\Delta$. Suppose also that $\Gamma \in \mathcal{E}(t, A, X, v)$, where A is a formula of L and X is a set of variables of L. We show $\Delta \in \mathcal{E}(t, A, X, v)$.

By definition of $\mathcal{E}$, v lives in Γ and $[t{:}_X A]\sigma_v \in \mathsf{form}(\Gamma)$. Axiom **B4** tells us $t{:}_X A \supset !t{:}_X t{:}_X A$ is an axiom, and so by the remarks above, $[t{:}_X A \supset !t{:}_X t{:}_X A]\sigma_v$ is provable, that is, $[t{:}_X A]\sigma_v \supset [!t{:}_X t{:}_X A]\sigma_v$. It follows that $[!t{:}_X t{:}_X A]\sigma_v \in \mathsf{form}(\Gamma)$, or equivalently, $!t{:}_{X\sigma_v}[t{:}_X A]\sigma_v \in \mathsf{form}(\Gamma)$. Then by definition of $\mathcal{R}$, $[t{:}_X A]\sigma_v \in \mathsf{form}(\Delta)$. Also v lives in Δ since v lives in Γ and $\mathsf{parm}(\Gamma) \subseteq \mathsf{parm}(\Delta)$. But then $\Delta \in \mathcal{E}(t, A, X, v)$.

Now the expected main result.

Theorem 6.1 (Truth Lemma) *For each formula A in the language L, for each $\Gamma \in \mathcal{G}$, and for each valuation v that lives in Γ,*

$$\mathcal{M}, \Gamma \Vdash_v A \Longleftrightarrow A\sigma_v \in \mathit{form}(\Gamma).$$

Proof The proof is by induction on formula degree. Much of this is familiar, so we only give two cases. In both, we assume v is a valuation that lives in Γ.

Justification Formulas Assume that $t{:}_X A$ is a formula in the language L and the Truth Lemma is known for A. One direction is very simple, just as it is propositionally.

Suppose $[t{:}_X A]\sigma_v \notin \mathsf{form}(\Gamma)$. Then $\Gamma \notin \mathcal{E}(t, A, X, v)$ by definition of $\mathcal{E}$, and so $\mathcal{M}, \Gamma \not\Vdash_v t{:}_X A$.

Next suppose $[t{:}_X A]\sigma_v \in \mathsf{form}(\Gamma)$. By definition of $\mathcal{E}$, $\Gamma \in \mathcal{E}(t, A, X, v)$. Further, let $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, and let w be a valuation that lives in Δ and agrees with v on members of X; we show $\mathcal{M}, \Delta \Vdash_w A$. A is a formula of L so its free variables are not parameters, and hence can be bound. Say the free variables of A that are not in X are $\vec{y}$. By repeated use of axiom **B5** we can produce a justification term u so that $t{:}_X A \rightarrow u{:}_X \forall\vec{y}A$ is provable. Since $[t{:}_X A]\sigma_v \in \mathsf{form}(\Gamma)$ then $[u{:}_X \forall\vec{y}A]\sigma_v \in \mathsf{form}(\Gamma)$. That is, $u{:}_{X\sigma_v}[\forall\vec{y}A]\sigma_v \in \mathsf{form}(\Gamma)$. Since $(\mathsf{form}(\Gamma))^\sharp \subseteq \mathsf{form}(\Delta)$ then $[\forall\vec{y}A]\sigma_v \in \mathsf{form}(\Delta)$. Now repeated use of the universal instantiation axiom establishes that $A\sigma_w \in \mathsf{form}(\Delta)$, so by the induction hypothesis, $\mathcal{M}, \Delta \Vdash_w A$. We now have enough to conclude $\mathcal{M}, \Gamma \Vdash_v t{:}_X A$.

Quantified Formulas Assume that $(\forall x)A(x)$ is a formula in the language L and the Truth Lemma is known for $A(x)$.

Suppose first that $[(\forall x)A(x)]\sigma_v \in \mathsf{form}(\Gamma)$. Let w be an arbitrary x-variant of v such that $w(x) \in \mathcal{D}(\Gamma)$, say $w(x) = p$ where $p \in \mathsf{parm}(\Gamma)$. Now $(\forall x)A(x) \supset A(p)$ is a provable formula of $L(\mathcal{P})$, so $[(\forall x)A(x) \supset A(p)]\sigma_v$ is also provable, but this is $[(\forall x)A(x)]\sigma_v \supset [A(x)]\sigma_w$, hence $[A(x)]\sigma_w \in \mathsf{form}(\Gamma)$. By the induction hypothesis, $\mathcal{M}, \Gamma \Vdash_w A(x)$. Since w was arbitrary, $\mathcal{M}, \Gamma \Vdash_v (\forall x)A(x)$.

Finally, suppose that $[(\forall x)A(x)]\sigma_v \notin \mathsf{form}(\Gamma)$. Since $\mathsf{form}(\Gamma)$ is E-complete, for some parameter p in $\mathsf{parm}(\Gamma)$, $[A(p)]\sigma_v \notin \mathsf{form}(\Gamma)$. Let w be the x-variant of v where $w(x) = p$. Then $[A(x)]\sigma_w \notin \mathsf{form}(\Gamma)$. By the induction hypothesis, $\mathcal{M}, \Gamma \not\Vdash_w A(x)$, and it follows that $\mathcal{M}, \Gamma \not\Vdash_v (\forall x)A(x)$.

■

Completeness now follows in the usual way. Since every formula is equi-provable with its universal closure, and also equi-valid with it, it is enough to prove completeness for closed formulas, so suppose closed formula A of L is not provable. Then $\{\neg A\}$ is consistent. Let $V \subseteq \mathcal{P}$ contain countably many parameters, while also omitting countably many parameters. Extend $\{\neg A\}$ to a set M that is maximally consistent and E-complete with respect to the language $L^*(V)$, with parameters serving as witnesses. Let $\Gamma = \langle M, V\rangle$. This is a possible world in the canonical model, and A will be false at it under any valuation that lives in Γ; the choice of valuation doesn't matter since A has no free variables.

7 Mkrtychev Models

For propositional LP, Mkrtychev models were the original non-arithmetic semantics. They carry over to FOLP in a direct way.

Definition 7.1 An *Mkrtychev FOLP model* is a structure, $\mathcal{M} = \langle\langle\mathcal{D}, \mathcal{I}\rangle, \mathcal{E}\rangle$ where $\langle\mathcal{D}, \mathcal{I}\rangle$ is a classical first-order model, and $\mathcal{E}$ is an evidence function. In detail, these are as follows.

$\mathcal{D}$ A non-empty set, the *domain* of the model.

$\mathcal{I}$ An *interpretation*, assigning to each n-place relation symbol of language L some n-ary relation on $\mathcal{D}$.

$\mathcal{E}$ An *evidence function*, mapping justification term t, formula A of L, finite set X of individual variables, and valuation v in $\mathcal{D}$, to a boolean truth value, $\mathcal{E}(t, A, X, v)$. An evidence function must meet the following conditions.

$\cdot$ **Condition** $\mathcal{E}(s, A \to B, X, v) \wedge \mathcal{E}(t, A, X, v) \to \mathcal{E}((s \cdot t), B, X, v)$.

$+$ **Condition** $\mathcal{E}(s, A, X, v) \vee \mathcal{E}(t, A, X, v) \to \mathcal{E}((s + t), A, X, v)$.

! **Condition** $\mathcal{E}(t, A, X, v) \to \mathcal{E}(!t, t{:}_X A, X, v)$.

Contraction Condition $\mathcal{E}(t, A, Xy, v) \to \mathcal{E}(t, A, X, v)$ provided $y \notin FVar(A)$.

Expansion Condition $\mathcal{E}(t, A, X, v) \to \mathcal{E}(t, A, Xy, v)$

gen_x **Condition** $\mathcal{E}(t, A, X, v) \to \mathcal{E}(\mathsf{gen}_x(t), \forall x A, X, v)$ provided $x \notin X$.

We write $\mathcal{M} \Vdash_v A$ to symbolize that formula A of language L is true in Mkrtychev model $\mathcal{M}$ under valuation v in $\mathcal{D}$. The truth conditions are as follows.

Atomic For an n place relation symbol R, $\mathcal{M} \Vdash_v R(x_1, \ldots, x_n) \Longleftrightarrow \langle v(x_1), \ldots, v(x_n)\rangle \in R^{\mathcal{I}}$.

Propositional $\mathcal{M} \Vdash_v (X \to Y) \Longleftrightarrow \mathcal{M} \not\Vdash_v X$ or $\mathcal{M} \Vdash_v Y$, and similarly for other connectives.

Quantifier $\mathcal{M} \Vdash_v \forall x A \Longleftrightarrow \mathcal{M} \Vdash_w A$ for every valuation w in $\mathcal{D}$ that is an x variant of v.

Justification Term $\mathcal{M} \Vdash_v t{:}_X A \Longleftrightarrow \mathcal{E}(t, A, X, v)$ and $\mathcal{M} \Vdash_w A$ for every valuation w in $\mathcal{D}$ that agrees with v on the members of X.

It is straightforward to check that Mkrtychev models are, essentially, one-world FOLP models, and so we have soundness with respect to them. It is also straightforward to check that each possible world in the canonical model is an Mkrtychev model, and completeness with respect to such models follows.

8 Fully Explanatory Models

Definition 8.1 A model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{A}, \mathcal{I}\rangle$ is *fully explanatory* if it meets the following condition. For each $\Gamma \in \mathcal{G}$, for each valuation v that lives in Γ, and for each formula A of language L with X being the set of its free variables, if $\mathcal{M}, \Delta \Vdash_v A$ for every $\Delta \in \mathcal{G}$ such that $\Gamma\mathcal{R}\Delta$, then $\mathcal{M}, \Gamma \Vdash_v t{:}_X A$ for some justification term t.

Theorem 8.2 *The canonical model is fully explanatory.*

Proof Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{A}, \mathcal{I}\rangle$ be the canonical model. Let $\Gamma \in \mathcal{G}$, and let v be a valuation that lives in Γ. Suppose A is a formula in the language L, with X as the set of its free variables, and $\mathcal{M}, \Gamma \not\Vdash_v t{:}_X A$ for every justification term t. We show that for some $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, $\mathcal{M}, \Delta \not\Vdash_v A$.

To keep notation simple, let $A' = A\sigma_v$ and let X' be $X\sigma_v$. Then for each t, $[t{:}_X A]\sigma_v = t{:}_{X\sigma_v}(A\sigma_v) = t{:}_{X'}A'$. By our assumption and the Truth Lemma, $[t{:}_X A]\sigma_v \notin \mathsf{form}(\Gamma)$ for every justification term t, that is, $t{:}_{X'}A' \notin \mathsf{form}(\Gamma)$. Using this we first show $(\mathsf{form}(\Gamma))^\sharp \cup \{\neg A'\}$ is consistent.

Well, suppose not, that is, suppose $(\mathsf{form}(\Gamma))^\sharp \cup \{\neg A'\}$ is inconsistent. We derive a contradiction. Assuming the inconsistency, there are $B_1, \ldots, B_n \in (\mathsf{form}(\Gamma))^\sharp$ so that $\vdash B_1 \to B_2 \to \ldots \to B_n \to$

A', where association is to the right. By Internalization, $\vdash w{:}_\emptyset(B_1 \to B_2 \to \ldots \to B_n \to A')$, for some justification term w. Also for each i, since $B_i \in (\mathsf{form}(\Gamma))^\sharp$ then $u_i{:}_{X_i}B_i \in \Gamma$ for some u_i, and X_i.

Let $Z = X_1 \cup X_2 \cup \ldots \cup X_n \cup X'$. Using axiom **A3** repeatedly we get $\vdash w{:}_Z(B_1 \to B_2 \to \ldots \to B_n \to A')$. Also, repeated use of **A3** gives us $\vdash u_i{:}_{X_i}B_i \to u_i{:}_Z B_i$, for each i, and since $\mathsf{form}(\Gamma)$ is maximally consistent, $u_i{:}_Z B_i \in \mathsf{form}(\Gamma)$ for each i.

Now, using axiom **B2**, each of the following is provable (association is to the left in the justification terms).

$$\begin{aligned} & w{:}_Z(B_1 \to B_2 \to \ldots \to B_n \to A') \\ u_1{:}_Z B_1 \to & (w \cdot u_1){:}_Z(B_2 \to \ldots \to B_n \to A') \\ u_1{:}_Z B_1 \to u_2{:}_Z B_2 \to & (w \cdot u_1 \cdot u_2){:}_Z(B_3 \to \ldots \to B_n \to A') \\ & \vdots \\ u_1{:}_Z B_1 \to u_2{:}_Z B_2 \to \ldots \to u_n{:}_Z B_n \to & (w \cdot u_1 \cdot u_2 \cdot \ldots \cdot u_n){:}_Z A' \end{aligned}$$

For each i, $u_i{:}_Z B_i \in \mathsf{form}(\Gamma)$. It follows that $(w \cdot u_1 \cdot u_2 \cdot \ldots \cdot u_n){:}_Z A' \in \mathsf{form}(\Gamma)$. Then repeated use of axiom **A2** gives us that $(w \cdot u_1 \cdot u_2 \cdot \ldots \cdot u_n){:}_{X'}A' \in \mathsf{form}(\Gamma)$, contradicting the fact that $t{:}_{X'}A' \notin \mathsf{form}(\Gamma)$ for every justification term t.

We have now shown that $(\mathsf{form}(\Gamma))^\sharp \cup \{\neg A'\}$ is consistent. Let $\mathsf{parm}(\Delta)$ extend $\mathsf{parm}(\Gamma)$ with the addition of a countable set of parameters, so that countably many parameters are still omitted. Extend the consistent set $(\mathsf{form}(\Gamma))^\sharp \cup \{\neg A'\}$ to a set $\mathsf{form}(\Delta)$ that is maximally consistent and E-complete with respect to $L^*(\Delta)$, with parameters as witnesses. Then $\Delta = \langle \mathsf{form}(\Delta), \mathsf{parm}(\Delta)\rangle \in \mathcal{G}$ and $\Gamma\mathcal{R}\Delta$. Since $\neg A' \in \mathsf{form}(\Delta)$, that is $\neg A\sigma_v \in \mathsf{form}(\Delta)$, by the Truth Lemma $\mathcal{M}, \Delta \not\Vdash_v A$, which completes the proof. ■

Corollary 8.3 *FOLP is complete with respect to the class of* fully explanatory *models.*

The definition of *fully explanatory* involves formulas of the form $t{:}_X A$ where X is the set of free variables of A. The following allows a broader range for X.

Theorem 8.4 *Assume the following:*

1. $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{A}, \mathcal{I}\rangle$ *is a fully explanatory model,*
2. $\Gamma \in \mathcal{G}$ *and v is a valuation that lives in* Γ,
3. A *is a formula of* L,
4. X *is a set of variables of L (not necessarily the set of free variables of A),*
5. *for each $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, and for each valuation w that lives in Δ and agrees with v on the members of X, $\mathcal{M}, \Delta \Vdash_w A$.*

Then for some justification term t, $\mathcal{M}, \Gamma \Vdash_v t{:}_X A$.

Proof Let X_0 be the set of free variables of A that are in X, and let $y_1, \ldots, y_n$ be the free variables of A that are not in X.

Let Δ be an arbitrary member of $\mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, and let w be a valuation that lives in Δ and agrees with v on X. Let w' be any y_n variant of w that lives in Δ. Then w' also agrees with v on members of X so by item 5, $\mathcal{M}, \Delta \Vdash_{w'} A$. Since w' was arbitrary, we have $\mathcal{M}, \Delta \Vdash_w \forall y_n A$.

We have shown that $\mathcal{M}, \Delta \Vdash_w \forall y_n A$ for every w that lives in Δ and agrees with v on the members of X; this is item 5 but for $\forall y_n A$ instead of for A. Then the argument can be repeated with this formula, to show that $\mathcal{M}, \Delta \Vdash_w \forall y_{n-1} \forall y_n A$, and so on. Finally we get that $\mathcal{M}, \Delta \Vdash_w \forall y_1 \cdots \forall y_n A$, for every w that lives in Δ and agrees with v on members of X. As a special case we can take $w = v$, and so $\mathcal{M}, \Delta \Vdash_v \forall y_1 \cdots \forall y_n A$.

The set of free variables of $\forall y_1 \cdots \forall y_n A$ is exactly X_0 and since Δ was arbitrary, by the definition of fully explanatory $\mathcal{M}, \Gamma \Vdash_v t{:}_{X_0} \forall y_1 \cdots \forall y_n A$, for some justification term t. It is a theorem of classical logic, and hence of FOLP, that $\forall y_1 \cdots \forall y_n A \to A$ so by Internalization, $u{:}_{\emptyset}(\forall y_1 \cdots \forall y_n A \to A)$ is provable for some justification term u, and hence also $u{:}_{X_0}(\forall y_1 \cdots \forall y_n A \to A)$, by repeated use of axiom **A3**. Then using axiom **B2**, $t{:}_{X_0} \forall y_1 \cdots \forall y_n A \to (u \cdot t){:}_{X_0} A$ is provable. By repeated use of axiom **A3**, $t{:}_{X_0} \forall y_1 \cdots \forall y_n A \to (u \cdot t){:}_X A$ is also provable. It follows that $\mathcal{M}, \Gamma \Vdash_v (u \cdot t){:}_X A$, which concludes the proof. ■

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