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# Multi Gluon Scattering : A String Based Calculation

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## Abstract

We suggest the use of open bosonic string theory to compute multigluon scattering in QCD, since gauge invariance ,cyclic symmetry etc. are manifest in this approach. The method is illustrated by the computation of five gluon scattering.

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The computation of multi gluon scattering cross sections is important for the study of hadronic jets. The amplitude and cross section calculations have been done upto six gluons[1]. Although considerable simplifications follow from the use of supersymmetry etc, these calculations are essentially done by summing up the appropriate set of Feynman diagrams. The number of Feynman diagrams involved grow rapidly with the number of external gluons. The intermediate steps are thus fairly complicated, but the final answer is extremely simple. It is clear from the form of the final answer that it is much simpler to think of quantum chromodynamics as the zero slope limit of some kind of string theory. In this letter, we show how this connection may be utilised to derive multigluon scattering amplitudes and cross sections in QCD.

We use an open bosonic string theory with $SU(3)$ Chan-Paton factors to take care of the non Abelian nature of the interactions. Since we are only interested in the tree level amplitudes, questions of consistency of the string theory such as the critical dimension will not be of concern. Also the tachyon state of the bosonic string will not contribute to vector particle scattering at the tree level. The string based calculation has considerable conceptual simplification. There is essentially one diagram and its non cyclic permutations for each process no matter how many external gluons we have. Although this gets split up into a number of terms upon taking the zero slope limit, this is still advantageous as cyclic symmetry, gauge invariance etc are preserved for each such amplitude. The cyclic symmetry of the final result is easily traced to the cyclic invariance of the individual string diagrams and is manifest at all intermediate steps. The various steps involved are quite straightforward and simple. The zero slope limit of the string amplitudes can be obtained with minimal investment of computer time. We shall illustrate the calculation by computing the five gluon amplitude and cross section. (The four point case is too trivial.) A longer paper with the results for higher numbers of external lines will be published elsewhere.

We start with the $n$ vector particle amplitude for the open bosonic string. This is obtained by evaluating the correlations for $n$ vertex operators $\int_R \; e^{ik_i \cdot X} \; \epsilon^i \cdot \partial X \; d\tau$ $(i = 1, 2...n)$ using the free action $S = \int_M \frac{1}{2}(\partial \; X)^2 \; d^2x$ defined on the upper half complex plane $M$. The vertex operators are integrated along the real axis $x_1 = \tau$. The parametric form of this amplitude for the $n = 5$ case we discuss and for fixed $\tau$ ordering of the vertex operators is [2,3]

$$\mathcal{A}(1,2,3,4,5) = \int d\tau_1 \cdots d\tau_5 \frac{|\tau_1 - \tau_4||\tau_4 - \tau_5||\tau_5 - \tau_1|}{d\tau_1 d\tau_4 d\tau_5} \prod_{1 \leq j < i \leq 5} (\tau_i - \tau_j)^{\alpha' k_i \cdot k_j}$$
$$\exp\left( \sum_{i>j} \frac{\epsilon_i \cdot \epsilon_j}{(\tau_i - \tau_j)^2} - \sum_{i>j} \sqrt{\alpha'} \frac{k_i \cdot \epsilon_j}{(\tau_i - \tau_j)} \right)|_{multilinear} \tag{1}$$

where the subscript *multilinear* means that we take only the multilinear part of the polarization vectors when we expand the exponential factors. Three of the parametric integrations, as indicated, are removed by fixing the $SL(2,R)$ conformal invariance of the upper half plane. The values of $\tau_1$ and $\tau_4$ are fixed to be 0 and 1 respectively, while $\tau_5 \to \infty$. A typical term of this amplitude will look like

$$\int_0^1 dy \int_0^y dx\, x^{\alpha_{21}} y^{\alpha_{31}} (1-x)^{\alpha_{42}} (1-y)^{\alpha_{43}} (y-x)^{\alpha_{32}} \tag{2}$$

with coefficient either $\epsilon_i \cdot \epsilon_j \epsilon_l \cdot \epsilon_l \epsilon_m \cdot k$ or $\epsilon_i \cdot \epsilon_j \epsilon_l \cdot k_a \epsilon_m \cdot k_b \epsilon_n \cdot k_c$ The $\alpha_{ij}$ is given as $\alpha' k_i \cdot k_j + integer$ The above integral can be done as

$$B(\alpha_{21}+1, \alpha_{32}+1) B(\alpha_{21}+\alpha_{31}+\alpha_{32}+2, \alpha_{43}+1)$$
$${}_3F_2(-\alpha_{42}, \alpha_{21}+1, \alpha_{21}+\alpha_{31}+\alpha_{32}+2; \alpha_{21}+\alpha_{32}+2, \alpha_{21}+\alpha_{31}+\alpha_{32}+\alpha_{43}+3; 1) \tag{3}$$

Here $B(a,b)$ is the beta function, and the ${}_3F_2(a,b,c;d,e;1)$ is the the generalized hypergeometric function evaluated at 1 [4].

The second step is to extract the zero slope limit of the amplitude. Because of the global $\sqrt{\alpha'}$ we only need the residues of the poles in the above expression. The non-vanishing terms of the amplitude as $\alpha' \to 0$ can be obtained as follows. Notice that so long as the power series expansions for ${}_3F_2(a,b,c;d,e;1)$ is convergent (which it will be for $a+b+c-d-e < 0$), the poles and finite pieces come from some finite number of leading terms. In the case where the series is divergent, this is somewhat more difficult, (although the logarithmically divergent case is straightforward, and could be done directly), but fortunately, there are identities which relate the hypergeometric functions for different values of the arguments in particular ${}_3F_2$ for $a+b+c-d-e > 0$ to ${}_3F_2$ with $a+b+$

$c - d - e < 0$. One can then use the series expansion. The relevant two identities, due to Bailey, Hardy, and Ramanujan [4], extract part of the divergence in leading $\Gamma$ functions:

$$
\begin{aligned}
{}_3F_2(a,b,c;d,e;1) &= \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-c)\Gamma(d+e-a-b)}\,{}_3F_2(d-a,d-b,c;d,d+e-a-b;1) \\
{}_3F_2(a,b,c;d,e;1) &= \frac{\Gamma(d)\Gamma(d-a-b)}{\Gamma(d-a)\Gamma(d-b)}\,{}_3F_2(a,b,e-c;a+b-d+1,e;1) \\
&\quad + \frac{\Gamma(d)\Gamma(e)\Gamma(a+b-d)\Gamma(d+e-a-b-c)}{\Gamma(a)\Gamma(b)\Gamma(e-c)\Gamma(d+e-a-b)} \\
&\quad\quad {}_3F_2(d-a,d-b,d+e-a-b-c;d-a-b+1,d+e-a-b;1)
\end{aligned}
\tag{4}
$$

We have checked that the zero slope limit so obtained is indeed cyclically invariant and has the expected gauge invariance, viz. it is invariant under $\epsilon^i \rightarrow \epsilon^i + c_i k_i$ (no summation over $i$). $c_i$ are arbitrary constants. Alternatively, one can utilise these invariances to reduce the calculation to a few terms. For the five point amplitude, at this stage there are about 150 terms in all. About fifty of these have the structure $\epsilon\cdot\epsilon\;\epsilon\cdot\epsilon\;\epsilon\cdot k$ while the remaining have three powers of $\epsilon\cdot k$. The symmetries, including equation (8) below, can be used to write the amplitdes in somewhat compact form. As an example we write the terms with one power of $\epsilon\cdot k$ as

$$
\begin{aligned}
\sum_{Z_5}\Bigg[&\frac{\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4}{2(1,2)(3,4)(5,1)(4,5)}\Big\{\,k_1\cdot\epsilon_5\{(4,5)(5,1)[2(3,4)-2(1,2)+(4,5)-(5,1)+(2,3)] \\
&\qquad\qquad +2(4,5)(1,2)[(2,3)+(3,4)]\} \\
&\qquad -k_2\cdot\epsilon_5(4,5)(5,1)[(4,5)+(5,1)-(2,3)] \\
&\qquad -(1\leftrightarrow 4, 2\leftrightarrow 3)\Big\} \\
&+\frac{\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4}{(4,5)(5,1)}\Big\{-k_1\cdot\epsilon_5(4,5)-k_4\cdot\epsilon_5(5,1)\Big\} \\
&+\frac{\epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3}{2(2,3)(5,1)(4,5)}\Big\{\,k_1\cdot\epsilon_5(4,5)[2(2,3)+2(3,4)-(5,1)] \\
&\qquad -k_2\cdot\epsilon_5(4,5)(5,1)(4,5) \\
&\qquad -(1\leftrightarrow 4, 2\leftrightarrow 3)\Big\}\Bigg]
\end{aligned}
\tag{5}
$$

In the above, $(1,2) = (k_1 + k_2)^2 = 2k_1\cdot k_2$ etc. and $Z_5$ denotes cyclic permutation of the external lines. The exact number of terms depends on the way the kinematical constraints $k_i^2 = 0$, $\sum k_i = 0$ and $\epsilon_i k_i = 0$ are used. The scattering matrix element is given as the sum over non cyclic permutations of the external lines of this amplitude with the insertion of Chan-Paton factors to take care of colour degrees of freedom. Define

$$\mathcal{M}(P) = tr(\lambda_{p(1)}\lambda_{p(2)}\lambda_{p(3)}\lambda_{p(4)}\lambda_{p(5)}) \;\; A(p(1), p(2), p(3), p(4), p(5)) \tag{6}$$

$A(1,2,3,4,5)$ denotes the $\alpha' \to 0$ limit of the amplitude in (1) and $P$ stands for a specific permutation of the external labels $(1,2,3,4,5)$. The scattering matrix element is given by $\sum_{non\ cyclic} \mathcal{M}(P)$.

The amplitudes $A(1,2,3,4,5)$ satisfy the following identities in addition to gauge invariance and cyclicity.

$$\sum_{Z_4} A(1,2,3,4,5) = 0 \tag{7}$$

$$A(1,2,3,4,5) = -A(5,4,3,2,1) \tag{8}$$

where $Z_4$ denotes cyclic permutations of the first four particles. This will be $Z_{n-1}$ for $n$ external lines. The origin of these relations is as follows. In string theory, the twist operator is given by $\sigma \to \pi - \sigma$ where $\sigma$ is the space like coordinate of the two dimensional world surface. Consider an amplitude as given in equation(1). $\mathcal{A}(4,5\ T\ 1,2,3)$ is given by the same integral with $\tau_2$, $\tau_3$ being integrated from 0 to $-1$. This does not give a full amplitude; by considering other twisted amplitudes one can complete the region of integration to give [2]

$$\mathcal{A}(4,5\ T\ 1,2,3) + \mathcal{A}(1,4\ T\ 2,3,5) + \mathcal{A}(5,1\ T\ 2,4,3) = \mathcal{A}(1,2,3,5,4). \tag{9}$$

On the particle states the twist operator corresponds to charge conjugation and the vector vertices are odd under this. Using this fact, in the low energy limit, equation (9) gives the identity (7). The second identity (8) follows from the operation $\sigma \to -\sigma$ applied to the correlation functions. For the $n$ point case, the amplitude with the order of the external particles reversed will be $(-1)^n$ times the original amplitude.

Consider the square of the matrix element now. The calculation of the group theory factors can be simplified by using U(N) rather than SU(N) as the gauge group since the extra U(1) gauge boson will not contribute in $\alpha' \to 0$ limit. There are only three independent group theory contributions. A priori in the squared matrix element one expects interference terms but the identities (7) and (8) reduce the square to a sum only involving diagonal terms, i.e. an incoherent sum.

$$|\sum_{P\ non\ cyclic} \mathcal{M}(P)|^2 = \frac{N^3(1-N^2)}{2^5} \sum_{P\ non\ cyclic} |A(P)|^2 \tag{10}$$

Finally we turn to the polarization vectors. We use the spinor helicity basis of reference [5]. i.e.

$$\epsilon^{\mu}_{\pm}(k,p) \;=\; \pm\frac{< k_{\pm}|\gamma^{\mu}|p_{\pm} >}{\sqrt{2} < p_{\pm}|k_{\pm} >} \tag{11}$$

where $\pm$ indicate positive and negtive helicities. $k$ is the momentum of the gluon and $p$ is a reference momentum. By virtue of the gauge invariance we discussed after equation (4), this is essentially arbitrary so long as $p^2 = 0,\ \ p\cdot k \neq 0$. In our calculation we used $p = k_4$ for particles 1,2,3 and $p = k_1$ for 4 and 5. The inner products are defined by

$$< p_-|q_+ > = \; \bar{u}(p)_- \; u(q)_+ \equiv s(p,q). \tag{12}$$

$$< q_+|p_- > = \; \bar{u}(q)_+ \; u(p)_- \equiv t(p,q). \tag{13}$$

where the spinors are massless free particle spinors of the helicities indicated. These are related to products of momenta by

$$| < p_-|q_+ > |^2 \;=\; 2p\cdot q \tag{14}$$

The following identities are useful in simplifying the calculations.

$$< A_+|\gamma^{\mu}|B_+ >< C_+|\gamma^{\mu}|D_+ > = \; 2 < C_+|A_- >< B_-|D_+ > \tag{15a}$$

$$< A_+|\gamma^{\mu}|B_+ > = < B_-|\gamma^{\mu}|A_- > \tag{15b}$$

The first of these is Fierz transformation while the second follows from charge conjugation. Using $k\cdot\gamma \;=\; |k_+ >< k_+| \;+\; |k_- >< k_-|$ and $\epsilon\cdot k \;=\; 0$ we can derive a number of relations among the spinor inner products. e.g.

$$s(1,2)t(2,4) + s(1,3)t(3,4) + s(1,5)t(5,4) = 0, \quad etc. \tag{16}$$

Once a specific choice of helicities is made for the gluons, the number of terms in the amplitude gets reduced to a few (less than about ten ). The identities (15) and (16) can then be used to reduce it further to a single term. Thus for three positive helicities and two negative helicities we get

$$A(1+,2+,3+,4-,5-) = 2\sqrt{2}\frac{g^3 s(4,5)^4}{s(1,2)s(2,3)s(3,4)s(4,5)s(5,1)} \tag{17}$$

An analogous expression can be written down for the case of three negative helicity gluons. These are the only non zero cases for the five point function. (We have also put in the coupling constant in this equation.)

The example of the five point amplitude shows how higher point amplitudes may be calculated. The parametric form of the amplitude again becomes a multiple hypergeometric function. Apart from extracting the zero slope limit of this function, the computer time required is minimal. We have double checked our results using MACSYMA and SMP independently. MACSYMA takes about an hour of cpu time for the reduction of the hypergeometric functions, SMP takes about 20 minutes. For higher point amplitudes cyclic invariance and the reflection property (8) can be used to significantly cut down the amount of computer time required. These and other results will be published elsewhere.

While this manuscript was being prepared, we received a Fermilab preprint by Mangano, Parke and Xu which overlaps with our work. Although the calculation is not done using string theory, the authors show how the Feynman amplitudes can be summed up in terms of string like sub-amplitudes which have cyclic and gauge invariance.

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