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Abstract

It is well and long known that random matrices tend to be well conditioned, and we employ them to advance some fundamental matrix computations. We begin with specifying and proving preconditioning properties of randomized additive preprocessing and randomized augmentation and then apply these results to outline new promising randomized algorithms for such fundamental matrix computations as preconditioning of an ill conditioned matrix that has a small numerical nullity or rank, its 2-by-2 block triangulation, numerical stabilization of Gaussian elimination with no pivoting, and approximation of a matrix by low-rank matrices and by structured matrices. According to both our formal study and numerical tests some of our algorithms significantly accelerate the known ones and improve their output accuracy. This should motivate further effort for advancing the presented approach. Besides the novel techniques of randomized preprocessing and the proof of their preconditioning power, our technical advances of potential independent interest include estimates for the condition numbers of random Toeplitz matrices and extension of the Sherman-Morrison-Woodbury formula.

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1 Introduction

It is well known that random matrices tend to be well conditioned [D88], [E88], [ES05], [CD05], [SST06], [B11], and we develop some formal and experimental support for employing this property toward advancing some fundamental matrix computations. Our analysis and experiments show substantial progress versus the known algorithms and should motivate further effort. Our technical advances include estimates for the condition numbers of random Toeplitz matrices, novel techniques

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of randomized preprocessing, a proof of their preconditioning power, and extension of the Sherman–Morrison–Woodbury formula.

1.1 Numerically safe Gaussian elimination with no pivoting

Hereafter "expected" and "likely" mean "with probability 1 or close to 1", $\sigma_j(A)$ denotes the jth largest singular value of an $n \times n$ matrix A, and the ratio $\kappa(A) = \sigma_1(A)/\sigma_\rho(A)$ for $\rho = \text{rank}(A)$ denotes its condition number. If this number is large (in context), then the matrix A is ill conditioned, that is lies near a rank deficient matrix. Numerical solution of a linear system of equations with such a coefficient matrix can be easily corrupted by rounding errors. Unless the number $\kappa(A)$ is large, the matrix A is well conditioned and then can be safely treated numerically with the IEEE standard double or single precision provided that the matrix has full rank and that the computations involve no ill conditioned or rank deficient auxiliary matrices.

To avoid dealing with such auxiliary matrices Gaussian elimination incorporates pivoting, that is row or column interchange. *Gaussian elimination with no pivoting* (hereafter we refer to it as *GENP*) can easily fail in numerical computations with rounding errors, except for some special classes of input matrices such as diagonally dominant and positive definite ones. For these matrix classes, GENP and its pivoting-free variations outperform Gaussian elimination with pivoting [GL96, page 119]. We dramatically expand these classes by proving in Corollary 4.2 that pre- as well as post-multiplication of a well conditioned coefficient matrix by a square Gaussian random matrix is expected to support safe numerical performance of GENP and block Gaussian elimination. Our tests have consistently supported such a preprocessed GENP using just circulant multipliers filled with the integers ± 1 for random signs $\pm$ (see Table 8.6). This extends our previous study in [PGMQ, Section 12.2] and [PQZa]. The tests in [PQZa, Table 2] show the power of this technique also for Householder multipliers filled with the integers ± 1 for random signs $\pm$.

1.2 Randomized preconditioning: the basic theorem

Given an ill conditioned matrix A, can we extend this advance by applying randomized multipliers X and Y to yield a much better conditioned matrix product XAY? No, because random square matrices X and Y are expected to be nonsingular and well conditioned [D88], [E88], [ES05], [CD05], [SST06], [B11] and because $\kappa(XAY) \geq \frac{\kappa(A)}{\kappa(X)\kappa(Y)}$. Approximate inverses of A are popular multipliers, but only for some important special classes of square matrices A for which computing approximate inverses is noncostly.

We can readily produce a well conditioned matrix C by applying additive preprocessing $A \Longrightarrow C = A + P$ (e.g., we can choose $P = I - A$), but it is not clear how this would help us to solve a linear system $A\mathbf{y} = \mathbf{b}$. (Here and hereafter I and I_n denote the $n \times n$ identity matrix.) Assume, however, that we are given a nonsingular ill conditioned $n \times n$ matrix A together with a small upper bound r on its numerical nullity, that is on the number of its singular values that are much smaller than the 2-norm $||A||_2$. Such matrices make up a large and important subclass in the class of nonsingular ill conditioned matrices (cf. [CDG03] and Remarks 2.1 and 5.2), and for this subclass we yield randomized additive preconditioning based on the following theorem. We prove it and extend it to the case of rectangular matrices A in Section 5. In our proof we supply detailed probabilistic estimates for the condition number $\kappa(C)$ in terms of the singular values of the matrix A.

Theorem 1.1. *Suppose A is a real $n \times n$ matrix, $||A||_2 = 1$, U and V are standard Gaussian random $n \times r$ matrices for $0 < r < n$, and either $U = V$ or all $2nr$ entries of the matrices U and V are independent of each other. Then the matrix $C = A + UV^T$ is singular if $r < \text{rank}\, A$, but otherwise is nonsingular with probability 1, and then its condition number $\kappa(C)$ is expected to have at most order $\sigma_1(A)/\sigma_{n-r}(A)$.*

Now suppose that the matrix A of Theorem 1.1 has positive numerical nullity at most r. Then the condition number $\kappa(A) = \sigma_1(A)/\sigma_n(A)$ is large, whereas the ratio $\sigma_1(A)/\sigma_{n-r}(A)$ and consequently the condition number $\kappa(C)$ are not expected to be large, and the matrix C is expected to be

nonsingular and well conditioned. Next we outline some promising directions for exploiting such *randomized preconditioning* toward advancing the known deterministic matrix algorithms.

1.3 Randomized preconditioning: some applications

Suppose we know the numerical nullity of the matrix A (cf. Section 7.1), set r equal to it, and combine the map $A \Longrightarrow C = A + UV^T$ with the Sherman–Morrison–Woodbury formula

$$A^{-1} = C^{-1} + C^{-1}UG^{-1}V^TC^{-1} \text{ for } G = I_r - V^TC^{-1}U, \tag{1.1}$$

hereafter referred to as the *SMW formula* [GL96, page 50], [S98, Corollary 4.3.2]. Surely inversion of an ill conditioned matrix A by any algorithm requires highly accurate computations, routinely achieved by using order n^3 flops (arithmetic operations) in high precision p_+, but if the matrix C is well conditioned, as is expected, then we can apply the SMW formula involving only $O(rn^2)$ flops in high precision, confined to the computations of the $r \times r$ matrix G. Our analysis in Section 7.6 specifies significant decrease of the computational cost estimates where this approach is restricted to the case of an ill conditioned matrix A of a large size having a small numerical nullity r (cf. Remarks 5.1 and 7.6).

The computation of the entries of the matrix G with high precision, however, requires extra care (cf. [ORO05], [PGMQ, Section 9]), and for our tests we selected the following alternative to the application of the SMW formula. Keep assuming that we have computed the exact values of the numerical rank $q < n$ and numerical nullity $r = n - q > 0$ and combine our techniques developed for proving Theorem 1.1 with some techniques from [PQ10]. This enables randomized computation of a 2×2 block triangulation of the matrix A whose two diagonal blocks have the sizes $r \times r$ and $q \times q$ and according to our formal study and tests are expected to be substantially better conditioned than the matrix A (see Section 7.4). As by-product of this approach we approximate the *trailing singular space* of the matrix A, associated with its r small singular values.

Our *dual SMW formula* and *dual additive preprocessing* enable us to extend our preconditioning techniques and in particular the latter algorithm for 2×2 block triangulation to any matrix A given with a small upper bound q on its numerical rank (see Section 7.5). Furthermore we readily approximate such matrices A by low-rank matrices; such approximations support many fundamental computations with matrices and tensors (cf. [T00], [MMD08], [OT09], [HMT11]). Our approach to low-rank approximation is distinct from the known methods.

If we agree to increase the size of an input matrix a little, we can alternatively apply randomized augmentation, defined, e.g., by the map $A \Longrightarrow K = \begin{pmatrix} I_r & V^T \\ -U & A \end{pmatrix}$. In the case of general $n \times n$ matrix A having norm 1 we choose standard Gaussian random $n \times r$ matrices U and V and deduce probabilistic upper bounds of order $||A||/\sigma_{n-r}(A)$ on the condition numbers $\kappa(K)$ and $\kappa(C)$ for $C = A + UV^T$ (see Section 6.1). This analysis enables distinct derivation of Theorem 1.1 and shows close link of augmentation to additive preprocessing, although. with augmentation we can a little better preserve sparseness and structure of the matrix A (cf. Section 6.2).

1.4 Condition numbers of random structured matrices and numerical experiments

Estimating the condition numbers of random structured matrices is a well known challenge [SST06]. We respond to it by estimating the condition numbers of Gaussian random Toeplitz matrices. In particular we prove in Section 3.4 that these numbers are not likely to grow to the infinity exponentially in n for $n \times n$ random Toeplitz matrices, even though the opposite has been proved in [BG05] for large and important subclass of the class of Toeplitz matrices. We obtain even stronger upper bounds on the condition numbers of random circulant matrices in Section 3.5. The estimates of both sections have been consistently supported by our tests (see Tables 8.1–8.4). We cannot extend our proof of Theorem 1.1 to the case of Gaussian random Toeplitz matrices U and V, but such an extension is consistent with the results of our extensive numerical tests, which otherwise are in good accordance with our theoretical estimates, e.g., in our tests application of randomized additive

preprocessing has dramatically decreased the condition numbers of ill conditioned matrices having small numerical nullity (see Table 8.5).

Furthermore we have numerically stabilized GENP by applying random circulant multipliers (see Table 8.6), accurately solved ill conditioned linear systems of equations by applying block triangulation based on randomization (see Tables 8.7–8.10), approximated trailing and leading singular spaces of ill conditioned matrices that have small numerical nullity or rank, and approximated a matrix by a low-rank matrix (see Tables 8.11–8.14). We have also matched the output accuracy of the customary algorithms for solving ill conditioned Toeplitz linear systems of equations but outperformed them in terms of the CPU time (see Table 8.15).

1.5 Organization of the paper and selective reading

We devote the next section to the definitions and basic results on matrix computations. We estimate the condition numbers of Gaussian random general, Toeplitz and circulant matrices in Section 3 and of randomized matrix products in Section 4. The latter estimate supports GENP with randomized multipliers. In Section 5 we prove that our randomized additive preprocessing of an ill conditioned matrix is expected to produce a well conditioned matrix. In Section 6 we study randomized augmentation, link it to randomized additive preprocessing, and solve ill conditioned Toeplitz linear systems of equations based on this study.

In Section 7 we propose a number of applications of randomized preprocessing to an ill conditioned matrix having a small numerical rank or nullity: we compute numerical rank and nullity of such a matrix, approximate its trailing and leading singular spaces, and extend this computation to the approximation by low-rank matrices and by structured matrices. We also cover randomized additive preconditioning based on the SMW formula, block triangulation, and structured preprocessing. Corollaries 4.1, 5.1 and 5.2 and the results of Section 3.4 and 6.1 show the power of these algorithms, but otherwise Section 7 can be read independently of Sections 3–6.

In Section 8 we cover numerical tests, which constitute the contribution of the second and the third authors. In Section 9 we comment on the related works, our technical novelties, and some directions for further study. In Appendix A we estimate the probability that a random matrix defined under the uniform probability distribution has full rank. In Appendix B we extend our probabilistic estimates to the case of complex matrices. In our paper, unlike its introduction, we cover the general case of rectangular input matrices A.

2 Some definitions and basic results

We assume computations in the field $\mathbb{R}$ of real numbers, and comment on the extension to the field $\mathbb{C}$ of complex numbers in Appendix B.

For two scalars a and b we write $a \ll b$ and $b \gg a$ if the ratio $|b/a|$ is large. We write $a \approx b$ if $|a - b| \ll |a| + |b|$.

Hereafter "flop" stands for "arithmetic operation"; "expected" and "likely" mean "with probability 1 or close to 1"; the concepts "large", "small", "near", "closely approximate", "ill conditioned" and "well conditioned" are quantified in the context.

We use and extend the customary definitions of matrix computations of [GL96], [S98].

2.1 Some basic definitions on matrix computations

$(B_1 \mid \ldots \mid B_k) = (B_j)_{j=1}^k$ is a $1 \times k$ block matrix with blocks $B_1, \ldots, B_k$.

$\mathrm{diag}(B_1, \ldots, B_k) = \mathrm{diag}(B_j)_{j=1}^k$ is a $k \times k$ block diagonal matrix with diagonal blocks $B_1, \ldots, B_k$.

I_n is the $n \times n$ identity matrix $(\mathbf{e}_1 \mid \ldots \mid \mathbf{e}_n)$. J_n is the $n \times n$ reflection matrix $(\mathbf{e}_n \mid \ldots \mid \mathbf{e}_1)$. $O_{k,l}$ is the $k \times l$ matrix filled with zeros. $\mathbf{0}_k$ is the vector $O_{k,1}$. We write I, J, O, and $\mathbf{0}$ where the size of a matrix or a vector is not important or is defined by context.

A^T is the transpose of a matrix A. A real matrix A is symmetric if $A = A^T$ and is symmetric positive definite if $A = B^T B$ for a nonsingular matrix B.

A matrix U is called *unitary*, *orthogonal* and *orthonormal* if $U^T U = I$ or $U U^T = I$.

Fact 2.1. *[GL96, Theorem 5.2.2]. QR factorization $A = QR$ of a matrix A having full column rank into the product of a unitary matrix $Q = Q(A)$ and an upper triangular matrix $R = R(A)$ is unique provided that the factor R is a square matrix with positive diagonal entries.*

2.2 Range, null space, rank, nullity, and nmbs

$\mathcal{R}(A)$ denotes the range of an $m \times n$ matrix A, that is the linear space $\{\mathbf{z} : \mathbf{z} = A\mathbf{x}\}$ generated by its columns. $\mathcal{N}(A)$ denotes its null space $\{\mathbf{v} : A\mathbf{v} = \mathbf{0}\}$, $\text{rank}(A) = \dim \mathcal{R}(A)$ its rank, and $\text{nul}(A) = \dim \mathcal{N}(A) = n - \text{rank}(A)$ its nullity or right nullity, whereas $\text{nul}\, A^T = m - \text{rank}(A)$ is the left nullity of A, equal to $\text{nul}\, A$ if and only if $m = n$. $\mathbf{v}$ is the null vector of A if $A\mathbf{v} = \mathbf{0}$.

Fact 2.2. *The set $\mathbb{M}$ of $m \times n$ matrices of rank ρ is an algebraic variety of dimension $(m+n-\rho)\rho$.*

Proof. Let M be an $m \times n$ matrix of a rank ρ with a nonsingular $\rho \times \rho$ leading block M_{00} and write $M = \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix}$. Then the $(m-\rho) \times (n-\rho)$ *Schur complement* $M_{11} - M_{10}M_{00}^{-1}M_{01}$ must vanish, which imposes $(m-\rho)(n-\rho)$ algebraic equations on the entries of M. Similar argument can be applied where any $\rho \times \rho$ submatrix of the matrix M (among $\binom{m}{\rho}\binom{n}{\rho}$ such submatrices) is nonsingular. Therefore $\dim \mathbb{M} = mn - (m-\rho)(n-\rho) = (m+n-\rho)\rho$. □

A matrix A having full column rank is said to be a *matrix basis* for the linear space $\mathcal{R}(A)$. Suppose a matrix B has full column rank and $\mathcal{R}(B) = \mathcal{N}(A)$. Then we call B a *null matrix basis* or a *nmb* for a matrix A and write $B = \text{nmb}(A)$. $\mathcal{N}(A^T)$ is the left null space of a matrix A, and similarly the map $A \to A^T$ defines left null vectors, left nmbs, and the left nullity of a matrix A; in particular an $m \times n$ matrix of a rank ρ has the left nullity $m - \rho$.

$A_k^{(k)}$ denotes the $k \times k$ leading, that is northwestern block submatrix of a matrix A.

A matrix of a rank ρ has *generic rank profile* if all its $i \times i$ leading blocks are nonsingular for $i = 1, \dots, \rho$. If such a matrix is nonsingular itself, then it is called *strongly nonsingular*.

2.3 Norms, SVD and perturbations

$||A||_h$ is the h-norm of a matrix $A = (a_{i,j})_{i,j=1}^{m,n}$. We write $||A|| = ||A||_2$ and $||\mathbf{v}|| = \sqrt{\mathbf{v}^T\mathbf{v}} = ||\mathbf{v}||_2$ and recall from [GL96, Section 2.3.2 and Corollary 2.3.2] that

$$\max_{i,j=1}^{m,n} |a_{i,j}| \le ||A|| = ||A^T|| \le \sqrt{mn}\, \max_{i,j=1}^{m,n} |a_{i,j}|,$$

$$\frac{1}{\sqrt{m}}||A||_1 \le ||A|| \le \sqrt{n}||A||_1, \quad ||A||_1 = ||A^T||_\infty, \quad ||A||^2 \le ||A||_1||A||_\infty, \tag{2.1}$$

$$||AB||_h \le ||A||_h||B||_h \text{ for } h = 1, 2, \infty \text{ and any matrix product } AB. \tag{2.2}$$

We write $A \approx B$ if $||A - B|| \ll ||A|| + ||B||$.

A matrix A is *normalized* if $||A|| = 1$. A normalized vector is unitary, and we call it *unit*.

$$A = S_A \Sigma_A T_A^T \tag{2.3}$$

is an *SVD* or *full SVD* of an $m \times n$ matrix A of a rank ρ provided $S_A S_A^T = S_A^T S_A = I_m$, $T_A T_A^T = T_A^T T_A = I_n$, $\Sigma_A = \text{diag}(\widehat{\Sigma}_A, O_{m-\rho,n-\rho})$, $\widehat{\Sigma}_A = \text{diag}(\sigma_j(A))_{j=1}^{\rho}$, $\sigma_j = \sigma_j(A) = \sigma_j(A^T)$ is the jth largest singular value of a matrix A. These values have the minimax property

$$\sigma_j = \max_{\dim(\mathbb{S})=j} \min_{\mathbf{x}\in\mathbb{S},\ ||\mathbf{x}||=1} ||A\mathbf{x}||,\ j = 1, \dots, \rho, \tag{2.4}$$

where $l = \min\{m, n\}$ and $\mathbb{S}$ denotes linear spaces [GL96, Theorem 8.6.1]. Consequently $\sigma_1 = \max_{||\mathbf{x}||=1} ||A\mathbf{x}|| = ||A||$ and $\sigma_j = 0$ for $j > \rho$.

Fact 2.3. *If A_0 is a submatrix of a matrix A, then $\sigma_j(A) \ge \sigma_j(A_0)$ for all j.*

Proof. [GL96, Corollary 8.6.3] implies Fact 2.3 where A_0 is any block of columns of the matrix A. Transposition of a matrix and permutations of its rows and columns do not change singular values, and thus we can extend the fact to all submatrices A_0. □

For every integer k in the range $1 \le k < \rho$ we define the partition $S_A = (S_{k,A} \mid S_{A,m-k})$ and $T_A = (T_{k,A} \mid T_{A,n-k})$ where the submatrices $S_{k,A}$ and $T_{k,A}$ are formed by the first k columns of the matrices S_A and T_A, respectively. We write $\mathbb{S}_{k,A} = \mathcal{R}(S_{k,A})$ and $\mathbb{T}_{k,A} = \mathcal{R}(T_{k,A})$. If $\sigma_k > \sigma_{k+1}$, then $\mathbb{S}_{k,A}$ and $\mathbb{T}_{k,A}$ are the left and right leading singular spaces, respectively, associated with the k largest singular values of the matrix A, whereas the orthogonal complements $\mathbb{S}_{A,m-k} = \mathcal{R}(S_{A,m-k})$ and $\mathbb{T}_{A,n-k} = \mathcal{R}(T_{A,n-k})$ are the left and right trailing singular spaces, respectively, associated with the other singular values of A. The pairs of subscripts $\{k, A\}$ versus $\{A, m-k\}$ and $\{A, n-k\}$ mark the leading versus trailing singular spaces. The left singular spaces of A are the right singular spaces of A^T and vice versa.

Finally we recall a result on perturbation norms.

Theorem 2.1. *Suppose C is a nonsingular matrix, E is a matrix of the same size, and $||C^{-1}E|| = \theta < 1$. Then $||I - (C+E)^{-1}C|| \le \frac{\theta}{1-\theta}$ and $|||(C+E)^{-1} - C^{-1}|| \le \frac{\theta}{1-\theta}||C^{-1}||$.*

Proof. See [S98, Corollary 1.4.19] for $P = -C^{-1}E$. □

2.4 Inverses, generalized inverses, SMW and a dual SMW formulae

$A^+ = T_A \operatorname{diag}(\widehat{\Sigma}_A^{-1}, O_{n-\rho,m-\rho}) S_A^T$ is the Moore–Penrose pseudo-inverse of the matrix A of (2.3), and

$$||A^+|| = 1/\sigma_\rho(A) \tag{2.5}$$

for a matrix A of a rank ρ. A^{+T} stands for $(A^+)^T = (A^T)^+$, and A^{-T} stands for $(A^{-1})^T = (A^T)^{-1}$.

An $n \times m$ matrix $X = A^{(I)}$ is a left inverse of an $m \times n$ matrix A if $XA = I$ and its right inverse if $AX = I$. A^+ is an $A^{(I)}$ if and only if a matrix A has full rank. $A^{(I)}$ is unique and is equal to A^{-1} if A is a nonsingular matrix.

Next suppose that $U_-, V_- \in \mathbb{R}^{n\times q}$, $A \in \mathbb{R}^{n\times n}$, A is a nonsingular matrix, and $0 < q < n$. Write

$$C_-^{-1} = A^{-1} + U_- V_-^T,\ H = I_q + V_-^T A U_-, \tag{2.6}$$

assume that one of these two matrices is nonsingular and deduce that both of them are nonsingular. Finally apply the SMW formula (1.1) to the matrix C_-^{-1} and obtain the *dual SMW formula*

$$C_- = A - AU_- H^{-1} V_-^T A \text{ for } H = I_q + V_-^T A U_-. \tag{2.7}$$

2.5 Condition number, numerical rank and numerical nullity

$\kappa(A) = \frac{\sigma_1(A)}{\sigma_\rho(A)} = ||A||\ ||A^+|\kappa(A^+)$ is the condition number of an $m \times n$ matrix A of a rank ρ. Such a matrix is *ill conditioned* if $\sigma_1(A) \gg \sigma_\rho(A)$ and is *well conditioned* otherwise. See [D83], [GL96, Sections 2.3.2, 2.3.3, 3.5.4, 12.5], [H02, Chapter 15], [KL94], and [S98, Section 5.3] on the estimation of norms and condition numbers. $\kappa(A) = ||A||\ ||A^{-1}||$ for a nonsingular matrix A. An $m \times n$ matrix A has *numerical rank* q, not exceeding $\operatorname{rank}(A)$, and has *numerical nullity* $r = n - q$ if the ratios $\sigma_j(A)/||A||$ are small for $j > q$ but not for $j \le q$. Actually this is the right numerical nullity of the matrix A, whereas its *left numerical nullity* equals the numerical nullity $m-q$ of the $n \times m$ transpose A^T and coincides with the right numerical nullity of A if and only if $m = n$.

Remark 2.1. *One can specify the adjective "small" above as "smaller than a fixed positive tolerance"; the choice of the tolerance can be a challenge, e.g., for the matrix* $\operatorname{diag}(1.1^{-j})_{j=0}^{999}$.

If an $m \times n$ well conditioned matrix A has a rank $\rho < l = \min\{m, n\}$, then almost all its close neighbours have full rank l, but all of them have numerical rank $q = \rho$. Conversely, suppose a matrix $\tilde{A}$ has a positive numerical rank q, set to 0 its all but q largest singular values and denote

the resulting matrix by A. Then $A \approx \tilde{A}$, $\operatorname{rank} A = q$, A is a well conditioned matrix, $\mathcal{R}(A) = \mathbb{T}_{q,\tilde{A}}$ and $\mathcal{N}(A) = \mathbb{T}_{\tilde{A},n-q}$.

A matrix has *generic conditioning profile* (cf. the end of Section 2.2) if it has a numerical rank q and if its $i \times i$ leading blocks are nonsingular and well conditioned for $i = 1, \dots, q$. If such a matrix has full rank and is well conditioned itself, then we call it *strongly well conditioned*.

Theorem 2.2. *[PQZa]. Suppose Gaussian elimination with no pivoting or block Gaussian elimination has been applied to a matrix A having a rank ρ (resp. numerical rank q) to compute LU factorizations of the leading block submatrices $A_j^{(j)}$ for $j = 1, \dots, \rho$ (resp. for $j = 1, \dots, q$). Then the computations involve no divisions by 0 (resp. by the values that are absolutely small relatively to the norm $||A||$) if and only if the matrix A has generic rank (resp. generic conditioning) profile.*

2.6 Toeplitz, Toeplitz-like, and circulant matrices

$m \times n$ *Toeplitz* matrix $T_{m,n} = (t_{i-j})_{i,j=1}^{m,n}$ is defined by its first row and column, that is by the vector $(t_h)_{h=1-n}^{m-1}$ of dimension $m + n - 1$. We write $T_n = T_{n,n} = (t_{i-j})_{i,j=1}^{n,n}$ (see the display below).

An $n \times n$ lower *triangular Toeplitz* matrix $Z(\mathbf{t}) = (t_{i-j})_{i,j=1}^{n}$ (where $t_k = 0$ for $k < 0$) is defined by its first column $\mathbf{t} = (t_h)_{h=0}^{n-1}$. We write $Z(\mathbf{t})^T = (Z(\mathbf{t}))^T$. $Z = Z(\mathbf{e}_2)$ is the $n \times n$ downshift matrix displayed below. We have $Z\mathbf{v} = (v_i)_{i=0}^{n-1}$ and $Z(\mathbf{v}) = \sum_{i=1}^{n} v_i Z^{i-1}$ for $\mathbf{v} = (v_i)_{i=1}^{n}$ and $v_0 = 0$,

$$T_n = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{1-n} \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & \cdots & t_1 & t_0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & & \dots & & 0 \\ 1 & \ddots & & & \\ \vdots & \ddots & \ddots & & \vdots \\ & & \ddots & 0 & \\ 0 & & \dots & 1 & 0 \end{pmatrix}.$$

Combine the equations $||Z(\mathbf{v})||_1 = ||Z(\mathbf{v})||_\infty = ||\mathbf{v}||_1$ with (2.1) to obtain

$$||Z(\mathbf{v})|| \le ||\mathbf{v}||_1. \tag{2.8}$$

Theorem 2.3. *Write $T_k = (t_{i-j})_{i,j=0}^{k-1}$ for $k = n, n+1$. (a) Let the matrix T_n be nonsingular and write $\mathbf{p} = T_n^{-1}\mathbf{e}_1$ and $\mathbf{q} = T_n^{-1}\mathbf{e}_n$. If $p_1 = \mathbf{e}_1^T\mathbf{p} \neq 0$, then $p_1 T_n^{-1} = Z(\mathbf{p})Z(J\mathbf{q})^T - Z(Z\mathbf{q})Z(ZJ\mathbf{p})^T$. In parts (b) and (c) below let the matrix T_{n+1} be nonsingular and write $\widehat{\mathbf{v}} = (v_i)_{i=0}^{n} = T_{n+1}^{-1}\mathbf{e}_1$, $\mathbf{v} = (v_i)_{i=0}^{n-1}$, $\mathbf{v}' = (v_i)_{i=1}^{n}$, $\widehat{\mathbf{w}} = (w_i)_{i=0}^{n} = T_{n+1}^{-1}\mathbf{e}_{n+1}$, $\mathbf{w} = (w_i)_{i=0}^{n-1}$, and $\mathbf{w}' = (w_i)_{i=1}^{n}$. (b) If $v_0 \neq 0$, then the matrix T_n is nonsingular and $v_0 T_n^{-1} = Z(\mathbf{v})Z(J\mathbf{w}')^T - Z(\mathbf{w})Z(J\mathbf{v}')^T$. (c) If $v_n \neq 0$, then the matrix $T_{1,0} = (t_{i-j})_{i=1,j=0}^{n,n-1}$ is nonsingular and $v_n T_{1,0}^{-1} = Z(\mathbf{w})Z(J\mathbf{v}')^T - Z(\mathbf{v})Z(J\mathbf{w}')^T$.*

Proof. See [GS72] on parts (a) and (b); see [GK72] on part (c). □

$Z_f = Z + f\mathbf{e}_1^T\mathbf{e}_n$ is the *f-circulant shift matrix* for a scalar $f \neq 0$. An *f-circulant matrix* $Z_f(\mathbf{v}) = \sum_{i=1}^{n} v_i Z_f^{i-1}$ is a special $n \times n$ Toeplitz matrix defined by its first column vector $\mathbf{v} = (v_i)_{i=1}^{n}$ and a scalar f. f-circulant matrix is called *circulant* if $f = 1$ and *skew circulant* if $f = -1$. By replacing f with 0 we arrive at a lower triangular Toeplitz matrix $Z(\mathbf{v})$. The following theorem implies that the products and inverses of f-circulant matrices (wherever defined) are f-circulant and can be computed in $O(n \log n)$ flops for $n \times n$ inputs.

Theorem 2.4. *(See [CPW74].) We have $Z_1(\mathbf{v}) = \Omega^{-1}D(\Omega\mathbf{v})\Omega$. More generally, for any $f \neq 0$, we have $Z_{f^n}(\mathbf{v}) = U_f^{-1}D(U_f\mathbf{v})U_f$ where $U_f = \Omega D(\mathbf{f})$, $\mathbf{f} = (f^i)_{i=0}^{n-1}$, $D(\mathbf{u}) = \operatorname{diag}(u_i)_{i=0}^{n-1}$ for a vector $\mathbf{u} = (u_i)_{i=0}^{n-1}$, and $\Omega = (\omega_n^{ij})_{i,j=0}^{n-1}$ is the $n \times n$ matrix of the discrete Fourier transform at n points, $\omega_n = \exp(\frac{2\pi}{n}\sqrt{-1})$ being a primitive n-th root of 1.*

Toeplitz-like $n \times n$ matrices T have a small *displacement rank* d, that is the rank of their displacements $Z_e T - T Z_f = GH^T$ for any pair of distinct scalars e and f; such matrices extend the class of Toeplitz matrices, for which $d \le 2$, and can be readily expressed via the $2dn$ entries of the matrix

pairs G and H, called displacement generators of length d for T. Such matrices T can be pairwise multiplied by using $O(d^2 n \log n)$ flops and, if nonsingular, can be inverted by using $O(d^2 n \log^2 n)$ flops provided every output has displacement rank at most $2d$ and is represented with at most $4dn$ parameters (cf. [P01]).

The classes of *Hankel* and *Hankel-like* matrices H can be defined by pre- or post-multiplying Toeplitz and Toeplitz-like matrices T by the reflection matrix J. The multiplication changes the pair of operator matrices $\{Z_e, Z_f\}$ into $\{Z_e, JZ_fJ\}$ or $\{JZ_eJ, Z_f\}$ but keeps the displacement rank. Clearly, $T^{-1} = JH^{-1}$ if $H = TJ$, $T^{-1} = H^{-1}J$ if $H = JT$, and in both cases $\kappa(H) = \kappa(T)$.

3 Ranks and conditioning of Gaussian random matrices

3.1 Random variables and Gaussian random matrices

Definition 3.1. *$F_\gamma(y) = Probability\{\gamma \le y\}$ for a real random variable γ is the* cumulative distribution function (cdf) *of X evaluated at y. $F_{g(\mu,\sigma)}(y) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{y} \exp(-\frac{(x-\mu)^2}{2\sigma^2})dx$ for a Gaussian random variable $g(\mu, \sigma)$ with a mean μ and a positive variance σ^2, and so*

$$\mu - 4\sigma \le y \le \mu + 4\sigma \text{ with a probability near 1.} \tag{3.1}$$

Definition 3.2. *A matrix or a vector is a* Gaussian random matrix or vector *with a mean μ and a positive variance σ^2 if it is filled with independent identically distributed Gaussian random variables, all having the mean μ and variance σ^2. $\mathcal{G}_{\mu,\sigma}^{m\times n}$ is the set of $m \times n$ Gaussian random matrices (which are* standard *for $\mu = 0$ and $\sigma^2 = 1$). By restricting this set to Toeplitz or f-circulant matrices we obtain the sets $\mathcal{T}_{\mu,\sigma}^{m\times n}$ and $\mathcal{Z}_{f,\mu,\sigma}^{n\times n}$ of* Gaussian random Toeplitz *and* Gaussian random f-circulant matrices*, respectively.*

Definition 3.3. *$\chi_{\mu,\sigma,n}(y)$ is the cdf of the norm $||\mathbf{v}|| = (\sum_{i=1}^{n} v_i^2)^{1/2}$ of a Gaussian random vector $\mathbf{v} = (v_i)_{i=1}^{n} \in \mathcal{G}_{\mu,\sigma}^{n\times 1}$. For $y \ge 0$ we have $\chi_{0,1,n}(y) = \frac{2}{2^{n/2}\Gamma(n/2)} \int_0^y x^{n-1} \exp(-x^2/2)dx$ where $\Gamma(h) = \int_0^\infty x^{h-1}\exp(-x)dx$, $\Gamma(n+1) = n!$ for nonnegative integers n.*

3.2 Nondegeneration of Gaussian random matrices

The total degree of a multivariate monomial is the sum of its degrees in all its variables. The total degree of a polynomial is the maximal total degree of its monomials.

Lemma 3.1. *[DL78], [S80], [Z79]. For a set Δ of cardinality $|\Delta|$ in any fixed ring let a polynomial in m variables have a total degree d and let it not vanish identically on this set. Then the polynomial vanishes in at most $d|\Delta|^{m-1}$ points.*

We assume that Gaussian random variables range over infinite sets Δ; consequently the lemma implies that a nonzero polynomial vanishes with probability 0. Thus Gaussian random general, Toeplitz and circulant matrices have full rank with probability 1 because the determinants of its square submatrices are polynomials in the entries. In particular a square Gaussian random general, Toeplitz or circulant matrix A is nonsingular with probability 1. Furthermore all entries of its adjoint, denoted $\operatorname{adj} A$, are subdeterminants and thus nonzeros with probability 1. Clearly this property of the adjoint can be extended to the inverse $A^{-1} = \frac{\operatorname{adj} A}{\det A}$. Hereafter, wherever this causes no confusion, we assume by default that *Gaussian random general, Toeplitz and circulant matrices have full rank, and their inverses (if defined) have nonzero entries.*

Similar properties with probability near 1 hold where the random variables are sampled under the uniform probability distribution from a finite set of a large cardinality (see Appendix A).

3.3 Conditioning of Gaussian random matrices

Besides having full rank with probability 1, Gaussian random matrices in Definition 3.2 are expected to be well conditioned [D88], [E88], [ES05], [CD05], [B11], and even the sum $M + A$ for $M \in \mathbb{R}^{m\times n}$ and $A \in \mathcal{G}_{\mu,\sigma}^{m\times n}$ is expected to be well conditioned unless the ratio $\sigma/||M||$ is small or large [SST06].

The following theorem states an upper bound on the probability (the cdf) $F_{1/||A^+||}(y)$ that the smallest positive singular value of a Gaussian random matrix A is at most a nonnegative scalar y (cf. (2.5)). Consequently this scalar itself can be viewed as a probabilistic lower bound on the smallest singular value of A. The bounds can be strengthened by a factor $y^{|m-n|}$ [ES05], [CD05].

Theorem 3.1. *Suppose $A \in \mathcal{G}_{\mu,\sigma}^{m\times n}$, $l = \min\{m,n\}$, and $y \geq 0$. Then $F_{1/||A^+||}(y) \leq 2.35\ \sqrt{l}y/\sigma$.*

Proof. For $m = n$ this is [SST06, Theorem 3.3]. Apply Fact 2.3 to extend it to any pair $\{m,n\}$. □

The following two theorems supply lower bounds on the probabilities that $||A|| \leq z$ and $\kappa(A) \leq y$ for two scalars y and z and a Gaussian random matrix A. The arguments y and z of the cdfs can also be viewed as probabilistic upper bounds on the norm $||A||$ and the condition number $\kappa(A)$, respectively. We do not use the second theorem, but state it for the sake of completeness and only for square $n \times n$ matrices A. The theorem implies that the function $1 - F_{\kappa(A)}(y)$ decays as $y \to \infty$ and that the decay is proportional to $\sqrt{\log y}/y$. For small values $y\sigma$ and a fixed n the lower bound becomes negative, in which case the theorem becomes trivial.

We assume that $\mu = 0$ when we estimate the cdfs of $||A||$ and consequently of $\kappa(A)$ for Gaussian random matrices A, but we make no such assumption when we estimate the cdf of $1/||A^+||$.

Theorem 3.2. *[DS01, Theorem II.7]. Suppose $A \in \mathcal{G}_{0,\sigma}^{m\times n}$, $l = \min\{m,n\}$ and $z \geq 2\sigma\sqrt{l}$. Then $F_{||A||}(z) \geq 1 - \exp(-(z - 2\sigma\sqrt{l})^2/(2\sigma^2))$.*

Theorem 3.3. *[SST06, Theorem 3.1]. Suppose $0 < \sigma \leq 1$, $y \geq 1$, $A \in \mathcal{G}_{0,\sigma}^{n\times n}$. Then the matrix A has full rank with probability 1 and $F_{\kappa(A)}(y) \geq 1 - (14.1 + 4.7\sqrt{(2\ln y)/n})n/(y\sigma)$.*

Proof. See [SST06, the proof of Lemma 3.2]. □

3.4 Conditioning of Gaussian random Toeplitz matrices

A matrix $T_n = (t_{i-j})_{i,j=1}^n$ is the sum of two triangular Toeplitz matrices

$$T_n = Z(\mathbf{t}) + Z(\mathbf{t}_-)^T,\ \mathbf{t} = (t_i)_{i=0}^{n-1},\ \mathbf{t}_- = (t'_{-i})_{i=0}^{n-1},\ t'_0 = 0. \tag{3.2}$$

If $T_n \in \mathcal{T}_{\mu,\sigma}^{n\times n}$, then T_n has $2n-1$ pairwise independent entries in $\mathcal{G}_{\mu,\sigma}$. Thus (2.8) implies that

$$\frac{1}{\sqrt{n}}||T_n|| \leq ||T_n||_1 \leq ||Z(\mathbf{t})||_1 + ||Z(\mathbf{t}_-)^T||_1 = ||\mathbf{t}||_1 + ||\mathbf{t}_-||_1 = ||(t_i)_{i=1-n}^{n-1}||_1 \leq \sqrt{n}||(t_i)_{i=1-n}^{n-1}||,$$

and so

$$F_{||T_n||}(y) \geq \chi_{\mu,\sigma,2n-1}(y/n). \tag{3.3}$$

Next we estimate the norm $||T_n^{-1}||$ for $T_n \in \mathcal{T}_{\mu,\sigma}^{n\times n}$, beginning with a basic lemma.

Lemma 3.2. *[SST06, Lemma A.2]. For a nonnegative scalar y, a unit vector $\mathbf{t} \in \mathbb{R}^{n\times 1}$, and a vector $\mathbf{b} \in \mathcal{G}_{\mu,\sigma}^{n\times 1}$, we have $F_{|\mathbf{t}^T\mathbf{b}|}(y) \leq y \leq \sqrt{\frac{2}{\pi}}\frac{y}{\sigma}$.*

Remark 3.1. *The latter bound is independent of μ and n; it holds for any μ even if all coordinates of the vector $\mathbf{b}$ are fixed except for a single coordinate in $\mathcal{G}_{\mu,\sigma}$.*

Theorem 3.4. *Given a matrix $T_n = (t_{i-j})_{i,j=1}^n \in \mathcal{T}_{\mu,\sigma}^{n\times n}$, assumed to be nonsingular (cf. Section 3.2), write $p_1 = \mathbf{e}_1^T T_n^{-1}\mathbf{e}_1$. Then $F_{1/||p_1 T_n^{-1}||}(y) \leq 2n\alpha\beta$ for two random variables α and β such that*

$$F_\alpha(y) \leq \sqrt{\frac{2n}{\pi}}\frac{y}{\sigma} \text{ and } F_\beta(y) \leq \sqrt{\frac{2n}{\pi}}\frac{y}{\sigma} \text{ for } y \geq 0. \tag{3.4}$$

Proof. By virtue of part (a) of Theorem 2.3 we have $p_1T_n^{-1} = Z(\mathbf{p})Z(J\mathbf{q})^T - Z(Z\mathbf{q})Z(ZJ\mathbf{p})^T$, and so $||p_1T_n^{-1}|| \le ||Z(\mathbf{p})||\ ||Z(J\mathbf{q})^T|| + ||Z(Z\mathbf{q})||\ ||Z(ZJ\mathbf{p})^T||$ for $\mathbf{p} = T_n^{-1}\mathbf{e}_1$, $\mathbf{q} = T_n^{-1}\mathbf{e}_n$, and $p_1 = \mathbf{p}^T\mathbf{e}_1$. It follows that $||p_1T_n^{-1}|| \le ||Z(\mathbf{p})||\ ||Z(J\mathbf{q})|| + ||Z(Z\mathbf{q})||\ ||Z(ZJ\mathbf{p})||$ since $||A|| = ||A^T||$ for all matrices A. Furthermore $||p_1T_n^{-1}|| \le ||\mathbf{p}||_1\ ||J\mathbf{q}||_1 + ||Z\mathbf{q}||_1\ ||ZJ\mathbf{p}||_1$ due to (2.8). Clearly $||J\mathbf{v}||_1 = ||\mathbf{v}||_1$ and $||Z\mathbf{v}||_1 \le ||\mathbf{v}||_1$ for every vector $\mathbf{v}$, and so (cf. (2.1))

$$||p_1T_n^{-1}|| \le 2||\mathbf{p}||_1\ ||\mathbf{q}||_1 \le 2n||\mathbf{p}||\ ||\mathbf{q}||. \tag{3.5}$$

By definition the vector $\mathbf{p}$ is orthogonal to the vectors $T_n\mathbf{e}_2, \dots, T_n\mathbf{e}_n$, whereas $\mathbf{p}^TT_n\mathbf{e}_1 = 1$ (cf. [SST06]). Consequenty the vectors $T_n\mathbf{e}_2, \dots, T_n\mathbf{e}_n$ uniquely define the vector $\mathbf{u} = \mathbf{p}/||\mathbf{p}||$, whereas $|\mathbf{u}^TT_n\mathbf{e}_1| = 1/||\mathbf{p}||$.

The last coordinate t_{n-1} of the vector $T_n\mathbf{e}_1$ is independent of the vectors $T_n\mathbf{e}_2, \dots, T_n\mathbf{e}_n$ and consequently of the vector $\mathbf{u}$. Apply Remark 3.1 to estimate the cdf of the random variable $\alpha = 1/||\mathbf{p}|| = |\mathbf{u}^TT_n\mathbf{e}_1|$ and obtain that $F_\alpha(y) \le \sqrt{\frac{2n}{\pi}}\frac{y}{\sigma}$ for $y \ge 0$.

Likewise the $n-1$ column vectors $T\mathbf{e}_1, \dots, T_{n-1}$ define the vector $\mathbf{v} = \beta\mathbf{q}$ for $\beta = 1/||\mathbf{q}|| = |\mathbf{v}^TT_n\mathbf{e}_n|$. The first coordinate t_{1-n} of the vector $T_n\mathbf{e}_n$ is independent of the vectors $T\mathbf{e}_1, \dots, T_{n-1}$ and consequently of the vector $\mathbf{v}$. Apply Remark 3.1 to estimate the cdf of the random variable β and obtain that $F_\beta(y) \le \sqrt{\frac{2n}{\pi}}\frac{y}{\sigma}$ for $y \ge 0$.

Finally combine these bounds on the cdfs $F_\alpha(y)$ and $F_\beta(y)$ with (3.5). □

By employing parts (b) and (c) of Theorem 2.3 instead of its part (a), we similarly deduce the bounds $||v_0T_{n+1}^{-1}|| \le 2\alpha\beta$ and $||v_nT_{n+1}^{-1}|| \le 2\alpha\beta$ for two pairs of random variables α and β satisfying (3.4) for $n+1$ replacing n. We have $p_1 = \frac{\det T_{n-1}}{\det T_n}$, $v_0 = \frac{\det T_n}{\det T_{n+1}}$, and $v_n = \frac{\det T_{0,1}}{\det T_{n+1}}$ for $T_{0,1} = (t_{i-j})_{i=0,j=1}^{n-1,n}$. Next we bound the geometric means of the ratios $|\frac{\det T_{h+1}}{\det T_h}|$ for $h = 1, \dots, k-1$. $1/|p_1|$ and $1/|v_0|$ are such ratios for $k = n-1$ and $k = n$, respectively, whereas the ratio $1/|v_n|$ is similar to $1/|v_0|$, under slightly distinct notation.

Theorem 3.5. *Let $T_h \ne O$ denote $h \times h$ matrices for $h = 1, \dots, k$ whose entries have absolute values at most t for a fixed scalar or random variable t, e.g. for $t = ||T||$. Furthermore let $T_1 = (t)$. Then the geometric mean $(\prod_{h=1}^{k-1} |\frac{\det T_{h+1}}{\det T_h}|)^{1/(k-1)} = \frac{1}{t}|\det T_k|^{1/(k-1)}$ is at most $k^{\frac{1}{2}(1+\frac{1}{k-1})}t$.*

Proof. The theorem follows from Hadamard's upper bound $|\det M| \le k^{k/2}t^k$, which holds for any $k \times k$ matrix $M = (m_{i,j})_{i,j=1}^k$ with $\max_{i,j=1}^k |m_{i,j}| \le t$. □

The theorem says that the geometric mean of the ratios $|\det T_{h+1}/\det T_h|$ for $h = 1, \dots, k-1$ is not greater than $k^{0.5+\epsilon(k)}t$ where $\epsilon(k) \to 0$ as $k \to \infty$. Furthermore if $T_n \in \mathcal{G}_{\mu,\sigma}^{n\times n}$ we can write $t = ||T||$ and apply (3.3) to bound the cdf of t.

3.5 Conditioning of Gaussian random circulant matrices

Next we estimate the norms of a random Gaussian f-circulant matrix and its inverse.

Theorem 3.6. *Assume $y \ge 0$ and an $n \times n$ circulant matrix $T = Z_1(\mathbf{v})$ for $\mathbf{v} \in \mathcal{G}_{\mu,\sigma}^{n\times 1}$. Then*
(a) $F_{||T||}(y) \ge \chi_{\mu,\sigma,n}(2y/n)$ for $\chi_{\mu,\sigma,n}(y)$ in Definition 3.3 and
(b) $F_{1/||T^{-1}||}(y) \le \sqrt{\frac{2}{\pi}}\frac{ny}{\sigma}$.

Proof. For the matrix $T = Z_1(\mathbf{v})$ we have both equation (3.2) and the bound $||\mathbf{t}_-||_1 \le ||\mathbf{t}||_1$, and so $||T||_1 \le 2||\mathbf{t}||_1$. Now part (a) of the theorem follows similarly to (3.3).

To prove part (b) recall Theorem 2.4 and write $B = \Omega T\Omega^{-1} = D(\Omega\mathbf{v})$, $\mathbf{u} = (u_i)_{i=0}^{n-1} = \Omega\mathbf{v}$. We have $\sigma_j(T) = \sigma_j(B)$ for all j because $\frac{1}{\sqrt{n}}\Omega$ and $\sqrt{n}\Omega^{-1}$ are unitary matrices.

By combining the equations $u_i = \mathbf{e}_i^T\Omega\mathbf{v}$, the bounds $||\Re(\mathbf{e}_i^T\Omega)|| \ge 1$ for all i, and Lemma 3.2, deduce that $F_{|\Re(u_i)|}(y) \le \sqrt{\frac{2}{\pi}}\frac{y}{\sigma}$ for $i = 1, \dots, n$. We have $F_{\sigma_n(B)}(y) = F_{\min_i |u_i|}(y)$ because $B = \operatorname{diag}(u_i)_{i=0}^{n-1}$. Clearly $|u_i| \ge |\Re(u_i)|$, and part (b) of the theorem follows. □

Remark 3.2. *Our extensive experiments suggest that the estimates of Theorem 3.6 are overly pessimistic (cf. Table 8.4).*

Combining Theorem 2.4 with minimax property (2.4) implies that

$$\frac{1}{g(f)}\sigma_j(Z_1(\mathbf{v})) \le \sigma_j(Z_f(\mathbf{v})) \le g(f)\sigma_j(Z_1(\mathbf{v}))$$

for all vectors $\mathbf{v}$, scalars $f \neq 0$, $g(f) = \max\{|f|, 1/|f|\}$, and $j = 1, \dots, n$. Thus we can readily extend the estimates of Theorem 3.6 to f-circulant matrices for $f \neq 0$. In particular the estimates do not change in the case of skew circulant matrices (for which $f = -1$), and furthermore Gaussian random f-circulant matrices tend to be well conditioned unless $f \approx 0$ or $1/f \approx 0$.

4 Conditioning of randomized matrix products and generic preconditioning

Next we deduce probabilistic lower bounds on the smallest singular values of the products of fixed and random matrices. We begin with three lemmas. The first two of them easily follow from minimax property (2.4).

Lemma 4.1. *Suppose* $\Sigma = \mathrm{diag}(\sigma_i)_{i=1}^n$, $G \in \mathbb{R}^{r\times n}$, *and* $H \in \mathbb{R}^{n\times r}$. *Then* $\sigma_j(G\Sigma) \ge \sigma_j(G)\sigma_n$, $\sigma_j(\Sigma H) \ge \sigma_j(H)\sigma_n$ *for all* j. *If also* $\sigma_n > 0$, *then* $\mathrm{rank}(G\Sigma) = \mathrm{rank}(G)$, $\mathrm{rank}(\Sigma H) = \mathrm{rank}(H)$.

Lemma 4.2. $\sigma_j(SM) = \sigma_j(MT) = \sigma_j(M)$ *for all* j *if* S *and* T *are square unitary matrices.*

Lemma 4.3. *[SST06, Proposition 2.2]. Suppose* $H \in \mathcal{G}_{\mu,\sigma}^{m\times n}$, $SS^T = S^TS = I_m$, $TT^T = T^TT = I_n$. *Then* $SH \in \mathcal{G}_{\mu,\sigma}^{m\times n}$ *and* $HT \in \mathcal{G}_{\mu,\sigma}^{m\times n}$.

Theorem 4.1. *Suppose* $G \in \mathcal{G}_{\mu,\sigma}^{r\times m}$, $H \in \mathcal{G}_{\mu,\sigma}^{n\times r}$, $M \in \mathbb{R}^{m\times n}$, $r(M) = \mathrm{rank}(M)$, *and* $y \ge 0$. *Then* $\max\{F_{1/||(GM)^+||}(y), F_{1/||(MH)^+||}(y)\} \le 2.35y\sqrt{\widehat{r}}/(\sigma_{r(M)}(M)\sigma)$ *for* $\widehat{r} = \min\{r, r(M)\}$.

Proof. With probability 1, the matrix MH has full rank $\widehat{r}$ because $H \in \mathcal{G}_{\mu,\sigma}^{n\times r}$. So (cf. (2.5))

$$F_{1/||(MH)^+||}(y) = F_{\sigma_{\widehat{r}}(MH)}(y). \tag{4.1}$$

Let $M = S_M\Sigma_M T_M^T$ be full SVD where $\Sigma_M = \mathrm{diag}(\widehat{\Sigma}_M, O) = \Sigma_M\,\mathrm{diag}(I_{r(M)}, O)$ and $\widehat{\Sigma}_M = \mathrm{diag}(\sigma_j(M))_{j=1}^{r(M)}$ is a nonsingular diagonal matrix.

We have $MH = S_M\Sigma_M T_M^T H$, and so $\sigma_j(MH) = \sigma_j(\Sigma_M T_M^T H)$ for all j by virtue of Lemma 4.2, because S_M is a square unitary matrix. Write $H_{r(M)} = (I_{r(M)} \mid O)T_M^T H$ and observe that $\sigma_j(\Sigma_M T_M^T H) = \sigma_j(\widehat{\Sigma}_M H_{r(M)})$ and consequently

$$\sigma_j(MH) = \sigma_j(\widehat{\Sigma}_M H_{r(M)}) \text{ for all } j. \tag{4.2}$$

Combine equation (4.2) for $j = \widehat{r}$ with Lemma 4.1 for the pair (Σ, H) replaced by $(\widehat{\Sigma}_M, H_{r(M)})$ and obtain that $\sigma_{\widehat{r}}(MH) \ge \sigma_{r(M)}(M)\sigma_{\widehat{r}}(H_{r(M)})$.

We have $T_M^T H \in \mathcal{G}_{\mu,\sigma}^{n\times r}$ by virtue of Lemma 4.3, since T_M is a square unitary matrix; consequently $H_{r(M)} \in \mathcal{G}_{\mu,\sigma}^{r(M)\times r}$. Therefore we can apply Theorem 3.1 for $A = H_{r(M)}$ and obtain the bound of Theorem 4.1 on $F_{1/||(MH)^+||}(y)$.

One can similarly deduce the bound on $F_{1/||(GM)^+||}(y)$ or can just apply the above bound on $F_{1/||(MH)^+||}(y))$ for $H = G^T$ and M replaced by M^T and then recall that $(M^TG^T)^T = GM$. ∎

Theorem 4.1 implies that multiplication by standard Gaussian random matrix is unlikely to decrease the smallest positive singular value of a matrix dramatically, even though $UV = O$ for some pairs of rectangular unitary matrices U and V.

By combining (2.2) with Theorems 3.2 and 4.1 one can probabilistically bound the condition numbers of randomized matrix products GM and MH.

The following corollary extends the bound of Theorem 4.1 for a randomized matrix product to the respective bounds for its leading blocks. This implies that *randomized multiplication of a well conditioned matrix is expected to be generic preconditioning*, that is, to ensure (with probability 1 or near 1) generic rank and conditioning profiles for the product.

Corollary 4.1. *Suppose j, k, m, n, q and s are integers, $1 \le j \le q$, $1 \le k \le s$, $M \in \mathbb{R}^{m\times n}$, $\sigma > 0$, $G \in \mathcal{G}_{\mu,\sigma}^{q\times m}$, $H \in \mathcal{G}_{\mu,\sigma}^{n\times s}$, $\mathrm{rank}(M_j) = j$ for $M_j = M\begin{pmatrix} I_j \\ O_{n-j,j}\end{pmatrix}$, $\mathrm{rank}(M^{(k)}) = k$ for $M^{(k)} = (I_k \mid O_{k,m-k})M$, and $y \ge 0$. Then (i) with probability 1 the matrix GM (resp. MH) has generic rank profile if $\mathrm{rank}(M) \ge q$ (resp. if $\mathrm{rank}(M) \ge s$). Furthermore (ii) $F_{1/||((GM)_j^{(j)})^+||}(y) \le 2.35y\sqrt{j}/(\sigma_j(M_j)\sigma)$ if $\mathrm{rank}(M) \ge j$, $F_{1/||((MH)_k^{(k)})^+||}(y) \le 2.35y\sqrt{k}/(\sigma_k(M^{(k)})\sigma)$ if $\mathrm{rank}(M) \ge k$.*

Proof. We immediately verify part (i) based on the techniques of Section 3.2. To prove part (ii) apply Theorem 4.1 for every j, replacing G by $(I_j \mid O_{j,q-j})G$ and M by $M\begin{pmatrix} I_j \\ O_{n-j,j}\end{pmatrix}$. For every k apply Theorem 4.1 replacing M by $(I_k \mid O_{k,m-k})M$ and H by $H\begin{pmatrix} I_k \\ O_{s-k,k}\end{pmatrix}$. □

Combining the latter results with (2.2) and Theorem 3.2 implies that for well conditioned inputs randomized multiplication is expected to serve as generic preconditioning, and if it does serve so, then Theorem 2.2 implies the following corollary, which shows that both Gaussian elimination with no pivoting and block Gaussian elimination are numerically safe.

Corollary 4.2. *Suppose M is a normalized $m \times n$ well conditioned matrix of full rank, $||M|| = 1$, $G \in \mathcal{G}_{0,1}^{m\times m}$ and $H \in \mathcal{G}_{0,1}^{n\times n}$. Then Gaussian elimination with no pivoting as well as block Gaussian elimination applied to the matrices GM and MH are expected to proceed using no divisions by absolutely small values.*

Remark 4.1. *We cannot extend Lemma 4.3 and consequently Theorem 4.1 and its corollaries to the case of Gaussian random Toeplitz matrices $G \in \mathcal{T}_{\mu,\sigma}^{r\times m}$ and $H \in \mathcal{T}_{\mu,\sigma}^{n\times r}$, but such extensions have been consistently supported empirically (cf. Tables 8.6 and 8.10).*

Remark 4.2. *We can write $H_0 = Q(MH)$ and compute the sequence $H_i = Q(MH_{i-1})$ of improved approximations $\mathcal{R}(H_i) \approx \mathbb{T}_{s,M}$ for $i = 1, 2, \ldots$.*

5 Randomized additive and dual additive preconditioning

In this section we first prove Theorem 1.1 and then extend it to the cases of rectangular $m \times n$ input matrices A for any pair $\{m, n\}$ and dual additive preprocessing, extending (2.6). The respective rank estimates are implicit in Section 3.2, and we only prove the norm and condition estimates.

5.1 Proof of Theorem 1.1

Theorem 5.1. *Suppose $A, C, S, T \in \mathbb{R}^{n\times n}$ and $U, V \in \mathbb{R}^{n\times r}$ for two positive integers r and n, $r \le n$, $A = S\Sigma T^T$ is full SVD of the matrix A (cf. (2.3)), S and T are unitary matrices, $\Sigma = \mathrm{diag}(\sigma_j)_{j=1}^n$, the matrix $C = A + UV^T$ is nonsingular, and so $\rho = n - r \ge \mathrm{rank}(A)$ and $\sigma_\rho > 0$. Write*

$$S^TU = \begin{pmatrix}\bar{U}\\ U_r\end{pmatrix},\ T^TV = \begin{pmatrix}\bar{V}\\ V_r\end{pmatrix},\ R_U = \begin{pmatrix} I_\rho & \bar{U}\\ O_{r,\rho} & U_r\end{pmatrix},\ R_V = \begin{pmatrix} I_\rho & \bar{V}\\ O_{r,\rho} & V_r\end{pmatrix},$$

where U_r and V_r are $r \times r$ matrices. Then

(a) $R_U\Sigma R_V^T = \Sigma$, whereas $R_U\,\mathrm{diag}(O_{\rho,\rho}, I_r)R_V^T = S^TUV^TT$, and so

$$C = SR_UDR_V^TT^T,\ D = \Sigma + \mathrm{diag}(O_{\rho,\rho}, I_r) = \mathrm{diag}(d_j)_{j=1}^n \tag{5.1}$$

where $d_j = \sigma_j$ for $j = 1, \ldots, \rho$, $d_j = \sigma_j + 1$ for $j = \rho + 1, \ldots, n$.

Furthermore suppose that the matrix A has been normalized so that $||A|| = 1$ and that the $r \times r$ matrices U_r and V_r are nonsingular, which holds with probability 1 where U and V are Gaussain random matrices (cf. Section 3.2). Write $p = ||R_U^{-1}||\ ||R_V^{-1}||$ and $f_r = \max\{1, ||U_r^{-1}||\}\ \max\{1, ||V_r^{-1}||\}$. Then

(b) the matrix C is nonsingular,

(c) $p \ge \sigma_\rho(A)/\sigma_n(C)$,

(d) $1 \le p \le (1 + ||U||)(1 + ||V||)f_r$,

(e) $||C|| \le ||A||(1 + ||U||\ ||V||)$ *and*

(f) $\sigma_\rho(A)/\sigma_n(C) \le (1 + ||U||)(1 + ||V||)f_r$.

Proof. Parts (a) and (b) are readily verified.

(c) Combine the equations $S^{-1} = S^T$, $T^{-1} = T$ and (5.1) and obtain $C^{-1} = TR_V^{-T}D^{-1}R_U^{-1}S^T$. It follows that $||C^{-1}|| = ||R_V^{-T}D^{-1}R_U^{-1}|| \le ||R_V^{-T}||\ ||D^{-1}||\ ||R_U^{-1}||$ because S and T are square unitary matrices. Substitute the equations $||D^{-1}|| = 1/\sigma_\rho(A)$ (implied by the equations $||A|| = 1$ and (5.1)) and $||C^{-1}|| = 1/\sigma_n(C)$ and obtain the claimed bound $p \ge \sigma_\rho(A)/\sigma_n(C)$.

(d) Observe that $R_U^{-1} = \begin{pmatrix} I_\rho & -\bar{U} \\ O & I_r \end{pmatrix} \begin{pmatrix} I_\rho & O \\ O & U_r^{-1} \end{pmatrix}$, $R_V^{-1} = \begin{pmatrix} I_\rho & -\bar{V} \\ O & I_r \end{pmatrix} \begin{pmatrix} I_\rho & O \\ O & V_r^{-1} \end{pmatrix}$, $||\bar{U}|| \le ||U||$ and $||\bar{V}|| \le ||V||$. Then combine these relationships.

(e) Combine the relationships $||C|| \le ||A|| + ||U||\ ||V^T||$, $||A|| = 1$ and $||V^T|| = ||V||$.

(f) Combine the bounds $\sigma_\rho(A) \le p\sigma_n(C)$ and $p \le (1+||U||)(1+||V||)f_r$ of parts (c) and (d). □

We have $\frac{\kappa(C)}{\kappa(A)} \le \frac{||C||}{||A||}\frac{\sigma_\rho(A)}{\sigma_n(C)}$, and so parts (e) and (f) together bound the ratio $\frac{\kappa(C)}{\kappa(A)}$ in terms of the norms $||U||$, $||V||$, $||U_r^{-1}||$ and $||V_r^{-1}||$ as follows,

$$\frac{\kappa(C)}{\kappa(A)} \le (1 + ||U||\ ||V||)(1 + ||U||)(1 + ||V||)\ \max\{1, ||U_r^{-1}||\}\ \max\{1, ||V_r^{-1}||\}, \tag{5.2}$$

and in particular

$$\frac{\kappa(C)}{\kappa(A)} \le (1 + ||U||^2)(1 + ||U||)^2 \max\{1, ||U_r^{-1}||^2\} \text{ if } U = V. \tag{5.3}$$

Let us estimate these norms where U and V are Gaussian random matrices.

Theorem 5.2. *Suppose $W \in \mathcal{G}_{0,\sigma}^{n\times r}$ and $y \ge 2\sigma\sqrt{r}$. Then $F_{||W||}(y) \ge 1 - \exp(-(y - 2\sigma\sqrt{r})^2/(2\sigma^2))$.*

Proof. The theorem follows from Theorem 3.2 applied for $A = W$ (in which case $l = r$). □

Theorem 5.3. *Suppose that A, U, V, U_r and V_r denote the five matrices of Theorem 5.1 where $U, V \in \mathcal{G}_{\mu,\sigma}^{n\times r}$. Then $\max\{F_{1/||U_r^{-1}||}(y), F_{1/||V_r^{-1}||}(y)\} \le 2.35\ y\sqrt{r}/\sigma$ for $y \ge 0$.*

Proof. Lemma 4.3 implies that $S^TU, T^TV \in \mathcal{G}_{\mu,\sigma}^{n\times r}$ because S and T are unitary matrices. Therefore $U_r, V_r \in \mathcal{G}_{\mu,\sigma}^{r\times r}$. Apply Theorem 3.1 for $A = U_r$ and $A = V_r$ where in both cases $m = n = r$. □

Combine bounds (5.2) and (5.3) with Theorems 5.2 and 5.3 and obtain Theorem 1.1.

Remark 5.1. *(Cf. Remark 7.6.) Theorem 1.1 can guide us in choosing the size $n \times r$ of the matrices U and V. Surely there is no point for applying the preprocessing $A \Longrightarrow C = A + UV^T$ to a well conditioned matrix A of full rank, whereas the output matrix C remains ill conditioned where the numerical nullity ν of the matrix A exceeds r. The theorem shows that for a normalized input matrix A and two independent standard Gaussian random matrices U and V as well as for standard Gaussian random matrix U and for $V = U$ the above preprocessing is expected to produce a well conditioned matrix C as long as $\nu \le r$. The best choice $r = \nu$ minimizes the size of the matrices U and V and leads to some other simplifications. These comments can be extended to the extensions of the theorem in Corollary 5.2 and Section 6.1.*

5.2 Extension of Theorem 1.1 to the case of rectangular matrices

Clearly our upper bound on the ratio $||C||/||A||$ holds for any pair of m and n. Furthermore we can apply Theorem 5.2 for n replaced by m and can immediately extend Theorem 5.3 to the case where $U \in \mathcal{G}_{\mu,\sigma}^{m\times r}$. Let us also relax the assumption that $m = n$ for our upper bound on $\sigma_\rho(A)/\sigma_n(C)$.

Theorem 5.4. *Suppose that $A \in \mathbb{R}^{m\times n}$, $U \in \mathcal{G}_{0,\sigma}^{m\times r}$, and $V \in \mathcal{G}_{0,\sigma}^{n\times r}$ for three positive integers m and n and assume that the matrix $C = A + UV^T$ has full rank $l = \min\{m, n\}$ (and thus is nonsingular where $m = n$). Keep all other assumptions of parts (a)–(f) of Theorem 5.1. Then the upper bound of part (f) of Theorem 5.1 can be extended, that is, $\sigma_\rho(A)/\sigma_l(C) \le (1 + ||U||)(1 + ||V||)f_r$.*

Proof. Let $M = S_M\Sigma_M T_M^T$ denote full SVDs for $M = A$ and $M = C$ (cf. (2.3)). W.l.o.g. we can assume that $S = I_m$, $T = I_n$ and $A = \Sigma_A$. Indeed substitute the SVDs of A and C into the matrix equation $C = A + UV^T$, pre-multiply the resulting equation by S_A^T, post-multiply it by T_A and obtain that $\tilde{C} = \Sigma_A + \tilde{U}\tilde{V}^T$ where $\tilde{C} = S_A^T C T_A$, $\sigma_j(\tilde{C}) = \sigma_j(C)$ and $\sigma_j(\Sigma_A) = \sigma_j(A)$ for all j, $\tilde{U} = S_A^T U \in \mathcal{G}_{\mu,\sigma}^{m\times r}$ and $\tilde{V} = T_A^T V \in \mathcal{G}_{\mu,\sigma}^{n\times r}$ because S_A and T_A are square orthogonal matrices (cf. Lemmas 4.2 and 4.3). Therefore it is sufficient to prove the theorem for $\tilde{C}$, Σ_A, $\tilde{U}$ and $\tilde{V}$ replacing C, A, U and V, respectively, or equivalently, to prove it under the original notation A, C, U and V, but under the assumption that $A = \Sigma_A$.

Now suppose that $m > n$ and estimate the ratio $\sigma_\rho(A)/\sigma_n(C)$ from above. Pre-multiply the equation $C = A+UV^T$ by the matrix $(I_n \mid O_{n,m-n})$ and note that $\widehat{U} = (I_n \mid O_{n,m-n})U \in \mathcal{G}_{0,\sigma}^{n\times r}$, $\widehat{A} = (I_n \mid O_{n,m-n})A = \mathrm{diag}(\sigma_j(A))_{j=1}^n$, and $\widehat{C} = (I_n \mid O_{n,m-n})C = \widehat{A}+\widehat{U}V^T$. Clearly $\sigma_j(\widehat{A}) = \sigma_j(A)$ for all j, whereas $\sigma_j(C) \ge \sigma_j(\widehat{C})$ for all j by virtue of Fact 2.3. Therefore $\sigma_\rho(\widehat{A})/\sigma_n(\widehat{C}) \ge \sigma_\rho(A)/\sigma_n(C)$, and so it is sufficient to estimate the ratio $\sigma_\rho(\widehat{A})/\sigma_n(\widehat{C})$ from above. $\widehat{A}$ and $\widehat{C}$ are square matrices, and so $\sigma_\rho(\widehat{A})/\sigma_l(\widehat{C}) \le (1 + ||\widehat{U}||)(1 + ||V||)\widehat{f}_r$ where $\widehat{f}_r = \max\{1, ||\widehat{U}_r^{-1}||\}\ \max\{1, ||V_r^{-1}||\}$. Clearly $||\widehat{U}|| \le ||U||$. Substitute this bound and obtain the theorem in the case where $m > n$.

For $m < n$ apply the same proof to the matrices A^T and C^T replacing A and C. □

In the above proof the transition $U \Longrightarrow \tilde{U} = S_A^T U$ and $V \Longrightarrow \tilde{V} = T_A^T V$ has kept the matrices in $\mathcal{G}_{\mu,\sigma}^{m\times r}$ and $\mathcal{G}_{\mu,\sigma}^{n\times r}$, respectively, and so we can extend Theorem 5.4 to probabilistic estimate for the ratio $\sigma_\rho(A)/\sigma_l(C)$ by applying Theorem 5.2 for $W = U$ and $n = m$ and for $W = V$ as well as applying Theorem 5.3. This yields the following corollary.

Corollary 5.1. *Keep all assumptions of parts (a)–(f) of Theorem 5.1 as well as Theorem 5.3, but let $\mu = 0$, assume that $A \in \mathbb{R}^{m\times n}$ and $U \in \mathcal{G}_{0,\sigma}^{m\times r}$ for any pair of positive integers m and n and that the matrix $C = A+UV^T$ has full rank $l = \min\{m, n\}$ (and thus is nonsingular where $m = n$). Then bound (5.2) still holds, the norms $||U||$ and $||V||$ still satisfy the randomized bounds of Theorem 5.2 (for $W = U$ and m replacing n as well as for $W = V$), and the reciprocals of the norms $||U_r^{-1}||$, and $||V_r^{-1}||$ still satisfy the randomized bounds of Theorem 5.3.*

Corollary 5.2. *Under the assiumptions of Corollary 5.1 the condition number $\kappa(C)$ is expected to have at most order $\kappa(A)||A||/\sigma_\rho(A)$, and so Theorem 1.1 can be extended to the case of matrices $A \in R^{m\times n}$, $U \in \mathcal{G}_{0,\sigma}^{m\times r}$ and $V \in \mathcal{G}_{0,\sigma}^{n\times r}$ for any pair of positive integers $\{m, n\}$.*

Remark 5.2. *How large is our class of $m\times n$ matrices $\tilde{A}$ having a numerical rank q? We characterize it indirectly, by noting that by virtue of Fact 2.2 the nearby matrices A of rank q form a variety of dimension $(m + n - q)q$, which increases as q increases.*

5.3 Dual additive preconditioning

For an $m \times n$ matrix A of full rank extend (2.6) to define the *dual additive preprocessing*

$$A^+ \Longrightarrow C_-^+ = A^+ + U_- V_-^T. \tag{5.4}$$

Our analysis implies that the value $\kappa(C_-^+)$ (equal to $\kappa(C_-)$) is expected to have order $\sigma_{q+1}(A)/\sigma_l(A)$ provided $l = \min\{m, n\}$, $U_- \in \mathcal{G}_{0,1}^{n\times q}$, $V_- \in \mathcal{G}_{0,1}^{m\times q}$, and the norm $||A^+||$ is neither large nor small.

The randomized algorithm of [D83] is expected to estimate the norm $||A^+||$ at a low computational cost. We can work with the $(m+1)\times(n+1)$ matrix $\widehat{A}=\mathrm{diag}(A,\epsilon)$ instead of the matrix A and choose a sufficiently small positive scalar ϵ such that $||\widehat{A}^+||=1/\epsilon$. Then we can scale the matrix $\widehat{A}$ to obtain that $||(\widehat{A}/\epsilon)^+||=1$.

6 Randomized augmentation

6.1 The case of general matrices

The solution of a nonsingular linear system of n equations, $A\mathbf{y}=\mathbf{b}$ can be readily recovered from a null vector $\begin{pmatrix}-1/\beta\\ \mathbf{y}\end{pmatrix}$ of the matrix $K=(\beta\mathbf{b}\mid A)$ for a nonzero scalar β. If the matrix A has numerical nullity 1 and if the ratio $||A||/||\beta\mathbf{b}||$ is neither large nor small, then the matrix K is well conditioned for the average vector $\mathbf{b}$ [PQa, Section 9.1]. The following simple theorem links additive preprocessing $A\Longrightarrow C=A+UV^T$ to an extension of such an augmentation technique.

Theorem 6.1. *Suppose* $K=\begin{pmatrix}I_r & V^T\\ -U & A\end{pmatrix}\in\mathbb{R}^{(m+r)\times(n+r)}$ *and* $C=A+UV^T$. *Then* $K=\widehat{U}\,\mathrm{diag}(C,I_r)\widehat{V}$ *for* $\widehat{U}=\begin{pmatrix}O_{r,m} & I_r\\ I_m & -U\end{pmatrix}$, $\widehat{V}=\begin{pmatrix}O_{n,r} & I_n\\ I_r & V^T\end{pmatrix}$, *and the matrix* C *is nonsingular if and only if the matrix* K *is nonsingular. Furthermore if the matrices* C *and* K *are nonsingular, then* $C^{-1}=(O_{n,r}\mid I_n)K^{-1}(O_{m,r}\mid I_m)^T$ *and* $K^{-1}=\bar{V}\,\mathrm{diag}(C^{-1},I_r)\bar{U}$ *for* $\bar{U}=\widehat{U}^{-1}=\begin{pmatrix}U & I_m\\ I_r & O_{r,m}\end{pmatrix}$ *and* $\bar{V}=\widehat{V}^{-1}=\begin{pmatrix}-V^T & I_r\\ I_n & O_{n,r}\end{pmatrix}$.

Now suppose that $m=n$, $U\in\mathcal{G}_{0,1}^{m\times r}$, $V\in\mathcal{G}_{0,1}^{n\times r}$, and $||A||\approx 1$, combine Theorem 6.1 and Corollary 5.2, and deduce that the matrix K is expected to have the condition number of order $\sigma_1(A)/\sigma_{n-r}(A)$. Thus the matrix K is expected to be well conditioned if the matrix A has numerical nullity at most r.

Let us prove similar preconditioning property for the general class of northwestern augmentations.

Theorem 6.2. *Suppose*

$$K=\begin{pmatrix}W & V^T\\ -U & A\end{pmatrix}, \tag{6.1}$$

$U,V\in\mathcal{G}_{0,1}^{n\times r}$, $W\in\mathbb{R}^{r\times r}$, $A\in\mathbb{R}^{n\times n}$, $||A||=1$, $||W||\le 1$, *and* $A=S_A\Sigma_A T_A^T$ *is the SVD. Write* $\bar{K}=\mathrm{diag}(I_r,S_A^T)K\,\mathrm{diag}(I_r,T_A)$. *Then*

(i) $\sigma_j(K)=\sigma_j(\bar{K})$ *for all* j *and*

(ii) $\bar{K}=\begin{pmatrix}W & \bar{V}^T\\ -\bar{U} & \Sigma_A\end{pmatrix}$ *where* $\bar{U},\bar{V}\in\mathcal{G}_{0,1}^{n\times r}$.

(ii) Furthermore write $\Sigma_A=\mathrm{diag}(\Sigma_{n-r},\Sigma_r)$, $\bar{U}=\begin{pmatrix}U_0\\ U_1\end{pmatrix}$ *and* $\bar{V}=\begin{pmatrix}V_0\\ V_1\end{pmatrix}$ *where* Σ_k *denotes the* $k\times k$ *diagonal submatrices of* Σ_A *for* $k=n-r$ *and* $k=r$; $U_0,V_0\in\mathcal{G}_{0,1}^{(n-r)\times r}$ *and* $U_1,V_1\in\mathcal{G}_{0,1}^{r\times r}$, *and so with probability* 1 *the matrices* U_1 *and* V_1 *are nonsingular. Suppose that indeed these matrices as well as the matrix* Σ_{n-r} *are nonsingular, whereas* $\Sigma_r=O$. *Then*

$$\bar{K}=\begin{pmatrix}W & V_1^T & V_0^T\\ -U_0 & \Sigma_{n-r} & O\\ -U_1 & O & O\end{pmatrix}\ \textit{and}\ \bar{K}^{-1}=\begin{pmatrix}O & O & -U_1^{-1}\\ O & \Sigma_{n-r}^{-1} & -\Sigma_{n-r}^{-1}U_0U_1^{-1}\\ V_1^{-T} & -V_1^{-T}V_0^T\Sigma_{n-r}^{-1} & V_1^{-T}(W+V_0^T\Sigma_{n-r}^{-1}U_0)U_1^{-1}\end{pmatrix}.$$

Proof. Parts (i) and (ii) follow because the matrices S_A and T_A are orthogonal (use Lemma 4.3 to prove part (ii)). Part (iii) is verified by inspection. □

We obtain $||K|| = ||\bar{K}|| \le 2 + \max\{||U||, ||V||\}$ from parts (i) and (ii), recall that $||W|| \le 1$, note that $||\Sigma_{n-r}^{-1}|| = 1/\sigma_{n-r}(A)$ and deduce from parts (i) and (iii) that

$$||K^{-1}|| = ||\bar{K}^{-1}|| \le \nabla/\sigma_{n-r}(A) + ||U_1^{-1}|| + ||V_1^{-1}|| + ||U_1^{-1}||\ ||V_1^{-1}||$$

where

$$\nabla = 1 + ||U_0||\ ||U_1^{-1}|| + ||V_0||\ ||V_1^{-1}|| + ||U_0||\ ||V_0||\ ||U_1^{-1}||\ ||V_1^{-1}||.$$

We can estimate the condition number $\kappa(K) = ||K||\ ||K^{-1}||$ by combining the above bounds on the norms $||K||$ and $||K^{-1}||$ and by applying Theorems 3.1 and 3.2 to estimate the norms $||U||$, $||V||$, $||V_0||$, $||U_0||$, $||U_1^{-1}||$ and $||V_1^{-1}||$. The resulting crude bound of order $||A||/\sigma_{n-r}(A)$ also holds under sufficiently small perturbations of the matrix A (cf. Theorem 2.1).

By combining this bound with Theorem 6.1 for $W = I_r$, we obtain a distinct derivation of Theorem 1.1 in the case of matrices A having numerical nullity r.

Next we extend the SMW formula to the augmentation. Under (6.1) let the matrices A, W and K be nonsingular, let $m = n$, and write $S = A + UW^{-1}V^T$ and $R = I - V^T S^{-1} U W^{-1}$. Then the matrix S is nonsingular, S^{-1} is the $n \times n$ trailing (southwestern) block of K^{-1}, and SMW formula (1.1), for C replaced by S, U by UW^{-1}, and G by R, implies that

$$A^{-1} = S^{-1} + S^{-1}UW^{-1}R^{-1}V^T S^{-1}. \tag{6.2}$$

Remark 6.1. *Here is an example of minor differences between the power of augmentation and additive preprocessing. The map* $A \Longrightarrow K = \begin{pmatrix} W & V^T \\ V & A \end{pmatrix}$ *cannot decrease the condition number* $\kappa(A)$ *if* K *is a symmetric and positive definite matrix; this follows from the Interlacing Property of the eigenvalues of* K *[GL96, Theorem 8.1.7]. In contrast scaled randomized symmetric additive preprocessing* $A \Longrightarrow C = A + VV^T$ *is expected to work as preconditioning for an ill conditioned symmetric positive definite matrix* A *having a small numerical nullity (cf. (5.3) and [W07]).*

6.2 A randomized Toeplitz solver

Let us apply Theorem 2.3 to support randomized augmentation for solving a nonsingular Toeplitz linear system $T\mathbf{y} = \mathbf{b}$ of n equations provided the matrix T has numerical nullity 1.

To compute the vector $\mathbf{y} = T^{-1}\mathbf{b}$, we first embed the matrix T into an $(n+1) \times (n+1)$ Toeplitz matrix $K = \begin{pmatrix} w & \mathbf{v}^T \\ \mathbf{f} & T \end{pmatrix}$. We write $w = \mathbf{e}_1^T T \mathbf{e}_1$ and fill the vectors $\mathbf{f} = (f_i)_{i=1}^n$ and $\mathbf{v} = (v_i)_{i=1}^n$ with appropriate entries of the matrix T except for the two coordinates f_n and v_n, which we choose at random and then scale to have the ratios $\frac{|f_n|}{||K||}$ and $\frac{|v_n|}{||K||}$ neither large nor small.

Part (b) of Theorem 2.3 expresses the inverse T^{-1} via the vectors $\mathbf{v} = K^{-1}\mathbf{e}_1$ and $\mathbf{w} = K^{-1}\mathbf{e}_{n+1}$.

In view of Section 3.2 and Appendix A, this policy is likely to produce a nonsingular matrix K whose inverse is likely to have a nonzero entry $\mathbf{e}_1^T K^{-1}\mathbf{e}_1$. Our tests were in very good accordance with these formal results and moreover consistently produced well conditioned matrices K.

To summarize, we reduce the solution of a nonsingular ill conditioned Toeplitz linear system $T\mathbf{y} = \mathbf{b}$ to computing highly accurate solutions of two linear systems $K\mathbf{x} = \mathbf{e}_1$ and $K\mathbf{z} = \mathbf{e}_{n+1}$, both expected to be well conditioned. High accuracy shall counter the magnification of the input and rounding errors, expected in the case of ill conditioned input.

In the important special case where a Toeplitz matrix T is real symmetric, we can choose real scalars w and $f_n = v_n$ to yield a real symmetric matrix $K = \begin{pmatrix} w & \mathbf{v}^T \\ \mathbf{v} & T \end{pmatrix}$. In this case the computation is simplified because $J_{n+1}K^{-1}J_{n+1} = K^{-1}$, $J_{n+1}\mathbf{e}_{n+1} = \mathbf{e}_1$, and so $K^{-1}\mathbf{e}_{n+1} = J_{n+1}K^{-1}\mathbf{e}_1$, and we only need to solve a single linear system with the matrix K. We refer to the resulting algorithm for the linear system $T\mathbf{y} = \mathbf{b}$ as **Algorithm 6.1**. In Section 8.6 we test this algorithm for solving an ill conditioned real symmetric Toeplitz linear system.

One can readily extend the approach of this section to the case of Toeplitz-like, Hankel and Hankel-like inputs as well as to augmenting the input matrix with r rows and r columns for $r > 1$. For the transition back to the solution of the original problem, we can employ expression (6.2) or Theorem 2.3.

7 Applications of randomized additive preprocessing and augmentation

In this section we outline some randomized matrix algorithms that utilize our results of the two previous sections. In the next subsection we apply Corollary 5.2 to estimate the numerical rank and numerical nullity of a matrix. In Section 7.2 we first recall the results of [PQ10] on computing a nmb and a left nmb of a rank deficient matrix and then extend these results, again based on Corollary 5.2, to randomized approximation of trailing and leading singular spaces of ill conditioned matrices. In Section 7.3 we show a simple alternative recipe for randomized approximation of leading singular spaces and then extend both of our recipes for this task to lower-rank approximation of an ill conditioned matrix. In Sections 7.4 and 7.5 we suppose that some approximate bases for trailing and leading singular spaces of a matrix are available (say supplied by the algorithms of Section 7.2) and then compute its 2×2 block triangulation to facilitate inversion and the solution of linear systems with this input matrix. In Section 7.6 we solve such linear systems by combining Corollary 5.2 with the SMW formula, and our analysis shows significant theoretical decrease of the estimated computational cost versus Gaussian elimination. We comment on using random structured matrices in these computations in Section 7.7 and on the links between our randomized computations and Newton's iteration for matrix inversion in Section 7.8. The results of Section 7.2 are used in Sections 7.3–7.5, but otherwise all subsections can be read independently of each other.

To simplify the notation we allow A to denote both rank deficient matrices of a fixed rank ρ and ill conditioned matrices having a fixed numerical rank q and possibly having full rank, and we still allow q (resp. r) to denote either the numerical rank (resp. numerical nullity) or its upper bound where this causes no confusion.

7.1 Estimation of numerical rank and numerical nullity

For an $m \times n$ matrix A that has norm 1 and numerical rank q, we expect to compute the integer $r = n - q$ by means of at most $2\lceil \log_2 r(r - r_-) \rceil$ search steps given a lower bound r_- on $r \geq 0$. In a search step we fix an integer s and a pair of matrices $U \in \mathcal{G}_{0,1}^{m \times s}$ and $V \in \mathcal{G}_{0,1}^{n \times s}$ and test whether the matrix $C = A + UV^T$ has full rank and is well conditioned. We apply the test for $s = 0, 1, 2, 4, \ldots$. If the matrix C passes the test for some $s = 2^h$ and $h > 1$, but not for $s = 2^{h-1}$, we expect to have r in the range $(2^{h-1}, 2^h]$ by virtue of Corollary 5.2. Then we apply binary search in this range for the integer r as the minimum integer for which the matrix C passes the test of having full rank and being well conditioned.

In the case of $q = l - r \ll r$ we begin our search for r with an upper bound r_+ on $r \leq l = \min\{m, n\}$, and in at most $2\lceil \log_2(r_+ - r) \rceil$ search steps we expect to compute $s = r$, again as the minimum integer for which the matrix C has full rank and is well conditioned. The minimum is attained where the ratio $\frac{||AC^{-1}U||}{||A||\ ||C^{-1}U||}$ is small (cf. [PQ10, Algorithm 6.7]).

Remark 7.1. *To facilitate the binary search, one can apply the power transforms $A \Longrightarrow B = (AA^T)^h A$ for positive integers h. They increase the gaps between all pairs of distinct singular values of A because $\sigma_j(B) = (\sigma_j(A))^{2h+1}$.*

7.2 Computation of nmbs and approximation of singular spaces

One can compute nmbs based on the algorithms of [PQa] or on the following theorem.

Theorem 7.1. *[PQ10, Theorem 3.1]. Suppose a matrix $A \in \mathbb{R}^{m \times n}$ has rank ρ, $0 < \rho < l = \min\{m, n\}$, $U \in \mathbb{R}^{m \times r}$, $V \in \mathbb{R}^{n \times r}$, $r = l - \rho$, and the matrix $C = A + UV^T$ has full rank l. Then the matrix C^+U is a* nmb(A) *if $m \geq n$, whereas the matrix $C^{+T}V$ is a left* nmb(A) *if $n \geq m$.*

We extend this result to the approximation of trailing and leading singular spaces in the case where the matrix A has numerical rank q.

Theorem 7.2. *Assume a matrix $A \in \mathbb{R}^{m \times n}$ having numerical rank q where $0 < q < l = \min\{m, n\}$.*

(a) Write $r = l - q$ and suppose $U \in \mathbb{R}^{m\times r}$, $V \in \mathbb{R}^{n\times r}$, and the matrix $C = A + UV^T$ has full rank and is well conditioned. Then there is a scalar c independent of A, U, V, m, n and q such that $||C^+UX - T_{A,r}|| \le c\sigma_{q+1}(A)$ for some matrix $X \in \mathbb{R}^{r\times r}$ if $m \ge n$, whereas $||C^{+T}VY - S_{A,r}|| \le c\sigma_{q+1}(A)$ for some matrix $Y \in \mathbb{R}^{r\times r}$ if $n \ge m$.

(b) Assume that the matrices $A \in \mathbb{R}^{m\times n}$, $U_- \in \mathbb{R}^{n\times q}$, $V_- \in \mathbb{R}^{m\times q}$, $H = I_q + V_-AU_-^T$, and $C_-^+ = A^+ + U_-V_-^T$ have full ranks, the matrix A^+ has numerical rank q, and the matrix C_-^+ is well conditioned. Then there exists a scalar c_- independent of A, U_-, V_-, m, n and q and such that $||C_-U_-X_- - S_{q,A}|| \le c_-\sigma_{q+1}(A)$ for some matrix $X_- \in \mathbb{R}^{q\times q}$ if $m \le n$, whereas $||C_-^TV_-Y_- - T_{q,A}|| \le c_-\sigma_{q+1}(A)$ for some matrix $Y_- \in \mathbb{R}^{q\times q}$ if $m \ge n$.

Proof. Part (a) has been proved in [PQ10, Section 7.1]. To extend the proof to part (b) rewrite the SVD $A = S_A\Sigma_AT_A^T$ as follows (cf. Section 2.3), $A = (S_{q,A} \mid S_{A,m-q})\,\mathrm{diag}(\Sigma_{q,A}, \bar{\Sigma}_A)(T_{q,A} \mid T_{A,n-q})^T$, where the matrices $S_{q,A}$ and $T_{q,A}$ are formed by the first q columns of the matrices S_A and T_A, respectively, whereas $\Sigma_{q,A}$ is the $q \times q$ leading block of the diagonal matrix Σ_A. Then apply part (a) to the matrix $A^+ = T_A\Sigma_A^+S_A^T = (T_{q,A} \mid T_{A,n-q})\,\mathrm{diag}(\Sigma_{q,A}^+, \bar{\Sigma}_A^+)(S_{q,A} \mid S_{A,m-q})^T$ observing that $\mathcal{R}(T_{q,A}) = \mathbb{S}_{A^+,q}$ for $m \ge n$, whereas $\mathcal{R}(S_{q,A}) = \mathbb{T}_{A^+,q}$ for $m \le n$. □

By virtue of Corollary 5.2 the matrix C of part (a) is expected to have full rank and to be well conditiined (as required in part (a)) provided $U \in \mathcal{G}^{m\times r}_{0,||A||}$ and $V \in \mathcal{G}^{n\times r}_{0,||A||}$, and the same properties hold for the matrix C_- in part (b) provided $U_- \in \mathcal{G}^{n\times q}_{0,||A^+||}$ and $V_- \in \mathcal{G}^{m\times q}_{0,||A^+||}$,

Part (a) of Theorem 7.2 shows that $\mathcal{R}(C^+U) \approx \mathbb{T}_{A,r}$ if $m \ge n$ and $\mathcal{R}(C^{+T}V) \approx \mathbb{S}_{A,r}$ if $m \le n$, that is, the linear spaces $\mathcal{R}(C^+U)$ for $m \ge n$ and $\mathcal{R}(C^{+T}V)$ for $m \le n$ approximate the right and left trailing singular spaces associated with the r smallest singular values of the matrix A, respectively. (Some of these values can vanish.) Likewise part (b) shows that $\mathcal{R}(C_-U_-) \approx \mathbb{S}_{q,A}$ if $m \le n$ and $\mathcal{R}(C_-^TV_-) \approx \mathbb{T}_{q,A}$ if $m \ge n$, that is, the linear spaces $\mathcal{R}(C_-U_-)$ for $m \le n$ and $\mathcal{R}(C_-^TV_-)$ for $m \ge n$ approximate the left and right leading singular spaces associated with the q largest singular values of the matrix A, respectively.

Alternatively we can approximate the leading singular spaces $\mathbb{S}_{q,A}$ by $\mathcal{R}(A^TM)$ for $M \in \mathcal{G}^{m\times q}_{0,1}$ and $\mathbb{T}_{q,A}$ by $\mathcal{R}(AN)$ for $N \in \mathcal{G}^{n\times q}_{0,1}$. Indeed Theorem 4.1 implies that $\mathcal{R}(A^TM) \approx \mathbb{S}_{q,A}$ and $\mathcal{R}(AN) \approx \mathbb{T}_{q,A}$.

We can extend the recipes of Remarks 4.2 and 7.1 to improve the approximations to the linear spaces $\mathbb{S}_{q,A}$, $\mathbb{T}_{q,A}$, $\mathbb{S}_{A,r}$, and $\mathbb{T}_{A,r}$ computed based on any of the two approaches.

In our tests in the next section we employed the following relationships,

$$X \approx B^+T_{A,r} \text{ and } C^+UX - T_{A,r} \approx BB^+T_{A,r} - T_{A,r}, \tag{7.1}$$

$$Y_- \approx B_-^+T_{q,A} \text{ and } C_-^TV_-Y_- - T_{q,A,} \approx B_-B_-^+T_{q,A} - T_{q,A}. \tag{7.2}$$

We obtain (7.1) from part (a) of Theorem 7.2 for $C^+U = B$ and $BX \approx T_{A,r}$ and obtain (7.2) from part (b) of Theorem 7.2 for $C_-^TV_- = B_-$ and $B_-Y_- \approx T_{q,A}$.

[PQa, Section 5.1] extends Theorem 7.1 under both bounds $m \ge n$ and $m < n$ as follows.

Theorem 7.3. *Assume two matrices $A \in \mathbb{R}^{m\times n}$ of a rank $\rho < n$ and $V \in \mathbb{R}^{r\times n}$ for $r = n - \rho$. Suppose the matrix $K = \begin{pmatrix} V \\ A \end{pmatrix}$ has full column rank n. Then $B = K^{(I)}\begin{pmatrix} I_r \\ O \end{pmatrix}$ is a nmb(A).*

Remark 7.2. *For $K^{(I)} = K^+$ and a matrix $A \in \mathbb{R}^{m\times n}$ having numerical rank $q < n$, we can extend the theorem to approximate the trailing singular space $\mathbb{T}_{A,n-q}$ by $\mathcal{R}(B)$, and we can improve the approximation by applying or extending the recursive techniques of Remarks 4.2 and 7.1. See other alternative techniques in [PQ10] and [PQa].*

Remark 7.3. *In the case where $m = n$ Theorems 7.1 and 7.2 define both left and right nmbs or approximations to both left and right trailing and leading singular spaces. We can reduce the computation of a null vector of a rectangular matrix A, its nmb and approximate nmb to the case of square inputs. Indeed (a) $\mathcal{N}(A) = \mathcal{N}(A^TA)$, (b) $\mathcal{N}(A) = \mathcal{N}(B^TA)$ if $A, B \in \mathbb{R}^{m\times n}$ and the matrix*

B has full rank $m \le n$, and (c) $(A \mid O_{n,m-n})\mathbf{u} = \mathbf{0}_m$ if and only if $A\widehat{\mathbf{u}} = \mathbf{0}_m$ provided $m \ge n$ and $\widehat{\mathbf{u}} = (I_n \mid O_{n,m-n})\mathbf{u}$, whereas $\mathbf{v}^T \begin{pmatrix} A \\ O_{n-m,n} \end{pmatrix} = \mathbf{0}_n^T$ if and only if $\widehat{\mathbf{v}}^T A = \mathbf{0}_n^T$ provided $m < n$ and $\widehat{\mathbf{v}} = \mathbf{v}^T \begin{pmatrix} I_m \\ O_{n-m,m} \end{pmatrix}$. Furthermore given an $m \times n$ matrix A for $m > n$, we can represent it as the sum $A = \sum_{i=1}^{h} (O, B_i^T, O)^T$ where $B_i = (O, I_{k_i}, O)^T A$ are $k_i \times n$ matrices for $i = 1, \dots, h$, $\sum_{i=1}^{h} k_i = m$. Then $\mathcal{N}(A) = \cap_{i=1}^{h} \mathcal{N}(B_i)$, and we can compute the intersection of nmbs based on [GL96, Theorem 12.4.1].

7.3 Approximation by low-rank matrices

Suppose that a matrix $A = S_A^T \Sigma_A T_A$ has numerical rank q, M is a matrix of rank q, and

$$\mathcal{R}(M) \approx \mathbb{T}_{q,A}. \tag{7.3}$$

Write $Q = Q(M)$. Then both matrices $AM(M^TM)^{-1}M^T$ and AQQ^T have rank q and approximate the matrix A. Thus the low-rank approximation of the matrix A is reduced to the approximation of a basis for its leading singular space $\mathbb{T}_{q,A}$.

One can deduce from Theorem 4.1 that the matrix $M = AG$ for $G \in \mathcal{G}_{0,1}^{n\times q}$ (and empirically even for $G \in \mathcal{T}_{0,1}^{n\times q}$) is expected to satisfy (7.3). Likewise the matrix $M = C_-^T V_-$ is expected to satisfy (7.3) for $C_- = (C_-^+)^+$, C_-^+ of (5.4), $\sigma \approx ||A^+||$, $U_- \in \mathcal{G}_{0,\sigma}^{n\times q}$ and $V_- \in \mathcal{G}_{0,\sigma}^{m\times q}$ (and empirically even for $U_- \in \mathcal{T}_{0,\sigma}^{n\times q}$ and $V_- \in \mathcal{T}_{0,\sigma}^{m\times q}$). In the case of a square nonsingular matrix A we can compute the matrix C_- by applying the dual SMW formula (2.7). In that case we can confine all divisions to the stages of the orthogonalization of the $n \times q$ matrix $C_-^T V_-$ and the inversion of the $q \times q$ matrix H. (The matrix H is nonsingular if so is the matrix $A^{-1} + U_- V_-^T$.) To increase the probability of obtaining a close approximation of the matrix A by the matrices $AM(M^TM)^{-1}M^T$ and AQQ^T at a slightly higher computational cost, one can write $M = AG$ for $G \in \mathcal{G}_{0,1}^{n\times q_+}$ and for q_+ slightly increasing above q (cf. [HMT11]), and similarly where $M = C_-^T V_-$, $U_- \in \mathcal{G}_{0,\sigma}^{n\times q_+}$ and $V_- \in \mathcal{G}_{0,\sigma}^{m\times q_+}$.

By applying the same algorithms to the displacement A of a matrix W having a numerical displacement rank q [BM01], that is lying near some matrices with displacement rank q, we approximate the matrix W by one of these matrices. For a sample application we can apply such approximations to simplify Newton's structured matrix inversion (see Section 7.8).

7.4 Block triangulation using approximate trailing singular spaces

Theorems 7.1–7.3 define randomized algorithms that compute nmbs of a rank deficient matrix and approximate trailing and leading singular spaces of its small norm perturbations. Next we extend these algorithms to block triangulation of such perturbations.

Theorem 7.4. *For a matrix $A \in \mathbb{R}^{m\times n}$ and $q < l = \min\{n, m\}$, let $\sigma_q(A) \gg \sigma_{q+1}(A)$ and write $r = n - q$ and $\bar{r} = m - q$.*

(a) Suppose $L_0 \in \mathcal{G}_{0,1}^{n\times q}$, $L_1 \in \mathbb{R}^{n\times r}$, $\mathcal{R}(L_1) = \mathbb{T}_{A,r}$, and $||L_1|| = 1$. Then $||AL_1|| \le \sigma_{q+1}(A)$ and the condition number $\kappa(AL_0)$ is expected to have order $\sigma_1(A)/\sigma_q(A)$.

(b) Suppose $K_0 \in \mathcal{G}_{0,1}^{q\times m}$, $K_1 \in \mathbb{R}^{\bar{r}\times m}$, $\mathcal{R}(K_1) = \mathbb{S}_{A,\bar{r}}$, and $||K_1|| = 1$. Then $||K_1A|| \le \sigma_{q+1}(A)$ and the condition number $\kappa(K_0A)$ is expected to have order $\sigma_1(A)/\sigma_q(A)$.

Proof. To estimate $\kappa(AL_0)$ and $\kappa(K_0A)$ combine Theorems 3.2 and 4.1. To estimate $||AL_1||$ substitute $A = S_A \Sigma_A T_A^T$ and $T_A^T L_1 = (O_{n,n-r}, T_{A,r})L_1$ to obtain $AL_1 = S\Sigma T_A^T L_1 = S\Sigma(O_{n,n-r}, T_{A,r})L_1$, $||AL_1|| \le ||S\Sigma(O_{n,n-r}, T_{A,r})||\ ||L_1|| = \sigma_{q+1}(A)$. Similarly estimate $||K_1A||$. □

Clearly, small perturbations of the matrices K_1 and L_1 little change the upper bounds on the norms $||K_1A||$ and $||AL_1||$, and we can extend Theorem 7.4 to the case where we are given approximate bases to the trailing singular spaces $\mathbb{S}_{A,\bar{r}}$ and $\mathbb{T}_{A,r}$.

Namely suppose that the assumptions of both parts (a) and (b) of the theorem hold for a small norm perturbation of the matrix A, that is $\mathcal{R}(K_1) \approx \mathbb{S}_{A,\bar{r}}$ and $\mathcal{R}(L_1) \approx \mathbb{T}_{A,r}$ for $r = n - q$,

$\bar{r} = m - q$, $K_0 \in \mathcal{G}_{0,1}^{q\times m}$, $L_0 \in \mathcal{G}_{0,1}^{n\times q}$, $K_1 \in \mathbb{R}^{\bar{r}\times m}$, $L_1 \in \mathbb{R}^{n\times r}$. Then Theorem 7.4 implies that the $q \times q$ leading block W_{00} of the $m \times n$ matrix $W = \begin{pmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \end{pmatrix} = KAL$, for $K = \begin{pmatrix} K_0 \\ K_1 \end{pmatrix}$ and $L = (L_0 \mid L_1)$, is still expected to dominate the three other blocks. Indeed by virtue of Theorem 7.4 we expect that the condition number $\kappa(W_{00})$ has order $\sigma_1(A)/\sigma_q(A)$, whereas the sum of the norms $||W_{01}|| + ||W_{10}|| + ||W_{11}||$ has at most the order $\sigma_{q+1}(A)$. Note that $\mathbf{y} = L\mathbf{x}$ provided $A\mathbf{y} = \mathbf{b}$ and $W\mathbf{x} = K\mathbf{b}$. If in addition the ratio $\sigma_1(A)/\sigma_q(A)$ is not large, that is if the matrix A has numerical rank q, then the matrix W_{00} is nonsingular and well conditioned. Let us specify this computation for $m = n$, $r = \bar{r}$, and the matrices K_1 and L_1 computed based on our randomized additive preconditioning.

Algorithm 7.1. Block triangulation using approximate trailing singular spaces.

INPUT: *A matrix $A \in \mathbb{R}^{n\times n}$ whose norm $||A||$ is neither large nor small, its numerical rank q satisfying $0 < q = n - r < n$, and a Subroutine* LIN·SOLVE *that either solves a linear system of equations if it is nonsingular and well condtioned or outputs FAILURE otherwise.*

OUTPUT: *FAILURE (with a low probability) or two random matrices K_0 and L_0 in $\mathbb{R}^{n\times q}$ and two matrices K_1 and L_1 in $\mathbb{R}^{n\times r}$ such that with a probability near 1 the $q \times q$ block submatrix $W_{00} = K_0^T A L_0$ of the matrix $W = (K_0 \mid K_1)^T A (L_0 \mid L_1) = \begin{pmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \end{pmatrix}$ is nonsingular, well conditioned, and strongly dominant, that is $\sigma_q(W_{00}) \gg \max\{||W_{01}||, ||W_{10}||, ||W_{11}||\}$.*

COMPUTATIONS:

1. *Generate four matrices $K_0, L_0 \in \mathcal{G}_{0,1}^{n\times q}$; $U, V \in \mathbb{G}_{0,1}^{n\times r}$. Output the matrices K_0 and L_0.*
2. *Compute the matrix $C = A + UV^T$, expected to be nonsingular and well conditioned.*
3. *Apply the Subroutine* LIN·SOLVE *to compute and to output the matrices $K_1 = C^{-T}V$ and $L_1 = C^{-1}U$. Stop and output FAILURE if so does the subroutine.*

Correctness of the algorithm follows from part (a) of Theorem 7.4. The algorithm fails with a low probability by virtue of Theorems 1.1 and 7.2 (part (a)) and the results of Section 7.2.

Block factorization

$$W = \begin{pmatrix} I & O \\ W_{10}W_{00}^{-1} & I \end{pmatrix} \begin{pmatrix} W_{00} & W_{01} \\ O & G \end{pmatrix}$$

for $G = W_{11} - W_{10}W_{00}^{-1}W_{01}$ implies that

$$W^{-1} = \begin{pmatrix} W_{00}^{-1} & -W_{00}^{-1}W_{01}G^{-1} \\ O & G^{-1} \end{pmatrix} \begin{pmatrix} I & O \\ -W_{10}W_{00}^{-1} & I \end{pmatrix}$$

and thus reduces the inversion of the matrices W and A and the solution of a linear system $A\mathbf{y} = \mathbf{b}$ to the similar operations with the matrices W_{00} and G of smaller sizes, where the matrix W_{00} is expected to be nonsingular and well conditioned.

Tables 8.11 and 8.12 demonstrate the power of this approach versus the customary algorithms.

Remark 7.4. *We expect to arrive at the matrices W_{01}, W_{10} and W_{11} having small norms. To counter the expected cancellation of the leading digits of the $2rn - r^2$ entries of these matrices, we should compute the matrices K_1 and L_1, their products by the blocks of the matrix A, and the Schur complement G with a high precision p_+. These computations involve $O(n^2r)$ flops, that is just a r/n fraction of order n^3 flops in high precision p_+ required by Gaussian elimination.*

Remark 7.5. *We can modify the block triangulation by replacing the random matrices K_0 with* $\mathrm{nmb}(K_1)$ *and L_0 with* $\mathrm{nmb}(L_1)$ *(cf. Algorithm 7.2 in the next subsection). E.g., we can first generate random matrices $G, H \in \mathcal{G}_{0,1}^{n\times q}$ and $U, V \in \mathbb{G}_{0,1}^{n\times r}$, all having full rank with probability 1, and then successively compute the matrices $C = A + UV^T$, $K_1 = C^{-T}V$, $L_1 = C^{-1}U$, $K_0 = (I_n - K_1(K_1^TK_1)^{-1}K_1^T)G$, and $L_0 = (I_n - L_1(L_1^TL_1)^{-1}L_1^T)H$. Apart from the well conditioned computation of the matrices $C^{-T}V$ and $C^{-1}U$, this takes $O(n^2r)$ flops, and we can extend our comments in the previous remark.*

7.5 Block triangulation using approximate leading singular spaces

Suppose a square matrix A has a small positive numerical rank q and define a dual variation of Algorithm 7.1 based on part (b) of Theorem 7.2. In this case matrix inversions are limited to the $q \times q$ matrices H, $K_0^T K_0$ and $L_0^T L_0$. Alternatively we can employ any other pair of approximate bases for the left and right leading singular spaces. E.g., we can compute them as the products $A^T V$ and AU for $V \in \mathcal{G}_{0,1}^{q \times m}$ and $U \in \mathcal{G}_{0,1}^{n \times r}$ (cf. Section 7.3). In our dual algorithm we assume that the nonsingular input matrix A has been scaled so that the norm $||A^{-1}||$ is neither large nor small; we can yield this property by applying the technique pointed out at the end of Section 5.

Algorithm 7.2. Block triangulation using approximate leading singular spaces.

INPUT: *A nonsingular ill conditioned matrix $A \in \mathbb{R}^{n \times n}$ scaled so that the norm $||A^{-1}||$ is neither large nor small; the numerical rank q of the matrix A such that $0 < q = n - r < n$, and a Subroutine* INVERT *that either inverts a matrix if it is nonsingular and well conditioned or outputs FAILURE otherwise.*

OUTPUT: *FAILURE (with a low probability) or four matrices $K_0,\ L_0 \in \mathbb{R}^{n \times q}$ and $K_1,\ L_1 \in \mathbb{R}^{n \times r}$ such that*

$$W = \begin{pmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \end{pmatrix} = (K_0 \mid K_1)^T A (L_0 \mid L_1)$$

and the block submatrix $W_{00} = K_0^T A L_0$ is expected to be nonsingular, well conditioned, and strongly dominant, such that $\sigma_q(W_{00}) \gg \max\{||W_{01}||,\ ||W_{10}||,\ ||W_{11}||\}$.

COMPUTATIONS**:**

1. *Generate two matrices U_- and V_- in $\mathcal{G}_{0,1}^{n \times q}$.*
2. *Compute the matrix $H = I_q + V_- A U_-^T$ of (2.7).*
3. *Apply the Subroutine* INVERT *to compute the matrix H^{-1}. Stop and output FAILURE if so does the subroutine.*
4. *Compute the matrix $C_- = A - AU_- H^{-1} V_-^T A$ of (2.7).*
5. *Compute and output the matrices $K_0 = C_- U_- / ||C_- U_-||$ and $L_0 = C_-^T V_- / ||C_-^T V_-||$.*
6. *Compute the matrices $M = \mathrm{nmb}(K_0^T)$ and $N = \mathrm{nmb}(L_0^T)$ (see our Section 7.2, [PQ10], and [PQa] on the computation of nmbs).*
7. *Compute and output the matrices $K_1 = M/||M||$ and $L_1 = N/||N||$.*

Correctness of this randomized algorithm follows from parts (b) of Theorems 7.2 and 7.4, and we can extend the comments in Remark 7.4. The algorithm fails with a low probability by virtue of Theorems 1.1 and 7.2 (part (b)) and the results of Section 7.2.

7.6 Randomized additive preconditioning with the SMW recovery

Suppose that we seek the solution $\mathbf{y} = A^{-1}\mathbf{b}$ of a real nonsingular ill conditioned linear system $A\mathbf{y} = \mathbf{b}$ of n equations where we are given a small upper bound r on the numerical nullity of A. Assume that the norm $||A||$ is neither large nor small. Then randomized additive preprocessing $A \Longrightarrow C = A + UV^T$ for $U, V \in \mathcal{G}_{0,1}^{n \times r}$ is expected to produce a well conditioned matrix C (cf. Corollary 5.2). The SMW formula (1.1) implies that $\mathbf{y} = C^{-1}\mathbf{b} + C^{-1}UG^{-1}V^T C^{-1}\mathbf{b}$ for $G = I_r - V^T C^{-1} U$. Substitute $X_U = C^{-1}U$ and $\mathbf{x_b} = C^{-1}\mathbf{b}$ and obtain

$$\mathbf{y} = \mathbf{x_b} + X_U G^{-1} V^T \mathbf{x_b} \text{ for } G = I_r - V^T X_U. \tag{7.4}$$

This reduces the computation of $\mathbf{y}$ essentially to the solution of the matrix equation $CX = (U \mid \mathbf{b})$ for $X = (X_U \mid x_\mathbf{b})$, computing the matrix G, and its inversion. The solution algorithm below incorporates iterative refinement.

Algorithm 7.3. Randomized Solution of a Linear System with Iterative Refinement.

INPUT: *a vector* $\mathbf{b} \in \mathbb{R}^{n\times 1}$*, a nonsingular ill conditioned matrix* $A \in \mathbb{R}^{n\times n}$*, and its numerical nullity* r *(cf. Remark 7.6).*

OUTPUT: $\tilde{\mathbf{y}} \approx A^{-1}\mathbf{b}$.

COMPUTATIONS*:*

1. *Generate two matrices* $U, V \in \mathcal{G}_{0,\sigma}^{n\times r}$.
2. *Compute the matrix* $C = A + UV^T$*, expected to be nonsingular and well conditioned.*
3. *Apply Gaussian elimination (or another direct algorithm) to compute an approximate inverse* $Y \approx C^{-1}$*. (Perform the computations in double precision. Application of the same algorithm to the original ill conditioned linear system* $A\mathbf{y} = \mathbf{b}$ *would require about as many flops but in extended precision.)*
4. *Apply sufficiently many loops of iterative refinement employing the approximate inverse* Y *to compute sufficiently accurate solution* X_U *of the matrix equation* $CX_U = U$*. (High accuracy is required to counter the cancelation of leading bits in the subsequent computation of the Schur complement* $G = I_r - V^TC^{-1}U$*.) Then recover a close approximation to the vector* $\mathbf{y} = A^{-1}\mathbf{b}$ *by applying equation (7.4).*

The algorithm reduces the original task of computations with ill conditioned matrix A to the computations with the well conditioned matrix C and matrices U and V of the smaller size $n \times r$. Handling an ill conditioned input A, we must perform computations with extended precision to counter magnification of rounding errors, but we can confine this essentially to computing the Schur complement $G = I_r - V^TC^{-1}U$. This is a small fraction of the computational time of the customary algorithms for a linear system $A\mathbf{y} = \mathbf{b}$ provided the ratio r/n is small and the precision required to handle the ill conditioned matrix A is high.

Let us supply some estimates. To support iterative refienment we must use a precision p exceeding $\log_2 \kappa(C)$; for well conditioned matrices C we can assume that $p > 2\log_2 \kappa(C)$, say. Then order $p - \log_2 \kappa(C)$ new correct bits are produced per an output value by a loop of iterative refinement (see [PGMQ, Sevtion 9]), which is reduced essentially to multiplication of the matrices C and Y by $2r$ vectors, that is to $(4n-2)nr$ flops in a low (e.g., double) precision p. The refinement algorithm outputs order rn values; by using advanced technique of [ORO05] one can accumulate them with high accuracy as the sums of sufficiently many low precision summands (cf. symbolic lifting in [P11]). Overall with this advanced implementation we only perform $O(rn^2p_+/p)$ flops in low precision p at Stage 4 of Algorithm 7.3.

For comparison Gaussian elimination uses $\frac{2}{3}n^3 + O(n^2)$ flops in extended precision $p_+ \approx p_{\text{out}} + \log_2 \kappa(A)$ to output the solution to the ill conditioned linear system $A\mathbf{y} = \mathbf{b}$ with a prescribed precision p_{out}. We compute an approximate inverse Y of the well conditioned matrix C at Stage 3 by using $\frac{2}{3}n^3 + O(n^2)$ flops as well, but in the low precision p. The cost of performing Stages 1 and 2 is dominated, and so our progress is significant where $np \gg rp_+$ and p_+ greatly exceeds p.

Finally, given a nonsingular $n \times n$ matrix A (with $||A^{-1}|| \approx 1$) and a small upper bound q on its numerical rank, we can define a dual variation of Algorithm 7.3 as follows: generate a pair of matrices $U_-, V_- \in \mathbb{G}_{0,1}^{n\times q}$ and then compute the matrices H and C_- of (2.6) to reduce the solution of a linear system of equations $A\mathbf{y} = \mathbf{b}$ to computing the vector $\mathbf{y} = (C_-^{-1} - U_-V_-^T)\mathbf{b}$.

Remark 7.6. *(Cf. Remark 5.1.) There is no point for applying Algorithm 7.3 where the matrix* A *is well conditioned or has numerical nullity exceeding* r*. In the former case the preconditioning is not needed, whereas in the latter case additive preprocessing would produce an ill conditioned matrix* C*. In both cases preprocessing would give no benefits but would involve extraneous computations and additional rounding errors. In the case where* r *is equal to the numerical nullity of* A*, however, these deficiencies are overwhelmed by the benefits where we avoid performing order* n^3 *high precision flops by applying additive preconditioning* $A \Longrightarrow C = A + UV^T$*. In principle one can extend the algorithm to the case where the input integer* r *is an upper bound on the unknown numerical nullity*

ν, but for $r > \nu$ the Schur complement G is expected to be ill conditioned [PGMQ, Section 7], which would complicate numerical computations.

Remark 7.7. *One can replace iterative refinement with the Conjugate Gradient or GMRES algorithms (cf. [GL96]). To their advantage they use no approximate inverse, but they are more sensitive to the success of preconditioning. In particular every Conjugate Gradient loop (essentially multiplication of the matrices C and C^T by two vectors) produces order of $1/\kappa(C)$ new correct bits per an output value versus $p - \log_2 \kappa(C)$ in iterative refinement. Thus we need stronger upper bounds on $\kappa(C)$ to ensure progress in the presence of rounding errors.*

7.7 Randomized structured preprocessing

Would the $n \times n$ preprocessed matrices $C = A + UV^T$ inherit the structure of an $n \times n$ matrix A where $U, V \in \mathbb{R}^{n \times r}$? For a small value r the adverse impact of adding the matrix UV^T on the structure is small, e.g., the displacement rank increases by $O(r)$ (cf. [P01]).

We can control this impact even for large values r by endowing the matrices U and V with proper structure and relying on ample empirical data that show preconditioning power of such structured additive preprocessing (see Remark 4.1 and Table 8.5).

In particular in the case of a nonsingular ill conditioned $n \times n$ Toeplitz-like matrix A having numerical nullity r and norm $||A|| \approx 1$ and given with a displacement generator of a small length d, we can choose a pair of $n \times r$ standard Gaussian random Toeplitz matrices U and V and obtain a displacement generator for the matrix $C = A + UV^T$ having length $d + O(1)$ and expected to be well conditioned. By exploiting the structure we can operate with this matrix in nearly linear arithmetic time, e.g., solve a nonsingular linear system $A\mathbf{y} = \mathbf{b}$ in $O(d^2 n \log^2 n)$ flops, even where r is large.

Randomized augmentation can preserve matrix structure even better and has about the same preconditioning power (see Section 6 and Table 8.15).

Alternative deterministic techniques of homotopy continuation also support inversion of nonsingular Toeplitz and various other structured matrices in nearly linear time (see [P01, Section 6.9], [P07], [P10]).

7.8 Preprocessing for Newton–Toeplitz iteration

Recall Newton's iteration for matrix inversion

$$X_{i+1} = X_i(2I - CX_i), \ i = 0, 1, \dots . \tag{7.5}$$

Its ith loop squares the residual $I - CX_i$, that is,

$$I - CX_{i+1} = (I - CX_i)^2 = (I - CX_0)^{2^{i+1}}. \tag{7.6}$$

Therefore

$$||I - CX_{i+1}|| \le ||I - CX_i||^2 = ||I - CX_0||^{2^{i+1}}, \ i = 0, 1, \dots , \tag{7.7}$$

so that the approximations X_i quadratically converge to the inverse C^{-1} right from the start provided that $||I - CX_0|| < 1$.

We can ensure that $||I - CX_0|| \le 1 - \frac{2n}{(\kappa(C))^2(1+n)}$ by choosing $X_0 = \frac{2nC^T}{(1+n)||C||_1||C||_\infty}$ [PS91].

Such a map $C \Longrightarrow X_0$ preserves the matrix structure of Toeplitz or Hankel type, but is the structure maintained throughout the iteration? Not automatically. In fact Newton's loop can triple the displacement rank of a matrix X_k. The structure can be maintained, however, via recursive compression of the displacement (also called *recompression*), in which case we arrive at *Newton's structured* (e.g., Newton–Toeplitz) iteration. In particular we can periodically set to 0 the smallest singular values of the displacements of the matrices X_i to keep the length of the displacements within a fixed tolerance, equal to or a little exceeding the displacement rank of the input matrix C. At this stage we can also apply the techniques of Section 7.3 to approximate the displacements of the matrices X_i by low-rank matrices. We refer the reader to [P01, Chapter 6], [P10] on the history, variations, and analysis of this approach, proposed in [P92], [P93], and [P93a] for Toeplitz-like

matrices. In [PBRZ99, Section 7.5.4] this iteration has been linked to iterative refinement combined with recursive updating of the input matrix; quadratic convergence of the resulting iteration has been proved in [PBRZ99]. In [BM01] the extension of this study has naturally led to the important concept of approximate displacement rank of a matrix.

According to the estimates in [P01], the Newton–Toeplitz iteration converges quadratically right from the start provided $||I - CX_0|| < \frac{1}{(1+||Z_e||+||Z_f||)\kappa(C)}||L^{-1}||$, $||L^{-1}|| \le c_{e,f}n$, L denotes the associated displacement operator $L: \ C \to Z_eC - CZ_f$ for $e \neq f$ or $L: \ C \to C - Z_eCZ_f^T$ for $ef \neq 1$, and $c_{e,f}$ is a constant defined by e and f. Similar bounds can be deduced for other classes of matrices with displacement structure [P01, Section 6.6], [PRW02].

Newton's iteration can be incorporated into our randomized algorithms. E.g., we can use it instead of Gaussian elimination in Algorithm 7.3.

Conversely, our preconditioning techniques are a natural tool for decreasing the initial residual norm $||I - CX_0||$ where it is close to 1.

We can concurrently apply Newton–Toeplitz iteration to a number of scaled randomized small rank modifications of the input matrix. As soon as one of these applications produces the inverse, we can recover the inverse of the original matrix via the SMW formula (1.1) or in case of augmentation via (6.2). Other than that no processor communication or synchronization are requiered.

8 Numerical Experiments

Our numerical experiments with random general, Hankel, Toeplitz and circulant matrices have been performed in the Graduate Center of the City University of New York on a Dell server with a dual core 1.86 GHz Xeon processor and 2G memory running Windows Server 2003 R2. The test Fortran code was compiled with the GNU gfortran compiler within the Cygwin environment. Random numbers were generated with the random_number intrinsic Fortran function, assuming the uniform probability distribution over the range $\{x : -1 \le x < 1\}$. The tests have been designed by the first author and performed by his coauthors.

8.1 Conditioning tests

We have computed the condition numbers of $n \times n$ random general matrices for $n = 2^k$, $k = 5, 6, \ldots$, with entries sampled in the range $[-1, 1)$ as well as complex general, Toeplitz, and circulant matrices whose entries had real and imaginary parts sampled at random in the same range $[-1, 1)$. We have performed 100 tests for each class of inputs, each dimension n, and each nullity r. Tables 8.2–8.4 display the test results. The last four columns of each table display the average (mean), minimum, maximum, and standard deviation of the computed condition numbers of the input matrices, respectively. Namely we have computed the values $\kappa(A) = ||A||\ ||A^{-1}||$ for general, Toeplitz, and circulant matrices A and the values $\kappa_1(A) = ||A||_1\ ||A^{-1}||_1$ for Toeplitz matrices A. We have computed and displayed in Table 8.3 the 1-norms of Toeplitz matrices and their inverses rather than their 2-norms to facilitate the computations in the case of inputs of large sizes. Relationships (2.1) link the 1-norms and 2-norms to one another, but the empirical data in Table 8.1 consistently show even closer links, in all cases of $n \times n$ general, Toeplitz, and circulant matrices A where $n = 32, 64, \ldots, 1024$.

8.2 Preconditioning tests

Table 8.5 covers our tests for the preconditioning power of additive preprocessing in [PIMR10]. We have tested the input matrices of the following classes.

1n. *Nonsymmetric matrices of type I with numerical nullity r.* $A = S\Sigma_r T^T$ are $n \times n$ matrices where S and T are $n \times n$ random orthogonal matrices, that is, the factors Q in the QR factorizations of random real matrices; $\Sigma_r = \mathrm{diag}(\sigma_j)_{j=1}^n$ is the diagonal matrix such that $\sigma_{j+1} \le \sigma_j$ for $j = 1, \ldots, n-1$, $\sigma_1 = 1$, the values $\sigma_2, \ldots, \sigma_{n-r-1}$ are randomly sampled in the semi-open interval $[0.1, 1)$, $\sigma_{n-r} = 0.1$, $\sigma_j = 10^{-16}$ for $j = n-r+1, \ldots, n$, and therefore $\kappa(A) = 10^{16}$ [H02, Section 28.3].

1s. *Symmetric matrices of type I with numerical nullity r.* The same as in part 1n, but for $S = T$.

The matrices of the six other classes were constructed in the form of $\frac{A}{||A||} + \beta I$ where the recipes for defining the matrices A and scalars β are specified below.

2n. *Nonsymmetric matrices of type II with numerical nullity r.* $A = (W \mid WZ)$ where W and Z are random orthogonal matrices of sizes $n \times (n-r)$ and $(n-r) \times r$, respectively.

2s. *Symmetric matrices of type II with numerical nullity r.* $A = WW^T$ where W are random orthogonal matrices of size $n \times (n-r)$.

3n. *Nonsymmetric Toeplitz-like matrices with numerical nullity r.* $A = c(T \mid TS)$ for random Toeplitz matrices T of size $n \times (n-r)$ and S of size $(n-r) \times r$ and for a positive scalar c such that $||A|| \approx 1$.

3s. *Symmetric Toeplitz-like matrices with numerical nullity r.* $A = cTT^T$ for random Toeplitz matrices T of size $n \times (n-r)$ and a positive scalar c such that $||A|| \approx 1$.

4n. *Nonsymmetric Toeplitz matrices with numerical nullity* 1. $A = (a_{i,j})_{i,j=1}^{n}$ is an $n \times n$ Toeplitz matrix. Its entries $a_{i,j} = a_{i-j}$ are random for $i - j < n - 1$. The entry $a_{n,1}$ is selected to ensure that the last row is linearly expressed through the other rows.

4s. *Symmetric Toeplitz matrices with numerical nullity* 1. $A = (a_{i,j})_{i,j=1}^{n}$ is an $n \times n$ Toeplitz matrix. Its entries $a_{i,j} = a_{i-j}$ are random for $|i - j| < n - 1$, whereas the entry $a_{1,n} = a_{n,1}$ is a root of the quadratic equation $\det A = 0$. We have repeatedly generated the matrices A until we arrived at the quadratic equation having real roots.

We have set $\beta = 10^{-16}$ for symmetric matrices A in the classes 2s, 3s, and 4s, so that $\kappa(A) = 10^{16} + 1$ in these cases. For nonsymmetric matrices A we have defined the scalar β by an iterative process such that $||A|| \approx 1$ and $10^{-18}||A|| \le \kappa(A) \le 10^{-16}||A||$ [PIMR10, Section 8.2].

Table 8.5 displays the average values of the condition numbers $\kappa(C)$ of the matrices $C = A + UU^T$ over 100,000 tests for the inputs in the above classes, $r = 1, 2, 4, 8$ and $n = 100$. We have defined the additive preprocessor UU^T by a normalized $n \times r$ matrix $U = U/||U||$ where $U^T = (\pm I \mid O_{r,r} \mid \pm I \mid O_{r,r} \mid \ldots \mid O_{r,r} \mid \pm I \mid O_{r,s})$, we have chosen the integer s to obtain $n \times r$ matrices U and have chosen the signs for the matrices $\pm I$ at random.

In our further tests the condition numbers of the matrices $C = A + 10^p UV^T$ for $p = -10, -5, 5, 10$ were steadily growing within a factor $10^{|p|}$ as the value $|p|$ was growing. This have showed the importance of proper scaling of the additive preprocessor UV^T.

8.3 Application of random circulant and Householder multipliers

Table 8.6 shows the results of our tests of the solution of a nonsingular well conditioned linear system $A\mathbf{y} = \mathbf{b}$ of n equations whose coefficient matrix had ill conditioned $n/2 \times n/2$ leading principal block for $n = 64, 256, 1024$. We have performed 100 numerical tests for each dimension n and computed the maximum, minimum and average relative residual norms $||A\mathbf{y} - \mathbf{b}||/||\mathbf{b}||$ as well as standard deviation. GENP applied to these systems has output corrupted solutions with residual norms ranging from 10 to 10^8. When we preprocessed the systems with circulant multipliers filled with ± 1 (choosing the n signs $\pm$ at random), the norms decreased to at worst 10^{-7} for all inputs. Table 8.6 also shows further decrease of the norm in a single step of iterative refinement. Table 2 in [PQZa] shows similar results of the tests where the input matrices were chosen similarly but so that all their leading blocks had numerical nullities 0 and 1 and where Householder multipliers $I_n - \mathbf{u}\mathbf{v}^T/\mathbf{u}^T\mathbf{v}$ replaced the circulant multipliers. Here $\mathbf{u}$ and $\mathbf{v}$ denote two vectors filled with integers 1 and -1 under random choice of the signs $+$ and $-$.

8.4 Approximation of the tails and heads of SVDs and low-rank approximation of a matrix

Table 8.7 shows the data from our tests on the approximation of trailing singular spaces of the SVD of an $n \times n$ matrix A having numerical nullity $r = n - q$ and on the approximation of this matrix with a matrix of rank $q = n - r$.

For $n = 64, 128, 256$ and $q = 1, 8, 32$ we have generated $n \times n$ random unitary matrices S and T and diagonal matrices $\Sigma = \mathrm{diag}(\sigma_j)_{j=1}^n$ such that $\sigma_j = 1/j,\ j = 1, \dots, q$, $\sigma_j = 10^{-10},\ j = q+1, \dots, n$ (cf. [H02, Section 28.3]). Then we computed the input matrices $A = S_A \Sigma_A T_A^T$, for which $||A|| = 1$ and $\kappa(A) = 10^{10}$. Furthermore we have generated pairs of $n \times r$ random matrices U and V for $r = 1, 8, 32$, scaled them to have $||UV^T|| \approx 1$, and computed the matrices $C = A + UV^T$, $B_r = C^{-1}U$, $Y_r = B_r^+ T_{A,r}$; $B_r Y$, $B_r Y_r - T_{A,r}$, $Q = Q(B_r)$, and $AQQ^T = A - A(I_n - QQ^T)$.

Table 8.7 summarizes the data on the values $\kappa(C)$ and the residual norms $\mathrm{rn}_1 = ||B_r Y_r - T_{A,r}||$ (cf. (7.1)) and $\mathrm{rn}_2 = ||AQQ^T||$ observed in 100 runs of our tests for every pair of n and q.

We have performed similar tests on the approximation of leading singular spaces of the SVDs of the same $n \times n$ matrices A having numerical rank q and numerical nullity $r = n - q$ and on the approximation of this matrix with a matrix of rank q. In some tests we employed dual additive preprocessing to approximate matrix bases for the leading singular spaces $\mathbb{T}_{q,A}$ of the matrices A. We have generated the pairs of $n \times q$ random matrices U_- and V_- for $q = 1, 8, 32$, scaled them to have $||U_- V_-^T|| \approx ||A^{-1}|| = 10^{10}$, and successively computed the matrices $H = I_q + V_-^T A U_-$ and $C_- = A - AU_- H^{-1} V_-^T A$ (cf. (2.7)), $B_{q,A} = C_-^T V_-$ (all three with extended precision), $Y_{q,A} = (B_{q,A})^+ T_{q,A}$; $B_{q,A} Y_{q,A}$, $B_{q,A} Y_{q,A} - T_{q,A}$, $Q_{q,A} = Q(B_{q,A})$, and $A - AQ_{q,A}(Q_{q,A})^T$. Table 8.8 summarizes the data on the condition numbers $\kappa(C_-)$ and the residual norms $\mathrm{rn}^{(1)} = ||B_{q,A} Y_{q,A} - T_{q,A}||$ (cf. (7.2)) and $\mathrm{rn}^{(2)} = ||A - AQ_{q,A}(Q_{q,A})^T||$ obtained in 100 runs of our tests for every pair of n and q.

We have also performed similar tests where we generated $n \times q$ random matrices U (for $q = 1, 8, 32$) and $n \times q$ random Toeplitz matrices $\bar{U}$ (for $q = 8, 32$) and then replaced the above matrix $B_{q,A}$ with the approximate matrix bases $Q_{q,A} = Q(AU)$ and $A\bar{U}$ for the leading singular space $\mathbb{T}_{q,A}$. Tables 8.9 and 8.10 display the results of these tests. In both cases the residual norms are equally small and are about as small as in Tables 8.7 and 8.8.

8.5 Solution of linear systems of equations based on approximation of trailing and leading singular spaces of the SVDs

For our further tests we have chosen $n = 32, 64$ and $r = 1, 2, 4$ and for every pair $\{n, r\}$ generated 100 instances of vectors $\mathbf{b}$ and matrices A, U, and V as follows.

We have generated (a) random vectors $\mathbf{b}$ of dimension n, (b) the matrices A as the error-free products $S\Sigma T^T$ where S and T were $n \times n$ random real orthonormal matrices (generated with double precision), $\Sigma = \mathrm{diag}(\sigma_j)_{j=1}^n$, $\sigma_{n-j} = 10^{-17}$ for $j = 0, 1, \dots, r-1$, and $\sigma_{n-j} = 1/(n-j)$ for $j = r, \dots, n-1$, and (c) $n \times r$ random matrices U and V such that $||A| = ||U|| = ||V|| = 1$. Note that $||A^{-1}|| = 10^{17}$.

For every choice of these matrices we have solved the linear systems $A\mathbf{y} = \mathbf{b}$ by applying Algorithm 7.1. We first generated $n \times (n-r)$ random matrices K_0 and L_0 and then computed the matrices $C = A + UV^T$ (which were always nonsingular and well conditioned in our tests), $K_1 = C^{-T}V$, $L_1 = C^{-1}U$, and $W = (K_0 \mid K_1)^T A (L_0 \mid L_1) = \begin{pmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \end{pmatrix}$. In all our tests the $(n-r) \times (n-r)$ leading principal $(n-r) \times (n-r)$ block $W_{00} = K_0^T A L_0$ was well conditioned and strongly dominated the three other blocks W_{01}, W_{10}, and W_{11} in the 2×2 block matrix W, as we expected to see in virtue of our analysis in Section 7.4. To solve the linear system $W\mathbf{x} = (K_0 \mid K_1)^T \mathbf{b}$, we first computed the dominated blocks W_{01}, W_{10}, and W_{11} with extended precision, then eliminated the subdiagonal block and computed the solution of the resulting block triangular linear system; the leading block W_{00} of its coefficient matrix was expected to be and consistently turned out to be well conditioned. Finally we computed and output the vector $\mathbf{y} = (L_0 \mid L_1)\mathbf{x}$.

Table 8.11 shows the average (mean) values of the relative residual norms $||A\mathbf{y} - \mathbf{b}||/||\mathbf{b}||$ of the output vectors $\mathbf{y}$ (these values range about 10^{-10}) as well as the minimums, maximums, and standard deviations in these tests.

For the same ill conditioned inputs the Subroutine MLDIVIDE(A,B) for Gaussian elimination from MATLAB has produced corrupted outputs, as can be seen from Table 8.12.

We have also performed similar tests for $n = 32, 64$ and $n \times n$ matrices A and vectors $\mathbf{b}$ generated as before, but for $q = n - r = 1, 2, 4$, and then we computed orthogonal matrices K_0, K_1, L_0 and L_1 by employing dual additive preprocessing and Algorithm 7.2. We first generated and scaled the

pairs of $n \times q$ random matrices U_- and V_- such that $||U_-|| \approx ||V_-|| \approx 3 * 10^8$, and so $||U_-||\ ||V_-|| \approx ||A^{-1}|| = 10^{17}$. Then we successively computed the matrices H and C_- (as in Section 8.4), C_-U_- and $C_-^T V_-$ (all with extended precision), $K_0 = Q(C_-U_-)$, $L_0 = Q(C_-^T V_-)$, $K_1 = Q(\mathrm{nmb}(K_0^T))$ and $L_1 = Q(\mathrm{nmb}(L_0^T))$, and continued as in the tests for Table 8.11. We displayed the results in Table 8.13, showing the residual norms of the order 10^{-9} on the average.

Furthermore we have performed similar tests where we first generated $n \times q$ random matrices U and V and then replaced the above matrices K_0 and L_0 by $K_0 = Q(A^T V)$ and $L_0 = Q(AU)$. Table 8.14 displays the results of these tests, showing the residual norms of the order 10^{-25} on the average.

Then again for the same ill conditioned inputs the Subroutine MLDIVIDE(A,B) for Gaussian elimination from MATLAB produced corrupted outputs, similarly to the results in Table 8.12.

8.6 Solution of a real symmetric Toeplitz linear system of equations with randomized augmentation

We have solved 100 real symmetric linear systems of equations $T\mathbf{y} = \mathbf{b}$ for each n where we used vectors $\mathbf{b}$ with random coordinates from the range $[-1, 1)$ and Toeplitz matrices $T = S + 10^{-9} I_n$ for an $n \times n$ singular symmetric Toeplitz matrices S having rank $n - 1$ and nullity 1 and generated according to the recipe in [PQ10, Section 10.1b].

Table 8.15 shows the average CPU time of the solution by our Algorithm 6.1 and, for comparison, based on the QR factorization and SVD, which we computed by applying the LAPACK procedures DGEQRF and DGESVD, respectively. To solve the auxiliary Toeplitz linear system $K\mathbf{x} = \mathbf{e}_1$ in Algorithm 6.1, we first employed the Toeplitz linear solver of [KV99], [V99], [VBHK01], and [VK98] and then applied iterative refinement with double precision.

The abbreviations "Alg. 6.1", "QR", and "SVD" indicate the respective algorithms. The last two columns of the table display the ratios of these data in the first and the two other columns.

We measured the CPU time with the mclock function by counting cycles. One can convert them into seconds by dividing their number by a constant CLOCKS_PER_SEC, which is 1000 on our platform. We marked the table entries by a "-" where the tests have run too long and were not completed.

We have obtained the solutions $\mathbf{y}$ with the relative residual norms of about 10^{-15} in all three algorithms, which showed that Algorithm 6.1 employing iterative refinement was as reliable as the QR and SVD based solutions but ran much faster.

We refer the reader to [PQZC, Table 3] on similar test results for the solution of ill conditioned homogeneous Toeplitz linear systems.

9 Related work, our technical novelties, and further study

Preconditioned iterative algorithms for linear systems of equations is a classical subject [A94], [B02], [G97]. The problem of creating inexpensive preconditioners for general use has been around for a long while as well.

On the study of conditioning of random matrices see [D88], [E88], [ES05], [CD05], [SST06], [B11], and the bibliography therein. Estimation of the condition numbers of random structured matrices was stated as a challenge in [SST06]. We provide such estimates for Gaussian random Toeplitz and circulant matrices in Sections 3.4 and 3.5. In particular the estimates show that the expected condition number of a Gaussian random $n \times n$ Toeplitz matrix does not grow exponentially fast as n grows to infinity. This can be surprising in view of [BG05].

Our present study of randomized preconditioning extends and substantially advances the works [PGMQ], [PIMR10], [PQZa], and [PQZC]. In our Sections 7 and 8 we outline and test some new applications of randomized preconditioning, whereas our Theorems 1.1 and 4.1 and Corollaries 4.2 and 5.2 provide formal support for such applications.

In particular our Theorem 4.1 and and Corollary 4.2 support the heuristic recipe of randomized multiplicative preconditioning versus pivoting, thus complementing its empirical support in [PQZa, Table 2]. Besides the cited estimates for the condition numbers of Gaussian random Toeplitz and

Table 8.1: The norms of random general, Toeplitz and circulant matrices and of their inverses

matrix A	n	$\|\|A\|\|_1$	$\|\|A\|\|_2$	$\frac{\|\|A\|\|_1}{\|\|A\|\|_2}$	$\|\|A^{-1}\|\|_1$	$\|\|A^{-1}\|\|_2$	$\frac{\|\|A^{-1}\|\|_1}{\|\|A^{-1}\|\|_2}$
General	32	1.9×10^1	1.8×10^1	1.0×10^0	4.0×10^2	2.1×10^2	1.9×10^0
General	64	3.7×10^1	3.7×10^1	1.0×10^0	1.2×10^2	6.2×10^1	2.0×10^0
General	128	7.2×10^1	7.4×10^1	9.8×10^{-1}	3.7×10^2	1.8×10^2	2.1×10^0
General	256	1.4×10^2	1.5×10^2	9.5×10^{-1}	5.4×10^2	2.5×10^2	2.2×10^0
General	512	2.8×10^2	3.0×10^2	9.3×10^{-1}	1.0×10^3	4.1×10^2	2.5×10^0
General	1024	5.4×10^2	5.9×10^2	9.2×10^{-1}	1.1×10^3	4.0×10^2	2.7×10^0
Toeplitz	32	1.8×10^1	1.9×10^1	9.5×10^{-1}	2.2×10^1	1.3×10^1	1.7×10^0
Toeplitz	64	3.4×10^1	3.7×10^1	9.3×10^{-1}	4.6×10^1	2.4×10^1	2.0×10^0
Toeplitz	128	6.8×10^1	7.4×10^1	9.1×10^{-1}	1.0×10^2	4.6×10^1	2.2×10^0
Toeplitz	256	1.3×10^2	1.5×10^2	9.0×10^{-1}	5.7×10^2	2.5×10^2	2.3×10^0
Toeplitz	512	2.6×10^2	3.0×10^2	8.9×10^{-1}	6.9×10^2	2.6×10^2	2.6×10^0
Toeplitz	1024	5.2×10^2	5.9×10^2	8.8×10^{-1}	3.4×10^2	1.4×10^2	2.4×10^0
Circulant	32	1.6×10^1	1.8×10^1	8.7×10^{-1}	9.3×10^0	1.0×10^1	9.2×10^{-1}
Circulant	64	3.2×10^1	3.7×10^1	8.7×10^{-1}	5.8×10^0	6.8×10^0	8.6×10^{-1}
Circulant	128	6.4×10^1	7.4×10^1	8.6×10^{-1}	4.9×10^0	5.7×10^0	8.5×10^{-1}
Circulant	256	1.3×10^2	1.5×10^2	8.7×10^{-1}	4.7×10^0	5.6×10^0	8.4×10^{-1}
Circulant	512	2.6×10^2	3.0×10^2	8.7×10^{-1}	4.5×10^0	5.4×10^0	8.3×10^{-1}
Circulant	1024	5.1×10^2	5.9×10^2	8.7×10^{-1}	5.5×10^0	6.6×10^0	8.3×10^{-1}

circulant matrices, our technical novelties include randomized multiplicative and additive preconditioning, the proof of their power, extension of the SMW formula, and block factorizations in Sections 7.4 and 7.5 based on randomized approximation of singular spaces.

Approximation by low-rank matrices (cf. Section 7.3) and the extensions to tensor decompositions are thriving research area, with numerous applications to matrix and tensor computations. Pne can trace its previous study through the papers [GTZ97], [GT01], [GOS08], [T00], [MMD08], [OT09], [HMT11], and the bibliography therein, but much earlier advances in this area appeared in the papers [BCLR79], [B80], [B85], [B86], [BC87], directed to estimating the border rank of matrices and tensors and initially motivated by the design of fast matrix multiplication algorithms. Presently, linking tensor and matrix computations for their acceleration is a fashionable subject with applications to many important areas of modern computing (see, e.g., [T00], [MMD08], [OT09], [KB09]), but then again its earliest examples appeared in the cited papers on border rank and in [P72]. The latter paper has introduced the technique of trilinear aggregation, a basic ingredient of all subsequent fast algorithms for matrix multiplication with the inputs of both immense sizes (far beyond any practical interest) [P79], [P81], [P84], [CW90], [S10], [VW12] and realistic moderate sizes [P81], [P84], [LPS92], [K04]; historically this work was the first example of the acceleration of fundamental matrix computations by means of tensor decomposition.

We expect that our present paper will motivate further efforts toward development and refinement of the outlined algorithms, as well as further study of randomized augmentation techniques, of its link to aggregation processes of [MP80], revealed in [PQa], and of specification of our methods to ill conditioned matrices that have the displacement or rank structures (cf. [KKM79], [P90], [GKO95], [P01], [VBHK01], [EG99], [VVM07], [VVM08]).

Advancing our progress in Section 3.4 toward formal support of randomized Toeplitz preconditioning is another research challenge. Comparison and combination of our novel techniques with each other and various known methods, such as the homotopy continuation methods and Newton's structured iteration (cf. Section 7.8, [PKRK], [P01, Chapter 6], and [P10] seem to be natural research directions as well.

Appendix

Table 8.2: The condition numbers $\kappa(A)$ of random matrices A

n	**input**	**min**	**max**	**mean**	**std**
32	real	2.4×10^1	1.8×10^3	2.4×10^2	3.3×10^2
32	complex	2.7×10^1	8.7×10^2	1.1×10^2	1.1×10^2
64	real	4.6×10^1	1.1×10^4	5.0×10^2	1.1×10^3
64	complex	5.2×10^1	4.2×10^3	2.7×10^2	4.6×10^2
128	real	1.0×10^2	2.7×10^4	1.1×10^3	3.0×10^3
128	complex	1.3×10^2	2.5×10^3	3.9×10^2	3.3×10^2
256	real	2.4×10^2	8.4×10^4	3.7×10^3	9.7×10^3
256	complex	2.5×10^2	1.4×10^4	1.0×10^3	1.5×10^3
512	real	3.9×10^2	7.4×10^5	1.8×10^4	8.5×10^4
512	complex	5.7×10^2	3.2×10^4	2.3×10^3	3.5×10^3
1024	real	8.8×10^2	2.3×10^5	8.8×10^3	2.4×10^4
1024	complex	7.2×10^2	1.3×10^5	5.4×10^3	1.4×10^4
2048	real	2.1×10^3	2.0×10^5	1.8×10^4	3.2×10^4
2048	complex	2.3×10^3	5.7×10^4	6.7×10^3	7.2×10^3

Table 8.3: The condition numbers $\kappa_1(A) = \frac{||A||_1}{||A^{-1}||_1}$ of random Toeplitz matrices A

n	**min**	**mean**	**max**	**std**
256	9.1×10^2	9.2×10^3	1.3×10^5	1.8×10^4
512	2.3×10^3	3.0×10^4	2.4×10^5	4.9×10^4
1024	5.6×10^3	7.0×10^4	1.8×10^6	2.0×10^5
2048	1.7×10^4	1.8×10^5	4.2×10^6	5.4×10^5
4096	4.3×10^4	2.7×10^5	1.9×10^6	3.4×10^5
8192	8.8×10^4	1.2×10^6	1.3×10^7	2.2×10^6

A Uniform random sampling and nonsingularity of random matrices

Uniform random sampling of elements from a finite set Δ is their selection from this set at random, independently of each other and under the uniform probability distribution on the set Δ.

Theorem A.1. *Under the assumptions of Lemma 3.1 let the values of the variables of the polynomial be randomly and uniformly sampled from a finite set Δ. Then the polynomial vanishes with a probability at most $\frac{d}{|\Delta|}$.*

Corollary A.1. *Let the entries of an $m \times n$ general or Toeplitz matrix have been randomly and uniformly sampled from a finite set Δ of cardinality $|\Delta|$ (in any fixed ring). Let $l = \min\{m, n\}$. Then (a) every $k \times k$ submatrix M for $k \le l$ is nonsingular with a probability at least $1 - \frac{k}{|\Delta|}$ and (b) is strongly nonsingular with a probability at least $1 - \sum_{i=1}^{k} \frac{i}{|\Delta|} = 1 - \frac{(k+1)k}{2|\Delta|}$. Furthermore (c) if the submatrix M is indeed nonsingular, then any entry of its inverse is nonzero with a probability at least $1 - \frac{k-1}{|\Delta|}$.*

Proof. The claimed properties of nonsingularity and nonvanishing hold for generic matrices. The singularity of a $k \times k$ matrix means that its determinant vanishes, but the determinant is a polynomial of total degree k in the entries. Therefore Theorem A.1 implies parts (a) and consequently (b). Part (c) follows because a fixed entry of the inverse vanishes if and only if the respective entry of the

Table 8.4: The condition numbers $\kappa(A)$ of random circulant matrices A

n	**min**	**mean**	**max**	**std**
256	9.6×10^0	1.1×10^2	3.5×10^3	4.0×10^2
512	1.4×10^1	8.5×10^1	1.1×10^3	1.3×10^2
1024	1.9×10^1	1.0×10^2	5.9×10^2	8.6×10^1
2048	4.2×10^1	1.4×10^2	5.7×10^2	1.0×10^2
4096	6.0×10^1	2.6×10^2	3.5×10^3	4.2×10^2
8192	9.5×10^1	3.0×10^2	1.5×10^3	2.5×10^2
16384	1.2×10^2	4.2×10^2	3.6×10^3	4.5×10^2
32768	2.3×10^2	7.5×10^2	5.6×10^3	7.1×10^2
65536	2.4×10^2	1.0×10^3	1.2×10^4	1.3×10^3
131072	3.9×10^2	1.4×10^3	5.5×10^3	9.0×10^2
262144	6.3×10^2	3.7×10^3	1.1×10^5	1.1×10^4
524288	8.0×10^2	3.2×10^3	3.1×10^4	3.7×10^3
1048576	1.2×10^3	4.8×10^3	3.1×10^4	5.1×10^3

adjoint vanishes, but up to the sign the latter entry is the determinant of a $(k-1)\times(k-1)$ submatrix of the input matrix M, and so it is a polynomial of degree $k-1$ in its entries. □

B Conditioning of random complex matrices

We have assumed dealing with real random matrices and vectors throughout the paper, but most of our study can be readily extended to the computations in the field $\mathbb{C}$ of complex numbers if we replace the transposes A^T by the Hermitian transposes A^H. All the results of Section 3.2 apply to this case equally well. Below we elaborate upon the respective extension of our probabilistic bounds on the norms and singular values.

Definition B.1. *The set $\mathcal{G}^{m\times n}_{\mathbb{C},\mu,\sigma}$ of $m \times n$ complex Gaussian random matrices with a mean μ and a variance σ is the set $\{A + B\sqrt{-1}\}$ for $(A \mid B) \in \mathcal{G}^{m\times 2n}_{\mu,\sigma}$ (cf. Definition 3.2).*

We can immediately extend Theorem 3.2 to the latter matrices. Let us extend Theorem 3.1. Its original proof in [SST06] relies on the following result.

Lemma B.1. *Suppose y is a positive number; $\mathbf{w} \in \mathbb{R}^{n\times 1}$ is any fixed real unit vector, $||\mathbf{w}|| = 1$, $A \in \mathcal{G}^{n\times n}_{\mu,\sigma}$ and therefore is nonsingular with probability* 1*. Then*

$$\text{Probability}\{||A^{-1}\mathbf{w}|| > 1/y\} \le \sqrt{\frac{2}{\pi}}\frac{y}{\sigma} \text{ for } j = 1, \dots, n.$$

The following lemma and corollary extend Lemmas 3.2 and B.1 to the complex case.

Lemma B.2. *The bound of Lemma 3.2 also holds provided $\mathbf{t} = \mathbf{q} + \mathbf{r}\sqrt{-1}$ is a fixed complex unit vector and $\mathbf{b} = \mathbf{f} + \mathbf{g}\sqrt{-1} \in \mathcal{G}^{n\times 1}_{\mathbb{C},\mu,\sigma}$ is a complex vector such that $\mathbf{f}$, $\mathbf{g}$, $\mathbf{q}$ and $\mathbf{r}$ are real vectors, $||\mathbf{t}|| = 1$, and the vectors $\mathbf{f}$ and $\mathbf{g}$ are in $\mathcal{G}^{n\times 1}_{\mu,\sigma}$.*

Proof. We have $\mathbf{t}^H\mathbf{b} = \mathbf{q}^T\mathbf{f} + \mathbf{r}^T\mathbf{g} + (\mathbf{q}^T\mathbf{g} - \mathbf{r}^T\mathbf{f})\sqrt{-1}$, and so $|\mathbf{t}^H\mathbf{b}|^2 = |\mathbf{q}^T\mathbf{f} + \mathbf{r}^T\mathbf{g}|^2 + |\mathbf{q}^T\mathbf{g} - \mathbf{r}^T\mathbf{f}|^2$. Hence $|\mathbf{t}^H\mathbf{b}| \ge |\mathbf{q}^T\mathbf{f} + \mathbf{r}^T\mathbf{g}| = |\mathbf{u}^T\mathbf{v}|$ where $\mathbf{u}^T = (\mathbf{q}^T \mid \mathbf{r}^T)$ and $\mathbf{v} = (\mathbf{f}^T \mid \mathbf{g}^T)^T$. Note that $\mathbf{v} \in \mathcal{G}^{1\times 2n}_{\mu,\sigma}$ and $||\mathbf{u}|| = ||\mathbf{t}|| = 1$ and apply Lemma 3.2 to real vectors $\mathbf{u}$ and $\mathbf{v}$ replacing $\mathbf{b}$ and $\mathbf{t}$. □

Corollary B.1. *Suppose y is a positive number; $A \in \mathcal{G}^{n\times n}_{\mathbb{C},\mu,\sigma}$ and therefore is nonsingular with probability* 1*. Then*

$$\text{Probability}\{||A^{-1}\mathbf{e}_j|| > 1/y\} \le \sqrt{\frac{2}{\pi}}\frac{y}{\sigma} \text{ for } j = 1, \dots, n.$$

Table 8.5: Preconditioning tests

Type	r	Cond (C)
1n	1	3.21E+2
1n	2	4.52E+3
1n	4	2.09E+5
1n	8	6.40E+2
1s	1	5.86E+2x
1s	2	1.06E+4
1s	4	1.72E+3
1s	8	5.60E+3
2n	1	8.05E+1
2n	2	6.82E+3
2n	4	2.78E+4
2n	8	3.59E+3
2s	1	1.19E+3
2s	2	1.96E+3
2s	4	1.09E+4
2s	8	9.71E+3
3n	1	2.02E+4
3n	2	1.53E+3
3n	4	6.06E+2
3n	8	5.67E+2
3s	1	2.39E+4
3s	4	1.69E+3
3s	8	6.74E+3
4n	1	4.93E+2
4n	2	4.48E+2
4n	4	2.65E+2
4n	8	1.64E+2
4s	1	1.45E+3
4s	2	5.11E+2
4s	4	7.21E+2
4s	8	2.99E+2

Proof. In the case of real matrices A the corollary is supported by the argument in the proof of [SST06, Lemma 3.2], which employs the well known estimate that we state as our Lemma 3.2. Now we employ Lemma B.2 instead of this estimate, otherwise keep the same argument as in [SST06], and arrive at Corollary B.1. □

Corollary B.2. *Under the assumptions of Corollary B.1 we have* $||A^{-1}|| \le \sum_{j=1}^{n} X_j$ *where* X_j *are nonnegative random variables such that*

$$\text{Probability}\{X_j > 1/y\} \le \sqrt{\frac{2}{\pi}}\frac{y}{\sigma} \text{ for } j = 1, \dots, n.$$

Proof. Recall that for any $n \times n$ matrix B we have $||B|| = ||B\mathbf{w}||$ for some unit vector $\mathbf{w} = \sum_{j=1}^{n} w_j \mathbf{e}_j$. We have $|w_j| \le ||\mathbf{w}|| = 1$ for all j. Substitute $B = A^{-1}$ and obtain $||A^{-1}|| = ||A^{-1}\mathbf{w}|| = ||\sum_{j=1}^{n} A^{-1} w_j \mathbf{e}_j|| \le \sum_{j=1}^{n} |w_j| \, ||A^{-1}\mathbf{e}_j||$, and so $||A^{-1}|| \le \sum_{j=1}^{n} X_j$ where $X_j = ||A^{-1}\mathbf{e}_j||$ for all j. It remains to combine this bound with Corollary B.1. □

The corollary implies that Probability$\{||A^{-1}|| > 1/y\}$ converges to 0 proportionally to y as $y \to 0$, which can be viewed as an extension of Theorem 3.1 to the case of complex inputs. One can deduce

Table 8.6: Relative residual norms of the solutions by GENP with randomized circulant multiplicative preprocessing (cf. [PQZa, Table 2])

dimension	iterations	min	max	mean	std
64	0	4.7×10^{-14}	8.0×10^{-11}	4.0×10^{-12}	1.1×10^{-11}
64	1	1.9×10^{-15}	5.3×10^{-13}	2.3×10^{-14}	5.4×10^{-14}
256	0	1.7×10^{-12}	1.4×10^{-7}	2.0×10^{-9}	1.5×10^{-8}
256	1	8.3×10^{-15}	4.3×10^{-10}	4.5×10^{-12}	4.3×10^{-11}
1024	0	1.7×10^{-10}	4.4×10^{-9}	1.4×10^{-9}	2.1×10^{-9}
1024	1	3.4×10^{-14}	9.9×10^{-14}	6.8×10^{-14}	2.7×10^{-14}

similar extensions of Theorems 3.4–3.6. The resulting estimates are a little weaker than in Section 3.3, being overly pessimistic; actually random complex matrices are a little better conditioned than random real matrices (see [E88], [ES05], [CD05] and our Table 8.2).

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Table 8.7: Approximation of the tails of the SVDs and lower-rank approximation of a matrix (cf. [PQ10])

r	$\kappa(C)$ or rrn_i	n	**min**	**max**	**mean**	**std**
1	$\kappa(C)$	64	$2.38 \times 10^{+02}$	$1.10 \times 10^{+05}$	$6.25 \times 10^{+03}$	$1.68 \times 10^{+04}$
1	$\kappa(C)$	128	$8.61 \times 10^{+02}$	$7.48 \times 10^{+06}$	$1.32 \times 10^{+05}$	$7.98 \times 10^{+05}$
1	$\kappa(C)$	256	$9.70 \times 10^{+02}$	$3.21 \times 10^{+07}$	$3.58 \times 10^{+05}$	$3.21 \times 10^{+06}$
1	rn_1	64	4.01×10^{-10}	1.50×10^{-07}	5.30×10^{-09}	1.59×10^{-08}
1	rn_1	128	7.71×10^{-10}	5.73×10^{-07}	1.58×10^{-08}	6.18×10^{-08}
1	rn_1	256	7.57×10^{-10}	3.2×10^{-07}	1.69×10^{-08}	5.02×10^{-08}
1	rn_2	64	4.01×10^{-10}	1.50×10^{-07}	5.30×10^{-09}	1.59×10^{-08}
1	rn_2	128	7.71×10^{-10}	5.73×10^{-07}	1.58×10^{-08}	6.18×10^{-08}
1	rn_2	256	7.57×10^{-10}	3.22×10^{-07}	1.69×10^{-08}	5.02×10^{-08}
8	$\kappa(C)$	64	$1.26 \times 10^{+03}$	$1.61 \times 10^{+07}$	$2.68 \times 10^{+05}$	$1.71 \times 10^{+06}$
8	$\kappa(C)$	128	$2.92 \times 10^{+03}$	$3.42 \times 10^{+06}$	$1.58 \times 10^{+05}$	$4.12 \times 10^{+05}$
8	$\kappa(C)$	256	$1.39 \times 10^{+04}$	$8.75 \times 10^{+07}$	$1.12 \times 10^{+06}$	$8.74 \times 10^{+06}$
8	rn_1	64	3.39×10^{-10}	2.27×10^{-06}	2.74×10^{-08}	2.27×10^{-07}
8	rn_1	128	4.53×10^{-10}	1.91×10^{-07}	1.03×10^{-08}	2.79×10^{-08}
8	rn_1	256	8.74×10^{-10}	1.73×10^{-07}	7.86×10^{-09}	1.90×10^{-08}
8	rn_2	64	1.54×10^{-09}	7.59×10^{-06}	8.87×10^{-08}	7.58×10^{-07}
8	rn_2	128	1.82×10^{-09}	7.27×10^{-07}	2.95×10^{-08}	8.57×10^{-08}
8	rn_2	256	2.62×10^{-09}	3.89×10^{-07}	2.27×10^{-08}	5.01×10^{-08}
32	$\kappa(C)$	64	$1.77 \times 10^{+03}$	$9.68 \times 10^{+06}$	$1.58 \times 10^{+05}$	$9.70 \times 10^{+05}$
32	$\kappa(C)$	128	$1.65 \times 10^{+04}$	$6.12 \times 10^{+07}$	$1.02 \times 10^{+06}$	$6.19 \times 10^{+06}$
32	$\kappa(C)$	256	$3.57 \times 10^{+04}$	$2.98 \times 10^{+08}$	$4.12 \times 10^{+06}$	$2.98 \times 10^{+07}$
32	rn_1	64	2.73×10^{-10}	3.29×10^{-08}	2.95×10^{-09}	4.93×10^{-09}
32	rn_1	128	3.94×10^{-10}	1.29×10^{-07}	7.18×10^{-09}	1.64×10^{-08}
32	rn_1	256	6.80×10^{-10}	4.00×10^{-07}	1.16×10^{-08}	4.27×10^{-08}
32	rn_2	64	2.10×10^{-09}	1.49×10^{-07}	1.55×10^{-08}	2.18×10^{-08}
32	rn_2	128	2.79×10^{-09}	3.80×10^{-07}	3.81×10^{-08}	6.57×10^{-08}
32	rn_2	256	5.35×10^{-09}	1.05×10^{-06}	5.70×10^{-08}	1.35×10^{-07}

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Table 8.8: Approximation of the heads of SVDs and low-rank approximation of a matrix by using dual additive preprocessing

q	$\kappa(C)$ or rrn_i	n	**min**	**max**	**mean**	**std**
1	$\kappa(C)$	64	$1.83 \times 10^{+02}$	$1.26 \times 10^{+06}$	$1.74 \times 10^{+04}$	$1.27 \times 10^{+05}$
1	$\kappa(C)$	128	$6.75 \times 10^{+02}$	$8.76 \times 10^{+05}$	$2.35 \times 10^{+04}$	$9.10 \times 10^{+04}$
1	$\kappa(C)$	256	$4.19 \times 10^{+03}$	$5.82 \times 10^{+05}$	$4.43 \times 10^{+04}$	$8.98 \times 10^{+04}$
1	$\mathrm{rn}^{(1)}$	64	2.43×10^{-10}	3.86×10^{-08}	2.55×10^{-09}	5.43×10^{-09}
1	$\mathrm{rn}^{(1)}$	128	4.36×10^{-10}	1.15×10^{-07}	4.45×10^{-09}	1.24×10^{-08}
1	$\mathrm{rn}^{(1)}$	256	6.40×10^{-10}	3.17×10^{-08}	4.00×10^{-09}	5.16×10^{-09}
1	$\mathrm{rn}^{(2)}$	64	8.30×10^{-10}	3.86×10^{-08}	2.81×10^{-09}	5.35×10^{-09}
1	$\mathrm{rn}^{(2)}$	128	1.21×10^{-9}	1.15×10^{-07}	4.80×10^{-09}	1.23×10^{-08}
1	$\mathrm{rn}^{(2)}$	256	1.72×10^{-9}	3.18×10^{-08}	4.53×10^{-09}	4.97×10^{-09}
8	$\kappa(C)$	64	$1.37 \times 10^{+03}$	$1.87 \times 10^{+06}$	$7.57 \times 10^{+04}$	$2.16 \times 10^{+05}$
8	$\kappa(C)$	128	$3.80 \times 10^{+03}$	$8.64 \times 10^{+06}$	$2.00 \times 10^{+05}$	$8.73 \times 10^{+05}$
8	$\kappa(C)$	256	$2.57 \times 10^{+04}$	$1.54 \times 10^{+07}$	$7.25 \times 10^{+05}$	$2.03 \times 10^{+06}$
8	$\mathrm{rn}^{(1)}$	64	1.87×10^{-9}	4.48×10^{-07}	2.29×10^{-08}	5.20×10^{-08}
8	$\mathrm{rn}^{(1)}$	128	3.04×10^{-09}	3.73×10^{-07}	2.72×10^{-08}	5.83×10^{-08}
8	$\mathrm{rn}^{(1)}$	256	3.78×10^{-09}	2.01×10^{-06}	4.81×10^{-08}	2.02×10^{-07}
8	$\mathrm{rn}^{(2)}$	64	1.30×10^{-09}	2.47×10^{-07}	1.09×10^{-08}	2.70×10^{-08}
8	$\mathrm{rn}^{(2)}$	128	1.85×10^{-09}	1.50×10^{-07}	1.36×10^{-08}	2.75×10^{-08}
8	$\mathrm{rn}^{(2)}$	256	2.19×10^{-09}	1.10×10^{-06}	2.36×10^{-08}	1.10×10^{-07}
32	$\kappa(C)$	64	$3.75 \times 10^{+03}$	$3.25 \times 10^{+07}$	$6.01 \times 10^{+05}$	$3.28 \times 10^{+06}$
32	$\kappa(C)$	128	$2.41 \times 10^{+04}$	$1.09 \times 10^{+08}$	$1.95 \times 10^{+06}$	$1.10 \times 10^{+07}$
32	$\kappa(C)$	256	$1.33 \times 10^{+05}$	$2.11 \times 10^{+10}$	$2.18 \times 10^{+08}$	$2.11 \times 10^{+09}$
32	$\mathrm{rn}^{(1)}$	64	7.78×10^{-09}	1.39×10^{-06}	8.17×10^{-08}	1.94×10^{-07}
32	$\mathrm{rn}^{(1)}$	128	9.81×10^{-09}	2.35×10^{-06}	1.17×10^{-07}	3.05×10^{-07}
32	$\mathrm{rn}^{(1)}$	256	2.05×10^{-08}	3.99×10^{-06}	1.91×10^{-07}	5.06×10^{-07}
32	$\mathrm{rn}^{(2)}$	64	1.84×10^{-09}	2.62×10^{-07}	1.85×10^{-08}	4.09×10^{-08}
32	$\mathrm{rn}^{(2)}$	128	2.47×10^{-09}	6.77×10^{-07}	2.93×10^{-08}	8.38×10^{-08}
32	$\mathrm{rn}^{(2)}$	256	5.05×10^{-09}	8.85×10^{-07}	4.38×10^{-08}	1.14×10^{-07}

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Table 8.9: Approximation of the heads of SVDs and low-rank approximation of a matrix by using randomized matrix products

q	$\kappa(C)$ or rrn_i	n	**min**	**max**	**mean**	**std**
1	$\mathrm{rn}^{(1)}$	64	2.35×10^{-10}	1.32×10^{-07}	3.58×10^{-09}	1.37×10^{-08}
1	$\mathrm{rn}^{(1)}$	128	4.41×10^{-10}	3.28×10^{-08}	3.55×10^{-09}	5.71×10^{-09}
1	$\mathrm{rn}^{(1)}$	256	6.98×10^{-10}	5.57×10^{-08}	5.47×10^{-09}	8.63×10^{-09}
1	$\mathrm{rn}^{(2)}$	64	8.28×10^{-10}	1.32×10^{-07}	3.86×10^{-09}	1.36×10^{-08}
1	$\mathrm{rn}^{(2)}$	128	1.21×10^{-09}	3.28×10^{-08}	3.91×10^{-09}	5.57×10^{-09}
1	$\mathrm{rn}^{(2)}$	256	1.74×10^{-09}	5.58×10^{-08}	5.96×10^{-09}	8.47×10^{-09}
8	$\mathrm{rn}^{(1)}$	128	2.56×10^{-09}	1.16×10^{-06}	4.30×10^{-08}	1.45×10^{-07}
8	$\mathrm{rn}^{(1)}$	256	4.45×10^{-09}	3.32×10^{-07}	3.40×10^{-08}	5.11×10^{-08}
8	$\mathrm{rn}^{(2)}$	64	1.46×10^{-09}	9.56×10^{-08}	5.77×10^{-09}	1.06×10^{-08}
8	$\mathrm{rn}^{(2)}$	128	1.64×10^{-09}	4.32×10^{-07}	1.86×10^{-08}	5.97×10^{-08}
8	$\mathrm{rn}^{(2)}$	256	2.50×10^{-09}	1.56×10^{-07}	1.59×10^{-08}	2.47×10^{-08}
32	$\mathrm{rn}^{(1)}$	64	6.80×10^{-09}	2.83×10^{-06}	1.01×10^{-07}	3.73×10^{-07}
32	$\mathrm{rn}^{(1)}$	128	1.25×10^{-08}	6.77×10^{-06}	1.28×10^{-07}	6.76×10^{-07}
32	$\mathrm{rn}^{(1)}$	256	1.85×10^{-08}	1.12×10^{-06}	1.02×10^{-07}	1.54×10^{-07}
32	$\mathrm{rn}^{(2)}$	64	1.84×10^{-09}	6.50×10^{-07}	2.30×10^{-08}	8.28×10^{-08}
32	$\mathrm{rn}^{(2)}$	128	3.11×10^{-09}	1.45×10^{-06}	2.87×10^{-08}	1.45×10^{-07}
32	$\mathrm{rn}^{(2)}$	256	4.39×10^{-09}	2.16×10^{-07}	2.37×10^{-08}	3.34×10^{-08}

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Table 8.10: Approximation of the heads of SVDs and low-rank approximation of a matrix by using random Toeplitz multipliers

q	$\mathrm{rrn}^{(i)}$	n	**min**	**max**	**mean**	**std**
8	$\mathrm{rrn}^{(1)}$	64	2.22×10^{-09}	7.89×10^{-06}	1.43×10^{-07}	9.17×10^{-07}
8	$\mathrm{rrn}^{(1)}$	128	3.79×10^{-09}	4.39×10^{-05}	4.87×10^{-07}	4.39×10^{-06}
8	$\mathrm{rrn}^{(1)}$	256	5.33×10^{-09}	3.06×10^{-06}	6.65×10^{-08}	3.12×10^{-07}
8	$\mathrm{rrn}^{(2)}$	64	1.13×10^{-09}	3.66×10^{-06}	6.37×10^{-08}	4.11×10^{-07}
8	$\mathrm{rrn}^{(2)}$	128	1.81×10^{-09}	1.67×10^{-05}	1.90×10^{-07}	1.67×10^{-06}
8	$\mathrm{rrn}^{(2)}$	256	2.96×10^{-09}	1.25×10^{-06}	2.92×10^{-08}	1.28×10^{-07}
32	$\mathrm{rrn}^{(1)}$	64	6.22×10^{-09}	5.00×10^{-07}	4.06×10^{-08}	6.04×10^{-08}
32	$\mathrm{rrn}^{(1)}$	128	2.73×10^{-08}	4.88×10^{-06}	2.57×10^{-07}	8.16×10^{-07}
32	$\mathrm{rrn}^{(1)}$	256	1.78×10^{-08}	1.25×10^{-06}	1.18×10^{-07}	2.03×10^{-07}
32	$\mathrm{rrn}^{(2)}$	64	1.64×10^{-09}	1.26×10^{-07}	9.66×10^{-09}	1.48×10^{-08}
32	$\mathrm{rrn}^{(2)}$	128	5.71×10^{-09}	9.90×10^{-07}	5.50×10^{-08}	1.68×10^{-07}
32	$\mathrm{rrn}^{(2)}$	256	4.02×10^{-09}	2.85×10^{-07}	2.74×10^{-08}	4.48×10^{-08}

Table 8.11: Relative residual norms for a linear system of equations via nmb approximation and block triangulation

n	**r**	**min**	**max**	**mean**	**std**
32	1	1.49×10^{-13}	1.36×10^{-9}	4.25×10^{-11}	1.56×10^{-10}
32	2	3.70×10^{-13}	2.13×10^{-8}	3.83×10^{-10}	2.35×10^{-9}
32	4	9.33×10^{-13}	1.08×10^{-8}	3.37×10^{-10}	1.26×10^{-9}
64	1	1.11×10^{-12}	6.87×10^{-9}	2.03×10^{-10}	7.49×10^{-10}
64	2	1.53×10^{-12}	1.21×10^{-8}	5.86×10^{-10}	1.77×10^{-9}
64	4	2.21×10^{-12}	1.27×10^{-7}	1.69×10^{-9}	1.28×10^{-8}

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Table 8.12: Relative residual norms for a linear system of equations with MLDIVIDE(A,B)

n	**r**	**min**	**max**	**mean**	**std**
32	1	6.34×10^{-3}	7.44×10^{1}	1.74×10^{0}	7.53×10^{0}
32	2	2.03×10^{-2}	1.32×10^{1}	9.19×10^{-1}	1.62×10^{0}
32	4	4.57×10^{-2}	1.36×10^{1}	1.14×10^{0}	1.93×10^{0}
64	1	3.82×10^{-3}	9.93×10^{0}	1.03×10^{0}	1.66×10^{0}
64	2	1.96×10^{-2}	1.27×10^{2}	3.09×10^{0}	1.40×10^{1}
64	4	7.13×10^{-3}	6.63×10^{0}	8.23×10^{-1}	1.20×10^{0}

Table 8.13: Relative residual norms for a linear system of equations by using dual additive preprocessing and block triangulation

n	q	**min**	**max**	**mean**	**std**
32	1	2.33×10^{-14}	2.28×10^{-06}	2.31×10^{-08}	2.28×10^{-07}
32	2	3.40×10^{-13}	4.93×10^{-08}	9.11×10^{-10}	5.71×10^{-09}
32	4	5.97×10^{-13}	1.63×10^{-07}	2.22×10^{-09}	1.64×10^{-08}
64	1	3.90×10^{-14}	2.78×10^{-05}	2.81×10^{-07}	2.78×10^{-06}
64	2	3.53×10^{-13}	3.76×10^{-08}	1.13×10^{-09}	4.72×10^{-09}
64	4	3.54×10^{-12}	2.53×10^{-07}	5.19×10^{-09}	2.83×10^{-08}

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Table 8.14: Relative residual norms for a linear system of equations based on multiplication by random matrices and block triangulation

n	q	**min**	**max**	**mean**	**std**
32	1	7.08×10^{-30}	4.00×10^{-23}	4.52×10^{-25}	4.01×10^{-24}
32	2	7.49×10^{-30}	2.29×10^{-21}	2.77×10^{-23}	2.33×10^{-22}
32	4	1.46×10^{-28}	1.63×10^{-07}	4.83×10^{-25}	2.73×10^{-24}
64	1	1.13×10^{-29}	1.01×10^{-24}	2.31×10^{-26}	1.11×10^{-25}
64	2	6.60×10^{-29}	6.90×10^{-24}	1.45×10^{-25}	7.73×10^{-25}
64	4	2.60×10^{-28}	1.41×10^{-21}	1.61×10^{-23}	1.42×10^{-22}

Table 8.15: The CPU time (in cycles) for solving an ill conditioned real symmetric Toeplitz linear system

n	**Alg. 6.1**	**QR**	**SVD**	**QR/Alg. 6.1**	**SVD/Alg. 6.1**
512	56.3	148.4	4134.8	2.6	73.5
1024	120.6	1533.5	70293.1	12.7	582.7
2048	265.0	11728.1	–	44.3	–
4096	589.4	–	–	–	–
8192	1304.8	–	–	–	–

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