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Abstract

In his dissertation of 1950, Nash based his concept of the solution to a game on the assumption that "a rational prediction should be unique, that the players should be able to deduce and make use of it." We study when such definitive solutions exist for strategic games with ordinal payoffs. We offer a new, syntactic approach: instead of reasoning about the specific model of a game, we deduce properties of interest directly from the description of the game itself. This captures Nash's deductive assumptions and helps to bridge a well-known gap between syntactic game descriptions and specific models which could require unwarranted additional epistemic assumptions, e.g., common knowledge of a model.

We show that games without Nash equilibria do not have definitive solutions under any notion of rationality, but each Nash equilibrium can be a definitive solution for an appropriate refinement of Aumann rationality. With respect to Aumann rationality itself, games with multiple Nash equilibria cannot have definitive solutions. Some games with a unique Nash equilibrium have definitive solutions, others don't, and the criteria for a definitive solution is provided by the iterated deletion of strictly dominated strategies.

1 Introduction

Some classical strategic games have definitive solutions which follow logically from the game description and plausible principles of knowledge and rationality. Here is a quote from Nash's dissertation [17] which raises the issue of a deductive approach to solving games:

> *We proceed by investigating the question: what would be a rational prediction of the behavior to be expected of rational[ly] playing the game in question? By*

> *using the principles that a rational prediction should be unique, that the players should be able to deduce and make use of it,* [...] *one is led to the concept of a solution* [...] *.*

Another quote from [17] explains this issue even further:

> [...] *we need to assume the players know the full structure of the game in order to be able to deduce the prediction for themselves.*

Some game theorists consider the aforementioned assumption, that the player can deduce the strategies of other players, as rarely met. Perhaps the following quote from Pearce [19] represents this skepticism fairly:

> *The rules of a game and its numerical data are seldom sufficient for logical deduction alone to single out a unique choice of strategy for each player. To do so one requires either richer information* [...] *or bolder assumptions about how players choose strategies.*

Foundations of Nash's approach have been widely studied from probabilistic positions, and it is not feasible to provide a representative survey of the corresponding literature within the limits of this paper; we mention [8, 10, 19], just to name few. There has also been a vast body of epistemic logic studies in the foundations of Game Theory, cf. [12, 23] for some recent surveys.

In this paper, we offer a logical analysis of Nash's assumption that the player can *deduce* the strategies of others. This concept is of the deductive logical character which is represented by the notion of a *definitive solution* of a game as a strategy profile s such that it logically follows from the game description, including the epistemic and rationality assumptions, that each player plays s. Here "logically follows" can be understood twofold: as a logical deduction by certain rules from a set of formalized postulates, or as a logical entailment of the semantic nature on a class of models. Due to the basic soundness and completeness theorems of logic, these two approaches are theoretically equivalent and it is up to the user to choose which approach to follow.

Now dominant, the semantic approach can be traced to Aumann's seminal "Agreeing to Disagree" paper [7] and the notion of Aumann structures that model both structural and epistemic sides of games. Though flexible and convenient, this semantical approach was not quite foundationally satisfactory, first of all due to assumptions that

1. a given model, including possible epistemic states of players, adequately represents the game, which is normally described syntactically, and

2. the model itself is common knowledge for the players.

As emphatically stressed by Aumann himself (cf. [9]), this created tension between the syntactic character of the game description and model-theoretic tools of studying games.

It does not appear problematic to assume that the rules of the game, let us call them *GAME*, in plain English (or an appropriate logical-mathematical formalism), are commonly known but this yields, generally speaking, neither (1) nor (2), since there can be many different models for a given game description[1]. Moreover, the standard method of producing a model of a syntactic set Γ, so-called canonical model construction, generally speaking, offers an infinite model not necessarily equivalent to the desired model $\mathcal{M}$. So, in a general setting, when we assume *GAME* and then study a specific model $\mathcal{M}$ of *GAME*, we lose information: if a property F holds in $\mathcal{M}$, it does not necessary follow from the game description *GAME* and could be an artifact of the chosen model $\mathcal{M}$. One way out of this predicament could be assuming $\mathcal{M}$ to be the definition of the game. This, however, renders the game description implausible. Aumann in [9] writes about this alternative: *Your mother won't understand if you try to say it semantically.*

How serious is this threat of non-categoricity? Many simple epistemic scenarios are categorical and yield a specific model, e.g., for regular strategic games without additional semantical constraints[2], the Muddy Children Puzzle, and others, the initial syntactic description is categorical and any model is equivalent (bi-similar) to the intended model. However, it is easy to offer a slight modification of the Muddy Children scenario which makes it non-categorical and the notion of the standard model meaningless. So, if we intend to consider games with general epistemic conditions, the categoricity consideration can become a serious matter.

The assumption about common knowledge of model $\mathcal{M}$ does not look plausible either: epistemic scenarios in plain English can be publicly announced to a group of players which makes *GAME* commonly known. However, publicly announcing even a simple Aumann structure does not seem realistic[3].

2 New format of reasoning about games

We suggest a modification of the format of reasoning about epistemic scenarios and games which retains the flexibility of the traditional model-theoretic approach and is free of its aforementioned deficiencies. The usual format of reasoning about games is

1. to assume the set of game rules *GAME*, including epistemic conditions given syntactically, in plain English or the usual logical-mathematical slang;

2. to pick a specific Aumann structure $\mathcal{A}$ that corresponds to *GAME*, often informally;

3. to tacitly assume 'common knowledge of $\mathcal{A}$' and not to bother about its justification (whereas a straightforward rigorous formalization of common knowledge of $\mathcal{A}$ requires

[1]This phenomenon is well known to logicians as 'non-categoricity.'

[2]cf. Section 7.4.

[3]This observation does not concern so-called computer knowledge when the programmer can program specific models to each computer.

tools outside $\mathcal{A}$);

4. to reason about $\mathcal{A}$ using (3).

The deficiency of this schema is obvious. Suppose you establish a property F using (4). Can you claim that F follows from the rules of the game? In general, no. Since $\mathcal{A}$ is a model of *GAME*, each fact that follows from *GAME* holds in $\mathcal{A}$, but not the other way around. It can easily happen that F holds in $\mathcal{A}$, but F does not follow from *GAME*. So this schema produces results about a specific model $\mathcal{A}$ of *GAME* with an additional and difficult-to-motivate assumption of the common knowledge of $\mathcal{A}$. Sometimes this schema works fine, e.g., when $\mathcal{A}$ is the only model of *GAME* (up to truth preserving bi-simulations). However, such categoricity analysis is not normally performed and it is easy to provide examples in which categoricity does not hold. If step (1) in this schema is omitted and $\mathcal{A}$ itself is assumed to be the definition of the game, then the categoricity objection becomes void, but the problem with common knowledge of $\mathcal{A}$ persists.

We suggest the following format for studying games:

1. Assume the set of game rules *GAME* as above and common knowledge of *GAME*. This can be arranged in the language of *GAME* that contains the common knowledge modality $\mathbf{C}$ - we just assume $\mathbf{C}(P)$ for each postulate P of *GAME*.

2. Logically reason from *GAME* directly, e.g., to establish that a fact F follows from the rules of the game. If *GAME* is categorical, then reasoning in a specific model is equivalent to logical reasoning in *GAME*.

3. Use specific models of *GAME*, if needed, to check the consistency of *GAME* or to establish that a certain F does not follow from *GAME*.

In this schema, both previous objections, possible non-categoricity of *GAME*, and dubious assumptions about common knowledge of $\mathcal{A}$, are eliminated. The categoricity requirement is replaced by a much lighter condition: soundness of *GAME* in $\mathcal{A}$, i.e., that all postulates of *GAME* hold in $\mathcal{A}$. Common knowledge of $\mathcal{A}$ is no longer required and is replaced by the assumption of common knowledge of *GAME*. Additional bonuses of this approach include the possibility of utilizing logical intuition and reasoning from *GAME* informally in the logic of knowledge; informal reasoning using the rules of the game is often quite efficient and produces shorter proofs than rigorous model reasoning[4]. Another attractive feature of this approach is greater flexibility in using models $\mathcal{A}$ that can be chosen specifically to be counter-models for F without commitment that $\mathcal{A}$ is a full description of the game.

All logical reasoning in this paper can be made completely formal within a framework of multi-agent epistemic modal logic although we don't see sufficient incentive for doing so[5].

[4]By the same token, we use rigorous yet informal reasoning to establish the Pythagorean theorem, though such a proof could be completely formalized and derived in an axiomatic geometry.

[5]We continue our analogy with the Pythagorean theorem: one could try to formalize its proof completely only as a challenge or an exercise. A normal mathematically rigorous proof of it is not formal.

We adopt the view that the reader possesses the robust intuition of epistemic reasoning and that reference to the S5-based principles of knowledge[6] and to common knowledge is sufficient.

3 Content

As a case study, we consider the class of finite strategic games with ordinal payoffs. Since, for this class of games, the concept of mixed strategies and expected payoff is not well-founded, we have to consider pure strategies only.

We observe that games without Nash equilibria lack definitive solutions under any notion of rationality and that each Nash equilibrium can be a definitive solution for an appropriate refinement of Aumann rationality.

We show that with respect to basic Aumann rationality, all games with two or more Nash equilibria, and some games with a unique Nash equilibrium, do not have definitive solutions either.

Perhaps some of these impossibility results do not come as a surprise for an experienced game theorist. For example, Theorem 3 states that no game with more than one Nash equilibrium under normal epistemic assumptions and Aumann rationality can have a definitive solution. This corresponds to the intuition that choosing between several Nash equilibria, in addition to Aumann rationality, in Pearce's words, "requires either richer information ... or bolder assumptions about how players choose strategies." However, there is a difference between empirically justified intuition and a rigorous proof. In computer programming, experts have no illusions that one could build a universal verifier which, for any given program P and input I, automatically decides whether P terminates on I. Turing's rigorous proof of this fact, known as the undecidability of the halting problem theorem, provided a basis for further fruitful studies of computability. If this paper is successful, this could be a step towards logical studies of consistency and impossibility in Game Theory.

Furthermore, we show that the criteria for Nash's definitive solution in strategic games with ordinal payoffs and Aumann rationality is provided not by Nash equilibria, but rather by iterated deletion of strictly dominated strategies.

This paper is an extended version of technical report [4] of 2010.

4 Logical presentation of strategic games

We consider strategic games with n players $1, 2, \ldots, n$. A strategy profile

$$s = (s_1, s_2, \ldots, s_n)$$

is a collection of strategies s_i for players $i = 1, 2, \ldots, n$. Each strategy profile s uniquely determines the *outcome* in which each move is made according to s. We assume that

[6] such as reflexivity and positive and negative introspection

everyone who knows the game can calculate i's payoff as determined by s. A strategy profile s is a **Nash equilibrium** if, given strategies of the other players, no player can profitably deviate (cf. [18] for rigorous definitions).

We assume that rules of the game are formally represented in an appropriate logical language as a set of formulas *GAME* as follows. Strategy j of player i is formally represented by a corresponding atomic sentence s_i^j stating that

player i has committed to strategy j.

For profile $s = (s_1, s_2, \ldots, s_n)$, its formula representation will be $s_1 \wedge s_2 \wedge \ldots \wedge s_n$ which we will also call s. In this setting,

$$\bigwedge_{k \neq l} \neg(s_i^k \wedge s_i^l)$$

means that player i chooses only one strategy, and

$$s_i^1 \vee s_i^2 \vee \ldots \vee s_i^{m_i}$$

where $s_i^1, s_i^2, \ldots, s_i^{m_i}$ is the list of propositions for all strategies for i, reflects the assumption that one of these strategies has to be played. The fact that a profile $s = (s_1, s_2, \ldots, s_n)$ is at least as preferable as $s' = (s'_1, s'_2, \ldots, s'_n)$ for player i can be represented by a special preference formula

$$s \geq_i s'$$

and all these preferences are supposed to be common knowledge.

In principle, *GAME* may be a (possibly infinite) set of logical formulas with additional preference relations on strategy profiles, which contains a comprehensive game description including its epistemic conditions. Note that *GAME* is not necessarily consistent.

By "logically follows" we mean here a logical deduction denoted as "$\vdash$." This notion is usually understood in logic as formal derivability. Our reasoning will not be completely formalized (cf. footnotes 4 and 5), but we assume could be if needed.

We use special logical symbols, **knowledge operators** (cf. [14]) $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_n$, to denote knowledge of players $1, 2, \ldots, n$ and assume the standard principles of knowledge, cf. [14, 18], a.k.a. S5 principles. For example, stating

$$\mathbf{K}_i(s_j^l)$$

says "i knows that j has chosen her strategy l."

We will use the "everybody knows" modality $\mathbf{E}$ as the abbreviation

$$\mathbf{E}F = \mathbf{K}_1 F \wedge \mathbf{K}_2 F \wedge \ldots \wedge \mathbf{K}_n F.$$

Common Knowledge of F, $\mathbf{C}F$, reflects a situation in which all propositions

$$\mathbf{K}_{i_1}\mathbf{K}_{i_2} \ldots \mathbf{K}_{i_m} F$$

hold for any $i_1, i_2, \ldots, i_m$.

An informal account of Aumann's notion of rationality can be found in ([8]). Aumann states that for a rational player i,

> *there is no strategy that i knows would have yielded him a conditional payoff ... larger than that which in fact he gets.*

Formal accounts of Aumann rationality based on epistemic models known as Aumann structures can be found in [8, 11] and other sources. Here we will adopt a syntactic formalization of Aumann rationality for strategic games.

A strategy profile $s = (s_1, s_2, \ldots, s_n)$ is **deemed possible** by player i if for each $j \neq i$,

$$\neg \mathbf{K}_i(\neg s_j).$$

This definition reflects the assumption that players consider all of their strategies possible whereas some of the other players' choices could be ruled out as impossible based on the rules and conditions of the game. Since we allow epistemic constraints in *GAMES*, the notion of possibility should be relativized: s is deemed possible by i at s' if

$$s' \rightarrow \text{“}s \textit{ is deemed possible by } i.\text{”}$$

In particular, s is possible for each i at s. Indeed, by laws of logic, $\mathbf{K}_i(\neg s_j) \rightarrow \neg s_j$, hence $s_j \rightarrow \neg \mathbf{K}_i(\neg s_j)$.

Let (s_{-i}, x) denote the strategy profile obtained from s by replacing s_i by x. It follows from definitions that

s is deemed possible by i iff (s_{-i}, x) is deemed possible by i for every strategy x of i.

Definition 1 Let $s = (s_1, s_2, \ldots, s_n)$ be a strategy profile. A formula $R_i(s)$ stating that **player i is Aumann-rational at s** is, by definition, the natural formalization of the following: *for i's any strategy x, there is a profile s' deemed possible by i at s such that*

$$(s'_{-i}, s_i) \geq_i (s'_{-i}, x).$$

A formula

$$R(s) = \bigwedge_i R_i(s)$$

states that all players $i = 1, 2, \ldots, n$ are Aumann-rational at s.

Note that quantified sentences “for any x ...” and “there is a profile s' ...” are represented by propositional formulas since there are only finite sets of possible x's and s's and there is no need to invoke quantifiers over strategies or strategy profiles.

Definition 1 formalizes the most basic, Aumann rationality though there are also other, more elaborate notions of rationality. However, we assume that any notion of rationality should be at least as strong as Aumann rationality on each profile. Technically, if $r_i(s)$ is a formula representing some notion of rationality of i at s, then

$$r_i(s) \rightarrow R_i(s)$$

is assumed in the *GAME*.

In addition to the game rules we consider a **solution predicate**, a formula $Sol(s)$ that specifies the conditions under which strategy profile s is considered to be a solution of the game. Given rationality predicates $r_i(s)$, $i = 1, 2, \ldots, n$ and

$$r(s) = \bigwedge_i r_i(s), \tag{1}$$

the typical cases of $Sol(s)$ are

1. $r(s)$ informally stating that s is a profile at which all players are rational;
2. $\mathbf{E}r(s)$ claiming that s is a profile at which players' rationality is mutually known;
3. $\mathbf{C}r(s)$ stating common knowledge of rationality at s.

We assume that a solution predicate $Sol(s)$ contains condition 1 that all players are rational, i.e., for each profile s,

$$GAME \vdash Sol(s) \rightarrow r(s).$$

Solution predicate 3 corresponds to the familiar assumption of common knowledge of rationality.

The natural characteristic principle for the solution condition is *if a profile is played, then it ought to satisfy the solution constraints*:

$$s \rightarrow Sol(s). \tag{2}$$

which may be more recognizable in its contrapositive form

$$\neg Sol(s) \rightarrow \neg s,$$

meaning *if s does not satisfy solution constraints, then s is not played.*

The familiar epistemic conditions: rationality, mutual knowledge of rationality and common knowledge of rationality, mean the corresponding conditions on the solution predicate.

Common knowledge of the game is formalized as common knowledge of each principle from *GAME*:

$$\textit{if } F \in GAME, \textit{ then } GAME \vdash \mathbf{C}F.$$

In particular, if *GAME* is commonly known, then

$$GAME \vdash \mathbf{E}(GAME).$$

We say that *GAME* has the **common knowledge of the game and rationality** property, *CKGR*, if *GAME* is commonly known and the solution predicate $Sol(s)$ contains common knowledge of rationality

$$GAME \vdash Sol(s) \rightarrow \mathbf{C}r(s)$$

for rationality predicates $r_1(s), r_2(s), \ldots, r_n(s)$ associated with the game and $r(s)$ being their conjunction as in (1).

Definition 2 A profile s is a **solution** of a game *GAME* if

$$GAME \vdash s.$$

For a consistent *GAME*, a solution, if it exists, is unique. The definition reflects the property that the rules of the game yield a unique strategy for each player.

For example, Prisoner's Dilemma has a solution, and the Battle of Sexes does not have a solution in pure strategies on the basis of Aumann rationality only[7].

Definition 3 A strategy profile s is a **definitive solution** of the game if

$$GAME \vdash \mathbf{E}(s).$$

The definition states that the rules of the game yield a unique strategy for each player and each player knows all these strategies. Each definitive solution is a solution. Hence, the definitive solution, if it exists, is unique.

Lemma 1 *For games with CKGR, each solution is a definitive solution.*

Proof. Indeed, since *CKGR*,

$$GAME \vdash \mathbf{E}(GAME).$$

By the rules of the (modal) logic of knowledge, from

$$GAME \vdash s,$$

one could conclude

$$\mathbf{E}(GAME) \vdash \mathbf{E}(s),$$

hence

$$GAME \vdash \mathbf{E}(s).$$

Informally, if the rules of the game yield a solution, and the rules are known to player i in full, then i knows this solution. □

[7]This obvious observation, technically speaking, follows from Theorem 3.

If rationality of players is not known, it is easy to provide an example of a solution which is not definitive. Consider Prisoner's Dilemma in which players are rational but do not know about each other's rationality. By the domination argument, each player plays '*defect*' but none knows the choice of the other player[8].

As another example on the notion of solution, consider the following **War and Peace dilemma**, W&P, introduced in [3].

> *Imagine two neighboring countries: a big powerful B, and a small S. Each player can choose to wage war or keep the peace. The best outcome for both countries is peace. However, if both countries wage war, B wins easily and S loses everything, which is the second-best outcome for B and the worst for S. In situation $(war_B, peace_S)$, B loses internationally, which is the second-best outcome for S. In $(peace_B, war_S)$, B's government loses national support, which is the worst outcome for B and the second-worst for S.*

The ordinal payoff matrix of this game is then

	war_S	$peace_S$
war_B	**2**,**0**	**1**,**2**
$peace_B$	**0**,**1**	**3**,**3** .

There is one Nash equilibrium,

$$(peace_B, peace_S). \tag{3}$$

Let us assume Aumann rationality and *CKGR*. We claim that strategy profile (3) is the definitive solution to W&P. Indeed, S has a dominant strategy $peace_S$ and as a rational player, has to commit to this strategy. This is known to B, since B knows the game and is aware of S's rationality. Therefore, as a rational player, B chooses $peace_B$. This reasoning can be carried out by any intelligent player. Hence it follows from the game description and *CKGR* that both players know solution (3) which is, therefore, the definitive solution of W&P.

Here is another game, W&P2, with the same payoff matrix in which players follow Aumann-Harsanyi rationality[9] that rules out Aumann-irrational strategies and then applies maximin[10] to make a choice. We assume that the payoff matrix is mutually known but players, though Aumann-Harsanyi-rational, are not aware of each other's rationality. In W&P2, S chooses $peace_S$ since it is S's dominant strategy. Since B considers both strategies for S, war_S and $peace_S$ possible[11], both strategies for B, war_B and $peace_B$ are Aumann-rational. Then B should follow the maximin strategy, hence choosing war_B. The resultant

[8] If player 1 knows that player 2 defects, then 1 knows that 2 is rational.

[9] which is equivalent to the knowledge-based rationality studied in [3].

[10] following Harsanyi's principle from [16] Sections 6.2 and 6.3, Postulate A1.

[11] otherwise, B would know that S is rational.

strategy profile

$$(war_B, peace_S) \tag{4}$$

is a unique solution of this game, but not a definitive solution. Indeed, B does not know S's choice $peace_S$.

5 Formalizing Nash reasoning

Theorem 1 *A definitive solution of a game with rational players is a Nash equilibrium.*

Proof. Consider a strategic game *GAME* with rationality predicates $r_i(x)$, $i = 1, 2, \ldots, n$, and solution predicate $Sol(x)$. Let $s = (s_1, s_2, \ldots, s_n)$ be a definitive solution of the game. We have to show that s is a Nash Equilibrium. Argue informally given *GAME*. We have

$$\mathbf{E}(s),$$

and hence for each i, j,

$$\mathbf{K}_i(s_j)$$

and hence for any strategy s'_j for j,

$$\mathbf{K}_i(\neg s'_j).$$

Therefore, s_j is the only strategy of j which is deemed possible by i.

Suppose s is not a Nash equilibrium, hence for some player i, the choice of strategy s_i is less preferable to some other choice x given the other players' strategies, so

$$s = (s_{-i}, s_i) <_i (s_{-i}, x)$$

for some strategy x of player i. Therefore, we found a player i and a strategy x of i such that for all possible for i profiles $(s_{-i}, s_i) <_i (s_{-i}, x)$, i.e., i is not Aumann-rational at s, i.e., $\neg R_i(s)$.

Since s is the solution of *GAME*, s should satisfy the solution condition $Sol(s)$ which yields that all players are rational at s, i.e., $r(s)$. Since $r(s) \rightarrow R(s)$, all players should be Aumann-rational at s, $R(s)$; a contradiction[12]. $\square$

The proof of Theorem 1 demonstrates not only that players cannot derive a definitive solution from the rules of a game that does not have Nash equilibria, but that the mere existence of such a solution known to all players is incompatible with the rules of the game. In particular, this yields that no refinement of rationality, as long as it respects Aumann rationality, can possibly lead to a definitive solution in such a game.

[12]Note that a default assumption that *GAME* is consistent is necessary since for inconsistent games, vacuously, each profile is a definitive solution.

Corollary 1 *No game with rational players and without Nash equilibria has a definitive solution under any notion of rationality.*

Note that this Theorem requires rationality but not any degree of *knowledge* of rationality. Roughly speaking, to spot that the solution is not a Nash equilibrium, we don't need players with knowledge of others' rationality: if the solution were not a Nash equilibrium in i's coordinate, it would directly contradict i-th player's rationality.

An analogue of Theorem 1 has been obtained by Aumann and Brandenburger in [10]: *Suppose that each player is rational, knows his own payoff function, and knows the strategy choices of the others. Then the players' choices constitute a Nash equilibrium in the game being played.* Theorem 1 differs from that in [10] by several accounts.

- The models of game and knowledge in [10] and this paper are fundamentally different. For Aumann and Brandenburger, a game is a probability distribution on the set of strategy profiles (a belief system), and knowledge of F is probability 1 of the event F. In this paper, we use a logic-based syntactic approach in which knowledge is represented symbolically by modal operators interpreted as strict, non-probabilistic knowledge. Since knowledge represented by modal operators is intrinsically linked to metareasoning (here a logical deduction from the game description, in accordance with Nash's aforementioned description of 1950), this logical model of game and knowledge allows us to make impossibility conclusions that do not appear to be within the scope of probabilistic methods.

- The notions of rationality in [10] and in this paper are of a quite different nature. In [10], rationality of i is maximization of i's expected payoff and is determined by the underlying belief system, whereas we allow as a rationality predicate any predicate which is at least as strong as Aumann rationality.

For the rest of the paper, we consider two extreme notions of rationality: the most general Aumann rationality, and a highly specialized notion of bullet rationality.

6 On stronger notions of rationality

In this section, we show that any Nash equilibrium can be a definitive solution for an appropriate notion of rationality.

Theorem 2 *Given a payoff matrix M and a Nash equilibrium e, not necessarily unique, there is a notion of rationality such that the corresponding game with CKGR has e as the definitive solution.*

Proof. Let $s = (s_1, s_2, \ldots, s_n)$ be an arbitrary strategy profile and $e = (e_1, e_2, \ldots, e_n)$ a Nash equilibrium profile. A **bullet e-rationality**[13] is, by definition, a set of predicates

[13]The name is analogous to "bullet voting," in which the voter can vote for multiple candidates but votes for only one.

$B_i^e(s)$ for $i = 1, 2, \ldots, n$ that hold at e and do not hold at any other strategy profile:

$$B_i^e(e) \in \mathit{GAME} \quad \text{and} \quad \neg B_i^e(s) \in \mathit{GAME} \text{ for any } s \neq e. \tag{5}$$

Informally, predicate $B_i^e(s)$ is used as a rationality of i predicate stating that

player i is rational only at profile e.

We define $B^e(s)$ as

$$B^e(s) = \bigwedge_i B_i^e(s).$$

Consider a strategic game GAME with

- payoff matrix M;
- bullet rationality $B_i^e(s)$ for $i = 1, 2, \ldots, n$;
- common knowledge of the game and rationality, CKGR;
- no other constraints except those explicitly mentioned above.

First, we show that GAME is consistent. For this, it suffices to find a model and a node at which all postulates of GAME hold. Consider the Aumann structure in which epistemic states are all strategy profiles,

$$\Omega = \{s \mid \; s \textit{ is a strategy profile}\}, \quad \mathbf{s}(s) = s,$$

knowledge partitions are singletons

$$\mathcal{K}_i(s) = \{s\},$$

i.e., each profile is common knowledge in itself, standard truth relation '$\Vdash$'

$$s \Vdash s_i^j \;\; \textit{iff} \;\; i\textit{-th strategy in profile } s \textit{ is } j,$$

and rationality predicates as defined in (5):

$$r_i(s) = B_i^e(s).$$

We claim that all assumptions of GAME hold at node e of the game model. All basic conditions on strategies and payoff preferences hold everywhere in the model. Rationality conditions (5) holds at e by definition and $B^e(e)$ is common knowledge at e. The solution predicate is, by assumption, $\mathbf{C}B^e(s)$ and it holds at $s = e$. Therefore, the solution condition $s \rightarrow \mathit{Sol}(s)$ also holds at $s = e$.

It is easy to see that all players are Aumann-rational at e, since for each player, the possible strategies of others are those from e, and i cannot improve her payoff by changing her own strategy because e is a Nash equilibrium. Therefore, for each profile s,

$$B^e(s) \rightarrow R(s),$$

hence $B^e(s)$ is a legitimate rationality predicate.

Now we show that

$$GAME \vdash e.$$

Indeed, since for each $s \neq e$, $GAME$ proves $\neg B^e(s)$ and $GAME$ proves $\neg \mathbf{C} B^e(s)$,

$$GAME \vdash \neg Sol(s)$$

and

$$GAME \vdash \neg s.$$

So, all profiles s different from e have been ruled out. However, $GAME$ assumes that each player has to choose a strategy: there should be at least one strategy profile chosen:

$$GAME \vdash \bigvee_s s$$

and e is the only remaining candidate,

$$GAME \vdash \bigwedge_{s \neq e} \neg s.$$

By propositional logic,

$$GAME \vdash (\bigwedge_{s \neq e} \neg s) \rightarrow e,$$

therefore

$$GAME \vdash e.$$

By Lemma 1,

$$GAME \vdash \mathbf{E}(e),$$

hence e is the definitive solution. □

Such "reverse engineered" bullet rationality is a technical notion which we do not offer as a viable practical notion of rationality. However, bullet rationality represents an epistemic condition which, when incorporated into the game description, can single out a given Nash equilibrium as a definitive solution. With a dash of good will, bullet rationality may be regarded as a theoretical prototype of Pearce's *bolder assumption about how players choose strategies* ([19]) that leads to a definitive solution in a multi-equilibrium situation.

Theorem 2 also has a nearby predecessor in [10] where it was stated that for any Nash equilibrium s, there is a belief system in which each player assigns probability 1 to s. Theorem 2 conveys basically the same message with "probability 1" replaced by the logical notion of strict knowledge which has helped to connect this result to Nash's notion of definitive solution.

7 Definitive solutions for Aumann rationality

Whereas Nash equilibria provide a general necessary condition for definitive solutions, a question of sufficient conditions, i.e., when definitive solutions actually exist, merits special analysis. Since tempering with the notion of rationality can render any Nash equilibrium a definitive solution, it makes sense to consider the definitive solution problem for a fixed notion of rationality. For the rest of the paper, we consider games with basic Aumann rationality.

We first show that under Aumann rationality, a game with two or more Nash equilibria cannot have a definitive solution.

7.1 Regular form of strategic games

A **regular strategic game** is a strategic game described by the following (finite) set of formulas *GAME*.

- **a**. Conditions on strategy propositions s_i^j stating '*player i chooses strategy j.*' These conditions express that each player i chooses one and only one strategy:
  $(s_i^1 \vee \ldots \vee s_i^{m_i})$ and $\neg(s_i^j \wedge s_i^l)$ for each $j \neq l$.
- **b**. A complete description of the preference relation for each player at each outcome.
- **c**. Knowledge of one's own strategy $s_i \rightarrow \mathbf{K}_i(s_i)$[14].
- **d**. The solution condition $s \rightarrow Sol(s)$ for each s where solution predicate $Sol(s)$ is the formula stating common knowledge of Aumann rationality at s, $\mathbf{C}R(s)$.
- **e**. Common knowledge of **a** – **d** above.

For example, in the regular form of the War and Peace dilemma W&P, we can demonstrate that $(peace_B, peace_S)$ is a definitive solution. Indeed, it suffices to logically derive $peace_B \wedge peace_S$ from *GAME* of W&P and argue that this derivation can be performed by any player, hence

$$\mathbf{K}_i[peace_B \wedge peace_S] \quad \text{for each } i \in \{B, S\}.$$

[14] This is the standard requirement of "measurability," cf. [8].

Here is a derivation of $peace_B \wedge peace_S$ from $GAME$ of W&P (an informal version of this derivation was presented in Section 4):

1. By (**b**) and (**e**), S knows that war_S is a strictly dominated strategy for S, hence S is not rational at all profiles with war_S;
2. by (**d**), none of these profiles can be a solution, hence $\neg war_S$;
3. by (**a**) and 2, $peace_S$;
4. by (**e**), B knows 1 and 2, which makes strategy war_S impossible for B;
5. from 4, B is not rational at $(war_B, peace_S)$, hence by (**d**), $\neg war_B$;
6. therefore $(peace_B \wedge peace_S)$, and, by (**e**), this conclusion is known to both players.

This example was intended to illustrate that the regular form of strategic games is sufficient for accommodating the usual epistemic reasoning in games.

7.2 Consistency Lemma

Lemma 2 *A regular strategic game GAME is consistent with the knowledge of any of its Nash equilibria: for each Nash equilibrium e,*

$$GAME + \mathbf{C}(e) \tag{6}$$

is consistent.

Proof. It suffices to present an Aumann structure $\mathcal{M}$ in which, at some node, both $GAME$ and $\mathbf{C}(e)$ hold. As in the proof of Theorem 2, we define $\mathcal{M}$ as

$$\Omega = \{s \mid \ s \textit{ is a strategy profile}\}, \ \ \mathbf{s}(s) = s,$$

knowledge partitions are singletons

$$\mathcal{K}_i(s) = \{s\},$$

the standard truth relation '$\Vdash$'

$$s \Vdash s_i^j \ \ \textit{iff} \ \ \textit{i-th strategy in profile s is j},$$

and Aumann rationality predicates $R_i(s)$.

By construction, $\mathcal{M}$ is omniscient, i.e., each fact which is true in a state is common knowledge in this state:

$$s \Vdash F \quad \textit{yields} \quad s \Vdash \mathbf{C}F.$$

Let $e = (e_1, e_2, \ldots, e_n)$ be a Nash equilibrium of the game. We claim that

$$e \Vdash GAME \wedge \mathbf{C}(e).$$

Since model $\mathcal{M}$ is omniscient, it suffices to check that

$$e \Vdash GAME \wedge e.$$

Since $e \Vdash e$ holds by definition of '$\Vdash$,' it remains to show that

$$e \Vdash GAME.$$

We will check conditions (**a** – **e**) one by one.

- (**a**), (**b**), and (**c**) hold at each node by definition of '$\Vdash$.'
- for (**d**) it suffices to check that each player is Aumann-rational at e. Player i knows all strategies of others, e_j with $j \neq i$, and deems any of i's own strategy x possible. However, by changing her strategy e_i at e, i cannot improve her payoff since e is a Nash equilibrium.
- (**e**) holds because model $\mathcal{M}$ is omniscient.

□

Alternatively, the consistency lemma (Lemma 2) can be also obtained by applying Proposition 5.4 (B) from [11] which pursues different goals.

The aforementioned result from [10] stating that for any Nash equilibrium e, there is a belief system in which each player assigns probability 1 to e, may be regarded as a natural probabilistic version of Lemma 2.

In this paper, we take one more step and draw impossibility conclusions from the consistency lemma, thus connecting it with Nash's definitive solutions programme (cf. Section 7.3).

Corollary 2 *A regular strategic game is consistent with playing any of its Nash equilibria e: set $GAME + e$ is consistent.*

Proof. Immediate from Lemma 6, since $\mathbf{C}(e) \rightarrow e$ in $GAME$. □

7.3 No definitive solutions to multi-equilibria regular games

Theorem 3 *No regular strategic game with more than one Nash equilibrium can have a definitive solution.*

Proof. Suppose otherwise, i.e., that some Nash equilibrium e is a definitive solution of $GAME$

$$GAME \vdash e$$

for some regular game that has another Nash equilibrium e'. By (**a**), two different profiles are incompatible, hence

$$GAME \vdash \neg e',$$

which yields that

$$GAME + e'$$

is inconsistent. This contradicts Corollary 2. □

Note that regular strategic games with some additional epistemic constraints can single out one of multiple Nash equilibria as a definitive solution. For example, take a regular game presented by *GAME* and let e be one of its multiple Nash equilibria. Consider a new game $GAME'$ consisting of *GAME* with an additional condition that e is common knowledge:

$$GAME' = GAME + \mathbf{C}(e).$$

By Lemma 2, $GAME'$ is consistent: it is easy to see that e is its definitive solution.

7.4 Definitive solutions of regular games via IDSDS

In this Section, we observe that definitive solutions of regular strategic games with ordinal payoffs are completely described by the procedure of the Iterated Deletion of Strictly Dominated Strategies (*IDSDS*) rather then a unique Nash equilibrium. *IDSDS* iteratively deletes strategies which are strictly dominated by other pure strategies. Let S^∞ denote the set of strategy profiles which survive *IDSDS*. By construction, $S^\infty \neq \varnothing$.

The role of IDSDS has been well studied (cf. [1, 11, 22]) and Theorem 4 mainly connects these studies to the definitive solution framework.

Theorem 4 *A strategic regular game with Aumann rationality has a definitive solution s if and only if s is the only strategy profile that survives IDSDS.*

Proof. (Sketch). We use the terminology of [11]. By Lemma 1, it suffices to prove the analogue of this theorem which speaks about "solution s" rather than "definitive solution s." Consider two cases.

Case 1. S^∞ contains states with different profiles, say s_1 and s_2. Then such a game does not have a definitive solution. Indeed, by Proposition 5.4 (B) from [11], there are states ω_1 and ω_2 corresponding to profiles s^1 and s^2 such that

$$\omega_i \Vdash GAME + s^i \quad \text{for } i = 1, 2.$$

Therefore, $GAME + s^i$ are consistent for $i = 1, 2$. Since s^1 and s^2 are incompatible,

$$GAME \nvdash s^i \quad \text{for } i = 1, 2.$$

Case 2. S^∞ is a singleton, e.g., $S^\infty = \{s\}$. We show that s is then the definitive solution. By the completeness theorem for the background modal logic of knowledge (normally, a multi-agent version of $\mathsf{S5}$), it is sufficient to establish that

$$GAME \rightarrow s \tag{7}$$

holds at each node of each model (e.g., each Aumann structure). By Proposition 5.4 (A) from [11], for each state ω, if $\mathbf{s}(\omega) \notin S^\infty$, then the statement of common knowledge of rationality fails in ω, hence

$$\omega \not\Vdash \mathbf{s}(\omega) \rightarrow Sol(\mathbf{s}(\omega))$$

and

$$\omega \not\Vdash GAME.$$

Consider an arbitrary node ω. If $\mathbf{s}(\omega) \in S^\infty$, then $\mathbf{s}(\omega) = s$, hence $\omega \Vdash s$. If $\mathbf{s}(\omega) \notin S^\infty$, then, as above, $\omega \not\Vdash GAME$. In either case, (7) holds at ω.

Since (7) holds in each model,

$$GAME \vdash s,$$

and, by Lemma 1,

$$GAME \vdash \mathbf{E}(s),$$

hence s is the definitive solution of $GAME$.

Case 2 can also be derived from [1] which shows that players will choose only strategies that survive the iterated delition of strictly dominated strategies. □

8 Unique Nash equilibrium does not yield a definitive solution

Consider the following game

$$\begin{bmatrix} \mathbf{1,2} & \mathbf{1,0} & \mathbf{0,1} \\ \\ \mathbf{0,0} & \mathbf{0,2} & \mathbf{1,1} \end{bmatrix}$$

It has a unique Nash equilibrium $(\mathbf{1,2})$, but no definitive solution within the scope of Aumann rationality, even if the game and rationality are commonly known. Indeed, each strategy in this game is Aumann-rational and hence cannot be deleted by *IDSDS*.

9 Discussion

We have seen that Nash's definitive solution paradigm is not at all universal: in many cases definitive solutions do not exist. For example, a direct count shows that though 75% of generic regular 2×2 games have definitive solutions, the proportion of solvable games quickly goes to 0 when the size of the game grows (the number of players or the number of strategies for each player). Even when definitive solutions exist, the notion of a Nash equilibrium does not provide sufficient criteria for them. In a way, the results of this paper support Aumann's views [9]:

- Equilibrium is not the way to look at games. The most basic concept should be: *to maximize your utility given your information*.
- The starting point for realization of this concept should be *Syntactic epistemic logic*.

In a game, one could expect epistemic and rationality conditions to be given, hence a methodologically correct way would be to consider whether a game has a definitive solution under given epistemic/rationality conditions.

For future work, one could apply similar methods for analyzing mixed strategies and settings with belief rather than knowledge conditions.

It makes sense to further explore the role of proof-theoretical methods in epistemic game theory. One possible avenue, along the lines of Johan van Benthem's 'rational dynamics' programme ([22, 23]), could be to add justifications – in particular, proofs as objects – to the logical analysis of games. The focus of such research could be to create a unified theory of reasoning and epistemic actions in the context of games. There is no action without reasoning for rational agents; reasoning is itself a kind of epistemic action, and takes other actions as inputs. A meaningful step in this direction was made by Renne in [20] in which he suggests interpreting proof terms t in the Logic of Proofs (cf. [2]) as strategies, so that $t{:}A$ may be read as

$$t \textit{ is a winning strategy on } A.$$

In this light, the Logic of Proofs may thus be seen as a logic containing in-language descriptions of winning strategies on its own formulas.

Other major issues in epistemic game theory that the Logic of Proofs could help to address are awareness and the logical omniscience problem. The standard semantics for the logics of proofs and justifications, Fitting models [15], is a more expressive dynamic extension of Fagin-Halpern awareness models [13]; awareness models are Fitting models corresponding to one fixed proof term [21]. A coherent general treatment of the logical omniscience problem on the basis of proof complexity has been offered in [5, 6].

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