2012

TR-2012012: Condition Estimation by Means of Power Method

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Abstract

We employ the Power Method (that is essentially a sequence of matrix-by-vector multiplications) to estimate the condition number of a matrix.

2000 Math. Subject Classification: 15A12, 65F35
Keywords: Condition estimation, Power Method

Assume a real symmetric nonnegative definite $n \times n$ matrix S and apply the Power Iteration

$$\mathbf{v}_k = S^k \mathbf{v} = S\mathbf{v}_{k-1},\ k = 1, 2, \ldots \tag{1}$$

for a random vector $\mathbf{v} = \mathbf{v}_0$ to approximate the largest eigenvalue $\lambda = \lambda(S)$ of the matrix S by the Rayleigh quotients $q_i = \mathbf{v}_k^T S\mathbf{v}_k/\mathbf{v}_k^T\mathbf{v}_k$. The paper [2] has proposed this technique and proved that $q_k \le \lambda\ \theta q_k$ with a probability at least $1 - 0.8\theta^{-k/2}n^{1/2}$ for any scalar $\theta > 1$. This estimate defines a stopping criterion for the iteration, and heuristically one can also stop where $q_i/q_{i-1} \approx 1$ or $||S\mathbf{v}_i - q_i\mathbf{v}_i||/(|q_i)|\ ||\mathbf{v}_i||) \le t$ for a fixed tolerance t. Instead of the Rayleigh quotients one can use the simple quotients $s_i = \mathbf{e}_i^T S\mathbf{v}_k/\mathbf{e}_i^T\mathbf{v}_k$ for the ith coordinate vectors $\mathbf{e}_i$ and fixed or random integers $i = i(k)$, $1 \le i \le n$ (cf. [1], [3]), [4]). Now assume an $m \times n$ matrix A for $m \ge n$, let $\sigma_j(A)$ denote its jth largest singlular value, and seek a crude estimates for $\sigma_1(A)$ and $\sigma_n(A)$, e.g., to decide whether the matrix is well conditioned. Apply the power iteration (1) to the matrix $S = A^TA$ to computed a close upper bound σ_+^2 on $\lambda(S) = ||A||^2 = \sigma_1^2(A)$. Then apply the power iteration (1) to the matrix $B = \sigma_+^2 I - A^TA$ to compute an approximation λ_+ to its largest eigenvalue and then obtain $\sigma_+^2 - \lambda_+ \approx \sigma_n^2(A)$. For $m \le n$ apply the same techniques to the matrix AA^T.

Acknowledgements: This research has been supported by NSF Grant CCF–1116736 and PSC CUNY Awards 4512–0042 and 65792–0043.

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