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U(1) CHERN–SIMONS THEORY AND $c=1$ CONFORMAL BLOCKS ☆

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The quantization of the U(1) Chern–Simons action in three dimensions is carried out in a coherent state representation. The wave functionals obtained are generating functionals for the current correlator blocks of two-dimensional $c=1$ rational conformal field theories, explicitly realizing a connection found by Witten.

A three-dimensional topological gauge theory with action given entirely by a Chern–Simons term was recently brought to prominence by Witten's demonstration that it provides a natural framework for setting up knot theory [1]. An essential step in the argument is the quantization of the theory in time-axial gauge on product manifolds $\mathbb{R}\times\Sigma$ for Σ compact and two-dimensional. Once the constraints due to gauge invariance are imposed, quantization of the reduced phase space produces a projectively flat vector bundle over the moduli space of complex structures of Σ. This is of considerable interest from a physical point of view because such structures are known to be at the heart of two-dimensional conformal field theory [2].

In this paper we reconsider the quantization of the three-dimensional Chern–Simons theory. We take a canonical approach, imposing the gauge constraints on the quantum wave functions rather than quantizing a constrained phase space as in ref. [1]. Our motivation for using this scheme is that it produces wave functions that are instantly recognized as generating functionals for two-dimensional current correlator blocks, thus making the connection with conformal field theory explicit. Only the U(1) Chern–Simons theory will be dealt with; this already illustrates the essential features. The non-abelian case adds certain technical complications, which we shall address elsewhere [3].

The action of the U(1) Chern–Simons theory on a three-dimensional manifold M is

$$S=-\frac{k}{4\pi}\int_{\mathrm{M}} \mathrm{d}^3x\, \epsilon^{\lambda\mu\nu}A_\lambda\,\partial_\mu A_\nu\,. \tag{1}$$

Taking M to be $\mathbb{R}\times\Sigma$, the product of (a segment of) the real line with a closed compact two-dimensional manifold Σ of genus g, and imposing the gauge $A_0=0$, this becomes

$$S=\frac{k}{4\pi}\int \mathrm{d}t\int_\Sigma \mathrm{d}^2x\,\epsilon^{ij}A_i\frac{\mathrm{d}}{\mathrm{d}t}A_j\,. \tag{2}$$

Since the lagrangian is linear in time derivatives it immediately leads to Poisson brackets

$$\{A_i(x),A_j(y)\}=-\frac{2\pi}{k}\epsilon_{ij}\delta^{(2)}(x-y)\,, \tag{3}$$

and gives a vanishing hamiltonian.

As suggested in ref. [1] we introduce a complex structure on Σ [#1], which furnishes local complex coordinates $(z,\bar z)$ in which $A=A_z\mathrm{d}z+A_{\bar z}\mathrm{d}\bar z$. In terms of A_z and $A_{\bar z}$ the Poisson brackets (3) lead to the non-vanishing quantum commutator

$$[A_z,A_{\bar z'}]=\frac{\pi}{k}\delta^{(2)}(z-z')\,. \tag{4}$$

It can be realized in the space of functionals of $A_{\bar z}$: the operator $A_{\bar z}$ is represented as multiplication by the

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[#1] See for example refs. [4,5], and references therein, for the relevant background.

variable $A_{\bar z}$ and A_z by functional differentiation, $A_z \equiv (\pi/k)(\delta/\delta A_{\bar z})$ [#2]. The classical relation $\bar{A}_z = A_{\bar z}$ persists as $A_{\bar z}^\dagger = A_{\bar z}$ with the inner product

$$(\Psi, \Psi') \equiv \int \mathscr{D}(A_z, A_{\bar z}) \exp\left(\frac{-\mathrm{i}k}{2\pi} \int \mathrm{d}^2 z\, A_z A_{\bar z}\right) \times \overline{\Psi[A_{\bar z}]}\, \Psi'[A_{\bar z}] , \tag{5}$$

the integration measure $\mathscr{D}(A_z, A_{\bar z})$ being defined by $\|\delta A\|^2 = \mathrm{i}\int \mathrm{d}^2 z\, \delta A_z\, \delta A_{\bar z}$. This is merely a functional analogue of the coherent state representation for the harmonic oscillator.

The transition from this unconstrained Hilbert space to the physical theory is achieved by imposing the residual gauge invariance of the time-axial gauge, the spatial gauge invariance on the manifold Σ. This may be done in two steps. First a subspace of admissible wave functions is selected by the requirement that the densities $\exp[(-\mathrm{i}k/2\pi)\int \mathrm{d}^2 z\, A_z A_{\bar z}] \times \overline{\Psi[A_{\bar z}]}\, \Psi'[A_{\bar z}]$ be invariant under gauge transformations. This vector space is then endowed with an inner product by restricting the integration $\int \mathscr{D}(A_z, A_{\bar z})$ to a submanifold in A-space intersecting every gauge orbit once.

We implement this program using a version of the Hodge parametrization for A. Let α_i, β_j, $i, j = 1, ..., g$, be a canonical set of closed contours in Σ generating its first homology, with intersection products $\#(\alpha_i, \alpha_j) = \#(\beta_i, \beta_j) = 0$, $\#(\alpha_i, \beta_j) = \delta_{ij}$. Let $\omega_i = \omega_i(z)\, \mathrm{d}z$ be the basis for the space of holomorphic one-forms determined by $\int_{\alpha_i} \omega_j = \delta_{ij}$ and set $\tau_{ij} = \int_{\beta_i} \omega_j$. Then also $\int \mathrm{d}^2 z\, \omega_i(z) \overline{\omega_j(z)} = -2\mathrm{i}\, \mathrm{Im}\, \tau_{ij}$; the $g \times g$ matrix τ is symmetric and has positive definite imaginary part. The $\bar\omega_i$ span the cokernel of the operator of antiholomorphic differentiation of scalars $\partial_{\bar z}$, so we may write

$$A_{\bar z}(z, \bar z) = \partial_{\bar z} \chi(z, \bar z) + \mathrm{i}\pi a \cdot (\mathrm{Im}\, \tau)^{-1} \cdot \overline{\omega(z)} , \tag{6}$$

where the g-vector a and the complex function χ are given by

$$a_i = \frac{1}{2\pi} \int \mathrm{d}^2 z\, \omega_i(z) A_{\bar z} ,$$

$$\chi(z, \bar z) = \frac{\mathrm{i}}{2} \int \mathrm{d}^2 w\, G(z, w) A_{\bar w} , \tag{7}$$

$$\partial_{\bar z} G(z, w) = \delta^{(2)}(z - w) - \overline{\omega(z)} \cdot (\mathrm{Im}\, \tau)^{-1} \cdot \omega(w) . \tag{7 cont'd}$$

The above definition determines $G(z, w)$ (up to a constant in z) to be $-(1/\pi)\partial_w \log E(z, w)$ minus its projection on the space of holomorphic one-forms in w; $E(w, z)$ is the prime form of Σ. $G(z, w)$ has the property

$$\partial_z G(z, w) = -\frac{1}{\pi} \partial_z \partial_w \log E(w, z) + \omega(z) \cdot (\mathrm{Im}\, \tau)^{-1} \cdot \omega(w) . \tag{8}$$

Gauge transformations are specified by smooth maps from Σ to U(1). Such maps have integer winding numbers around homologically non-trivial curves in Σ, and the winding numbers for arbitrary closed curves are determined by those around the primitive cycles α_i, β_j. The general form of a U(1) gauge transformation $R(z, \bar z)$ is therefore

$$R(z, \bar z) = \exp\left[\mathrm{i}\left(\lambda(z, \bar z) + \mathrm{i}\pi m \cdot (\mathrm{Im}\, \tau)^{-1} \cdot \int_{z_0}^{z} (\bar\omega - \omega) + \mathrm{i}\pi n \cdot (\mathrm{Im}\, \tau)^{-1} \cdot \int_{z_0}^{z} (\tau \cdot \bar\omega - \bar\tau \cdot \omega)\right)\right] , \tag{9}$$

where $\lambda(z, \bar z)$ is a smooth real function on Σ and n_i, $m_j \in \mathbb{Z}$. Under $A_{\bar z} \to A_{\bar z} - \mathrm{i}R^{-1} \partial_{\bar z} R$,

$$\chi(z, \bar z) \to \chi(z, \bar z) + \lambda(z, \bar z) , \quad a \to a + m + \tau \cdot n , \tag{10}$$

so a and χ are sensitive only to large and small gauge transformations, respectively; gauge transformations do not mix them. Moreover, variations in χ and a are orthogonal for the measure in (5) and the integrand factorizes:

$$\exp\left(\frac{-\mathrm{i}k}{2\pi} \int \mathrm{d}^2 z\, A_z A_{\bar z}\right) = \exp\left(\frac{-\mathrm{i}k}{2\pi} \int \mathrm{d}^2 z\, \partial_z \bar\chi \partial_{\bar z} \chi\right) \times \exp[-k\pi \bar a \cdot (\mathrm{Im}\, \tau)^{-1} \cdot a] . \tag{11}$$

[#2] We shall take k to be positive; if necessary by switching the roles of A_z and $A_{\bar z}$.

Consequently the constrained wave functions appear in product form, $\Psi[A_{\bar z}]=\Xi[\chi]\Phi[a]$.

Looking at the χ-dependence first, we note that invariance of $\exp[(-\mathrm{i}k/2\pi)\int \mathrm{d}^2z\, \partial_z\bar\chi\partial_{\bar z}\chi]\overline{\Xi[\chi]}\Xi[\chi]$ under an infinitesimal gauge transformation $\epsilon(z,\bar z)$ requires

$$\Xi[\chi+\epsilon]=\Xi[\chi]\exp\left(\frac{-\mathrm{i}k}{2\pi}\int \mathrm{d}^2z\,\epsilon\partial_z\partial_{\bar z}\chi\right). \quad (12)$$

This is solved by

$$\Xi[\chi]=\exp\left(\frac{\mathrm{i}k}{4\pi}\int \mathrm{d}^2z\,\partial_z\chi\partial_{\bar z}\chi\right), \quad (13)$$

which is invariant under finite small gauge transformations as well. Actually (12) is determined up to a character of the group of small gauge transformations; the phase choice here amounts to $\partial_{\bar z}(\delta/\delta A_{\bar z})\Psi[A_{\bar z}]=(k/\pi)\partial_z A_{\bar z}\Psi[A_{\bar z}]$, i.e. $F_{z\bar z}\Psi[A_{\bar z}]=0$ in our representation.

We turn now to the zero modes a_i. Invariance of $\exp[-k\pi\bar a(\mathrm{Im}\,\tau)^{-1}a]\overline{\Phi[a]}\Phi[a]$ under large gauge transformations can be factorized holomorphically (in a) to

$$\begin{aligned}\Phi[a+m+n\cdot\tau]=\Phi[a]\exp[&k\pi(m+n\cdot\bar\tau)\cdot(\mathrm{Im}\,\tau)^{-1}\cdot a\\ &+\tfrac{1}{2}k\pi(m+n\cdot\bar\tau)\cdot(\mathrm{Im}\,\tau)^{-1}\cdot(m+n\cdot\tau)\\ &+\mathrm{i}km\cdot n+\mathrm{i}\phi_1\cdot m+\mathrm{i}\phi_2\cdot n], \end{aligned}\quad (14)$$

where ϕ_1, ϕ_2 are g-vectors of phases, provided k is an integer; otherwise it is impossible to obtain a transformation rule that obeys the group composition rule.

A convenient basis for the corresponding vector space is obtained via identification of a commuting subset of the algebra of observables. Of the operators built out of the field A only gauge invariant ones are admissible in the constrained theory. This restriction leaves the holonomies around closed curves in Σ, $\exp(\mathrm{i}\int_{\mathrm{C}}A)$. The A-commutators imply

$$\exp\left(\mathrm{i}\int_{\mathrm{C}}A\right)\exp\left(\mathrm{i}\int_{\mathrm{C}'}A\right)=\exp\left(\frac{\pi\mathrm{i}}{k}\#(\mathrm{C},\mathrm{C}')\right)\exp\left(\mathrm{i}\int_{\mathrm{C}+\mathrm{C}'}A\right). \quad (15)$$

The effect of the holonomies in the physical subspace is found to be

$$\int_{\mathrm{C}}A\Phi[a]\Xi[\chi]=\int_{\mathrm{C}}\left(\mathrm{d}z\frac{-\mathrm{i}}{k}\omega(z)\cdot\frac{\partial}{\partial a}+\mathrm{d}\bar z\,\mathrm{i}\pi a\cdot(\mathrm{Im}\,\tau)^{-1}\cdot\overline{\omega(z)}\right)\Phi[a]\Xi[\chi]. \quad (16)$$

It is clear from this expression that the action of the operator $\exp(\mathrm{i}\int_{\mathrm{C}}A)$ in the constrained space depends only on the homology class of the curve C, so we obtain a single operator O_{pq} for every class $\sum_i(p_i\alpha_i+q_i\beta_i)$, p_i, $q_j\in\mathbb{Z}$. According to (15) these operators satisfy the algebra

$$\mathrm{O}_{pq}\mathrm{O}_{p'q'}=\exp\left(\frac{\pi\mathrm{i}}{k}(p\cdot q'-q\cdot p')\right)\mathrm{O}_{p+p',q+q'}. \quad (17)$$

A unitary implementation of the O_{pq} in the physical Hilbert space will amount to a projective unitary representation of the first integer homology group of Σ. This can be seen as a trivial example of Wigner's theorem; since the hamiltonian is zero, every O_{pq} is a symmetry. The adjoint representation of the algebra (17)

$$\mathrm{O}_{pq}\mathrm{O}_{p'q'}\mathrm{O}_{-p,-q}=\exp\left(\frac{2\pi\mathrm{i}}{k}(p\cdot q'-q\cdot p')\right)\mathrm{O}_{p'q'} \quad (18)$$

shows that the transformations induced by the O_{pq} are discrete shifts of the holonomies.

It is obvious from (17) that all the operators O_{pq} can be constructed from the finite subset consisting of the $\mathrm{O}_{e^j,0}$ and O_{0,e^j} (e^j is the unit vector in the j-direction), i.e. in terms of $\exp(\mathrm{i}\int_{\alpha_j}A)$ and $\exp(\mathrm{i}\int_{\beta_j}A)$. The $\exp(\mathrm{i}\int_{\alpha_j}A)$ are a commuting subset, the common eigenvectors of which will form a basis for the physical Hilbert space.

In view of (16) the eigenvector problem for the $\int_{\alpha_j}A$ is

$$\left(-\frac{\mathrm{i}}{k}\frac{\partial}{\partial a_j}+\mathrm{i}\pi a\cdot(\mathrm{Im}\,\tau)_j^{-1}\right)F_\lambda[a]=\lambda_jF_\lambda[a], \quad (19)$$

with λ a real vector of dimension g. It is solved by

$$F_\lambda[a]=\exp[\mathrm{i}k\lambda\cdot a+\tfrac{1}{2}k\pi a\cdot(\mathrm{Im}\,\tau)^{-1}\cdot a]. \quad (20)$$

A common eigenvector of $\exp(\mathrm{i}\int_{\alpha_j}A)$ is of the form $\sum_{v\in\mathbb{Z}^g}C_{\lambda,v}F_{\lambda+2\pi v}[a]$. Imposing (14) leads to the requirement

$$C_{\lambda,v+n}=C_{\lambda,v}\exp[\mathrm{i}k\pi n\cdot\tau\cdot n+\mathrm{i}k(\lambda+2\pi v)\cdot(m+\tau\cdot n)$$
$$-\mathrm{i}\phi_1\cdot m-\mathrm{i}\phi_2\cdot n]\,. \tag{21}$$

Inspection of the m-dependence of this relation shows that consistency requires $k\lambda$ to equal ϕ_1 modulo $2\pi\mathbb{Z}^g$, say $k\lambda=\phi_1+2\pi r$. The corresponding solutions may be compared with the theta function with characteristics α, $\beta\in\mathbb{R}^g/\mathbb{Z}^g$:

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z|\tau)=\sum_{n\in\mathbb{Z}^g}\exp[\mathrm{i}\pi(n+\alpha)\cdot\tau\cdot(n+\alpha)$$
$$+2\pi\mathrm{i}(n+\alpha)\cdot(z+\beta)] \tag{22}$$

and written (with a suitable choice of prefactor)

$$\Phi_{\phi_1,\phi_2,r}[a]=\vartheta\begin{bmatrix}\phi_1/2\pi k+r/k\\ -\phi_2/2\pi\end{bmatrix}(ka|k\tau)$$
$$\times\exp[\tfrac{1}{2}k\pi a\cdot(\mathrm{Im}\,\tau)^{-1}\cdot a]\,. \tag{23}$$

Clearly two different wave functions $\Phi_{\phi_1,\phi_2,r}$ and $\Phi_{\phi_1',\phi_2',r'}$ will be compatible in the inner product density only if $\phi_1=\phi_1'$ and $\phi_2=\phi_2'$; the physical Hilbert space is characterized by fixed values of ϕ_1 and ϕ_2. Since $\Phi_{\phi_1,\phi_2,r+ke^j}=\Phi_{\phi_1,\phi_2,r}$ this space will be k^g-dimensional. It carries an irreducible representation of the algebra (17) determined by

$$\exp\left(\mathrm{i}\int_{\alpha_j}A\right)\Phi_{\phi_1,\phi_2,r}[a]\Xi[\chi]$$
$$=\exp\left(\frac{\mathrm{i}(\phi_1+2\pi r)_j}{k}\right)\Phi_{\phi_1,\phi_2,r}[a]\Xi[\chi]\,,$$
$$\exp\left(\mathrm{i}\int_{\beta_j}A\right)\Phi_{\phi_1,\phi_2,r}[a]\Xi[\chi]$$
$$=\exp\left(\frac{\mathrm{i}(\phi_2)_j}{k}\right)\Phi_{\phi_1,\phi_2,r+1}[a]\Xi[\chi]\,. \tag{24}$$

The matrices of (17) are unitary in the basis $(\Phi_{\phi_1,\phi_2,r}\Xi)$. Obviously the phases ϕ_1 and ϕ_2 merely vary the representation in its group cohomology class.

From a geometric point of view (see e.g. ref. [6]), the large gauge transformations in (10) effectively reduce the a, $\bar{a}$ phase space M to a g-dimensional complex torus (the so-called jacobian variety); the Poisson brackets for the A's give a symplectic form on that space, $\Omega=-\mathrm{i}\pi k\,\mathrm{d}\bar{a}\cdot(\mathrm{Im}\,\tau)^{-1}\cdot\mathrm{d}a$. The "prequantum" Hilbert space consists of the smooth sections ψ of a line bundle over M with curvature Ω and an inner product given by

$$\langle 1|2\rangle=\int\psi_1^*\psi_2\,\rho(\Omega)\,, \tag{25}$$

where $\rho(\Omega)$ is the Liouville measure based on Ω. The Hilbert space is obtained by imposing a polarization condition on the ψ's, i.e. by restricting their dependence on half of the coordinates of M. Introducing a symplectic potential α for Ω (such that $\mathrm{d}\alpha=\Omega$)

$$\alpha=-\tfrac{1}{2}\pi k(\bar{a}-a)\cdot(T\cdot\mathrm{d}\bar{a}-\bar{T}\cdot\mathrm{d}a) \tag{26}$$

(where $T=(\mathrm{Im}\,\tau)^{-1}\cdot\tau\cdot(\mathrm{Im}\,\tau)^{-1}$), and defining covariant derivatives ∇ on the ψ's with α as connection, we may choose

$$\nabla_{\bar{a}}\psi=[\partial_{\bar{a}}+\tfrac{1}{2}\mathrm{i}\pi k(\bar{a}-a)\cdot T]\psi=0\,, \tag{27}$$

which can be solved as

$$\psi=\exp[-\tfrac{1}{4}\mathrm{i}\pi k(\bar{a}-a)\cdot T\cdot(\bar{a}-a)]f(a)\,. \tag{28}$$

Under a large gauge transformation $a\to a+m+\tau\cdot n$ as in (10), the connection α changes by $\mathrm{d}\Lambda_{mn}$, where

$$\Lambda_{mn}=\mathrm{i}\pi kn\cdot[\tau\cdot(\mathrm{Im}\,\tau)^{-1}\cdot\bar{a}-\bar{\tau}\cdot(\mathrm{Im}\,\tau)^{-1}\cdot a] \tag{29}$$

consequently, the transition functions for the ψ's are $\exp(\mathrm{i}\Lambda_{mn})$, $\psi(a+m+\tau\cdot n)=\exp(\mathrm{i}\Lambda_{mn})\psi(a)$. This is essentially the same as eq. (21), and leads to the theta functions, as in (23). The existence of the prequantum Hilbert space requires that the integral of Ω (representing its Chern class) be an integer, which amounts to integrality of k.

Within this framework, gauge inequivalent connections corresponding to the same Ω are parametrized by a phase for each of the homology cycles α_j and β_j; explicitly, $\alpha(\phi_1,\phi_2)=\alpha+\mathrm{d}\Lambda_0$ with

$$\Lambda_0=\tfrac{1}{2}\mathrm{i}\{(\bar{a}-a)\cdot(\mathrm{Im}\,\tau)^{-1}\cdot\phi_2$$
$$-\phi_1\cdot[\tau\cdot(\mathrm{Im}\,\tau)^{-1}\cdot\bar{a}-\bar{\tau}\cdot(\mathrm{Im}\,\tau)^{-1}\cdot a]\}\,.$$

Λ_0 is not a gauge transformation, as it is not well defined on M but only on its covering space. The transition functions now acquire additional phases $\exp(\mathrm{i}\phi_1\cdot m+\mathrm{i}\phi_2\cdot n)$ in agreement with (21).

We turn now to the inner product. The formal restriction of (5) to the χ-sector is

$$\int \mathscr{D}(\partial_z\bar{\chi}, \partial_{\bar{z}}\chi) \exp\left(\frac{-ik}{2\pi}\int d^2z\, \partial_z\bar{\chi}\partial_{\bar{z}}\chi\right)\overline{\Xi[\chi]}\Xi[\chi] .$$

Gauge fixing can be performed by insertion of $\delta(G[\chi, \bar{\chi}])\det\{G, F\}$, which makes the integral independent of the specific gauge function G. Note that the curvature F is independent of the complex structure. This leads to

$$\int \mathscr{D}(\partial_z\bar{\chi}, \partial_{\bar{z}}\chi)\, \delta(G[\chi, \bar{\chi}])\det\{G, F\}$$
$$\times \exp\left(\frac{-ik}{\pi}\int d^2z\, \partial_z \operatorname{Im}\chi\, \partial_{\bar{z}} \operatorname{Im}\chi\right) . \tag{30}$$

Gauge transformations affect the real part of χ, so a good choice for G is $\chi+\bar{\chi}$. Now $\det\{\chi+\bar{\chi}, F_{z\bar{z}}\}=1$ and the jacobian for the change of variables $\partial_z\bar{\chi}, \partial_{\bar{z}}\chi\to \operatorname{Re}\chi, \operatorname{Im}\chi$ is the determinant of the scalar laplacian, $(\det'\Delta)/\int\sqrt{g}$. (As usual, the prime and the compensating area factor $\int\sqrt{g}$ denote the omission of the zero mode. It is understood that the integral is covariantly regularized.) The remaining integral over $\operatorname{Im}\chi$ is equal to the inverse square root of the same determinant and the final result is $[(\det'\Delta)/\int\sqrt{g}]^{1/2}$.

Writing $a=\varphi+\tau\cdot\theta$, the integration measure for the zero modes is $\prod_j d\varphi_j d\theta_j$. The gauge orbit space can be obtained by restricting the range of each φ_j and θ_j to $[0, 1]$. The inner product inherited from (5) then becomes the standard inner product for theta-functions,

$$\int \prod d\varphi_j d\theta_j \exp(-2k\pi\theta\cdot\operatorname{Im}\tau\cdot\theta)$$
$$\times \overline{\vartheta\begin{bmatrix}\phi_1/2\pi k+s/k\\ -\phi_2/2\pi\end{bmatrix}(k\varphi+k\tau\cdot\theta|k\tau)}$$
$$\times \vartheta\begin{bmatrix}\phi_1/2\pi k+r/k\\ -\phi_2/2\pi\end{bmatrix}(k\varphi+k\tau\cdot\theta|k\tau) . \tag{31}$$

The φ-integrals set s equal to r and the θ-integrals may then be done via a Poisson transformation. The result is, up to a factor independent of τ, $\delta_{rs}(\det\operatorname{Im}\tau)^{-1/2}$. (Here (31) can be identified with the prequantum Hilbert space inner product (25); the exponent in (31), which is twice the imaginary part of the exponent in (28), is a Kähler potential for Ω.)

Combining the above results, we obtain in the physical Hilbert space

$$(\Phi_s\Xi, \Phi_r\Xi)=\delta_{rs}\left(\frac{\det'\Delta}{\int\sqrt{g}\det\operatorname{Im}\tau}\right)^{1/2} . \tag{32}$$

We would now like to factorize the right hand side holomorphically in τ, so that we may normalize the wave functionals. However, it has a well-known anomalous dependence on the scale of the metric g. For a particular metric in the conformal class of the complex structure (say the constant curvature one), the requisite normalizing factor, which we shall denote by $\mathscr{C}$, has a gravitational anomaly. Therefore, the normalized wave functionals

$$\Psi_r[A_{\bar{z}}]\equiv\mathscr{C}^{-1}\Xi[\chi]\Phi_r[a] , \tag{33}$$

when considered as local sections of a vector bundle over moduli space, define a connection that is only projectively flat [1,2].

As for the global aspects of this bundle, in view of the topological nature of the theory global diffeomorphisms of Σ should be implemented unitarily in the Hilbert space. Once a complex structure is chosen, the diffeomorphisms affecting the canonical homology basis are modular transformations,

$$\tau\to(A\tau+B)(C\tau+D)^{-1} , \quad a\to(C\tau+D)^{t,-1}a ,$$
$$\begin{pmatrix}D & C\\ B & A\end{pmatrix}\in \mathrm{Sp}(2g, \mathbb{Z}) . \tag{34}$$

Ξ is not affected by these operations. From the transformation properties of the $\Phi[a]$ one finds that if k is odd, it is not possible to choose the phases ϕ so as to obtain modular invariance for every genus g. For k even, modular transformations are seen to be unitary transformations of the Hilbert space provided the $(\phi_1)_j$ and $(\phi_2)_j$ equal zero.

With k and the ϕ's thus constrained, the connection with two-dimensional conformal field theory is manifest from the form of the Ψ_r themselves, since they are generating functionals for the holomorphic current blocks of $c=1$ U(1)-symmetric rational conformal field theories. These can be realized by compactified free scalar fields with rational values of the square of the radius of compactification; in the conventions of ref. [7], if $R^2=2p/q$, then $k=2pq$, with p, q integer. In particular, the $\Psi_r[A_{\bar{z}}=0]$ are characters of the relevant chiral algebras, which are U(1) current algebras extended by chiral vertex operators.

For example, on the torus, with the usual coordinates $z \equiv z+m+n\tau$ in the complex plane, $\mathscr{C}=\eta(\tau)$ and the blocks of the (unnormalized) two-current correlator are given by

$$\langle J_z J_w \rangle_r = -\tfrac{1}{4}\pi^2 \frac{\delta}{\delta A_{\bar z}} \frac{\delta}{\delta A_{\bar w}} \Psi_r[A_{\bar z}]\bigg|_{A=0}$$

$$= \tfrac{1}{4}k[\partial_z\partial_w \log E(z,w)\chi_r(\tau)$$

$$+\, 4\pi i \partial_\tau \log \eta(\tau)\chi_r(\tau) + 4\pi i \partial_\tau \chi_r(\tau)]\,,$$

$$\chi_r(\tau) = \frac{1}{\eta(\tau)}\, \vartheta\begin{bmatrix} r/k \\ 0 \end{bmatrix}(0|k\tau)\,. \tag{35}$$

From this two-dimensional point of view, the functional equation for the χ-sector, $F_{z\bar z}\Psi=0$, imposes the Ward identities of the current algebra [8] on the functional Ψ. The solution of these identities is obfuscated by the zero modes of the currents (the number of which equals the genus g). The missing information is supplied by the a-dependence of the Ψ's, which relates the current blocks to the characters occurring in the two-dimensional partition function, effectively by the Sugawara–Sommerfield construction. (This may also be understood in the language of ref. [9], where the twists in the character-valued expectation values correspond to our $\exp(2\pi i a)$.) The Chern–Simons approach naturally produces the operators considered by Verlinde [10], as observables corresponding to the holonomies.

The considerations given here for a single U(1) gauge field can be generalized to larger compact abelian groups. If an N-dimensional abelian group is realized as a quotient $\mathbb{R}^N/\Lambda$, where Λ is an N-dimensional lattice, the Chern–Simons action may be normalized such that quantization is possible only if Λ is integral and even.

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