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Mixing Angles, B-Meson Life-Time in the Six-Quark Model Based on SO(10) X U(1)PQ

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MIXING ANGLES, B-MESON LIFE-TIME IN THE SIX-QUARK MODEL BASED ON SO(10) × $U(1)_{PQ}$

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We show that a recently proposed multigenerational, grand unified model with three parameters (top-quark mass and two phases of Higgs vacuum expectation values) yields results in good agreement with the most recent phenomenological limits placed on the weak mixing angles of the quarks. These limits take into account the experimental value for B-meson life-time.

The recent determination [1] of B-meson life-time has provided an important piece of experimental information which enables one to put stringent restrictions on the Kobayashi–Maskawa mixing angles in the six-quark scheme. Revised phenomenological fits [2,3] [‡1] show that the bottom–charm (b–c) quark transition matrix element compared with up–strange (u–s) transition is very small, implying that the weak mixing angles θ_2 and θ_3 are quite small. This poses a serious challenge to theoretical models, because the implied hierarchy seems to be in an apparent contradiction with the expectations for the values of the mixing angles in terms of quark mass ratios.

The purpose of this note is to discuss in some detail the consequences of a recently proposed model [5] on this question. Three generations of quarks and leptons are considered within the framework of a grand unified theory based on the groun SO(10) combined with the global, axial, U(1) Peccei–Quinn [6] symmetry [$U(1)_{PQ}$]. The Peccei–Quinn symmetry plays a dual role in the model; (i) it eliminates the strong *CP*-violation problem: (ii) it acts as horizontal flavor symmetry, and distinguishes the different generations. We shall confine ourselves only to a few relevant features of the model here. The interested reader should consult ref. [5] for further details.

With each generation of fermions belonging to a **16**-dimensional spinorial representation of SO(10), and with the choice of **10** (complexified) and **126** representations for the Higgs scalars that couple to the fermions, the quark mass matrices are complex, symmetry matrices with the generic form M,

$$M = \begin{pmatrix} 0 & Ae^{i\alpha} & 0 \\ Ae^{i\alpha} & 0 & Be^{i\beta} \\ 0 & Be^{i\beta} & Ce^{i\gamma} \end{pmatrix}, \qquad (1)$$

which can be written as

$$M = PXP\,, \qquad (2)$$

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[‡1] For a comprehensive review of both the theoretical and phenomenological background and references, see ref. [4]. Earlier limits on the mixing angles are given in this report.

where

$$P=\mathrm{diag}(e^{i(\alpha-\beta+\gamma/2)}, e^{i(\beta-\gamma/2)}, e^{i\gamma/2}) , \qquad X=\begin{pmatrix} 0 & A & 0 \\ A & 0 & B \\ 0 & B & C \end{pmatrix} . \tag{3}$$

Thus X is a real symmetric matrix with every element being positive. A real orthogonal matrix O diagonalizes X,

$$OXO^{\mathrm{T}} = \mathrm{diag}(m_1, -m_2, m_3) , \tag{4}$$

where $0 < m_1 < m_2 < m_3$. They are the values of the current quark masses. *The most important feature* [7][‡2] *of the model is that both X and O can be expressed in terms of* m_1, m_2, m_3,

$$A=[m_1m_2m_3/(m_1-m_2+m_3)]^{1/2}, \qquad B=[(m_3+m_1)(m_3-m_2)(m_2-m_1)/(m_1-m_2+m_3)]^{1/2},$$

$$C=m_3-m_2+m_1 , \tag{5}$$

$$O=\begin{bmatrix} \left[\dfrac{m_2m_3(m_3-m_2)}{(m_3-m_1)(m_2+m_1)(m_3-m_2+m_1)}\right]^{1/2} & \left[\dfrac{m_1(m_3-m_2)}{(m_3-m_1)(m_2+m_1)}\right]^{1/2} & -\left[\dfrac{m_1(m_2-m_1)(m_3+m_1)}{(m_3-m_1)(m_2+m_1)(m_3-m_2+m_1)}\right]^{1/2} \\ \left[\dfrac{m_1m_3(m_3+m_1)}{(m_3+m_2)(m_2+m_1)(m_3-m_2+m_1)}\right]^{1/2} & -\left[\dfrac{m_2(m_3+m_1)}{(m_3+m_2)(m_2+m_1)}\right]^{1/2} & \left[\dfrac{m_2(m_3-m_2)(m_2-m_1)}{(m_3+m_2)(m_2+m_1)(m_3-m_2+m_1)}\right]^{1/2} \\ \left[\dfrac{m_1m_2(m_2-m_1)}{(m_3+m_2)(m_3-m_1)(m_3-m_2+m_1)}\right]^{1/2} & \left[\dfrac{m_3(m_2-m_1)}{(m_3+m_2)(m_3-m_1)}\right]^{1/2} & \left[\dfrac{m_3(m_3-m_2)(m_3+m_1)}{(m_3-m_2+m_1)(m_3+m_2)(m_3-m_1)}\right]^{1/2} \end{bmatrix} \tag{6}$$

Then, as in ref. [5], let $X^{(\mathrm{d})}$ and $X^{(\mathrm{u})}$ correspond to the mass matrices $M^{(\mathrm{d})}$ and $M^{(\mathrm{u})}$ in the down- and up-charge sectors,

$$M^{(\mathrm{d})}=P^{(\mathrm{d})}X^{(\mathrm{d})}P^{(\mathrm{d})}, \qquad M^{(\mathrm{u})}=P^{(\mathrm{u})}X^{(\mathrm{u})}P^{(\mathrm{u})}, \tag{7}$$

where $P^{(\mathrm{d})}$ and $P^{(\mathrm{u})}$ are the corresponding diagonal, pure phase matrices. If $O^{(\mathrm{d})}$ and $O^{(\mathrm{u})}$ are the desired orthogonal matrices that diagonalise $X^{(\mathrm{d})}$ and $X^{(\mathrm{u})}$,

$$O^{(\mathrm{d})}X^{(\mathrm{d})}O^{(\mathrm{d})\mathrm{T}}=\mathrm{diag}(m_{\mathrm{d}}, -m_{\mathrm{s}}, m_{\mathrm{b}}), \qquad O^{(\mathrm{u})}X^{(\mathrm{u})}O^{(\mathrm{u})\mathrm{T}}=\mathrm{diag}(m_{\mathrm{u}}, -m_{\mathrm{c}}, m_{\mathrm{t}}) , \tag{8,9}$$

where $m_{\mathrm{d}}, ..., m_{\mathrm{t}}$ denote the masses of the indicated quarks, then the Cabibbo–Kobayashi–Maskawa matrix in the charged current interaction, namely,

$$U_{\mathrm{c}}=U_{\mathrm{L}}^{(\mathrm{u})}U_{\mathrm{L}}^{(\mathrm{d})\dagger} , \tag{10}$$

is given by

$$U_{\mathrm{c}}=Q[O^{(\mathrm{u})}P^{(\mathrm{u})*}P^{(\mathrm{d})}O^{(\mathrm{d})\mathrm{T}}]R , \tag{11}$$

with the matrix in the rectangular parantheses determined completely by the parameters that enter in the mass matrix (vacuum expectation values, coupling constants) and Q and R being two arbitrary diagonal pure phase matrices. They reflect the arbitrary phases of the quark fields. We define

$$P=P^{(\mathrm{u})*}P^{(\mathrm{d})}=\mathrm{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}) , \qquad c_{ij}=\sum_k O_{ik}^{(\mathrm{u})}O_{jk}^{(\mathrm{d})}\cos\phi_k , \qquad s_{ij}=\sum_k O_{ik}^{(\mathrm{u})}O_{jk}^{(\mathrm{d})}\sin\phi_k , \tag{12}$$

[‡2] The authors of ref. [7] discuss a model based on O(10) and arrive at the generic forms in (1) for the quark mass matrices. They carry out the diagonalization in the appendix of the paper.

and choose Q and R so that U_c reduces to the standard Kobayashi–Maskawa form. Then the required mixing angles $\theta_1, \theta_2, \theta_3$ and $\mathrm{Im}(U_c)_{12}$ that is related to the weak *CP*-violation parameter are given by

$$\cos\theta_1 = (U_c)_{11} = (c_{11}^2 + s_{11}^2)^{1/2} , \tag{13}$$

$$-\sin\theta_1 \cos\theta_2 = (U_c)_{21} = -(c_{21}^2 + s_{21}^2)^{1/2} , \quad \sin\theta_1 \cos\theta_3 = (U_c)_{12} = (c_{12}^2 + s_{12}^2)^{1/2} , \tag{14, 15}$$

$$-\sin\theta_2 \sin\theta_3 \sin\delta = \mathrm{Im}(U_c)_{22} = \frac{(s_{11}c_{22} + c_{11}s_{22})(c_{12}c_{21} - s_{12}s_{21}) - (c_{11}c_{22} - s_{11}s_{22})(s_{12}c_{21} + c_{12}s_{21})}{\cos\theta_1 \cos\theta_2 \cos\theta_3 \sin^2\theta_1} . \tag{16}$$

From (12)–(16), it follows that the mixing angles and *CP*-violation depend on the six quark masses and two phase angle differences. As only the topquark mass is unknown, the model contains only three unknown parameters – the top quark mass m_t and two phase differences, say

$$\alpha = (\phi_1 - \phi_2), \quad \beta = (\phi_2 - \phi_3) \quad (\text{then } \phi_1 - \phi_3 = \alpha + \beta) . \tag{17}$$

The mixing angles and $\mathrm{Im}(U_c)_{22}$ are given by

$$\cos^2\theta_1 = K_0 + K_1 \cos\alpha + K_2 \cos(\alpha + \beta) + K_3 \cos\beta , \tag{18}$$

$$\sin^2\theta_1 \cos^2\theta_2 = L_0 + L_1 \cos\alpha + L_2 \cos(\alpha + \beta) + L_3 \cos\beta , \tag{19}$$

$$\sin^2\theta_1 \cos^2\theta_3 = N_0 + N_1 \cos\alpha + N_2 \cos(\alpha + \beta) + N_3 \cos\beta , \tag{20}$$

$$\cos\theta_1 \cos\theta_2 \cos\theta_3 \sin^2\theta_1 \,\mathrm{Im}(U_c)_{22} = A_1 \sin\alpha + A_2 \sin(\alpha + \beta) + A_3 \sin\beta + \sin\alpha[A_4 \cos(\alpha + \beta) + A_5 \cos\beta]$$
$$+ \sin(\alpha + \beta)(A_6 \cos\alpha + A_7 \cos\beta) + \sin\beta[A_8 \cos\alpha + A_9 \cos(\alpha + \beta)] , \tag{21}$$

where K_i, L_i, N_i, and A_i are all given functions of masses. We take the standard typical values for the quark masses, namely,

$$m_d = 7.5\ \mathrm{MeV}, \quad m_s = 150\ \mathrm{MeV}, \quad m_b = 5000\ \mathrm{MeV}, \quad m_u = 5\ \mathrm{MeV}, \quad m_c = 1250\ \mathrm{MeV}, \quad m_t > 30\ \mathrm{GeV} ,$$

and compute these functions for varying values of m_t. The results for $K_0 .. N_3$ are given in table 1. The results for A's are similar. Notice that they are very slowly varying smooth functions of m_t. Having these numbers at our disposal, we study the choice of other parameters.

The most stringent requirement arises from the very well determined Cabibbo angle [2],

$$\sin\theta_1 = 0.231 \pm 0.003 .$$

First of all, *it rules out the choice* $\alpha = \beta = 0$, *which would have implied that* $\mathrm{Im}(U_c)_{22} = 0$. *In other words, there would have been no weak CP-violation in the conventional way due to gauge bosons in the charged interactions.* One then had to appeal to the Higgs sector for *CP*-violation. Secondly, the study of the numbers in table 1 shows that K_1 contributes most dominantly, K_2 and K_3 being relatively of no significance to the value of θ_1. The value $\alpha = 90^\circ$ leads to

$$\sin\theta_1 = 0.2264 ,$$

for $m_t = 30$–100 GeV. This is remarkably close to the lower limit $\sin\theta_1 = 0.228$ set by experiments. The variation of $\sin\theta_1$ from 0.228–0.234 allows the variation of α from 91.5° to 97°. We take $\alpha = 94^\circ$ to yield

$$\sin\theta_1 = 0.231 , \tag{22}$$

independent of m_t when it is varied from 30 GeV to 100 GeV. Having fixed α this way, we vary β to set limits on its variation for various values of m_t. For this purpose, we assume the limits provided in ref. [2],

Table 1

Dependence of the coefficients $K_0, ..., K_3, L_0, ..., L_3, N_0, ..., N_3$ on the top-quark mass, See eqs. (18)–(20) for the definitions.

m_t (GeV)	K_0	K_1	K_2	K_3	L_0	L_1	L_2	L_3	N_0	N_1	N_2	N_3
30	0.9487	0.0259	9.578×10^{-4}	1.307×10^{-5}	0.0481	−0.0259	-9.193×10^{-4}	3.136×10^{-3}	0.0510	−0.0259	-9.285×10^{-4}	2.534×10^{-4}
40	0.9487	0.0260	8.250×10^{-4}	1.132×10^{-5}	0.0485	−0.0261	-7.999×10^{-4}	2.744×10^{-3}	0.0510	−0.0261	-7.997×10^{-4}	2.195×10^{-4}
50	0.9487	0.0261	7.355×10^{-4}	1.012×10^{-5}	0.0488	−0.0261	-7.175×10^{-4}	2.469×10^{-3}	0.0511	−0.0261	-7.130×10^{-4}	1.963×10^{-4}
60	0.9487	0.0262	6.700×10^{-4}	9.242×10^{-6}	0.0490	−0.0262	-6.563×10^{-4}	2.263×10^{-3}	0.0511	−0.0262	-6.495×10^{-4}	1.792×10^{-4}
70	0.9487	0.0262	6.193×10^{-4}	8.556×10^{-6}	0.0491	−0.0262	-6.084×10^{-4}	2.101×10^{-3}	0.0511	−0.0262	-6.004×10^{-4}	1.659×10^{-4}
80	0.9487	0.0262	5.787×10^{-4}	8.003×10^{-6}	0.0492	−0.0262	-5.697×10^{-4}	1.970×10^{-3}	0.0511	−0.0263	-5.610×10^{-4}	1.552×10^{-4}
90	0.9487	0.0263	5.451×10^{-4}	7.546×10^{-6}	0.0493	−0.0263	-5.376×10^{-4}	1.860×10^{-3}	0.0511	−0.0263	-5.284×10^{-4}	1.463×10^{-4}
100	0.9487	0.0263	5.168×10^{-4}	7.158×10^{-6}	0.0493	−0.0263	-5.104×10^{-4}	1.767×10^{-3}	0.0511	−0.0263	-5.010×10^{-4}	1.388×10^{-4}

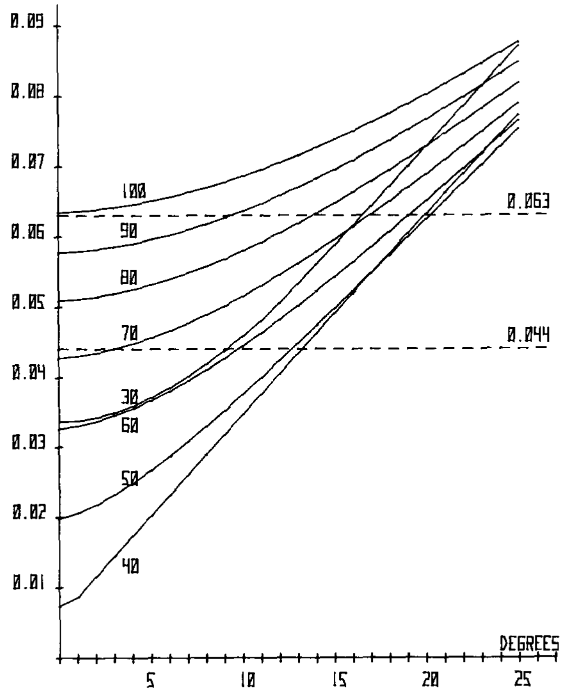

Fig. 1. $|(U_c)_{bc}|$ as a function of β in degrees for various values of m_t in GeV. The dotted horizontal lines represent the phenomenological constraints, eq. (24).

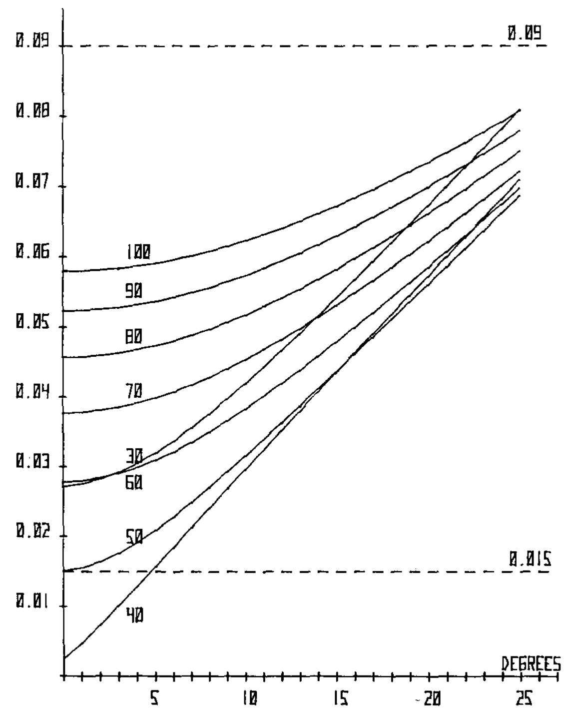

Fig. 2. $\sin\theta_2$ as a function of β in degrees for various values of m_t in GeV. The dotted horizontal lines represent the phenomenological constraints. $\sin\theta_3$ is always less than 0.04. We do not plot it, but give its values in table 2.

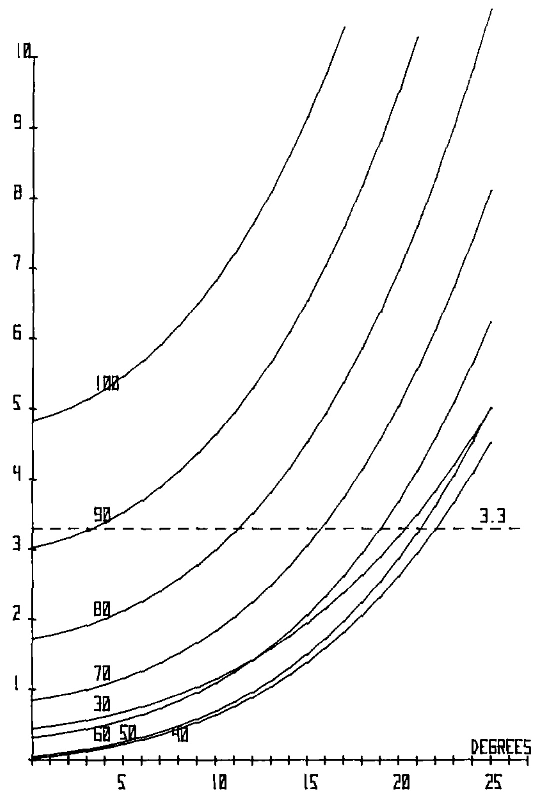

Fig. 3. c Im $\mathcal{M}$ as a function of β in degrees for various values of m_t in GeV. The dotted horizontal line represents the constraint cB Im $\mathcal{M} \leqslant 1$, where $B = 1/3$.

$$0.015 < \sin\theta_2 < 0.09, \quad \sin\theta_3 < 0.04\,, \tag{23}$$

which include the recent experiments [1],

$$|(U_c)_{bc}| = 0.053^{+0.010}_{-0.009}\,, \tag{24}$$

on B-meson life-time in their analysis. The results are plotted in fig. 1 and fig. 2. They show that for m_t = 30–100 GeV, we can find β such that all the experimental constraints are satisfied quite well.

Finally, we come to the *CP*-violation effect predicted by the model. The K^0–$\bar{K}^0$ transition matrix M_{12} from the standard relevant box graph [4] is given by

$$M_{12} = -\frac{G_F^2 M_W^2}{16\pi^2}\left(\sum_{i,j=u,c,t} \lambda_i \lambda_j A_{ij}\right) \mathcal{M}_{12,\mathrm{vac}} B\,, \tag{25}$$

where

$$\mathcal{M}_{12,\mathrm{vac}} = \tfrac{4}{3} f_K^2 m_K\,, \tag{26}$$

with $f_K \simeq 1.23 m_\pi$, is the vacuum insertion contribution and B is a constant characterizing the deviation of the vacuum-insertion calculation from unity [8] [‡3]. The other quantities appearing in (25) are defined in ref. [4]. We calculate the quantity

[‡3] $B = 0.33$ in the evaluation of ref. [8]. This is regarded as the most model independent evaluation of the constant. For earlier evaluations of B based on bag model, see ref. [4].

Table 2
By restricting $|(U_c)_{bc}|$, we find allowed ranges for β, $\sin\theta_2$, $\sin\theta_3$, $\sin\delta$, and $c\,\mathrm{Im}\mathcal{M}$.

m_t (GeV)	β	$\sin\theta_2$	$\sin\theta_3$	$c\,\mathrm{Im}\mathcal{M}$	$\sin\delta$
30	9°–16°	0.0397–0.0567	0.0177–0.0225	1.013 –2.101	0.9994–0.9994
40	13°–20°	0.0379–0.0572	0.0160–0.0214	1.019 –2.552	0.9862–0.9927
50	13°–20°	0.0387–0.0560	0.0148–0.0200	1.055 –2.797	0.9123–0.9852
60	10°–19°	0.0382–0.0563	0.0132–0.0192	1.064 –3.233	0.9360–0.9790
70	3°–17°	0.0385–0.0566	0.0109–0.0185	0.9731–3.666	0.9092–0.9740
80	0°–14°	0.0457–0.0568	0.0120–0.0177	1.686 –4.055	0.9383–0.9698
90	0°–9°	0.0523–0.0564	0.0138–0.166	2.961 –4.259	0.9563–0.9653
100	–	–	–	–	–

$$\mathcal{M} = \sum \lambda_i \lambda_j A_{ij} , \quad (27)$$

for m_t = 30–100 GeV. The results for $\mathrm{Im}M_{12}$ [‡4] are plotted in fig. 3. Inserting the value of $\mathcal{M}_{12,\mathrm{vac}}$, we find

$$\mathrm{Im}M_{12} = -(0.114 \times 10^{-13}\ \mathrm{MeV}) \times cB\,\mathrm{Im}\,\mathcal{M} , \quad (28)$$

where $c = 1.0223 \times 10^7$. In order that we do not conflict with experiments, $cB\,\mathrm{Im}\,\mathcal{M} \leqslant 1$.

The main points of our results can be read from the figures. For convenience, we summarize them in table 2. The most stringent limits on β are provided by eq. (24). We note that the values $m_t > 90$ GeV are excluded, both from the constraint of eq. (24) and $cB\,\mathrm{Im}\,\mathcal{M} \leqslant 1$, if we take $B \simeq \frac{1}{3}$. For each value of m_t, an allowed range of β emerges from fig. 1. In this range, $\sin\theta_2$, $\sin\theta_3$, $\sin\delta$, and $c\,\mathrm{Im}\,\mathcal{M}$ are slowly varying, increasing functions of β. We have given in table 2 the values of the above mentioned quantities for the end points of the allowed range of β. It is worth noting that $(U_c)_{bc}$ involves a specific combination of sines and cosines of all the mixing angles and the Kobayashi–Maskawa phase δ. By restricting the absolute value of this matrix element, the model predicts, for all the investigated values of m_t, limits for $\sin\theta_2$ and $\sin\theta_3$,

$$0.038 \leqslant \sin\theta_2 \leqslant 0.057 , \qquad 0.011 \leqslant \sin\theta_3 \leqslant 0.022 .$$

These limits are more stringent than the current phenomenological constraints. The model also predicts $\sin\delta$ to lie between $0.909 \leqslant \sin\delta \leqslant 0.999$. Thus, for each value of the top-quark mass, the complete Kobayashi–Maskawa matrix is known within certain limits. Consequently, the model provides a rich body of results that can be compared with experiments, once the value of m_t is known.

In conclusion, the generic form for the mass matrix in (1) seems to provide a good and satisfactory description of low energy parameters including the new piece of information concerning B-meson lifetime. Such a form for the mass matrix was suggested a long time ago by Fritzsch [10] from heuristic considerations. Here it is derived within the framework of a grand unified theory combined with Peccei–Quinn symmetry which eliminates the strong *CP*-violation problem. The axion can be made to be phantom axion [11]. Thus the model is a realistic one and the results obtained show that it merits a serious study. We have used a minimal model consisting of only **10** and **126** Higgs representations that couple to the fermions. Addition of **120** representation introduces an antisymmetric component in the mass matrix; Stech [12] has recently analysed such a general situation. The results in the two cases appear, at least qualitatively, the same. The presently established experimental results do not warrant the addition of an antisymmetric component.

We note that the results are very sensitive to numerical approximations. The assumed quark masses and hence the hierarchies in the quark mass ratios suggest linear expansions in terms of the mass ratios which have often been

‡4 We only consider $\mathrm{Im}M_{12}$, because the real part is quite sensitive to low energy contributions, see ref. [9]. As stated later in our paper, our model does not warrant a detailed calculation of the *CP*-violation effect, until we analyze the Higgs sector.

used in the literature evaluating the Kobayashi–Maskawa matrix elements. However, comparison of such expansions with exact numerical evaluations used in this paper shows that there are serious discrepancies. There are delicate cancellations in the first order expansions leading to significant contributions from higher orders in mass ratios. In such models, therefore it seems advisable to do careful numerical work without resorting to approximation. We also note that we have deliberately avoided a very careful study of the weak *CP*-violation including both the real and imaginary parts of M_{21}. Our reason for this is that the model has a rich Higgs structure, and there is bound to be *CP*-violation due to Higgs exchanges. A full understanding of the latter mechanism requires a detailed study of the Higgs potential of the model, which includes in addition to the representations **10** and **126**, those that are necessary to break the SO(10) symmetry down to that of the Lie algebra SU(3) × SU(2) × U(1). This and other implications of the model (rare decay modes, charm and B-meson physics) are currently under study.

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References

[1] E. Fernandez et al., Phys. Rev. Lett. 51 (1983) 1022;
N.S. Lockyer et al., Phys. Rev. Lett. 51 (1983) 1316.
[2] K. Kleinknecht and B. Renk, Phys. Lett. 130B (1984) 459, and references contained therein.
[3] P.H. Ginspor, S.L. Glashow and M.B. Wise, Phys. Rev. Lett. 50 (1983) 1415.
[4] L.-L. Chau, Phys. Rep. 95 (1983) 1.
[5] A. Davidson, V.P. Nair and K.C. Wali, Syracuse University preprints SU-4217-257, SU-4222-265 (1983), submitted for publication in Phys. Rev.
[6] R. Peccei and H. Quinn, Phys. Rev. Lett. 38 (1979) 1440.
[7] H. Georgi and D.V. Nanopolous, Nucl. Phys. B109 (1979) 51.
[8] J.F. Donohue, E. Golowich and B.R. Holstein, Phys. Lett. 119B (1982) 412.
[9] D. Hochberg and R.G. Sachs, Phys. Rev. D27 (1983) 606;
see also L. Bergström et al., CERN preprint TH.3659 (1983).
[10] H. Fritzsch, Phys. Lett. 73B (1978) 317; Nucl. Phys. B155 (1979) 189.
[11] M. Dine, W. Fishler and M. Srednicki, Phys. Lett. 104B (1981) 199;
M.B. Wise, H. Georgi and S.L. Glashow, Phys. Rev. Lett. 47 (1981) 402.
[12] B. Stech, Phys. Lett. 130B (1983) 189.