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Abstract

The name Generic Common Knowledge (GCK) was suggested by Artemov to capture a state of a multi-agent epistemic system that yields iterated knowledge $I(\varphi)$: 'any agent knows that any agent knows that any agent knows... φ' for any number of iterations. The generic common knowledge of φ, $GCK(\varphi)$, yields $I(\varphi)$,

$$GCK(\varphi) \rightarrow I(\varphi)$$

but is not necessarily logically equivalent to $I(\varphi)$. Modal logics with GCK were suggested by McCarthy and Artemov. It has been shown that in the usual epistemic scenarios, GCK can replace the conventional common knowledge. Artemov noticed that such epistemic actions as public announcements of atomic sentences, generally speaking, yield GCK rather than the conventional common knowledge. In this paper we introduce logics with explicit GCK and show that they realize corresponding modal systems, i.e., GCK, along with the individual knowledge modalities, can be always made explicit.

1 Introduction

Common knowledge C is perhaps the most studied form of shared knowledge. It is often cast as equivalent to iterated knowledge I, "everyone knows that everyone knows that..." [10, 13]. However there is an alternate view of common knowledge, generic common knowledge (GCK), which has advantages. The characteristic feature of GCK is that it implies, but not equivalent to, iterated knowledge I. Logics with this type of common knowledge have already been seen ([8, 16, 17]) but this new term "GCK" clarifies this distinction ([4]). Generic Common Knowledge can be used in many situations where C has traditionally been used ([2, 6, 4]) and has a technical asset in that the cut rule can be eliminated.[1]

Moreover, Artemov pointed out in [4] that public announcements of atomic sentence – a prominent vehicle for attaining common knowledge – generally speaking, leads to GCK rather than to the conventional common knowledge. Artemov also argues in [6] that in the analysis

[1]See details in [1] as to why the finitistic cut-elimination in traditional common knowledge systems may be seen as unsatisfactory.

of perfect information games in the belief revision setting, Aumann's "no irrationality in the system" condition is fairly represented by some kind of generic common knowledge rather than conventional common knowledge, and that this distinction lies in the heart of the well-known Aumann–Stalnaker controversy.

We assume that the aforementioned arguments provide sufficient motivation for mathematical logical studies of the generic common knowledge and its different forms.

Another research thread we consider is Justification Logic. In the generative justification logic LP, logic of proofs, knowledge and reasoning are made explicit with proof terms representing evidence for facts and new logic atoms $t{:}F$ are introduced with the reading "t is (sufficient) evidence for knowing F" or simply "t is a proof of F."

In this paper we consider justification logic systems with multiple knowers and generic common knowledge. As the standard example, we assume that all knowers as well as their *GCK* system are confined to LP. We call the resulting system $\mathsf{LP}_n(\mathsf{LP})$ which symbolically indicates n LP-type agents with an LP-type common knowledge evidence system.

Multi-agent justification logic systems were first considered in [20], but without any common knowledge component. Systems with the explicit equivalent of the traditional common knowledge were considered in [12, 11]; capturing common knowledge explicitly proved to be a serious technical challenge and the desirable realization theorem has not yet been obtained.

Generic common knowledge in the context of modal epistemic logic, in which individual agents' knowledge is represented 'implicitly' by the standard epistemic modalities was considered by Artemov in [8]. In the resulting modal epistemic logic $\mathsf{S4}_n^J$, sentences may be known, but specific reasons are not. This is a multi-agent logic augmented with a *GCK* operator J (previously termed justified common knowledge in [8] and elsewhere). Artemov reconstructed $\mathsf{S4}_n^J$-derivations in $\mathsf{S4}_n\mathsf{LP}$ via a Realization algorithm which makes the generic common knowledge operator J explicit, but does not realize individual knowledge modalities.

The current paper takes a natural next step by offering a realization of the entire *GCK* system $\mathsf{S4}_n^J$ in the corresponding explicit knowledge system $\mathsf{LP}_n(\mathsf{LP})$, In particular, all epistemic operators in $\mathsf{S4}_n^J$, not only J, become explicit in such a realization.

2 Explicit Epistemic Systems with *GCK*

Here we introduce an explicit generic common knowledge operator into justification logics in the context of a multi-agent logic of explicit justifications to form a logic $\mathsf{LP}_n(\mathsf{LP})$. The "($\mathsf{LP}$)" corresponds to *GCK*.

Definition 1. $\mathcal{L}_{\mathsf{LP}_n(\mathsf{LP})}$, the *language of* LP_n*(*LP*)*, is an extension of the propositional language:

$$\mathcal{L}_{\mathsf{LP}_n(\mathsf{LP})} := \{\text{Var, pfVar, pfConst}, \vee, \wedge, \rightarrow, \neg, +, \cdot, !, \text{Tm}\} \ .$$

Var is propositional variables $(p, q, \dots)$. *Justification terms* Tm are built from pfVar and pfConst, proof variables $(x, y, z, \dots)$ and constants $(c, d, \dots)$, by the grammar

$$t := x \mid c \mid t + t \mid t{\cdot}t \mid\, !t \ .$$

Formulas (Fm) are defined by the grammar, for $i \in \{0, 1, 2, \dots, n\}$,

$$\varphi := p \mid e \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid \neg\varphi \mid t{:}_i\, \varphi \ .$$

The formulas $t{:_i}\varphi$ have the intended reading of "t is a justification of φ for agent i." Index $i = 0$ is reserved for explicit generic common knowledge, for which we will also use the alternative notation $[t]\varphi$ for better readability.

Definition 2. The *axioms and rules of* $\mathsf{LP}_n(\mathsf{LP})$:

CLASSICAL PROPOSITIONAL LOGIC:

A. axioms of classical propositional logic

R. modus ponens

LP AXIOMS FOR ALL $n+1$ AGENTS, $i \in \{0, 1, 2, \ldots, n\}$:

L1. $t{:_i}(\varphi \to \psi) \to (s{:_i}\varphi \to (t\cdot s){:_i}\psi)$

L2. $t{:_i}\varphi \to (t+s){:_i}\varphi$ and $t{:_i}\varphi \to (s+t){:_i}\varphi$

L3. $t{:_i}\varphi \to \varphi$

L4. $t{:_i}\varphi \to !t{:_i}(t{:_i}\varphi)$

CONNECTION PRINCIPLE:

C. $[t]\varphi \to t{:_i}\varphi$.

Term operators mirror properties of justifications: "$\cdot$" is *application* for deduction; "+", *sum*, maintains that justifications are not spoiled by adding (possibly irrelevant) evidence; and "!" is *inspection* and stipulates that justifications themselves are justified. This last operator appears only in justification logics with L4, whose corresponding modal logic contains the modal axiom 4 ($\Box\varphi \to \Box\Box\varphi$), as shown in [7]. A multitude of justification logics of a single agent corresponding to standard modal logics have been developed ([7]). Yavorskaya has investigated versions of LP with two agents in which agents can check each other's proofs ([20]).

Definition 3. A *constant specification* for each agent, $i \in \{0, 1, \ldots, n\}$, $\mathcal{CS}_i$ is a set of sentences of sort $c :_i A$ where c is a constant and A an axiom of $\mathsf{LP}_n(\mathsf{LP})$. The intuitive reading of these sentences is 'c is a proof of A for agent i.' Let

$$\mathcal{CS} = \{\mathcal{CS}_1, \ldots, \mathcal{CS}_n\}$$

and $\mathcal{CS}_0 \subseteq \mathcal{CS}_i$ for all $i \in \{1, 2, \ldots, n\}$. By $\mathsf{LP}_{n,\mathcal{CS}}(\mathsf{LP}_{\mathcal{CS}_0})$ we mean the system with the postulates A, R, L1–L4, C above, plus $\mathcal{CS}_0$ and $\mathcal{CS}$ as additional axioms. As formulas in a constant specification are taken as axioms, they themselves may be used to form other formulas in a $\mathcal{CS}$ so that it's possible to have $c{:_1}(d{:_2}A) \in \mathcal{CS}_1$ if $d{:_2}A \in \mathcal{CS}_2$.

The constant specification represents assumptions about proofs of basic postulates that are not further analyzed. If $\mathcal{CS}_i = \emptyset$, agent i is totally skeptical; no formulas are justified. If this is so for all agents, it would be denoted $\mathsf{LP}_{n,\{\emptyset\}}(\mathsf{LP}_\emptyset)$. Constant Specifications of different types have been studied: schematic, injective, full, etc. and have been defined with various closure properties. See [7] for a fuller discussion of constant specifications. The *total constant specification* for any agent, $\mathcal{TCS}_i$, is the union of all possible $\mathcal{CS}_i$. Henceforth we will assume each agent's constant specification is total and will abbreviate this to $\mathsf{LP}_n(\mathsf{LP})$.

Definition 4. A *modular model of* $\mathsf{LP}_n(\mathsf{LP})$ is $\mathcal{M} = (W, R_0, R_1, R_2, \ldots, R_n, *, \Vdash)$ where

1. • W is a nonempty set,

- $R_i \subseteq W \times W$ are reflexive for $i \in \{0, 1, 2, \ldots, n\}$. R_0 is the designated accessibility relation for GCK.
- $* : W \times \mathrm{Var} \to \{0, 1\}$ and $* : W \times \{0, 1, 2, \ldots, n\} \times \mathrm{Tm} \to 2^{\mathrm{Fm}}$

i.e., for each agent i at node u, $*(u, i, t)$ is a set of formulas t justifies. We write $t_u^{*,i}$ for $*(u, i, t)$. We assume that GCK evidence is everybody's evidence:

$$t_u^{*,0} \subseteq t_u^{*,i}, \quad \text{for} \quad i \in \{0, 1, 2, \ldots, n\} \ .$$

2. For each agent i and node u, $*$ is closed under the following conditions:

$$\begin{aligned} \textit{Application:} &\quad s_u^{*,i} \cdot t_u^{*,i} \subseteq (s \cdot t)_u^{*,i} \\ \textit{Sum:} &\quad s_u^{*,i} \cup t_u^{*,i} \subseteq (s + t)_u^{*,i} \\ \textit{Inspection:} &\quad \{t :_i \varphi \mid \varphi \in (t_u^{*,i})\} \subseteq (!t)_u^{*,i} \end{aligned}$$

where $s^* \cdot t^* = \{\psi \mid \varphi \to \psi \in s^* \text{ and } \varphi \in t^* \text{ for some } \varphi\}$, the set of formulas resulting from applying modus ponens to implications in s^* whose antecedents are in t^*.

3. For $p \in \mathrm{Var}$, we define forcing $\Vdash$ for atomic formulas at node u as $u \Vdash p$ if and only if $*(u, p) = 1$. To define the truth value of all formulas, extend forcing $\Vdash$ to compound formulas by Boolean laws, and define

$$u \Vdash t :_i \varphi \quad \Leftrightarrow \quad \varphi \in t_u^{*,i} \ .$$

4. 'justification yields belief' (JYB), i.e., for $i \in \{0, 1, 2, \ldots, n\}$, $u \Vdash t :_i \varphi$ yields $v \Vdash \varphi$ for all v such that uR_iv.

Modular models, first introduced for the most basic justification logic in [5], are useful for their clear semantical interpretation of justifications as sets of formulas. For modular models of some other justification logics refer to [15]. For a detailed discussion of the relationship between modular models and Mkrtychev–Fitting models for justification logics, see [5].

A model *respects* $\mathcal{CS}_0, \ldots, \mathcal{CS}_n$, if each $c :_i \varphi$ in these constant specifications holds (at each world u) in the model.

Theorem 1 (soundness and completeness)**.** $\mathsf{LP}_{n,\mathcal{CS}}(\mathsf{LP}_{\mathcal{CS}_0}) \vdash F$ *iff* F *holds in any basic modular model respecting* $\mathcal{CS}_i$*,* $i \in \{0, 1, 2, \ldots, n\}$*.*

Proof. Soundness – by induction on the derivation of F, for $i \in \{0, 1, 2, \ldots, n\}$.

- Constant Specifications: If $c :_i \varphi \in \mathcal{CS}_i$, then $u \Vdash c :_i \varphi$ as the model respects $\mathcal{CS}_i$.
- Boolean connectives: hold by definition of the truth of formulas.
- Application: Suppose $u \Vdash s :_i (F \to G)$ and $u \Vdash t :_i F$. Then by assumption, $(F \to G) \in s_u^{*,i}$ and $F \in t_u^{*,i}$. Then $G \in s_u^{*,i} \cdot t_u^{*,i} \subseteq (s \cdot t)_u^{*,i}$; thus $u \Vdash (s \cdot t) :_i G$.
- Sum: Suppose $u \Vdash t :_i F$. Then $F \in t_u^{*,i}$ and so $F \in s_u^{*,i} \cup t_u^{*,i} \subseteq (s + t)_u^{*,i}$. Thus $u \Vdash (s + t) :_i F$. Likewise, $u \Vdash (t + s) :_i F$.

- Modus Ponens: Suppose $u \Vdash F \to G$. Then by the definition of the connectives either $u \nVdash F$ or $u \Vdash G$. So if also $u \Vdash F$, then $u \Vdash G$.

- Factivity: Suppose $u \Vdash t{:}_i F$. By the 'justification yields belief' condition, $v \Vdash F$ for all v such that uR_iv. As each R_i is reflexive, uR_iu, so also $u \Vdash F$. Inspection: Suppose $u \Vdash t{:}_i F$. Then $F \in t_u^{*,i}$ so $t{:}_i F \in (!t)_u^{*,i}$. Thus $u \Vdash !t{:}_i (t{:}_i F)$.

- Connection Principle: Suppose $u \Vdash t{:}_0 F$. Then $F \in t_u^{*,0} \subseteq t_u^{*,i}$ so $u \Vdash t{:}_i F$.

Completeness – by the maximal consistent set construction. For $i \in \{0, 1, 2, \ldots, n\}$, let

- W the set of all maximal consistent sets,

- $\Gamma R_i \Delta$ iff $\Gamma^{i,\#} \subseteq \Delta$ where $\Gamma^{i,\#} = \{F \mid t{:}_i F \in \Gamma\}$,

- For $p \in Var$, $*(\Gamma, p) = 1$ iff $p \in \Gamma$,

- $t_\Gamma^{*,i} = \{F \mid t{:}_i F \in \Gamma\}$ (i.e., for $X = p, t{:}_i F,\ \Gamma \Vdash X$ iff $X \in \Gamma$) .

To confirm that these comprise a modular model, the R_i need to be reflexive, the GCK and closure conditions must be checked, and the model must satisfy 'justification yields belief'. As each world is maximally consistent $\Gamma^{i,\#} \subseteq \Gamma$, hence $\Gamma R_i \Gamma$ by L3, so each R_i is reflexive. The GCK conditions $t_\Gamma^{*,0} \subseteq t_\Gamma^{*,i}$ for $i \in \{0, 1, 2, \ldots, n\}$ follow from the C axiom $t{:}_0 F \to t{:}_i F$ for $i \in \{1, 2, \ldots, n\}$. Closure conditions for $\cdot$, $+$, and ! follow straightforwardly from the axioms L1, L2, and L4. It remains to check the JYB condition, following the Truth Lemma.

Lemma 1 (Truth Lemma). *$\Gamma \Vdash X$ iff $X \in \Gamma$, for each Γ and X.*

Proof. Induction on X. The atomic and Boolean cases are standard. The only interesting cases are $X = t{:}_i F$. Note that $\Gamma \Vdash t{:}_i F$ iff $F \in t_\Gamma^{*,i}$ by the definition of modular models. Moreover, under the evaluation particular to this model, $F \in t_\Gamma^{*,i}$ iff $t{:}_i F \in \Gamma$. Thus $\Gamma \Vdash t{:}_i F$ iff $t{:}_i F \in \Gamma$. □

Now to see the JYB condition, suppose $\Gamma \Vdash t{:}_i F$ and consider an arbitrary Δ such that $\Gamma R_i \Delta$. By the definition of this model, $t{:}_i F \in \Gamma$, hence $F \in \Gamma^{i,\#}$, hence $F \in \Delta$. By the Truth Lemma, $\Delta \Vdash F$.

To finish the proof of completeness, let $\mathsf{LP}_{n,\mathcal{CS}}(\mathsf{LP}_{\mathcal{CS}_0}) \nvdash G$, hence $\{\neg G\}$ is consistent and has a maximal consistent extension, Φ. Since $G \notin \Phi$, by the Truth Lemma, $\Phi \nVdash G$. □

Corollary 1. *The canonical model of the completeness proof is transitive.*

Proof. Suppose $\Gamma R_i \Delta$ and $\Delta R_i \Theta$. If $t{:}_i F \in \Gamma$, then $!t{:}_i (t{:}_i F) \in \Gamma$ as Γ is maximal consistent. As $!t{:}_i (t{:}_i F) \in \Gamma$ and $\Gamma R_i \Delta$, by the definition of the R_i, $t{:}_i F \in \Delta$. As $t{:}_i F \in \Delta$ and $\Delta R_i \Theta$, $F \in \Theta$. Thus if $t{:}_i F \in \Gamma$, then $F \in \Theta$ that is, $\Gamma R_i \Theta$, hence R_i is transitive in the model of the completeness proof. □

Corollary 2. *Modular models for* LP *(i.e.,* $\mathsf{LP}_0(\mathsf{LP})$*) are* $M = (W, R, *, \Vdash)$ *where*

1.
   - *W is nonempty*
   - *R is reflexive*

- $* : W \times \mathrm{Var} \to \{0,1\}$, $* : W \times \mathrm{Tm} \to 2^{\mathrm{Fm}}$;

2. $*$ *closure conditions for* $\cdot$, $+$, *and* $!$;

3. $u \Vdash p \Leftrightarrow *(u,p) = 1$ *and forcing* $\Vdash$ *extends a truth value to all formulas by Boolean laws and* $u \Vdash t{:}F \Leftrightarrow F \in t^*_u$.

4. *justification yields belief (JYB):* $u \Vdash t{:}F$ *yields* $v \Vdash F$ *for all* v *such that* uRv.

These modular models for LP differ from those by Kuznets and Studer in [15] as no transitivity is required of R, which enlarges the class of modular models for LP. Artemov suggests (personal communication) this modular model for $LP_\emptyset$ which satisfies Definition 4 and is not transitive and hence ruled out by the formulation offered in [15]:

- $W = \{a, b, c\}$
- $R = \{(aa), (bb), (cc), (ab), (bc)\}$
- $*$ is arbitrary on propositional variables, t^*_a, t^*_b, t^*_c are all empty.

Of course, one could produce more elaborate examples as well, e.g., on the same non-transitive frame, fix a propositional variable p and have $t_1 : t_2 : \ldots : t_n : p$ hold for all proof terms $t_1, \ldots, t_n$, for all n, at any node (in particular, make p true at a, b, c).

While it does not appear to be justified to confine consideration a priori to transitive modular models, the exact role of transitivity of accessibility relations in modular models is still awaiting a careful analysis.

3 Realizing Generic Common Knowledge

We show that $\mathsf{LP}_n\mathsf{LP}$, a logic of explicit knowledge using proof terms, has a precise modal analog in the epistemic logic with GCK, $\mathsf{S4}^J_n$.

Definition 5. The *axioms and rules of* $\mathsf{S4}^J_n$:

	CLASSICAL PROPOSITIONAL LOGIC:
A.	axioms of classical proposition logic
R1.	modus ponens
	S4-KNOWLEDGE PRINCIPLES FOR EACH K_i, $i \in \{0, 1, \ldots, n\}$, ($J$ may be used in place of K_0):
K.	$K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$
T.	$K_i\varphi \to \varphi$
4.	$K_i\varphi \to K_iK_i\varphi$
R2.	$\vdash \varphi \Rightarrow \vdash K_i\varphi$
	CONNECTION PRINCIPLE:
C1.	$J\varphi \to K_i\varphi$.

In $\mathsf{S4}_n^J$, the common knowledge operator J is indeed generic as $J(\varphi) \to C(\varphi)$ while $C(\varphi) \not\to J(\varphi)$, as illustrated in [2]. McCarthy et al. provide Kripke models for one of their logics in [17], see also [8]. In Kripke models, a distinction between generic and conventional common knowledge is clear. The accessibility relation for C, R_C, is the exact transitive closure of the union of all other agents' accessibility relations R_i. R_J, the accessibility relation for J is any transitive and reflexive relation which *contains* the union of all other agents' relations, thus

$$R_{GCK} = R_J \supseteq R_C.$$

This means that generally speaking, there is flexibility in choosing R_J while R_C is unique in each given model. Note that in the case where we have explicit proof terms and not just modalities of implicit knowledge, we also have this multiplicity of options for generic common knowledge: there may be many evaluations $*$ such that $t_u^{*,0}$ that satisfiest $t_u^{*,0} \subseteq t_u^{*,i}$ for all i.

We now have $\mathsf{LP}_n(\mathsf{LP})$ and $\mathsf{S4}_n^J$, each is a multi-agent epistemic logic with generic common knowledge, where all justifications are explicit in the former and implicit in the latter. By proving the Realization Theorem, we will establish that $\mathsf{LP}_n(\mathsf{LP})$ is the exact explicit version of $\mathsf{S4}_n^J$.

Definition 6. The *forgetful projection* is a translation

$$\circ : \mathcal{L}_{\mathsf{LP}_n(\mathsf{LP})} \to \mathcal{L}_{\mathsf{S4}_n^J}$$

defined inductively as follows:

- $p^\circ = p$, for $p \in$Var
- $(\neg\psi)^\circ = \neg(\psi^\circ)$
- $\circ$ commutes with binary Boolean connectives: $(\psi \wedge \varphi)^\circ = \psi^\circ \wedge \varphi^\circ$ and $(\psi \vee \varphi)^\circ = \psi^\circ \vee \varphi^\circ$
- $(t :_i \psi)^\circ = K_i(\psi^\circ)$ for $i \in \{0, 1, \ldots, n\}$.

Proposition 1. $[\mathsf{LP}_n(\mathsf{LP})]^\circ \subseteq \mathsf{S4}_n^J$.

Proof. The $\circ$ translations of all the $\mathsf{LP}_n(\mathsf{LP})$ axioms and rules are easily seen to be theorems of $\mathsf{S4}_n^J$. $\square$

We want to show that these two logics are really correspondences and that

$$\mathsf{S4}_n^J \subseteq [\mathsf{LP}_n(\mathsf{LP})]^\circ$$

also holds. This is much more involved. Theorem 3 shows that a derivation of any $\mathsf{S4}_2^J$ theorem σ can yield an $\mathsf{LP}_2(\mathsf{LP})$ theorem τ such that $\tau^\circ = \sigma$. This process, the converse of the $\circ$-translation, is a *Realization* r.

Definition 7. A realization r is *normal* if all negative occurrences of modalities (whether a K_i or J) are realized by distinct proof variables.

To provide an algorithm r for such a process, we first give the Gentzen system for $\mathsf{S4}_n^J$ and the Lifting Lemma (Proposition 2).

Definition 8. $\mathsf{S4}_n^J\mathsf{G}$, the *Gentzen version of* $\mathsf{S4}_n^J$, is the usual propositional Gentzen rules (i.e., system $\mathsf{G1c}$ in [18]) with addition of $n+1$ pairs of rules, where $\Box$ is J or some K_i:

$$\frac{\varphi, \Gamma \ \Rightarrow \ \Delta}{\Box\varphi, \Gamma \ \Rightarrow \ \Delta}\,(\Box, \Rightarrow) \qquad \textit{and} \qquad \frac{J\Gamma, \Box\Delta \ \Rightarrow \ \varphi}{J\Gamma, \Box\Delta \ \Rightarrow \ \Box\varphi}\,(\Rightarrow, \Box) \ .$$

As usual, capital letters are multisets and $\Box\{\varphi_1, \ldots, \varphi_n\} = \{\Box\varphi_1, \ldots, \Box\varphi_n\}$. In the special case of no K_i modalities in the premise, the second rule can be read as

$$\frac{J\Gamma \ \Rightarrow \ \varphi}{J\Gamma \ \Rightarrow \ \Box\varphi}\,(\Rightarrow, \Box) \ ,$$

where $\Box$ may be J or some K_i.

Theorem 2. *$\mathsf{S4}_n^J\mathsf{G}$ is equivalent to $\mathsf{S4}_n^J$ and admits cut-elimination.*

Proof. See Artemov's proof in Section 6 of [8]. □

Let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$, $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ be finite lists of formulas, $\vec{y}, \vec{z}$ finite lists of proof variables of matching length, respectively. Then $[\vec{y}\,]\Gamma = [y_1]\gamma_1, \cdots, [y_m]\gamma_m$ and $\vec{z}:_i\Sigma = z_1:_i \sigma_1, \cdots, z_n:_i\sigma_n$, $i \in \{0, 1, 2, \ldots, n\}$.

Proposition 2 (Lifting Lemma)**.** *In $\mathsf{LP}_n(\mathsf{LP})$, for $i \in \{0, 1, 2, \ldots, n\}$ and each $\Gamma, \Sigma, \vec{y}, \vec{z}$,*

$$\frac{[\vec{y}\,]\Gamma, \vec{z}:_i\Sigma \vdash \varphi}{[\vec{y}\,]\Gamma, \vec{z}:_i\Sigma \vdash f(\vec{y}, \vec{z}):_i\varphi}$$

for the corresponding proof term $f(\vec{y}, \vec{z})$.

Proof. By induction on the derivation of φ.

- φ is an axiom of $\mathsf{LP}_n(\mathsf{LP})$, then as $\mathsf{LP}_n(\mathsf{LP})$ has $\mathcal{TCS}$, for any constant c, $c:_i\varphi$ so let $f(\vec{y}, \vec{z}) = c$. As $\vdash_{\mathsf{LP}_n(\mathsf{LP})} c:_i\varphi$, also $[\vec{y}\,]\Gamma, \vec{z}:_i\Sigma \vdash_{\mathsf{LP}_n(\mathsf{LP})} c:_i\varphi$. (Here, any $\mathcal{CS}$ in which each axiom has a justification would suffice.)

- φ is $[y_j]\gamma_j$ for some $[y_j]\gamma_j \in [\vec{y}]\Gamma$, then

  $$[\vec{y}\,]\Gamma, \vec{z}:_i\Sigma \vdash_{\mathsf{LP}_n(\mathsf{LP})} [y_j]\gamma_j \ ,$$

  hence

  $$[\vec{y}\,]\Gamma, \vec{z}:_i\Sigma \vdash_{\mathsf{LP}_n(\mathsf{LP})} [!y_j]([y_j]\gamma_j) \ ,$$

  and

  $$[\vec{y}\,]\Gamma, \vec{z}:_i\Sigma \vdash_{\mathsf{LP}_n(\mathsf{LP})} !y_j:_i([y_j]\gamma_j) \ .$$

  So,

  $$[\vec{y}\,]\Gamma, \vec{z}:_i\Sigma \vdash_{\mathsf{LP}_n(\mathsf{LP})} !y_j:_i\varphi \ ,$$

  and we can put $f(\vec{y}, \vec{z}) = !y_j$.

- φ is $z_j :_i \sigma_j$ for some $z_j :_i \sigma_j \in \vec{z} :_i \Sigma$, then as $!z_j :_i (z_j :_i \sigma_j)$ is given,

$$[\vec{y}\,]\Gamma, \vec{z} :_i \Sigma \vdash_{\mathsf{LP}_n(\mathsf{LP})} !z_j :_i \varphi \ .$$

  So let $f(\vec{y}, \vec{z}) = !z_j$.

- φ is derived by modus ponens from ψ and $\psi \to \varphi$. By the Induction Hypothesis, there exists $t :_i \psi$ and $u :_i (\psi \to \varphi)$ (where $t = f_t(\vec{y}, \vec{z})$ and $u = f_u(\vec{y}, \vec{z})$). Since $u :_i (\psi \to \varphi) \to (t :_i \psi \to (u \cdot t) :_i \varphi)$, by modus ponens $(u \cdot t) :_i \varphi$. So let $f(\vec{y}, \vec{z}) = (u \cdot t)$.

- φ is $c :_i A \in \mathcal{TCS}$. Since $c :_i A \to !c :_i (c :_i A)$ and $\vdash_{\mathsf{LP}_n(\mathsf{LP})} c :_i A$, also $\vdash_{\mathsf{LP}_n(\mathsf{LP})} \ !c :_i (c :_i A)$ thus

$$[\vec{y}\,]\Gamma, \vec{z} :_i \Sigma \vdash_{\mathsf{LP}_n(\mathsf{LP})} !c :_i \varphi.$$

  So let $f(\vec{y}, \vec{z}) = !c$.

$\square$

Theorem 3 (Realization Theorem). *If $\mathsf{S4}_n^J \vdash \varphi$, then $\mathsf{LP}_n(\mathsf{LP}) \vdash \varphi^r$ for some normal realization r.*

Proof. The proof follows closely the realization proof from [9] with adjustments to account for the Lifting Lemma.

If $\mathsf{S4}_n^J \vdash \varphi$, then by Theorem 2 there is a cut-free derivation $\mathcal{D}$ of the sequent $\Rightarrow \varphi$ in $\mathsf{S4}_n^J\mathsf{G}$. We now construct a normal realization algorithm r that runs on $\mathcal{D}$ and returns an $\mathsf{LP}_n(\mathsf{LP})$ theorem $\varphi^r = \psi$ such that $\psi^\circ = \varphi$.

In φ, positive and negative modalities are defined as usual. The rules of $\mathsf{S4}_n^J\mathsf{G}$ respect these polarities so that $(\Rightarrow, \Box)$ introduces positive occurrences and $(\Box, \Rightarrow)$ introduces negative occurrences of $\Box$, where $\Box$ is J or some K_i. Call the occurrences of $\Box$ *related* if they occur in related formulas in the premise and conclusion of some rule: the same formula, that formula boxed or unboxed, enlarged or shrunk by $\wedge$ or $\vee$, or contracted. Extend this notion of related modalities by transitivity. Classes of related $\Box$ occurrences in $\mathcal{D}$ naturally form disjoint *families* of related occurrences. An *essential* family is one which at least one of its members arises from the $(\Rightarrow, \Box)$ rule, these are clearly positive families.

Now the desired r is constructed by the following three steps so that negative and non-essential positive families are realized by proof variables while essential families will be realized by sums of functions of those proof variables.

Step 1. For each negative family and each non-essential positive family, replace all $\Box$ occurrence so that $J\alpha$ becomes $[x]\alpha$ and $K_i\alpha$ becomes $y :_i \alpha$. Choose new and distinct proof variables x and y for each of these families.

Step 2. Choose an essential family f. Count the number n_f of times the $(\Rightarrow, \Box)$ rule introduces a box to this family. Replace each $\Box$ with a sum of proof terms so that for $i \in \{0, 1, 2, \ldots, n\}$, $K_i\alpha$ becomes

$$(w_1 + w_2 + \cdots + w_{n_f}) :_i \alpha,$$

with each w_j a fresh provisional variable. Do this for each essential family. The resulting tree $\mathcal{D}'$ is now labeled by $\mathsf{LP}_n(\mathsf{LP})$-formulas.

Step 3. Now the provisional variables need to be replaced, starting with the leaves and working toward the root. By induction on the depth of a node in $\mathcal{D}'$ we will show that after the process passes a node, the sequent at that level becomes derivable in $\mathsf{LP}_n(\mathsf{LP})$ where

$$\Gamma \Rightarrow \Delta$$

is read as provability of

$$\Gamma \vdash_{\mathsf{LP}_n(\mathsf{LP})} \bigvee \Delta.$$

Note that axioms $p \Rightarrow p$ and $\bot \Rightarrow$ are derivable in $\mathsf{LP}_n(\mathsf{LP})$. For each move down the tree other than by the rule $(\Rightarrow, \Box)$, the concluding sequent is $\mathsf{LP}_n(\mathsf{LP})$-derivable if its premises are; for rules other that this one, do not change the realization of formulas. For a given essential family f, for the occurrence numbered j of the $(\Rightarrow, \Box)$ rule, the corresponding node in $\mathcal{D}'$ is labeled

$$\frac{[\vec{z}]\Gamma,\ \vec{q}{:_i}\Sigma \Rightarrow \alpha}{[\vec{z}]\Gamma,\ \vec{q}{:_i}\Sigma \Rightarrow (u_1 + \cdots + u_{n_f}){:_i}\alpha}\ , \text{ for } \Box \text{ is } K_i, \text{ for } i \in \{0, 1, 2, \ldots, n\}$$

where the z's and q's are proof variables and the u's are evidence terms, with u_j a provisional variable. By the Induction Hypothesis, the premise is derivable in $\mathsf{LP}_n(\mathsf{LP})$. By the Lifting Lemma (Proposition 2), construct a justification term $f(\vec{z}, \vec{q})$ for α where

$$[\vec{z}]\Gamma,\ \vec{q}{:_i}\Sigma \vdash f(\vec{z}, \vec{q}){:_i}\alpha\ .$$

Now we will replace the provisional variable u_j as follows

$$[\vec{z}]\Gamma,\ \vec{q}{:_i}\Sigma \vdash (u_1 + \cdots + u_{j-1} + f(\vec{z}, \vec{q}) + u_{j+1} + \cdots + u_{n_f}){:_i}\alpha\ .$$

Substitute each u_j with $f(\vec{z}, \vec{q})$ everywhere in $\mathcal{D}'$. There is now one fewer provisional variable in the tree as $f(\vec{z}, \vec{q})$ has none. The conclusion to this j^{th} instance of the rule $(\Rightarrow, \Box)$ becomes derivable in $\mathsf{LP}_n(\mathsf{LP})$, completing the induction step.

Eventually all provisional variables are replaced by terms of non-provisional variables, establishing that the root sequent of $\mathcal{D}$, φ^r, is derivable in $\mathsf{LP}_n(\mathsf{LP})$. The realization constructed in this manner is normal. $\Box$

Corollary 3. $\mathsf{S4}_n^J$ *is the forgetful projection of* $\mathsf{LP}_n(\mathsf{LP})$.

Proof. A straightforward consequence of Proposition 1 and Theorem 3. $\Box$

We see that the common knowledge component of $\mathsf{LP}_n(\mathsf{LP})$ indeed corresponds to the generic common knowledge J and hence can be regarded as the *explicit GCK*.

4 Realization Example

Here we demonstrate a realization of an $\mathsf{S4}_2^J$ theorem in $\mathsf{LP}_2(\mathsf{LP})$.

Proposition 3. $\mathsf{S4}_2^J \vdash J\neg\phi \rightarrow K_2\neg K_1\phi$.

Proof. Here is an $\mathsf{S4}_2^J\mathsf{G}$ derivation of the corresponding sequent.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi}{K_1\phi \Rightarrow \phi}\,(\Box,\Rightarrow)}{\neg\phi, K_1\phi \Rightarrow}\,(\neg,\Rightarrow)}{\neg\phi \Rightarrow \neg K_1\phi}\,(\Rightarrow,\neg)}{J\neg\phi \Rightarrow \neg K_1\phi}\,(\Box,\Rightarrow)}{J\neg\phi \Rightarrow K_2\neg K_1\phi}\,(\Rightarrow,\Box)}{\Rightarrow J\neg\phi \to K_2\neg K_1\phi}\,(\Rightarrow,\to)
$$

□

Now we follow the realization algorithm to end up with an $\mathsf{LP}_2(\mathsf{LP})$ theorem. In the sequent proof, the J in the conclusion is in negative position and all the Js in this derivation are related and form a negative family. The occurrences of the K_1 modality are all related and they too form a negative family. The two occurrences of K_2 form an essential positive family with $n_f = 1$ as there is one use of the $(\Rightarrow, \Box)$ rule.

Step 1. Replace all J occurrences with '$[x]$' and K_1 occurrences with '$y{:}_1$'.

Step 2. Replace all K_2 occurrences with a '$w{:}_2$' with w a provisional variable. Since here $n_f = 1$, a sum is not required. At this stage the derivation tree looks like this, where '$\Rightarrow$' is read as '$\vdash$' in $\mathsf{LP}_2(\mathsf{LP})$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi}{y{:}_1\phi \Rightarrow \phi}\,(\Box,\Rightarrow)}{\neg\phi, y{:}_1\phi \Rightarrow}\,(\neg,\Rightarrow)}{\neg\phi \Rightarrow \neg y{:}_1\phi}\,(\Rightarrow,\neg)}{[x]\neg\phi \Rightarrow \neg y{:}_1\phi}\,(\Box,\Rightarrow)}{[x]\neg\phi \Rightarrow w{:}_2(\neg y{:}_1\phi)}\,(\Rightarrow,\Box)}{\Rightarrow [x]\neg\phi \to w{:}_2(\neg y{:}_1\phi)}\,(\Rightarrow,\to)
$$

Step 3. The one instance of the $(\Rightarrow, \Box)$ rule calls for the Lifting Lemma to replace w with $f(x)$ so that

$$[x]\neg\phi \vdash f(x){:}_2(\neg y{:}_1\phi)$$

in $\mathsf{LP}_2(\mathsf{LP})$. The proof of the Lifting Lemma is constructive and provides a general algorithm of finding such f. To skip some routine computations we will use the trivial special case of Lifting Lemma: if F is proven from the axioms of $\mathsf{LP}_2(\mathsf{LP})$ by classical propositional reasoning, then there is a ground[2] term g such that $g{:}_i F$ is also derivable in $\mathsf{LP}_2(\mathsf{LP})$ for each $i \in \{0, 1, 2\}$, without specifying g.

Consider the following Hilbert-style derivation in $\mathsf{LP}_2(\mathsf{LP})$, line 7 in particular.

[2]Ground proof terms are built from constants only and do not contain proof variables.

1. $y:_1\phi \to \phi$ — L3 axiom for agent 1
2. $\neg\phi \to \neg y:_1\phi$ — from 1. by contraposition
3. $[g](\neg\phi \to \neg y:_1\phi)$ — for some ground term g
4. $[g](\neg\phi \to \neg y:_1\phi) \to ([x]\neg\phi \to [g\cdot x]\neg y:_1\phi)$ — L1 axiom for GCK
5. $[x]\neg\phi \to [g\cdot x]\neg y:_1\phi$ — from 3. and 4. by modus ponens
6. $[g\cdot x]\neg y:_1\phi \to (g\cdot x):_2 \neg y:_1\phi$ — connection principle
7. $[x]\neg\phi \to (g\cdot x):_2 \neg y:_1\phi$ — from 5. and 6. by propositional reasoning

So, it suffices to put $f(x) = g\cdot x$ where g is a ground proof term from line 3.[3] Note the forgetful projection of the $\mathsf{LP}_2(\mathsf{LP})$ theorem line 7.,

$$[[x]\neg\phi \to (g\cdot x):_2 \neg y:_1\phi]^\circ = J\neg\phi \to K_2\neg K_1\phi\ ,$$

is the original $\mathsf{S4}_n^J$ theorem which was Realized.

5 Conclusion

The family of Justification Logics offers a robust and flexible setting in which to investigate explicit reasons for knowing: $t\!:\!F$, "F is know for reason t", in contrast to a modal approach in which $\Box F$ or $\mathbf{K}F$ represent implicit knowledge of F, where reasons are not specified. The addition of generic common knowledge opens these systems to numerous epistemic applications ([2, 6, 4]). The Realization Theorem for $\mathsf{S4}_n^J$ allows for all modalities, including GCK (J), to be made explicit in $\mathsf{LP}_n(\mathsf{LP})$, allowing reasoning to be tracked.

The construction of $\mathsf{LP}_n(\mathsf{LP})$ can serve as a template to construct other multi-agent explicit justification logics with GCK, even in cases where not all the agents' reasoning may be factive. If other justification logics such as J, JT, and $\mathsf{J4}$ ([7]) were augmented with GCK to form the logics $\mathsf{J}_n(\mathsf{LP})$, $\mathsf{JT}_n(\mathsf{LP})$, $\mathsf{J4}_n(\mathsf{LP})$, it is assumed that they too would correspond to implicit modal logics, presumably to K_n^J, KT_n^J, $\mathsf{K4}_n^J$, a respectively, but this has yet to be shown. There are also justification logics such as $\mathsf{J45}$, $\mathsf{JD45}$, and $\mathsf{JT45}$ with the negative introspection axiom $(\neg t:_i F \to ?t:_i(\neg t:_i F))$, whose forgetful projection is the modal ($\mathsf{5}$) axiom $\neg K_iF \to K_i(\neg K_iF)$ ([7]). On the modal side, GCK must be strong enough to be considered knowledge (usually $\mathsf{S4}$) but at least as strong as any agent ([3]) and so logics with the $\mathsf{5}$ axiom may use $\mathsf{S5}$ as the generic common knowledge. While there are realization theorems for $\mathsf{S5}$, such as Fitting's semantic approach ([14]), it may be more delicate to establish correspondences between these multi-agent GCK systems with negative introspection.

In the $\mathsf{LP}_n(\mathsf{LP})$ case presented here all agent reasoning represents knowledge. While it is useful to track the justifications, in the knowledge domain, each justification is a proof and so yields truth. However, in a belief setting, justifications are not necessarily sufficient to yield truth. In these situations it may become even more crucial, essential, to track specific evidence in order to analyze their reliability and compare justifications arriving from different sources. Logics of belief with GCK can be constructed: without factivity (L3) belief rather than knowledge is modeled. Investigating multi-agent logics of belief with GCK will likely

[3]Here g is built from the constant that proves this instance of L3 and those used in the derivation of the contrapositive from an implication.

also yield a rich source of models in which to analyze several traditional epistemic scenarios and may also offer an entry to considering an explicit version of common belief.

Even within the knowledge content, it may be also be worthwhile investigating other levels of group knowledge in an explicit setting. In [13] a hierarchy of group knowledge is presented, from distributed knowledge at the weakest, to "everybody knows", to finite iterative knowledge I, and finally common knowledge. Understanding these from an explicit, justification logic, standpoint could enrich the field. However, currently we see that generic common knowledge is a useful choice for modeling many epistemic situations and here we have presented what has yet to be shown for conventional common knowledge: that a modal epistemic logic with generic common knowledge can be made fully explicit. This is done through the introduction of the justification logic $\mathsf{LP}_n(\mathsf{LP})$ with explicit GCK and the Realization algorithm.

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