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The classical amplitude for scattering of two positive helicity and $n-2$ negative helicity (or the other way) gauge bosons is shown to be generated by a Wess–Zumino–Witten (WZW) model for $N=4$ supersymmetric gauge theory; i.e. the current algebra of the WZW model (with central charge $k=1$) gives a Kac–Moody algebra as the symmetry behind these amplitudes.

Gauge theories have a deep and as yet not completely understood geometrical origin. It is therefore of some interest to identify symmetries governing their dynamics beyond the manifest gauge symmetry. Of course this might be possible only for certain sectors of the theory or for some supersymmetric extensions. The self-dual sector of non-abelian gauge theories has been shown to possess a Kac–Moody symmetry [1]. However, progress beyond this has been difficult. Recently there have been calculations of scattering amplitudes for several gauge bosons at the classical level. These are done by summing up the appropriate number of Feynman diagrams [2]. The number of diagrams for each process is quite large but the final results are very simple. One can define a set of sub-amplitudes which are gauge invariant and cyclically symmetric. The full amplitude is then the sum over non-cyclic permutations of the external lines. The sub-amplitudes correspond to the low energy limit of similar scattering amplitudes of a string theory [3]. Although these amplitudes have many of the nice properties of the string amplitudes infinite-dimensional symmetries such as the Virasoro and Kac–Moody symmetries are lost upon taking the low energy limit. In this note we make the observation that there is a Kac–Moody-type symmetry behind some of the scattering amplitudes. More specifically, the amplitude for the scattering of $n-2$ negative helicity and 2 positive helicity (or the other way) gauge bosons can be constructed in terms of the two-dimensional Wess–Zumino–Witten (WZW) model [4]. The correspondence is at the classical level and for the specific choice of polarizations indicated above. Extension to other processes seems possible at the expense of introducing non-local terms in the operator product expansion of the WZW currents. The correspondence is actually for the $N=4$ supersymmetric model. The superpartners do not contribute to scattering classically and thus the result applies to a non-supersymmetric theory also.

The relation to two-dimensional theories arises as follows. The gauge bosons are massless and thus their momenta are null vectors. Apart from an overall scale a null vector defines a two-sphere. One can construct a WZW model over this two-manifold and we show that this generates the scattering amplitudes of the four-dimensional theory. The momentum p_μ of a massless particle can be written as $p_\mu = \pi_{\dot{A}}\pi_A$ where the Weyl spinors π_A and $\pi_{\dot{A}}$ are determined by p_μ up to a phase. One example for the choice of π is

$$\pi_A = \left(\frac{k_1 - \mathrm{i}k_2}{\sqrt{k_0-k_3}}, \sqrt{k_0-k_3}\right). \tag{1}$$

The following properties are true for the spinor momenta:

$$\langle 12\rangle = \pi_1\cdot\pi_2 = \pi_A(p_1)\pi_B(p_2)\epsilon^{AB} = -\pi_2\cdot\pi_1,$$

$$p_1\cdot p_2 = |\pi_1\cdot\pi_2|^2$$

for two null vectors p_1 and p_2. The Lorentz invariant phase space measure $\mathrm{d}^3p/2p_0$ for a massless particle can be expressed as

$$\frac{\mathrm{d}^3p}{2p_0} = \frac{1}{2\mathrm{i}}\left(\pi\cdot\mathrm{d}\pi\,\mathrm{d}^2\bar{\pi} - \bar{\pi}\cdot\mathrm{d}\bar{\pi}\,\mathrm{d}^2\pi\right),$$

where $\bar{\pi}$ denotes the complex conjugate of π. The amplitude for n gauge boson scattering can be expressed as follows. Define sub-amplitudes $A(1, 2, ..., n)$ as

$$A(1,2,...,n)=(\sqrt{2})^n \mathrm{i} g^{n-2} \times \mathrm{Tr}(t^{a_1}t^{a_2}...t^{a_n}) \frac{\langle IJ\rangle^4}{\langle 12\rangle\langle 23\rangle...\langle n1\rangle}, \tag{2}$$

where I, J refer to the positive helicity bosons. t^a's are the matrix representatives of the group generators in the fundamental representation of the group. g is the gauge coupling constant. This result has been checked by explicit calculations up to $n=6$. For two negative helicity bosons it is believed to be correct for any n since it satisfies the Ward identities and the Altarelli–Parisi equations [2]. Up to $n=5$, the general amplitude has this form. For $n=6$, $(+++---)$ is also an allowed choice of helicities. The amplitude for this has the form

$$A(1,2,3,4,5,6)=\frac{1}{t_{123}s_{12}s_{23}s_{45}s_{56}}+... \tag{3}$$

up to a multiplicative constant and the sum is over various permutations of particles. $t_{123}=(p_1+p_2+p_3)^2$, $s_{12}=(p_1+p_2)^2$, etc. Our aim will be to generate amplitudes like (2) in terms of a WZW model. The full amplitude is given by the sum of (2) over the non-cyclic permutations. Using the expression given above for the phase space factor $d^3p/2p_0$ it is possible to write the transition rates and cross sections also in terms of spinor momenta.

Now apart from an overall scale, a null vector defines a two-sphere. This can also be thought of as the set of light cones around a point. Consider a point in spacetime with coordinates x_μ. The light cone is then given by $x_\mu+p_\mu\tau$, where p_μ is a null vector. The overall scale of p_μ can be absorbed by rescaling of the affine parameter τ. Thus the set of null cones around a point can be parametrized by a two-sphere. One can also consider the set of null cones as defining a bundle over compactified Minkowski space, an S^2 or CP^1 bundle. This is one of the ways of defining twistor space [5]. The spinor momenta π_A can be thought of as the homogeneous coordinates of this CP^1. The spinor has two complex numbers, the two components. The overall scale is removed as before, an overall phase can also be removed or fixed to any specific value since p_μ does not determine it. Thus one gets CP^1.

A twistor is usually defined as the doublet $Z_\alpha=(\omega_A, \pi_{\dot{A}})$ of Weyl spinors which transforms as the four-dimensional representation of SU(2, 2). The four complex numbers Z_α define $\mathbb{C}^4$. One is generally interested in functions of Z_α with degree of homogeneity zero, i.e. functions on CP^3 defined by the equivalence $Z_\alpha=\lambda Z_\alpha$, $\lambda\in\mathbb{C}$. Spacetime coordinates are defined via the identification $\omega_A=x_{A\dot{A}}\pi^{\dot{A}}$. The CP^3 defined by the Z_α's can be thought of as a CP^1 bundle over (compactified) spacetime. The fibre CP^1 is given by $\pi^{\dot{A}}$ with the identification of $\pi^{\dot{A}}$ and $\lambda\pi^{\dot{A}}$. Since we want to identify π with the spinor momentum, we are interested, in momentum space description, in functions purely of π. The Lorentz generator on such functions is

$$J_{AB}=\tfrac{1}{2}(\pi_A\partial_B+\pi_B\partial_A) \tag{4}$$

and similarly for $J_{\dot{A}\dot{B}}$, where ∂_B denotes differentiation with respect to π^B. The Pauli–Lubanski spin operator is then, with $p_{A\dot{A}}=\pi_A\pi_{\dot{A}}$,

$$S_{A\dot{A}}=\pi_A\pi_{\dot{A}}(\tfrac{1}{2}\pi^B\partial_B)=p_{A\dot{A}}(\tfrac{1}{2}\pi^B\partial_B). \tag{5}$$

This identifies the helicity operator as $(\frac{1}{2})$ the degree of homogeneity in π's. The operator corresponding to a gauge boson and its partners of momentum $p_{A\dot{A}}=\pi_A\pi_{\dot{A}}$ is given by

$$A^a(p)=\bar{\psi}t^a\psi\phi(\pi). \tag{6}$$

The index a gives the internal (color) degree of freedom. ψ and $\bar{\psi}$ denote two-dimensional fermions defined on the CP^1 with homogeneous coordinates π_A. The operator $\bar{\psi}t^a\psi$ has degree of homogeneity in π equal to -2 and thus the π-independent term in $\phi(\pi)$ describes a gauge boson of helicity -1. One can expand ϕ in powers of π to obtain higher helicities. The easiest way to do this, and in particular to stop with helicity $+1$, is to introduce four Grassmann parameters θ^{Ai}, $i=1, 2, 3, 4$, i.e., $N=4$ extended twistor space [6]. Let $\xi^i=\theta^{Ai}\pi_A$. We can then define ϕ as a polynomial in ξ:

$$\phi(\pi)=a_{-1}+\xi^i a_i+(\tfrac{1}{2})\xi^i\xi^j a_{ij} +(\tfrac{1}{6})\xi^i\xi^j\xi^k\bar{a}^l\epsilon_{ijkl}+\xi^1\xi^2\xi^3\xi^4\epsilon_{ijkl}a_{+1}. \tag{7}$$

a_{+1} and a_{-1} are free creation operators for gauge bosons of the helicities indicated and momenta corresponding to π^A. (We can take all gauge bosons to be outgoing; other amplitudes are obtained by appropriate conjugation and crossing symmetry relations.) The remaining terms in (7) can be interpreted as the fermions and scalars of the $N=4$ supersymmetric model. Notice that ϕ is very similar to the light-cone superfield of the $N=4$ theory [7].

In terms of the operator $A^a(p)$ of eq. (4), the amplitude for the scattering of n gauge bosons is given as

$$\mathscr{A}=(\sqrt{2})^n \mathrm{i}g^{n-2}$$
$$\times \int \mathrm{d}^4x\, \mathrm{d}^2\theta_1\, \mathrm{d}^2\theta_2\, \mathrm{d}^2\theta_3\, \mathrm{d}^2\theta_4 \prod_i \exp(\mathrm{i}p_i\cdot x)$$
$$\times \langle A^{a_1}(p_1)A^{a_2}(p_2)\ldots A^{a_n}(p_n)\rangle, \tag{8}$$

where the expectation values are calculated by

$$\langle \psi_r(\pi)\bar{\psi}_s(\pi')\rangle = \frac{\delta_{rs}}{\pi\cdot\pi'}, \tag{9}$$

r, s are color labels.

Consider n external gauge bosons in eq. (8). In order to saturate the Grassmann integrations of eq. (8) one has to have two positive helicity and $n-2$ negative helicity gauge bosons. It is easy to see that the amplitude (8) agrees with explicit calculations; the sub-amplitudes are correctly reproduced and the sum over permutations as well.

The above approach obviously does not give amplitudes with more than two positive helicity gauge bosons, except for the conjugate with $n-2$ positive helicity bosons. Thus for $n=6$, we can have the choice $(+++---)$. The proliferation of Grassmann numbers gives zero for this amplitude if one uses a local (in x) expression such as (8), in contradiction to the explicit calculation (3). One can obtain the latter by non-local combinations like

$$\mathscr{A}(1_+, 2_+, 3_-, J_-)\frac{1}{p_J^2}\mathscr{A}(J_+, 4_+, 5_-, 6_-)$$

with $p_J^2=(p_1+p_2+p_3)^2$. We do not yet have a systematic way of generating such terms.

We now consider the WZW model defined on CP^1. If we denote the local complex coordinate of CP^1 as z, the model is defined by currents $J^a(z)$ which have the operator product expansion (OPE) [8]

$$J^a(z)J^b(z')$$
$$=\frac{-k}{2}\frac{\delta^{ab}}{(z-z')^2}+\frac{f^{abc}J^c(z')}{(z-z')}+\ldots, \tag{10}$$

where ... denotes non-singular terms. The integer k is the central extension of the corresponding Kac–Moody algebra. Introduce homogeneous coordinates π^A for CP^1. The OPE can then be written as

$$J^a(\pi)J^b(\pi')$$
$$=\frac{-k}{2}\frac{\delta^{ab}}{(\pi\cdot\pi')^2}+\frac{f^{abc}J^c(\pi')}{(\pi\cdot\pi')}+\ldots. \tag{11}$$

We can recover the former version by defining $\pi_1=\pi_1'$ and $z=\pi_2/\pi_1$. The currents are related by $J^a(z)=J^a(\pi)\pi_1^2$. The OPE (11) has manifest projective invariance. The π's transform linearly as a doublet of the projective group SL(2, $\mathbb{C}$). In interpreting π as the spinor momentum of a massless particle, the group SL(2, $\mathbb{C}$) of projective transformations becomes the four-dimensional Lorentz group.

In constructing the amplitudes we have used free fermions on CP^1. Eq. (9) for the vacuum expectation value can be expressed as an OPE

$$\psi_r(\pi)\bar{\psi}_s(\pi')=\frac{\delta_{rs}}{\pi\cdot\pi'}+\ldots. \tag{12}$$

In the amplitudes only the currents $J^a=\bar{\psi}t^a\psi$ enter. Using (12), the OPE for these currents is seen to be identical to (11) with $k=1$. Thus we have shown that a WZW model with $k=1$ can generate the scattering amplitudes of the $N=4$ model, classically and for some choices of polarizations.

The equations of motion for the $N=4$ supersymmetric gauge theory can be shown to be equivalent to the integrability of vector bundles on supertwistor space when restricted to light-like lines, i.e., the curvature vanishes along light-like directions [9]. As in the case of other equations of motion which can be expressed as integrability conditions, e.g. instantons, sine-Gordon theory, etc., one can naturally expect an infinite-dimensional symmetry for the $N=4$ theory, at least classically. Since the curvature vanishes, the gauge potentials are pure gauge along light-like directions in supertwistor space. The algebra of gauge transformations thus determines the classical dynamics. The Kac–Moody algebra (11) with $k=1$ that we use is part of this symmetry. An interesting question that arises is whether other two-dimensional conformal theories can be used to generate amplitudes for four-dimensional theories. The OPEs for the two-dimensional theories can be easily written in terms of the homogeneous coordinates π^A of the CP^1 of the null momenta. The question whether the correlation functions can be consistently interpreted as the momentum space amplitudes in four dimensions is under investigation.

References

[1] See e.g. L. Dolan, Phys. Rep. 109 (1984) 1.
[2] S. Parke and T. Taylor, Phys. Rev. Lett. 56 (1986) 2459;
Z. Xu, D.H. Zhang and L. Chang, Nucl. Phys. B 291 (1987) 392;
F.A. Berends and W. Giele, Nucl. Phys. B 294 (1987) 700.
[3] D. Kosower, V.P. Nair and B.H. Lee, Phys. Lett. B 201 (1988) 85;
M. Mangano, S. Parke and Z. Xu, Fermilab preprint Pub-87/52-T (June 1987).
[4] E. Witten, Commun. Math. Phys. 92 (1984) 451;
A. Polyakov and P.B. Wiegmann, Phys. Lett. B 131 (1983) 121.
[5] R. Penrose and M.A.H. MacCallum, Phys. Rep. 6 (1972) 241;
R. Penrose and W. Rindler, Spinors and spacetime, Vol. 1 (Cambridge U.P., Cambridge, 1984).
[6] A. Ferber, Nucl. Phys. B 132 (1978) 55.
[7] S. Mandelstam, Nucl. Phys. B 213 (1983) 149.
[8] See e.g. V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B 247 (1984) 83;
D. Gepner and E. Witten, Nucl. Phys. B 278 (1986) 493.
[9] E. Witten, Phys. Lett. B 77 (1978) 394; Nucl. Phys. B 266 (1986) 245.