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Sergei N. Artemov

Tatiana Yavorskaya (Sidon)

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Sergei N. Artemov
CUNY Graduate Center
365 Fifth Avenue, rm. 4319
New York, NY 10016, USA
sartemov@gc.cuny.edu

Tatiana Yavorskaya (Sidon)*
Department of Mathematics
Moscow University
Moscow, 119899, Russia
tanya@lpcs.math.msu.su

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Abstract

The standard first-order reading of modality does not bind individual variables, i.e., if x is free in $F(x)$, then x remains free in $\Box F(x)$. Accordingly, if $\Box$ stands for 'provable in arithmetic,' $\forall x \Box F(x)$ states that $F(n)$ is provable for any given value of $n = 0, 1, 2, \ldots$; this corresponds to a *de re* reading of modality. The other, *de dicto* meaning of $\Box F(x)$, suggesting that $F(x)$ is derivable as a formula with a free variable x, is not directly represented by a modality, though, semantically, it could be approximated by compound constructions, e.g., $\Box \forall x F(x)$.

We introduce the first-order logic $\mathsf{FOS4}^*$ in which modalities can bind individual variables and, in particular, can directly represent both *de re* and *de dicto* modalities. $\mathsf{FOS4}^*$ extends first-order $\mathsf{S4}$ and is the natural forgetful projection of the first-order logic of proofs FOLP. The same method of introducing binding modalities obviously works for other modal logics as well.

1 Introduction

Let $A(x)$ be a formula with a parameter x. Then, in the standard reading of the modality in first-order language, formula $\Box A(x)$ also has x as a

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parameter. In the provability interpretation of $\Box$[1], this reflects the reading of $\Box A(x)$ as

given a natural parameter $x = n$, formula $A(n)$ is provable,

whereas the alternative meaning

formula $A(x)$ with a free variable x is provable

does not have a direct modal representation.

We extend the first-order modal language with a construction that makes it possible. Namely, we replace the modality $\Box$ by a family of modalities $\Box_X$, each labeled by a finite set of individual variables X. In a formula $\Box_X A$, the modality $\Box_X$ binds all variables in A except those in X. We call $\Box_X$ *binding modalities.*

In Section 2, we introduce the logic $\mathsf{FOS4}^*$ which is a version of first-order modal logic $\mathsf{S4}$[2] in the richer language with binding modalities. In Section 3, we present the sequent calculus for $\mathsf{FOS4}^*$ that admits cut-elimination. In Section 4, we discuss the connections between $\mathsf{FOS4}^*$ and first-order $\mathsf{S4}$. In Section 6, we show that the first-order logic of proofs FOLP is able to realize $\mathsf{FOS4}^*$ which, in combination with the results of the previous sections, yields another proof of the Realization Theorem for $\mathsf{FOS4}$ and FOLP. Section 5 is devoted to the Kripke-style semantics for $\mathsf{FOS4}^*$.

2 Modal logic with binding modalities

By X, Y, etc., we denote finite sets of individual variables. If y is an individual variable, then we will write Xy for $X \cup \{y\}$. An additional convention: notation Xy means, in part, that $y \notin X$.

Definition 1 Let $\mathcal{L}^*$ denote the first-order modal language that contains individual variables $x_0, x_1, x_2, \ldots$, Boolean connectives, quantifiers over individual variables, predicate symbols Q_i^n of any arity n ($i, n = 0, 1, 2, \ldots$), and the family of modal operators $\Box_X$ where X ranges over finite sets of individual variables.

[1]For informal and formal provability interpretations of modality see, e.g., [1, 2, 3, 4, 5].

[2]First-order modal logic $\mathsf{S4}$ has postulates of first-order classical logic and modal postulates of $\mathsf{S4}$: $\Box(F \to G) \to (\Box F \to \Box G)$, $\Box F \to \Box\Box F$, $\Box F \to F$, and the necessitation rule '$\vdash F$ yields $\vdash \Box F$.'

Formulas are defined in the standard way. The only difference is in the definition of open and closed occurrences of variables: the modality $\Box_X$ binds all variables except those from X, namely,

- All occurrences of individual variables in an atomic formula are free.
- Boolean connectives preserve free and bound occurrences of variables.
- $\forall x$ binds all occurrences of x in $\forall x A$ and preserves free and bound occurrences of all other variables.
- Free occurrences of variables from X in A, and only they, are free in $\Box_X A$.

The set of free variables of a formula A is denoted by $FVar(A)$.

Note also that in $\Box_X$, variables from X are merely a syntactic label of this modality and are not considered occurrences of variables.

Definition 2 The logic $\mathsf{FOS4}^*$ is axiomatized by the following schemas. Here A, B are formulas, X is a set of individual variables, and y is an individual variable.

A0 classical axioms of first-order logic
A1 $\Box_{Xy} A \rightarrow \Box_X A, \quad y \notin FVar(A)$
A2 $\Box_X A \rightarrow \Box_{Xy} A$
A3 $\Box_X A \rightarrow \Box_X \forall x A, \quad x \notin X$
B1 $\Box_X (A \rightarrow B) \rightarrow (\Box_X A \rightarrow \Box_X B)$
B2 $\Box_X A \rightarrow \Box_X \Box_X A$
B3 $\Box_X A \rightarrow A$

$\mathsf{FOS4}^*$ has the following inference rules:

R1	$\vdash A, A \rightarrow B \;\Rightarrow\; \vdash B$	*Modus Ponens*
R2	$\vdash A \;\Rightarrow\; \vdash \forall x A$	*generalization*
R3	$\vdash A \;\Rightarrow\; \vdash \Box_\emptyset A$	*necessitation.*

We define derivations in $\mathsf{FOS4}^*$ and derivations from the hypothesis in the standard manner. In particular, in a derivation from the set of hypotheses Γ, the generalization rule may not be applied to variables that are free in Γ.

Note that each $\Box_X$ is a normal propositional modality. In particular, the following standard lemma holds:

Lemma 1 *1.* $\mathsf{FOS4}^* \vdash \Box_X(A \wedge B) \leftrightarrow (\Box_X A \wedge \Box_X B)$.
2. If $\mathsf{FOS4}^* \vdash A \leftrightarrow B$ *then* $\mathsf{FOS4}^* \vdash \Box_X A \leftrightarrow \Box_X B$.

Modality $\Box_\emptyset$, which considers all individual variables as local and admits unlimited quantification under its scope, represents *de dicto* modality.

The following lemma shows that the principle

$$\Box_X A \rightarrow \Box_X \forall x A \quad \text{with } x \notin X,$$

(e.g., axiom A3) is derivable from the rest of $\mathsf{FOS4}^*$. However, we decide to keep it in the list of $\mathsf{FOS4}^*$ postulates to underline the significance of this principle. Furthermore, the proof of Lemma 2 relies on transitivity axiom B2 and hence does not work for logics without transitivity, e.g., binding modalities versions of first-order logics K or T which, however, should contain A3.

Lemma 2 $\mathsf{FOS4}^* \vdash \Box_X A \leftrightarrow \Box_X \forall x A$, *provided* $x \notin X$.

Proof. The left-to-right direction.

1. $\Box_X A \rightarrow A$ — axiom B3;
2. $\forall x \Box_X A \rightarrow \forall x A$ — from 1 by first-order logic;
3. $\Box_X A \rightarrow \forall x \Box_X A$ — by first-order logic, since $x \notin FVar(\Box_X A)$;
4. $\Box_X A \rightarrow \forall x A$ — from 2,3 by first-order logic;
5. $\Box_\emptyset(\Box_X A \rightarrow \forall x A)$ — from 4 by R3;
6. $\Box_X(\Box_X A \rightarrow \forall x A)$ — from 5 in view of A2;
7. $\Box_X \Box_X A \rightarrow \Box_X \forall x A$ — from 6 by B1 and propositional logic;
8. $\Box_X A \rightarrow \Box_X \Box_X A$ — axiom B2;
9. $\Box_X A \rightarrow \Box_X \forall x A$ — from 7, 8 by propositional logic.

The right-to-left direction.

1. $\forall x A \rightarrow A$ — by first-order logic;
2. $\Box_\emptyset(\forall x A \rightarrow A)$ — from 1 by R3;
3. $\Box_X(\forall x A \rightarrow A)$ — from 2 in view of A2;
4. $\Box_X \forall x A \rightarrow \Box_X A$ — from 3 in view of B1. □

A straightforward iteration of Lemma 2 yields

Lemma 3 *Let* A *be a formula in the language* $\mathsf{FOS4}^*$. *By* A^c *we denote a new formula obtained from* A *by replacing each subformula of* A *of the form* $\Box_X B$ *by* $\Box_X \forall y_1 \ldots \forall y_n B$ *where* $Y = \{y_1, \ldots, y_n\} = FVar(B) \setminus X$. *Then* $\mathsf{FOS4}^* \vdash A \leftrightarrow A^c$.

3 Cut-elimination

We define a sequent calculus for $\mathsf{FOS4}^*$ similar to that of the sequent calculus for $\mathsf{FOS4}$ (see [7], Section 9.1.3). In addition to structural rules (including the cut-rule), the sequential calculus for $\mathsf{FOS4}^*$ denoted by $\mathcal{G}\mathsf{FOS4}^*$ contains the following axioms:

$$\bot \Rightarrow \qquad \text{and} \qquad P(\vec{x}) \Rightarrow P(\vec{x})$$

and logical rules:

$$\frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ (R\to), \qquad \frac{\Gamma \Rightarrow \Delta, A, \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}\ (L\to),$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \forall x\ A}\ (R\forall), \qquad \frac{\Gamma, A(y/x) \Rightarrow \Delta}{\Gamma, \forall x\ A \Rightarrow \Delta}\ (L\forall),$$

$$\frac{\Box_{X_1} A_1, \ldots, \Box_{X_n} A_n \Rightarrow A}{\Box_{X_1} A_1, \ldots, \Box_{X_n} A_n \Rightarrow \Box_X A}\ (R\Box), \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\ (L\Box).$$

In $R\forall$, we suppose that $x \notin FVar(\Gamma, \Delta)$. In $R\Box$, we assume that

$$(\bigcup_{i=1}^{n} X_i) \cap FVar(A) \subseteq X.$$

The following connection between $\mathsf{FOS4}^*$ and its Gentzen-style version $\mathcal{G}\mathsf{FOS4}^*$ takes place:

Lemma 4

$$\mathcal{G}\mathsf{FOS4}^* \vdash \Gamma \Rightarrow \Delta \qquad \textit{iff} \qquad \mathsf{FOS4}^* \vdash \bigwedge \Gamma \to \bigvee \Delta.$$

Proof. In order to prove left–to–right implication, reason by induction on the derivaion in $\mathcal{G}\mathsf{FOS4}^*$. The only nontrivial cases here are the modal rules $R\Box$ and $L\Box$.

For $L\Box$, by the induction hypothesis, $\mathsf{FOS4}^* \vdash A \wedge (\bigwedge \Gamma) \to \bigvee \Delta$. Since $\mathsf{FOS4}^* \vdash \Box_X A \to A$, we obtain $\mathsf{FOS4}^* \vdash \Box_X A \wedge (\bigwedge \Gamma) \to \bigvee \Delta$.

For $R\Box$, the induction hypothesis yields that $\mathsf{FOS4}^* \vdash (\bigwedge \Box_{X_i} A_i) \to B$. Then for $X' = \bigcup_i X_i$ we have

1. $\Box_\emptyset((\bigwedge \Box_{X_i} A_i) \to B)$ by R3;
2. $\Box_{X'}(\bigwedge \Box_{X_i} A_i) \to B)$ from 1 in view of A2;
3. $\Box_{X'}(\bigwedge \Box_{X_i} A_i) \to \Box_{X'} B$ from 2 in view of B1;
4. $\Box_{X'}(\bigwedge \Box_{X_i} A_i) \leftrightarrow \bigwedge(\Box_{X'} \Box_{X_i} A_i)$ by Lemma 1;
5. $\Box_{X_i} A_i \to \Box_{X_i} \Box_{X_i} A_i$ by B2;
6. $\Box_{X_i} \Box_{X_i} A_i \to \Box_{X'} \Box_{X_i} A_i$ by A2;
7. $\Box_{X_i} A_i \to \Box_{X'} \Box_{X_i} A_i$ from 5, 6;
8. $\bigwedge_i \Box_{X_i} A_i \to \Box_{X'}(\bigwedge \Box_{X_i} A_i)$ from 4, 7;
9. $\Box_{X'} B \to \Box_X B$ by A1;
10. $\bigwedge_i \Box_{X_i} A_i \to \Box_X B$ by 3, 8, 9.

In order to prove right–to–left implication it suffices to show that

$$\mathcal{G}\mathsf{FOS4}^* \vdash \ \Rightarrow A$$

where A is any axiom of $\mathsf{FOS4}^*$. Below are the proofs for axioms A1 and A2; axioms B1–B3 are derived similarly with the standard $\mathsf{FOS4}$.

Axiom A1. If $y \notin FVar(A)$, then the tree below is a valid derivation in $\mathcal{G}\mathsf{FOS4}^*$:

$$\dfrac{\dfrac{A \Rightarrow A}{\Box_{Xy} A \Rightarrow A}}{\Box_{Xy} A \Rightarrow \Box_X A.}$$

Axiom A2.

$$\dfrac{\dfrac{A \Rightarrow A}{\Box_X A \Rightarrow A}}{\Box_X A \Rightarrow \Box_{Xy} A.}$$

$\Box$

Theorem 1 (Cut-elimination) *If $\mathcal{G}\mathsf{FOS4}^* \vdash \Gamma \Rightarrow \Delta$, then $\Gamma \Rightarrow \Delta$ can be derived in $\mathcal{G}\mathsf{FOS4}^*$ without using the cut-rule.*

Proof. We repeat the standard proof of the cut-elimination theorem for $\mathcal{G}\mathsf{FOS4}$ by joint induction on the complexity of the cut and its rank (see [7]). $\Box$

4 Connection to first-order modal logic

In this section, we study the interconnection between $\mathsf{FOS4}^*$ and the standard modal logic $\mathsf{FOS4}$.

Definition 3 Let A be a first-order modal formula. By A^* we mean a formula of the language of $\mathsf{FOS4}^*$ that is obtained from A by replacing all occurrences of subformulas of A of the form $\Box B$ by $\Box_X B$ where $X = FVar(B)$.

Definition 4 Let $\vec{x}$ be a finite string of individual variables $\{x_1, \ldots, x_n\}$. By $\vec{x}, \vec{y}$ we denote the concatenation of $\vec{x}$ and $\vec{y}$. By $\forall \vec{x} A$ and $\exists \vec{x} A$ we mean the formulas

$$\forall x_1, \ldots, \forall x_n A \quad \text{and} \quad \exists x_1, \ldots, \exists x_n A$$

respectively.

Definition 5 For each $\mathsf{FOS4}^*$-formula F, we define its translation to the language of $\mathsf{FOS4}$, denoted by $F^\sharp$, by induction. For atomic formulas, we stipulate $F^\sharp = F$, $\sharp$ commutes with Boolean connectives and quantifiers, and

$$(\Box_X F)^\sharp = \Box \forall \vec{y} F^\sharp,$$

where $\vec{y} = \{y_0, \ldots, y_k\} = FVar(F) \setminus X$.

Note that $FVar(A^*) = FVar A$ and $FVar(F^\sharp) = FVar(F)$.

Lemma 5 *If* $\mathsf{FOS4} \vdash F$*, then* F^* *is derivable in* $\mathsf{FOS4}^*$.

Proof. By induction on a derivation in $\mathsf{FOS4}$. For the base of induction consider axioms of $\mathsf{FOS4}$.

- First-order axioms remain the same axioms.
- $(\Box(A \to B) \to (\Box A \to \Box B))^* = \Box_Z(A^* \to B^*) \to (\Box_X A^* \to \Box_Y B^*)$ where $X = FVar(A)$, $Y = FVar(B)$, and $Z = X \cup Y$; then in $\mathsf{FOS4}^*$ we reason as follows.
  1. $\Box_Z(A^* \to B^*) \to (\Box_Z A^* \to \Box_Z B^*)$ — axiom B1
  2. $\Box_Z A^* \leftrightarrow \Box_X A^*$ — axioms A1 and A2
  3. $\Box_Z B^* \leftrightarrow \Box_Y B^*$ — axioms A1 and A2
  4. $\Box_Z(A^* \to B^*) \to (\Box_X A^* \to \Box_Y B^*)$ — from 1–3 by propositional logic

- $(\Box A \to \Box\Box A)^* = \Box_X A^* \to \Box_X \Box_X A^*$ for $X = FVar A$; axiom B2.

- $(\Box A \to A)^* = \Box_X A^* \to A^*$ for $X = FVar A$; as the axiom B3.

For the induction step, we note that the translation $(\cdot)^*$ preserves *Modus Ponens* and the generalization rule.

In the case of the necessitation rule, $\Box A$ is derived from A in FOS4, then by the induction hypothesis $\mathsf{FOS4}^* \vdash A^*$; therefore in $\mathsf{FOS4}^*$

$\Box_\emptyset A^*$ by R3;

$\Box_X A^*$ by A2 and *Modus Ponens* for $X = FVar(A)$.

□

Theorem 2 *If* $\mathsf{FOS4}^* \vdash A$, *then* $\mathsf{FOS4} \vdash A^\sharp$.

Proof. Induction on the derivation of A in $\mathsf{FOS4}^*$. For the base of induction, consider axioms of $\mathsf{FOS4}^*$.

- Translation $(\cdot)^*$ respects axioms of first-order logic.

- Axiom A1. If $y \notin FVar(A)$, then $(\Box_{Xy} A)^\sharp$ coincides with $(\Box_X A)^\sharp$.

- Axiom A2. If $y \notin FVar(A)$, then $(\Box_{Xy} A)^\sharp$ coincides with $(\Box_X A)^\sharp$ as above. If $y \in FVar(A)$, then $(\Box_X A)^\sharp = \Box(\forall y, \vec{y} A^\sharp)$ and $(\Box_{Xy} A)^\sharp = \Box(\forall \vec{y} A^\sharp)$; it is clear that the latter formula follows from the former one.

- Axiom B1. Let $\vec{y}, \vec{u} = \{y_1, \dots, y_k, u_1, \dots, u_l\}$ be $FVar(A) \setminus X$, and $\vec{z}, \vec{u} = \{z_1, \dots, z_m, u_1, \dots, u_l\}$ be $FVar(B) \setminus X$, and $\vec{u} = \{u_1, \dots, u_l\}$ be $(FVar(B) \cap FVar(A)) \setminus X$. Then $FVar(A \to B) \setminus X$ is $\vec{y}, \vec{z}, \vec{u}$. Therefore $(\Box_X(A \to B) \to (\Box_X A \to \Box_X B))^\sharp$ is

$$\Box(\forall \vec{y}, \vec{z}, \vec{u}(A^\sharp \to B^\sharp)) \to \Box(\forall \vec{y}, \vec{u} A^\sharp) \to \Box(\forall \vec{z}, \vec{u} B^\sharp)),$$

  which is clearly provable in FOS4 since no y_i occures in $B^\sharp$ and no z_i occures in $A^\sharp$.

- Axiom B2. Let $\vec{y} = \{y_1, \dots, y_k\}$ be $FVar(A) \setminus X$. Since $FVar(\Box_X A) = X$, we obtain $(\Box_X A)^\sharp = \Box \forall \vec{y} A^\sharp$ and $(\Box_X \Box_X A)^\sharp = \Box\Box \forall \vec{y} A^\sharp$; therefore the translation of B2 is an instance of the transitivity axiom in $S4$.

- Axiom B3. $(\Box_X A \to A)^\sharp = (\Box \vec{y} A^\sharp \to A^\sharp)$; the last formula is clearly derivable in FOS4.

□

Corollary 1 $\mathsf{FOS4} \vdash A$ *if and only if* $\mathsf{FOS4}^* \vdash A^*$.

Proof. The left–to–right direction is proven by Lemma 5. To prove the right–to–left direction, we assume that $\mathsf{FOS4}^* \vdash A^*$. Since for each subformula of A^* of the form $\Box_X B$ one has $X = FVar(B)$, we conclude that $(A^*)^\sharp$ coincides with A. By Theorem 2, from $\mathsf{FOS4}^* \vdash A^*$ we obtain $\mathsf{FOS4} \vdash (A^*)^\sharp$, thus $\mathsf{FOS4} \vdash A$.

□

Corollary 2 $\mathsf{FOS4}^* \vdash A$ *if and only if* $\mathsf{FOS4} \vdash A^\sharp$.

Proof. The left–to–right direction is proven by Theorem 2. To prove the right–to–left direction let us assume $\mathsf{FOS4} \vdash A^\sharp$. Then by Lemma 5, we obtain $\mathsf{FOS4}^* \vdash (A^\sharp)^*$. By induction on A it can be easily proven that $\mathsf{FOS4}^* \vdash (A^\sharp)^* \leftrightarrow A$, therefore $\mathsf{FOS4}^* \vdash A$. □

5 Semantics

Definition 6 An $\mathsf{FOS4}^*$-model is a standard Kripke model for $\mathsf{FOS4}$. Namely, a frame $\mathcal{F} = (W, R, D)$ for $\mathsf{FOS4}^*$ consists of a nonempty set of possible worlds $W \neq \emptyset$, a transitive reflexive accessibility relation $R \subseteq W^2$, and a domain function D which assigns a nonempty set $D(u)$ to each element u of W monotonically, i.e., uRv yields $D(u) \subseteq D(v)$ for all $u, v \in W$.

A model $\mathcal{M} = (\mathcal{F}, V)$ is a frame F supplied with the validity function V which assigns each n-ary predicate letter P and each possible world u a subset of $D(u)^n$ (on which P is true at node u).[3]

For each formula A of the language $\mathsf{FOS4}^*$, each substitution of σ of the values from

$$U = \bigcup_{u \in W} D(u)$$

for free variables, and each $u \in W$, we define $\mathcal{M}, u \models A\sigma$ by induction on the complexity of A:

[3]For comparison, Kripke-style semantics for FOLP in [6] treats both variable binding operators and proof terms.

- if A is atomic, that is, $A = P(x_1, \dots, x_n)$ for a predicate letter P, then $\mathcal{M}, u \models A\sigma$ iff $(x_1\sigma, \dots, x_n\sigma) \in V(P, u)$;
- quantifiers and Boolean connectives are treated in the usual way;
- if $A = \Box_X B$, then $\mathcal{M}, u \models A\sigma$ iff $\mathcal{M}, v \models B\sigma'$ for each v accessible from u and each substitution σ' which coincides with σ on variables from X and assigns arbitrary elements from $D(v)$ to the remaining free variables of B.[4]

We say that A is valid in a model $\mathcal{M}$ ($\mathcal{M} \models A$) if $\mathcal{M}, u \models A$ for each $u \in W$.

Theorem 3 $\mathsf{FOS4}^* \vdash A$ *iff* A *is valid in all models.*

Proof. Soundness follows easily by induction on derivations in $\mathsf{FOS4}^*$. Completeness follows immediately from the connection to $\mathsf{FOS4}$ established in Section 4. Indeed, suppose $\mathsf{FOS4}^* \not\vdash F$. By Corollary 2, $\mathsf{FOS4} \not\vdash A^\sharp$. By Completeness, there is an $\mathsf{FOS4}$-model $\mathcal{M}$ in which $A^\sharp$ does not hold in the sense of $\mathsf{FOS4}$, hence $A^\sharp$ does not hold in $\mathcal{M}$ in the sense of $\mathsf{FOS4}^*$. $\Box$

6 Connection to the first-order logic of proofs

Definition 7 (see [4] for details). The language of the first-order logic of proofs FOLP in the extension of the first-order language with a countable set of predicate symbols of any arity, without functional symbols and equality by special means to represent proofs and proof assertions, namely,

- proof variables and constants;
- functional symbols for operations on proofs: binary $+$, $\cdot$ and unary $!$ and gen_x for each individual variable x;
- an operational symbol $(\cdot){:}_X(\cdot)$ for each finite set X of individual variables.

[4] It is easy to see that $\mathcal{M}, u \models A$ if and only if $\mathcal{M}, u \models A^\sharp$ in the usual sense of $\mathsf{FOS4}$.

Proofs are representes by *proof terms*, constructed from proof constants and variables by means of functional symbols. *Formulas* are defined in the standard way with an additional clause for proof assertions:

- if t is a proof term, A is a formula, and X is a finite set of individual variales, then $t{:}_X A$ is a formula.

The intended semantics of $t{:}_X A$ is

$$t \textit{ is a proof of } A \textit{ with parameters } X$$

has been made precise in [4].

In addition to axioms and rules of first-order logic, FOLP has the following axioms and rules.

- $t{:}_{Xy} A \rightarrow t{:}_X A, \quad y \notin FVar(A)$
- $t{:}_X A \rightarrow t{:}_{Xy} A$
- $t{:}_X A \rightarrow A$
- $s{:}_X (A \rightarrow B) \rightarrow (t{:}_X A \rightarrow [s \cdot t]{:}_X B)$
- $t{:}_X A \rightarrow [t+s]{:}_X A, \quad s{:}_X A \rightarrow [t+s]{:}_X A$
- $t{:}_X A \rightarrow\ !t{:}_X t{:}_X A$
- $t{:}_X A \rightarrow \mathsf{gen}_x(t){:}_X \forall x A,\ x \notin X$
- $\vdash c{:}A$, where A is an axiom, c is a proof constant.

Definition 8 Let A be an $\mathsf{FOS4}^*$–formula. By *realization* of a formula A, we mean a formula A^r of the language of FOLP that is obtained from A by replacing all occurrences of subformulas of A of the form $\Box_X B$ by $t{:}_X B$ for some proof terms t. To avoid unnecessary formalism, we suggest thinking of a realization as a result of an iterated procedure which always replaces an innermost $\Box B$ by $t{:}_X B$. A realization is *normal* if all negative occurrences of $\Box$ are assigned proof variables.

Definition 9 The forgetful projection $(\cdot)^0$ of FOLP to the language $\mathsf{FOS4}^*$ erases all terms in an FOLP-formula, namely, it replaces all subformulas of the form $t{:}_X B$ by $\Box_X B$:

$$(t{:}_X F)^0 = \Box_X F^0.$$

Lemma 6 *If* FOLP $\vdash F$*, then* F^0 *is derivable in* FOS4**.*

Proof. By straightforward induction on derivations in FOLP. □

Remark 1 Note that the forgetful projection of an FOLP–formula A to the language of FOS4 (as defined in [4]) is a superposition of the forgetful projection of A to the language of FOS4* and the translation $\sharp$ of FOS4* to FOS4.

Similarly, a realization of an FOS4–formula A in FOLP is a composition of the translation $*$ of A to FOS4* and a realization in FOLP.

Theorem 4 [Realization Theorem] *If* FOS4* $\vdash A$*, then there is a normal realization* A^r *such that* FOLP $\vdash A^r$*.*

Proof. Induction on the cut-free proof in the Gentzen variant for FOS4* similar to that in [4]. □

7 Conclusions

Binding modalities system FOS4* naturally contains both the traditional *de re* modality $\Box F$, which is represented by $(\Box F)^*$ i.e., $\Box_X F$ with X being the set of all free variables of F, and the *de dicto* modality, which is represented by $\Box_\emptyset$.

This extended set of modalities allows FOS4* to directly express and derive new principles of modal logic. For example, FOS4* supports the Barcan Formula for $\Box_\emptyset$:

$$\forall x \Box_\emptyset A \rightarrow \Box_\emptyset \forall x A,$$

which does not hold for the usual S4 modality rather than $\Box_\emptyset$.

One could argue that FOS4* is as expressive as traditional FOS4 since the binding modality $\Box_X F$ can be semantically encoded by $\Box\forall\vec{y}F$ where $\Box$ is the FOS4 modality and $\vec{y}$ is the list of all free variables of F that are not in X. However, such an encoding changes both formulas and derivations: for example, converting $\Box_\emptyset A$ to $\Box\forall x A$ renders the aforementioned Barcan Formula meaningless within the context of FOS4. In FOS4*, however, this is a revealing and meaningful principle. We want to believe that the structural and semantic properties of the *de dicto* modality deserve direct study and a designated formal logic system and that $\mathsf{FOS4}^*$ is such a system.

$\mathsf{FOS4}^*$ captures the *de dicto* modality in a coherent first-order manner, offers a natural axiom system, sequent formulation with cut-elimination, and Kripke model theory. It makes reasoning with variables in the modal context explicit and hence a subject of formal analysis which could be of interest for studies of modalities and applications.

In summary, $\mathsf{FOS4}^*$ can be regarded as a version of $\mathsf{FOS4}$ in a refined language with an explicit mechanism of binding/unbinding variables which yet enjoys the convenient features of $\mathsf{FOS4}$, such as cut-elimination and Kripke completeness.

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