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I. Introduction

Compactification of the Twisted Heterotic String

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ABSTRACT

We consider compactifications of the recently discovered tachyon free non-supersymmetric string on tori of Lorentzian self dual lattices. The one loop cosmological constant is computed as a function of the parameters characterizing the lattices. When the lattice parameters are treated as dynamical variables, their equations of motion require extremization of Λ . This extremization determines the allowed gauge groups and mass spectra. We explicitly find some extrema of Λ , valid to all orders in perturbation theory. A very general duality relation between compactifications is demonstrated, which leads to an equivalence of compactifications on tori of radii R and $1/R$.

Recent progress in understanding string theories has brought to light a number of new constructions for producing both new string theories [1-3] and new compactifications [4-6]. Whether these constructions truly differ in a fundamental way, or are merely expansions of a single (or a few) string field theories around different vacua is an important issue, for hope of deriving observable realities from considerations of consistency at the Planck mass must largely rely on there being a severely limited choice of options. To settle this issue it will ultimately be necessary to go well beyond perturbative expansions, and to determine if constructions which look quite different in perturbation theory are somehow connected. Even if they do turn out to be connected to one another an important practical question will remain: which vacuum, if any, is the best starting point for a perturbative expansion? This is at least partially a question of stability; the false vacua will decay to the correct one.

In this note we combine two of the new constructions, to define still more consistent compactifications. The nature of these compactifications permits us to make a modest start on the questions of connectedness and stability. Specifically, we combine Narain's idea of compactification on Lorentzian lattices [4] with the recent ideas of Dixon and Harvey [1], and of Alvarez-Gaumé, Ginsparg, Moore, and Vafa [2] for constructing non-supersymmetric string theories. Narain's construction leads to classes of models which are continuously connected to one another. Indeed, this construction has been reinterpreted [7] as the presence of certain constant background fields on the 16-dimensional torus of the usual heterotic string [8]. It is very reasonable, then, to think that these models all sit within the same class — namely they are all different compactifications of the heterotic string. At the same time, the twisting construction has the important new feature of producing tachyon free models *without* supersymmetry. In particular, a finite non-zero cosmological term appears in these models due to imperfect cancellation between boson and fermion zero point energies.

When we combine the two constructions, we find that the magnitude of the cos-

mological term depends on the form of the Lorentzian lattice (or, equivalently, on the magnitudes of the background fields). Stability requires that this term be extremized. This occurs only for a discrete set of compactifications.

A crucial feature of our analysis is a remarkable duality between tori with radii R and $\frac{1}{R}$ (in units of string tension). We show that this duality implies a complete isomorphism between S -matrix elements on the different tori and has a generalization to arbitrary Lorentzian even self dual lattices. This duality is intimately related to several other dualities: the classic $s \leftrightarrow t$ channel duality and modular invariance, Kramers-Wannier duality of two-dimensional theories [9], and electric-magnetic or charge-monopole duality [10]. In the present context, this duality is especially important because it allows us to identify stationary lattices, i.e. lattices for which the cosmological term has zero derivative to all orders in string perturbation theory.

II. The twisted heterotic string on Lorentzian lattices

We begin with a brief review of Narain's heterotic string theories and the twisted ten dimensional heterotic string. Then, after defining a twist projection and enumerating the sectors in twisted lower-dimensional theories, we compute the partition function and the cosmological constant Λ . Asymptotic properties of Λ are discussed, and local extrema are found. Modular invariance is proved in the appendix.

In the ten dimensional heterotic string [8], a 26-dimensional left moving closed bosonic string is fused to a right moving 10-dimensional closed superstring. The sixteen "internal" left moving dimensions are compactified on a 16-torus \mathbb{R}^{16}/Γ , where the lattice Γ is required to be integral, even and self dual by momentum quantization, σ -translation invariance, and modular invariance, respectively. Such lattices exist only in 8n dimensions, and there are only two in 16 dimensions. These are the root lattices of $E_8 \times E_8$ and $Spin(32)/Z_2$. Actions of these groups on the string Hilbert space are realized when the internal dimensions are compactified on the corresponding tori. Remarkably, these same groups are singled out by the requirement that the resulting low energy field theory be anomaly free [11].

Narain realized that the heterotic construction may be extended to dimensions $D < 10$ by compactifying $(26-D)$ left moving and $(10-D)$ right moving string degrees of freedom on a Lorentzian even self dual lattice Γ in $\mathbb{R}^{26-D, 10-D}$ [4]. Classically, the compactified left and right movers then have the mode expansions

$$\begin{aligned} \tilde{x}^I(\sigma+t) &= \tilde{x}^I + \tilde{L}^I(\sigma+t) + \sum_{n \neq 0} \frac{i}{2n} \tilde{\alpha}_n^I e^{in(\sigma+t)} \\ \tilde{x}^A(\sigma-t) &= \tilde{x}^A + \tilde{L}^A(\sigma-t) + \sum_{n \neq 0} \frac{i}{2n} \tilde{\alpha}_n^A e^{in(\sigma-t)} \end{aligned} \quad (1)$$

where $(L^I, \tilde{L}^A) \in \Gamma$ is the winding vector, and σ ranges from 0 to 2π . In the quantum theory, the left and right moving mass operators are

$$\begin{aligned} \frac{1}{4} m_L^2 &= \frac{1}{2} L^2 + N - 1 \\ \frac{1}{4} m_R^2 &= \frac{1}{2} \tilde{L}^2 + \tilde{N} - c \end{aligned}$$

c is a normal ordering constant which equals 0 for the Ramond sector and $\frac{1}{2}$ for the Neveu-Schwarz sector. The physical states obey the constraint

$$m_L^2 = m_R^2 \quad (2)$$

Modular invariance requires that the tachyon at level $-\frac{1}{2}$ be projected out of the physical spectrum. Thus m_R^2 is never negative and since $m^2 = m_L^2 + m_R^2 = 2m_R^2$, these theories are tachyon free.

The massless gauge boson states correspond to winding vectors $(L, 0) \in \Gamma$ with $L^2 = 2$ in addition to the standard Kaluza-Klein $U(1)$ gauge bosons. If the set of such winding vectors contains the root lattice of a Lie group G , then G is a gauge group of the theory. The Kaluza-Klein $U(1)$ bosons correspond to the Cartan subalgebra of the gauge group. Narain has found lattices leading to several gauge groups, including $SO(52-2D)$ and $E_8 \times E_8 \times G_{10-D}$ where G_{10-D} is any simply laced Lie group of rank $(10-D)$.

The above construction gives a $(26-D)(10-D)$ parameter family of compactifications of the heterotic string to D dimensions. This is because any two Lorentzian

even self dual lattices in $\mathbb{R}^{26-D, 10-D}$ are related by an $SO(26-D, 10-D)$ rotation. In general, such rotations will change the mass spectrum, unless both m_L^2 and m_R^2 are left unchanged. Physically distinct compactifications therefore correspond to "Lorentz boosts," i.e., points in the manifold

$$M = \frac{SO(26-D, 10-D)}{SO(26-D) \times SO(10-D)}$$

The other modification of the heterotic string mentioned above is obtained by defining a "twisted fermion number operator" α , with $\alpha^2 = 1$, and projecting in a modular invariant fashion onto states with $\alpha = 1$. The α defined in [1] is

$$\alpha = (-1)^G \cdot e^{2\pi i p \cdot \delta}$$

where $2\delta \in \Gamma$ and G , the G-parity, is defined as follows. In the antiperiodic (Neveu-Schwarz) sector constructed from oscillators of half-integral level number, G is defined to be worldsheet fermion number minus one. In the periodic (Ramond) sector, G is the chirality operator γ . The GSO projection corresponds to $\alpha = (-1)^G$ and yields the original heterotic string. Adding in the extra Z_2 translation in the compactification torus leads to radically different theories.

The particular choice of $\delta^2 = 2$, realized by

$$\delta_0 = \begin{cases} (1, 0^7; 1, 0^7) & \text{in } \Gamma_8 \oplus \Gamma_8 \\ \left(\left(\frac{1}{2} \right)^8, 0^8 \right) & \text{in } \Gamma_{16} \end{cases}$$

results in a chiral, tachyon free theory with $O(16) \times O(16)$ symmetry and no spacetime supersymmetry. Briefly, the projection $\frac{1}{2}(1 + \alpha)$ applied to either of the original heterotic strings results in two sectors: an even G-parity Neveu-Schwarz bosonic sector with $\delta \cdot L \in \mathbb{Z}$ and a Ramond sector with $\delta \cdot L \in \mathbb{Z} + \frac{1}{2}$ (and negative helicity, with our convention for G). In addition, modular invariance demands that we include two twisted sectors of strings which close only up to a transformation by $\alpha : X(2\pi) = \alpha X(0) \alpha^{-1}$ (which amounts to replacing L by $L + \delta$ in the mode expansion). These are an NS sector with $\delta \cdot L \in \mathbb{Z} + \frac{1}{2}$, and a R sector with $\delta \cdot L \in \mathbb{Z}$. The only

remaining symmetry group generators correspond to $L^2 = 2$ with $\delta \cdot L \in \mathbb{Z}$ and $G = O(16) \times O(16)$. Furthermore, the lowest mass level in the left moving twisted sector is at $\frac{1}{2}m_L^2 = \frac{1}{2}(L + \delta)^2 + N - 1 = \frac{1}{2}\delta^2 - 1$, so $\delta^2 = 2$ and the constraint (2) ensure that the theory contains no tachyon.

Now consider applying such a twist to the theories of Narain. If Γ is a Lorentzian lattice, let

$$\alpha \equiv (-1)^G (-1)^{2(L \cdot \delta - \tilde{L} \cdot \tilde{\delta})} \quad (3)$$

where $(2\delta, 2\tilde{\delta}) \in \Gamma$ and $\delta^2 - \tilde{\delta}^2 \in \mathbb{Z}$. In particular, we shall focus on the case $\delta^2 - \tilde{\delta}^2 = 2$, which always leads to a tachyon free theory. Such vectors $(\delta, \tilde{\delta})$ always exist, because any Lorentzian lattice is an $SO(26-D, 10-D)$ rotation of $\Gamma_8 \oplus \Gamma_8 \oplus (10-D) \cdot \sigma_1$. (Here σ_1 is the Lorentzian lattice in $\mathbb{R}^{1,1}$, generated by the null vectors $\frac{1}{\sqrt{2}}(\hat{i} \pm \hat{z})$.) We can just take $(\delta, \tilde{\delta})$ to be the very same rotation of the δ_0 . Other such twists for $\Gamma_8 \oplus \Gamma_8 \oplus (10-D)\sigma_1$ are possible, such as

$$\delta_1 = \left(\overbrace{1, 0^7; 0^8}^{\delta}, \overbrace{\left(\frac{1}{\sqrt{2}} \right)^2, 0^{8-D}, 0^{10-D}}^{\tilde{\delta}} \right)$$

$$\delta_2 = \left(\overbrace{0^8; 0^8}^{\delta}, \overbrace{\left(\frac{1}{\sqrt{2}} \right)^4, 0^{6-D}, 0^{10-D}}^{\tilde{\delta}} \right)$$

and δ_0, δ_1 or δ_2 plus any perpendicular light-like vector twice which is in Γ . These exhaust the inequivalent twists with $\delta^2 - \tilde{\delta}^2 = 2$. In addition, there are other inequivalent twists with $\delta^2 - \tilde{\delta}^2 \neq 2$ leading to tachyon free theories. ($\delta^2 - \tilde{\delta}^2$ is in general constrained to be an integer in order to have matched left and right moving mass levels in accordance with equation (2).) However, in what follows, we consider only $\delta^2 - \tilde{\delta}^2 = 2$.

Like the $O(16) \times O(16)$ string, these theories all have twisted and untwisted fermionic and bosonic sectors. We shall enumerate the states with $\alpha = 1$ in each of these four sectors. The untwisted bosonic sector is the product of a right moving even G-parity NS sector and a left moving bosonic sector with $L \cdot \delta - \tilde{L} \cdot \tilde{\delta} \in \mathbb{Z}$. So

the untwisted bosonic partition function is

$$P_B^{un} = \text{Tr} z^{\frac{1}{2}m_L^2} z^{\frac{1}{2}m_R^2} \\ = \frac{1}{2z^{\frac{1}{2}}} \prod_{n=1}^{\infty} \left(\frac{1}{1-z^n} \right)^{24} \left[\prod_{n=1}^{\infty} \left(\frac{1+z^{n-\frac{1}{2}}}{1-z^n} \right)^8 - \prod_{n=1}^{\infty} \left(\frac{1-z^{n-\frac{1}{2}}}{1-z^n} \right)^8 \right] \\ \cdot \sum_{(L, \tilde{L}) \in \Gamma} z^{\frac{1}{2}L^2} z^{\frac{1}{2}\tilde{L}^2} \frac{1}{2} (1 + (-1)^{2(L \cdot \delta - \tilde{L} \cdot \tilde{\delta})}) \quad (4)$$

where the trace is over mass eigenstates in the projected sector. In this expression,

$$\frac{1}{2z^{\frac{1}{2}}} \prod_n \left(\frac{1}{1-z^n} \right)^{24} \left[\prod_n \left(\frac{1+z^{n-\frac{1}{2}}}{1-z^n} \right)^8 - \prod_n \left(\frac{1-z^{n-\frac{1}{2}}}{1-z^n} \right)^8 \right]$$

is the usual partition function of the bosonic (NS) sector of an uncompactified heterotic string, with a minus sign in the bracketed factor for projection onto even G-parity states. The compactification is accounted for by the sum over lattice vectors, with the projection operator

$$\frac{1}{2} (1 + (-1)^{2(L \cdot \delta - \tilde{L} \cdot \tilde{\delta})})$$

serving to pick out states with $L \cdot \delta - \tilde{L} \cdot \tilde{\delta} \in \mathbb{Z}$.

The untwisted fermions have $L \cdot \delta - \tilde{L} \cdot \tilde{\delta} \in \mathbb{Z} + \frac{1}{2}$ so similarly

$$P_F^{un} = \frac{8}{z} \prod_{n=1}^{\infty} \left(\frac{1}{1-z^n} \right)^{24} \left(\frac{1+z^n}{1-z^n} \right)^8 \sum_{\Gamma} z^{\frac{1}{2}L^2} z^{\frac{1}{2}\tilde{L}^2} \frac{1}{2} (1 - (-1)^{2(L \cdot \delta - \tilde{L} \cdot \tilde{\delta})}) \quad (5)$$

In the twisted NS sector, modular invariance will require projection onto states with half-integer level number, i.e., odd G-parity. Modular invariance will also imply that the path integral over the twisted NS sector must be summed with weight +1, indicating that this sector is bosonic. The mass operators are also different, since compactified momenta now lie in the shifted lattice $\Gamma + \delta$:

$$\frac{1}{4} m_L^2 = \frac{1}{2} (L + \delta)^2 + N - 1 \\ \frac{1}{4} m_R^2 = \frac{1}{2} (\tilde{L} + \tilde{\delta})^2 + \tilde{N} - \frac{1}{2}$$

Therefore

$$P_B^{tw} = \frac{1}{2z^{\frac{1}{2}}} \prod_{n=1}^{\infty} \left(\frac{1}{1-z^n} \right)^{24} \left[\prod_{n=1}^{\infty} \left(\frac{1+z^{n-\frac{1}{2}}}{1-z^n} \right)^8 + \prod_{n=1}^{\infty} \left(\frac{1-z^{n-\frac{1}{2}}}{1-z^n} \right)^8 \right] \\ \cdot \sum_{(L, \tilde{L}) \in \Gamma} z^{\frac{1}{2}(L+\delta)^2} z^{\frac{1}{2}(\tilde{L}+\tilde{\delta})^2} \frac{1}{2} (1 - (-1)^{2(L \cdot \delta - \tilde{L} \cdot \tilde{\delta})}) \quad (6)$$

Similarly,

$$P_F^{tw} = \frac{8}{z} \prod_{n=1}^{\infty} \left(\frac{1}{1-z^n} \right)^{24} \left(\frac{1+z^n}{1-z^n} \right)^8 \sum_{(L, \tilde{L}) \in \Gamma} z^{\frac{1}{2}(L+\delta)^2} z^{\frac{1}{2}(\tilde{L}+\tilde{\delta})^2} \\ \cdot \frac{1}{2} (1 + (-1)^{2(L \cdot \delta - \tilde{L} \cdot \tilde{\delta})}) \quad (7)$$

Putting everything together and using the standard θ -function identity $\theta_2^4 + \theta_4^4 = \theta_3^4$ we find

$$P_B - P_F = \frac{1}{z} \prod_{n=1}^{\infty} \left(\frac{1}{1-z^n} \right)^{24} \left\{ 8 \prod_{n=1}^{\infty} \left(\frac{1+z^n}{1-z^n} \right)^8 \sum_{\Gamma} z^{\frac{1}{2}L^2} z^{\frac{1}{2}\tilde{L}^2} (-1)^{2(L \cdot \delta - \tilde{L} \cdot \tilde{\delta})} \right. \\ \left. - \frac{1}{2z^{\frac{1}{2}}} \prod_{n=1}^{\infty} \left(\frac{1+z^{n-\frac{1}{2}}}{1-z^n} \right)^8 \sum_{\Gamma} z^{\frac{1}{2}(L+\delta)^2} z^{\frac{1}{2}(\tilde{L}+\tilde{\delta})^2} (-1)^{2(L \cdot \delta - \tilde{L} \cdot \tilde{\delta})} \right. \\ \left. + \frac{1}{2z^{\frac{1}{2}}} \prod_{n=1}^{\infty} \left(\frac{1-z^{n-\frac{1}{2}}}{1-z^n} \right)^8 \sum_{\Gamma} z^{\frac{1}{2}(L+\delta)^2} z^{\frac{1}{2}(\tilde{L}+\tilde{\delta})^2} \right\} \quad (8)$$

We now discuss the massless sectors of these theories. First consider the untwisted bosonic sector, with $|0\rangle = |0\rangle_L |0\rangle_R$ the product of the left moving vacuum with $N=1$ and the right moving Neveu-Schwarz vacuum at level $\tilde{N} = -\frac{1}{2}$. The lowest lying states satisfying the constraint $m_L^2 = m_R^2$ are the massless states

$$\alpha_{-1, L}^i b_{-\frac{1}{2}, R}^j |0\rangle \quad \alpha_{-1, L}^i b_{-\frac{1}{2}, R}^j |0\rangle \\ \alpha_{-1, L}^A b_{-\frac{1}{2}, R}^j |0\rangle \quad \alpha_{-1, L}^A b_{-\frac{1}{2}, R}^j |0\rangle \\ b_{-\frac{1}{2}, R}^j |(L, 0)\rangle \quad b_{-\frac{1}{2}, R}^j |(L, 0)\rangle \\ (2 \leq i < D, D \leq I < 10, D \leq A < 26)$$

where $|(L, 0)\rangle$ is the lowest state with $\delta \cdot L \in \mathbb{Z}$, $L^2 = 2$, and $(L, 0) \in \Gamma$. The α 's are left moving bosonic oscillators and b 's are right moving fermionic oscillators. The untwisted fermionic states, with the Ramond vacuum transforming as an 8_s under $O(8)$, are simply

$$|a, (L, 0)\rangle$$

where now $\delta \cdot L \in \mathbb{Z} + \frac{1}{2}$, $L^2 = 2$ and a is the 8_s index.

The twisted NS sector contains the massless scalar bosons

$$|(L + \delta, \tilde{L} + \tilde{\delta})\rangle'$$

where $(L + \delta)^2 = 2$, $(\bar{L} + \bar{\delta})^2 = 1$ and $L \cdot \delta - \bar{L} \cdot \bar{\delta} \in \mathbb{Z} + \frac{1}{2}$. (We label all twisted states with a prime.) The twisted R sector fermions transform as an $8'_s$:

$$|a', (L + \delta, 0)'\rangle$$

where $(L + \delta)^2 = 2$, $L \cdot \delta \in \mathbb{Z}$, and $(L, 0) \in \Gamma$. Note that the massless state spectrum is nonchiral for $D < 10$.

As an example, consider the lattice Γ_{\max} generated by the vectors in $\mathbb{R}^{26-D, 10-D}$

$$\begin{aligned} (\pm e_i \pm e_j, 0) & \quad 1 \leq i < j \leq 26 - D \\ (0, \pm \bar{e}_i \pm \bar{e}_j) & \quad 1 \leq i < j \leq 10 - D \\ (\omega_s, \bar{\omega}_s) & = \left(\frac{1}{2}(e_1 + \dots + e_{26-D}), \frac{1}{2}(\bar{e}_1 + \dots + \bar{e}_{10-D}) \right) \\ (\omega_v, \bar{\omega}_v) & = (e_1, \bar{e}_1) \end{aligned} \quad (9)$$

The vectors $(k, 0)$ in Γ_{\max} with $k^2 = 2$ generate the root lattice of $SO(52 - 2D)$, which is therefore the gauge group of the heterotic string theory compactified on Γ_{\max} . $(\omega_s, \bar{\omega}_s)$ and $(\omega_v, \bar{\omega}_v)$ are spinor and vector weights of $SO(52 - 2D)$, $SO(20 - 2D)$. For the twist $(\delta, \bar{\delta}) = \left(\left(\frac{1}{2}\right)^8; 0^8, 0^{10-D}, \bar{0}^{10-D} \right)$ this group is broken down to $G = O(16) \times O(36 - 2D)$. The massless field theory has bosons which transform under $O(D - 2) \times O(16) \times O(36 - 2D)$ as

$$[(10 - D) \cdot (1, 1, 1) \oplus (D - 2, 1, 1)]_R \oplus [(D - 2, 1, 1) \oplus (1, Adj, 1) \oplus (1, 1, Adj)]_L$$

These bosons include $O(16) \times O(36 - 2D)$ gauge fields as well as $(10 - D)$ scalars in the adjoint of G , $(10 - D)$ $U(1)$ vector bosons, and $g^{\mu\nu}$, $B^{\mu\nu}$ and ϕ . The twisted and untwisted fermions transform under $O(8) \times O(16) \times O(36 - 2D)$ * as

$$(8_s, 16, 36 - 2D) \quad \text{and} \quad (8'_s, 128, 1)$$

for $D < 10$. What about other twists? If $(\delta, \bar{\delta}) = \left(\left(\frac{1}{2}\right)^9, 0^{18-D}, \frac{1}{2}, \bar{0}^{9-D} \right)$, then the maximal symmetry group is $O(18) \times O(34 - 2D)$, and the massless twisted sector disappears. Clearly, by choosing appropriate twists, we can get any maximal symmetry

* Of course, $O(8)$ is broken down to $O(D - 2)$ but it is more convenient notationally to group the $O(D - 2)$ representations together.

group

$O(2n) \times O(2m)$, where $2m, 2n \geq 16$ and $n + m = 26 - D$.

The methods of reference 12 allow us to write down the cosmological constant for the compactified theory without further ado. Let F be the fundamental region $\{\tau = \tau_1 + i\tau_2 \in \mathbb{C} : \tau_2 > 0, |\tau_1| < \frac{1}{2}, |\tau| \geq 1\}$. Then the cosmological constant in a ten dimensional uncompactified theory is

$$\Lambda_{1-loop} = -c \int_F \frac{d^2\tau}{\tau_2} \tau_2^{-5} [P_B(z, \bar{z}) - P_F(z, \bar{z})]$$

where c is a positive numerical factor which depends on the string tension and the factor τ_2^{-5} arises from the integration over D -dimensional momenta for $D = 10$. In our case, we simply replace τ_2^{-5} with $\tau_2^{-D/2}$ and obtain

$$\Lambda_{1-loop} = -c' \int_F \frac{d^2\tau}{\tau_2} \tau_2^{-(D-2)/2} [P_B(z, \bar{z}) - P_F(z, \bar{z})] \quad (10)$$

We now turn to the question of modular invariance of Λ . Our argument will be along the lines of reference 3 and details are in the appendix. The partition function (8) can be written as

$$P = -B \sum_{\alpha\beta} \bar{A}_{\alpha\beta} C_{\alpha\beta} \quad (11)$$

where

$$B = z^{-1} \bar{z}^{-\frac{1}{2}} \prod_n \frac{1}{(1 - z^n)^{24} (1 - \bar{z}^n)^8} = \eta(\tau)^{-24} \bar{\eta}(\tau)^{-8} \quad (12)$$

and η is the Dedekind η function. The $A_{\alpha\beta}$'s are in the notation of reference 15, i.e., $(\alpha\beta)$ specifies (σ, t) boundary conditions with (anti-) periodicity denoted by $+$ ($-$):

$$\begin{aligned} A_{--} &= \frac{1}{2} z^{-\frac{1}{4}} \prod_n (1 + z^{n-\frac{1}{2}})^8 \\ A_{+-} &= -8z^{\frac{1}{2}} \prod_n (1 + z^n)^8 \\ A_{-+} &= -\frac{1}{2} z^{-\frac{1}{4}} \prod_n (1 - z^{n-\frac{1}{2}})^8 \end{aligned} \quad (13)$$

The bar over $A_{\alpha\beta}$ denotes complex conjugation. The $C_{\alpha', \beta'}$'s are sums over lattice

vectors with twists put in.

$$\begin{aligned}
C_{--} &= \sum z^{\frac{1}{2}(L+\delta)} z^{\frac{1}{2}(\bar{L}+\bar{\delta})} (-1)^{2(L\delta-\bar{L}\bar{\delta})} \\
C_{+-} &= \sum z^{\frac{1}{2}L} z^{\frac{1}{2}\bar{L}} (-1)^{2(L\delta-\bar{L}\bar{\delta})} \\
C_{-+} &= \sum z^{\frac{1}{2}(L+\delta)} z^{\frac{1}{2}(\bar{L}+\bar{\delta})}
\end{aligned} \tag{14}$$

The correspondence for the C 's is as follows: in the (σ, t) notation, antiperiodicity in σ corresponds to summing over lattice vectors with a shift, i.e. $(L, \bar{L}) \rightarrow (L+\delta, \bar{L}+\bar{\delta})$, while antiperiodicity in t corresponds to the insertion of the factor $(-1)^{2(L\delta-\bar{L}\bar{\delta})}$.

From the amplitudes $A_{\alpha\beta}$ and $C_{\alpha'\beta'}$ one can construct 16 possible products. Certain appropriately weighted sums of these products correspond to unitary, modular invariant string theories [1-3]. In our case, the partition function is the sum of the diagonal terms, where the spin structure $(\alpha\beta)$ is identified with the twist structure $(\alpha'\beta')$ of the lattice sums. From the transformation of the amplitudes given in the appendix, A is easily verified to be modular invariant.

There are also other combinations of the 16 amplitudes which are modular invariant, e.g., $(\sum A_{\alpha\beta}) \cdot (\sum C_{\alpha'\beta'})$. This is zero since $\sum A = 0$. Since the diagonal sum is modular invariant, we see that the nondiagonal terms in the above sum also give a modular invariant combination corresponding to a theory with a cosmological constant of the opposite sign to the diagonal one. This theory has tachyons, however, and we shall not discuss it further.

We shall conclude this section with some remarks on the question of modular invariance for higher loop amplitudes. For ℓ loop string calculations, we have to consider sums over surfaces with 2ℓ noncontractible loops or homology cycles. This gives $2^{2\ell}$ spin structures or choices for fermionic boundary conditions. The difference between two spin structures corresponds to the assignment of relative \pm signs corresponding to periodicity or antiperiodicity. Since the values of X differing by lattice vectors are identified, one can also associate winding numbers (lattice vectors) and twist vectors with each cycle. There is thus a one to one correspondence between spin structures and twist structures; for each spin structure, there is a corresponding twist structure

for the lattice sums and the amplitudes $A_{\alpha_1\beta_1\dots}$ and $C_{\alpha'_1\beta'_1\dots}$ can be constructed. Presumably the diagonal sum will give a modular invariant amplitude.

III. Extrema of Λ and equations of motion

In section II we have described $(26-D)(10-D)$ parameter families of string compactifications which are degenerate at the string tree level. We now wish to determine which of these remain solutions at the one loop level. In terms of string diagrams, this requires that the one loop one-point functions vanish [13]. These are given by the expectation values of vertex operators on the world sheet torus. In principle, one might have to consider an infinite number of vertex operators. In practice this is avoided because we have a small parameter at our disposal, namely the dilaton expectation value ϕ_0 , which governs the strength of string loop corrections. At string tree level, this is a free parameter, and it must be assumed to be small for the consistency of our analysis. This small parameter justifies ignoring one point functions for massive fields — they at most shift their expectation values by a small amount.* The relevant massless modes are the lattice moduli (which correspond to deformations of the metric and the gauge and antisymmetric fields [7]) and the ten dimensional dilaton which couples to the Euler density of the world sheet manifold.

Consider first the $(26-D)(10-D)$ vertex operators that create the massless modes associated with the lattice moduli. In general, the one point function associated with a massless deformation of the background is simply the linearized variation of the functional integral under the deformation. In our case, the one point functions vanish at $\Gamma = \Gamma_0$ if

$$\left. \frac{\delta}{\delta\Gamma} \Lambda(\Gamma) \right|_{\Gamma=\Gamma_0} = 0$$

We shall show that Λ has several critical points as a function on the lattice moduli space $SO(p, q)/SO(p) \times SO(q)$ ($p = 26 - D$, $q = 10 - D$), a consequence of a certain duality relating the different theories on this space.

* In particular one can ignore at this order the tadpole which might tell us to shift to de Sitter space [14]. The scale of the relevant de Sitter geometry will be large relative to the string scale.

As usual, let Γ be a Lorentzian lattice with $(L, \tilde{L}) \in \Gamma$. We consider a one parameter family of $SO(p, q)$ transformations which mix the vectors L and \tilde{L} . We shall first examine the case of compactification to $D = 9$, i.e. $q = 1$. In this case L_i is a 17 dimensional vector and we can write

$$L'_i = \cosh \beta_i L_i + \sinh \beta_i \tilde{L}$$

$$\tilde{L}' = \sinh \beta_i L_i + \cosh \beta_i \tilde{L}$$

β_i parameterizes the "boost" which mixes the i th component of L with \tilde{L} .

Γ is an extremal point of Λ if $\Lambda(\Gamma(\beta_i)) = \Lambda(\Gamma(-\beta_i))$ for infinitesimal β_i for all 17 choices of i , since these 17 boosts form a local basis for the space $\frac{SO(17,1)}{SO(17) \times SO(1)}$. From the transformation equations above, we see that $\beta_i \rightarrow -\beta_i$ can be compensated for by $\tilde{L} \rightarrow -\tilde{L}$. Since the expression for the masses is invariant under $\tilde{L} \rightarrow -\tilde{L}$, Γ will be a critical point of Λ_{1-loop} provided $(L, -\tilde{L})$ is always a lattice vector. (Then the summation $\sum_{L, \tilde{L}}$ also runs over the same range.)

Consider now the general case of compactification to D dimensions. The lattices may now be parametrized by $SO(p, q)$ elements. We shall call lattices *symmetric* if $(L, \tilde{L}) \in \Gamma \Leftrightarrow (L, -\tilde{L}) \in \Gamma$, and denote them by Γ_{sym} . (Note that a Lorentzian self dual lattice is symmetric if and only if it is also Euclidean self dual (although it need not be Euclidean even). Hence the number of symmetric lattices in D dimensions is finite.) For $S \in SO(p, q)$, let $\Gamma_{\text{sym}}(S)$ be the lattice obtained by boosting Γ_{sym} by S . It is straightforward to see that $\Lambda_{1-loop}[\Gamma_{\text{sym}}(S^T)] = \Lambda_{1-loop}[\Gamma_{\text{sym}}(S^{-1})]$. Also, from the definition of Γ_{sym} that for S near the identity, the vector in $\frac{SO(p, q)}{SO(p) \times SO(q)}$ corresponding to S^{-1} is minus that corresponding to S^T . Thus Γ_{sym} is an extremum of Λ_{1-loop} .

For the lattice Γ_{max} of equation (9) leading to the maximal gauge group, we have weights of $SO(4k)$ for even D , where $2k = 26 - D$. In this case, if (w_s, \tilde{w}_s) is in Γ , then so is $(w_s, -\tilde{w}_s)$. Γ_{max} is thus symmetric and is a critical point of Λ_{1-loop} . Another critical point is the lattice $\Gamma_8 \oplus \Gamma_8 \oplus n\sigma_1$, which corresponds to the gauge group $O(16) \times O(16) \times SU(2)^n$. In the next section, we shall show that the theories defined by $\Gamma_{\text{sym}}(S^T)$ and $\Gamma_{\text{sym}}(S^{-1})$ formally have the same cosmological constant to

all orders in string perturbation theory and so symmetric lattices are extrema to all orders as well.

The formulation of these theories in terms of constant background values of massless fields also leads us to expect the existence of a large number of critical points for Λ . The topology of the moduli space is quite complicated since there are various periodic shifts of the background fields which leave the theory physically unchanged. This means that the Betti numbers of the moduli space will be large and Morse theory suggests a correspondingly large number of critical points.

We now turn to the ten dimensional dilaton. Its one loop one point function vanishes if and only if the cosmological constant Λ does. $\Lambda = 0$ at the string tree level by virtue of the conformal invariance of the σ -model [13]. Since the one loop cosmological constant is generically nonzero, our approximate solutions will in general develop some slow (for small ϕ) time dependence controlled by the value of Λ at the extrema. For Λ positive, ϕ will roll in to weak coupling, while if Λ is negative it will roll to strong coupling. Nevertheless for small ϕ we have approximate solutions for a long time.

Of particular interest is the point at "infinity" corresponding to "decompactification." By this we mean the limit in which all the radii of the $(10 - D)$ dimensional compactification torus go to infinity, while other parameters are held finite. (This limit is most easily understood if one interprets the Lorentzian lattice compactifications in terms of expectation values for background massless fields. It then corresponds to an infinite constant rescaling of the metric.) We will show that this point is a local minimum of the D dimensional energy. For large radii, the field theory approximation should be valid. The dynamics are described by the ten dimensional effective action

$$S_{10} = \int d^{10}x \sqrt{-g^{(10)}} (R^{(10)} - \Lambda^{(10)})$$

where the numerical value of $\Lambda^{(10)}$ has been computed in [2] and is positive. Compactifying to D dimensions on a flat torus, one obtains the D dimensional effective

action

$$S_D = \int d^D x \sqrt{-g^{(D)}} (V R^{(D)} - V \Lambda^{(10)}) \quad (15)$$

where V is the volume of the torus. (This effective action is relevant on scales large relative to the string scale $\sqrt{\alpha'}$ but small relative to $\phi_0^{-1} \sqrt{\alpha'}$. At the latter scale, the curvature of the background geometry becomes relevant.) $V \Lambda^{(10)}$ increases as $V \rightarrow \infty$. However it is wrong to conclude from this that the dynamics force us to smaller V . In fact the opposite is true. To see this, define a new metric

$$\tilde{g}_{ab} = g_{ab} V^{\frac{1}{D-3}}$$

S_D then takes the canonical form

$$S_D = \int d^D x \sqrt{-\tilde{g}^{(D)}} (\tilde{R}^{(D)} - \frac{(D-1)}{(D-2)} \tilde{g}^{ab} \nabla_a \ln V \nabla_b \ln V - \Lambda V^{-\frac{1}{D-3}}) \quad (16)$$

$\ln V$ is a scalar field in D dimensions with a canonical kinetic term (up to a constant factor.) Its potential pushes it out to larger V for any finite value of V , and $V = +\infty$ is a stable point for $\Lambda > 0$. This is in accord with the general behavior described in reference 15.

IV. Duality

We have seen in the last section that the cosmological constant has the symmetry $\Lambda(\Gamma_{\text{sym}}(S^T)) = \Lambda(\Gamma_{\text{sym}}(S^{-1}))$. Now all the theories we consider can be obtained by $SO(p, q)$ transformations on a given lattice which can be taken as one of the symmetric ones. The above result thus shows that, at least as far as the one loop vacuum energy is concerned, there is a duality between the theories $\Gamma_{\text{sym}}(S^T)$ and $\Gamma_{\text{sym}}(S^{-1})$, in the sense that they give the same Λ . (A duality of this type was noticed in the context of bosonic strings in [16].) This result, as we shall show below, generalizes to the full theory.

For the case of $D = 9$, the parameter e^β is proportional to the radius R of the torus involved in the compactification, so the duality gives a mapping between physics at large and small radii. More generally we have a duality between $\Gamma_{\text{sym}}(S^T)$

and $\Gamma_{\text{sym}}(S^{-1})$, where the parameters of $S \in SO(p, q)$ give a measure of the radii of the various tori as well as expectation values for the other fields. This duality generalizes to all the scattering amplitudes and provides a complete mapping between the two theories. (In fact, it applies equally well to the untwisted theories considered by Narain.)

The duality mapping can be defined in terms of the vertex operators. These are labelled by $(k^I, \tilde{k}^I) \in \Gamma$ in addition to the oscillator labels $\{n\}$ and the D dimensional momentum p . The vertex $V(k^I, \tilde{k}^I, \{n\}, p)$ of $\Gamma_{\text{sym}}(S^T)$ is computationally equivalent to $V(k^I, -\tilde{k}^I, \{n\}, p)$ of $\Gamma_{\text{sym}}(S^{-1})$ because the minus sign in front of \tilde{k}^I is compensated for by a minus sign in the \tilde{X}^I mode expansion on S^T relative to S^{-1} . * The amplitudes can be calculated in a covariant operator formalism. The invariance of the propagator together with the above equivalence of the vertex operators shows that there is a corresponding mapping of the various scattering amplitudes, i.e., the tree level S -matrix elements on $\Gamma_{\text{sym}}(S^T)$ and $\Gamma_{\text{sym}}(S^{-1})$ are related by

$$A(k^I, \tilde{k}^I, \{n\}, p, S^T) = A(k^I, -\tilde{k}^I, \{n\}, p, S^{-1}) \quad (17)$$

The mapping relates states of equal mass. The higher loop amplitudes can be constructed from these by use of unitarity relations and factorization. Thus the duality relation (17) extends to all orders in string perturbation theory.

It is instructive to consider this duality in a little more detail for the simplest case, viz. compactification on a circle of radius R to $D = 9$. For large values of R , we can use the field theory approximation. As $R \rightarrow \infty$, the massive Kaluza-Klein states move down to zero mass and the theory becomes effectively ten dimensional. How can we get the same spectrum, as implied by duality, for $R \rightarrow 0$? In this case, the Kaluza-Klein states get more and more massive. However, the monopole or solitonic states which wrap around the circle move down to zero mass and the theory again

* The vertex operators, in addition to polarization tensors and $e^{ik \cdot X}$, involve the operator cocycles $C(L, \tilde{L})$. From the definition of $C(L, \tilde{L})$, one can see that they can be chosen to be invariant, i.e., $C(L, -\tilde{L}) = C(L, \tilde{L})$.

becomes effectively ten dimensional. These solitonic states are related by the duality transformation to the low mass Kaluza-Klein states at large R .

The duality we find is similar to the Krauss-Wannier duality of two dimensional spin systems where there is a mapping between high and low temperatures (rather than large and small radii) [8]. The theory is self dual at the critical point. It is also reminiscent of charged particle-monopole duality where inversions of the type $R \rightarrow \frac{1}{R}$ play a key role. In field theory examples where one has such a duality, e.g. $N = 4$ and $N = 2$ supersymmetric gauge theories, the mapping also relates strong and weak coupling regimes and the self dual point is again a critical point, i.e. the β function is zero [9]. We expect our duality to extend beyond the toroidal compactifications considered here and to be of more general validity in string theories.

This research was supported in part by National Science Foundation grants PHY-82-17853 and PHY-8313324, and by funds from the National Aeronautics and Space Administration.

Appendix : Modular Invariance

This appendix provides some mathematical details in the proof of modular invariance of the one loop partition function. As always, let Γ be a Lorentzian even self dual lattice and let $\delta^2 - \bar{\delta}^2$ be even. Recall the definitions given in the text:

$$\begin{aligned} A_{--}(\tau) &\equiv \frac{1}{2} z^{-1/8} \prod_{n=1}^{\infty} (1 + z^{n-\frac{1}{2}})^8 = \frac{1}{2} (2\pi)^{4/3} \frac{\theta_3(0, \tau)^4}{\theta_1'(0, \tau)^{4/3}} \\ A_{+-}(\tau) &\equiv -8z^{1/3} \prod_{n=1}^{\infty} (1 + z^n)^8 = -\frac{1}{2} (2\pi)^{4/3} \frac{\theta_2(0, \tau)^4}{\theta_1'(0, \tau)^{4/3}} \\ A_{-+}(\tau) &\equiv -\frac{1}{2} z^{-1/8} \prod_{n=1}^{\infty} (1 - z^{n-\frac{1}{2}})^8 = -\frac{1}{2} (2\pi)^{4/3} \frac{\theta_4(0, \tau)^4}{\theta_1'(0, \tau)^{4/3}} \end{aligned} \quad (A1)$$

$$B(\tau) \equiv \frac{1}{z\bar{z}^{1/3}} \prod \left(\frac{1}{1-z^n} \right)^{24} \left(\frac{1}{1-\bar{z}^n} \right)^8 = \frac{\eta(\tau)^{-24} \overline{\eta(\tau)}^{-8}}{\eta(\tau)^{-24} \overline{\eta(\tau)}^{-8}} \quad (A2)$$

$$C_{--}(\tau) \equiv \sum_{(L, \bar{L}) \in \Gamma} z^{\frac{1}{2}(L+\delta)^2} \bar{z}^{\frac{1}{2}(\bar{L}+\bar{\delta})^2} (-1)^{2(L\delta - \bar{L}\bar{\delta})}$$

$$\begin{aligned} C_{+-}(\tau) &\equiv \sum_{\Gamma} z^{\frac{1}{2}L^2} \bar{z}^{\frac{1}{2}\bar{L}^2} (-1)^{2(L\delta - \bar{L}\bar{\delta})} \\ C_{-+}(\tau) &\equiv \sum_{\Gamma} z^{\frac{1}{2}(L+\delta)^2} \bar{z}^{\frac{1}{2}(\bar{L}+\bar{\delta})^2} \\ x &\equiv e^{2\pi i \tau} \end{aligned} \quad (A3)$$

From the properties of θ functions discussed in reference 17, it easily follows that

$$A_{-+}(\tau) = A_{+-} \left(-\frac{1}{\tau} \right) \quad (A4)$$

$$A_{--}(\tau) = A_{--} \left(-\frac{1}{\tau} \right) \quad (A4)$$

$$B(\tau) = \tau^{12} \bar{\tau}^4 B \left(-\frac{1}{\tau} \right) \quad (A5)$$

We can determine the transformation rules for C_{-+} and C_{+-} using the identity

$$\begin{aligned} \sum_{(L, \bar{L})} \exp \left[i\pi r (L - \frac{Q}{r})^2 - i\pi r (\bar{L} - \frac{\bar{Q}}{r})^2 \right] &= (-ir)^{-\frac{(2\delta-D)}{2}} (i\bar{r})^{-\frac{(2\bar{\delta}-D)}{2}} \\ &\cdot \sum_{(L, \bar{L})} \exp \left[\frac{-i\pi}{r} (L - Q)^2 + \frac{i\pi}{\bar{r}} (\bar{L} - \bar{Q})^2 \right] \exp \left[\frac{i\pi}{r} Q^2 - \frac{i\pi}{\bar{r}} \bar{Q}^2 \right] \end{aligned} \quad (A6)$$

for any vector Q . (The derivation for Euclidean self dual lattices in reference 1 is easily extended to the Lorentzian case.) We find

$$C_{-+}(\tau) = (-ir)^{-\frac{2\delta-D}{2}} (i\bar{r})^{-\frac{2\bar{\delta}-D}{2}} C_{+-} \left(-\frac{1}{r} \right) \quad (A7)$$

The last equation we need is

$$C_{--}(\tau) = (-ir)^{-\frac{2\delta-D}{2}} (i\bar{r})^{-\frac{2\bar{\delta}-D}{2}} C_{--} \left(-\frac{1}{r} \right). \quad (A8)$$

Assuming this for the moment, we prove modular invariance of Λ , as given in equations (8) and (9). The measure $\tau_2^{-2} d^2\tau$ is invariant under both $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -\frac{1}{\tau}$. From equations (A1) to (A5), the integrand,

$$\tau_2^{-\frac{D-1}{2}} B \cdot \{ \bar{A}_{--} C_{--} + \bar{A}_{-+} C_{-+} + \bar{A}_{+-} C_{+-} \}$$

is invariant under $\tau \rightarrow -\frac{1}{\tau}$ since $\tau_2 \rightarrow \frac{\tau_2}{|\tau|^2}$. It is also easy to see that $\tau \rightarrow \tau + 1$ has the sole effect of interchanging the $(--)$ and $(-+)$ terms.

Finally, we wish to prove equation (A8). For simplicity, take Γ to be a Euclidean self dual lattice in \mathbb{R}^N . Then (A8) amounts to

$$\sum_{\Gamma} s^{\frac{1}{2}}(L+\delta)^2 (-1)^{2L\cdot\delta} = (-ir)^{-N/2} \sum_{\Gamma} s'^{\frac{1}{2}}(L+\delta)^2 (-1)^{2L\cdot\delta} \quad (\text{A9})$$

where $s' = \exp\left(\frac{-2\pi i}{r}\right)$.

We shall need to use the lattice $\Gamma_{\text{even}} = \{L | L \cdot \delta \in \mathbb{Z}\}$, where $2\delta \in \Gamma$ and $\delta^2 \in \mathbb{Z}$. Its dual is simply the lattice generated by δ and the basis vectors of Γ , namely,

$$\Gamma_{\text{even}}^* = \Gamma \cup \{\Gamma + \delta\} \quad (\text{A10})$$

Consider the following function on \mathbb{R}^N :

$$\begin{aligned} f(X) &= \sum_{L \in \Gamma} \exp i\pi r(L+X)^2 \cdot \frac{1}{2} [1 + \exp i2\pi L \cdot \delta] \\ &= \sum_{L \in \Gamma_{\text{even}}} \exp i\pi r(L+X)^2 \end{aligned} \quad (\text{A11})$$

f is periodic on Γ_{even} so we can take its Fourier transform, defined on $2\pi\Gamma_{\text{even}}^*$:

$$\begin{aligned} \hat{f}(2\pi L^*) &\equiv \sum_{L \in \Gamma_{\text{even}}} \frac{1}{\text{vol}(\Gamma_{\text{even}})} \int_{F_{\text{even}}} d^N X \exp i\pi r(L+X)^2 \exp i2\pi L^* \cdot X \\ &= \frac{1}{\text{vol}(\Gamma_{\text{even}})} \int_{\mathbb{R}} d^N X \exp i\pi r X^2 \cdot \exp i2\pi L^* \cdot X \\ &= \frac{1}{2} \cdot (-ir)^{-N/2} \exp -\frac{i\pi}{r} (L^*)^2 \end{aligned}$$

where $\text{vol}(\Gamma_{\text{even}}) = 2$ is the volume of the fundamental cell F_{even} of Γ_{even} . (Recall that $\text{vol}(\Gamma) = 1$.) Reverting and using (A10), we obtain

$$\begin{aligned} f(X) &= \frac{1}{2} (ir)^{-N/2} \sum_{L^* \in \Gamma_{\text{even}}^*} \exp -\frac{i\pi}{r} (L^*)^2 \cdot \exp -i2\pi L^* \cdot X \\ &= \frac{1}{2} (-ir)^{-N/2} \left\{ \sum_{L \in \Gamma} \exp -i\pi r L^2 \cdot \exp(-i2\pi L \cdot X) \right. \\ &\quad \left. + \sum_{L \in \Gamma} \exp -\frac{i\pi}{r} (L+\delta)^2 \cdot \exp -i2\pi(L+\delta) \cdot X \right\} \end{aligned}$$

We can now extract equation (A9) by setting $X = \delta$:

$$\begin{aligned} f(\delta) &= \frac{1}{2} (-ir)^{-N/2} \sum_{L \in \Gamma} \exp \left(-\frac{i\pi}{r} L^2 \right) \exp(-i2\pi L \cdot \delta) \\ &\quad + \frac{1}{2} (-ir)^{-N/2} \sum_{L \in \Gamma} \exp \left(-\frac{i\pi}{r} (L+\delta)^2 \right) \exp(-i2\pi L \cdot \delta) \quad (\text{A12}) \\ &\equiv \sum_{L \in \Gamma} \exp i\pi r(L+\delta)^2 \cdot \frac{1}{2} [1 + \exp i2\pi L \cdot \delta] \end{aligned}$$

We already know from (A6) that

$$\frac{1}{2} \sum_{L \in \Gamma} \exp i\pi r(L+\delta)^2 = \frac{1}{2} (-ir)^{-N/2} \sum_{L \in \Gamma} \exp -\frac{i\pi}{r} L^2 \cdot \exp i2\pi L \cdot \delta$$

Subtracting this from both sides of (A12) yields

$$\sum_{\Gamma} \exp i\pi r(L+\delta)^2 (-1)^{2L\cdot\delta} = (-ir)^{-N/2} \sum_{\Gamma} \exp -\frac{i\pi}{r} (L+\delta)^2 (-1)^{2L\cdot\delta}$$

which is precisely (A9). The extension of this argument to the Lorentzian case is straightforward.

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