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Motivic integration over nilpotent structures

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Abstract

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This thesis concerns developing the notion of Motivic Integration in such a way that it captures infinitesimal information yet reduces to the classical notion of motivic integration for reduced schemes. Moreover, I extend the notion of Motivic Integration from a discrete valuation ring to any complete Noetherian ring with residue field κ, where κ is any field. Schoutens' functorial approach (as opposed to the traditional model theoretic approach) allows for some very general notions of motivic integration. However, the central focus is on using this general framework to study generically smooth schemes, then non-reduced schemes, and then, finally, formal schemes. Finally, a computational approach via Sage for computing the equations defining affine arc spaces is introduced and implemented.

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Contents

Chapter 1

Introduction

This thesis is concerned with constructing a motivic integral for separated κ-schemes of finite type over the field κ, which will go by the nomenclature of *Schemic Motivic Integration*. This chapter serves as both an introduction to the field of motivic integration as a whole and the specific problem worked on in this thesis. Roughly speaking, the structure of this chapter is as follows. In §1.1, we motivate the idea of motivic integration and give a brief introduction to various types of motivic integrals. In §1.2, we go into much more detail as to rigorously describe the geometric motivic integral. In this section, we also describe the constructible motivic integral. The final section (§1.3) of this chapter contains a brief summary of the results of this thesis in the sense of the progress I made toward developing a schemic motivic integration theory.

1.1 The origins of motivic integration

The goal of this dissertation is to adequately define the notion of *schemic motivic integration over infinite arcs*. In this section, we motivate the problem by putting it in context with the history and development of the motivic integral. We use this as an opportunity to go over some of the technical knowledge necessary in order for the reader to understand the problem.

We make it clear from the start that by a variety we mean a reduced separated scheme of finite type over a field κ. We denote the full subcategory of schemes whose objects are varieties by $\mathbf{Var}_\kappa$.

1.1.1 Zeta function of a curve

Let p be a prime number and denote by $\mathbb{F}_p$ the field with p elements. The finite extensions of $\mathbb{F}_p$ are the fields with p^n elements where $n \geq 1$ which we will denote by $\mathbb{F}_{p^n}$. Lets consider the set $\mathbb{Cur}_{\mathbb{F}_p}$ of separated schemes of finite type over $\mathbb{F}_p$ of dimension 1. We introduce a counting function $\#_n : \mathbb{Cur}_{\mathbb{F}_p} \to \mathbb{N}$ for each $n \in \mathbb{N}$ by defining

$$\#_n(C) = \text{ the number of } \mathbb{F}_{p^n} - \text{rational points of } C \ . \tag{1.1.1}$$

It is natural to renormalize to obtain the function $\bar{\#}_n : \mathbb{Cur}_{\mathbb{F}_p} \to \mathbb{N}$ defined by $\bar{\#}_n(C) = \frac{\#_n(C)}{n}$. As usual, one can put this combinatorial information into a generating series. We therefore let $g_C(t)$ be the element of $\mathbb{Z}[[t]]$ defined by

$$g_C(t) := \sum_{n=1}^{\infty} \bar{\#}_n(C)t^n. \tag{1.1.2}$$

Intuitively, it seems likely that the coefficients $\bar{\#}_n(C)$ grow logarithmically as n becomes large. Hence, one would expect that $Z_C(t) := \exp(g_C(t))$ is a rational function[1]. Indeed, André Weil (cf. [We2]) proved the following:

1.1.1 Theorem. *If $C \in \mathbb{Cur}_{\mathbb{F}_p}$ is smooth and projective with genus g. Then,*

$$Z_C(t) = \frac{P(t)}{(1-t)(1-pt)} \tag{1.1.3}$$

where $P(t)$ is a polynomial in $\mathbb{Z}[t]$ of degree $2g$.

1.1.2 p-adic integration

Lets generalize the situation in §1.1.1 a little bit. Fix a prime p and let κ be a field with valuation $\mathrm{ord} : \kappa^\times \to \mathbb{Z}$. We always extend the valuation to a function $\mathrm{ord} : \kappa \to \mathbb{Z} \cup \{+\infty\}$ by defining $\mathrm{ord}(0) = +\infty$. We assume that the residue field $\Bbbk = \mathcal{O}_\kappa/\mathcal{M}_\kappa$ of κ is isomorphic to $\mathbb{F}_p$ and that κ is complete with respect to the norm defined by $|x| = p^{-ord(x)}$ for all $x \in \kappa$. This implies that $(\kappa^n, +)$ is a locally compact abelian group and thus has a unique Haar measure μ_κ^n with the property that $\mu_\kappa^n(\mathcal{O}_\kappa^n) = 1$. Suppose that $f : \kappa^n \to \kappa$ is a κ-analytic map such that

[1] For the record, $Z_C(t)$ is known as the *zeta function* of the curve C.

the fibers $A_m := (\text{ord} \circ f)^{-1}(m)$ are measurable with respect to μ_κ^n for all $m \in \mathbb{N}$. Then, we may define the *p-adic integral of f at s* to be the following sum:

$$\int_{\kappa^n} |f|^s d\mu_\kappa^n := \sum_{m \in \mathbb{N}} \mu_\kappa^n(sA_m)p^{-ms}, \tag{1.1.4}$$

where s is any complex number and whenever the summation on the right hand side converges.

Now consider a separated scheme X of finite type over $\mathbb{Z}$ and let $X_p := X \times_{\mathbb{Z}} \mathbb{F}_p$ for each prime p. In other words, X_p is the closed fiber of X over the closed point (p) in $\mathrm{Spec}(\mathbb{Z})$. Assume that there is a scheme $\bar{X}_p \to \mathrm{Spec}(\mathcal{O}_\kappa)$ such that its special fiber is isomorphic to X_p. Assume further that $\bar{X}_p$ is separated and integral over κ so that we may identify its set of κ-rational points as an algebraic subset A of κ^n for some n.

Now, consider a κ-rational point of $\bar{X}_p$. This determines a polynomial function $f : A \to \kappa$. One defines the *local p-adic zeta function* of f by

$$Z_f(s) = \int_A |f|^s d\mu_\kappa^n. \tag{1.1.5}$$

Weil proved that it is rational in [We2]. The interested reader may also care to refer to [G4].

There is a fundamental relationship between the local p-adic zeta function and another important type of generating function. As we will speak about a motivic version of this generating function in Chapter 3, we take the opportunity here to introduce it. For each $m \in \mathbb{N}$, we define

$$N_m(f) := \{x \in \mathbb{F}_{p^m}^n \mid f_m(x) = 0\} \,, \tag{1.1.6}$$

where f_m is the residue class of f modulo $(p^m) \cdot \mathbb{Z}_p$. We define the *p-adic Poincaré series* of f to be

$$P_f(t) = \sum_{m \in \mathbb{N}} \text{card}(N_m) t^m \,. \tag{1.1.7}$$

Note that this power series is related to the local p-adic zeta series of f via the equation

$$Z_f(s) = P_f(\frac{1}{p^{n+s}})(1 - p^s) + p^s. \tag{1.1.8}$$

Igusa proved this in [Ig].

Now, we restrict to the case where κ is a finite field extension of $\mathbb{Q}_p$ of ramification index m. We assume that $\bar{X}$ is smooth of relative dimension n over $\mathcal{O}_\kappa$ equipped with a gauge[2] form $\omega \in \Omega_{\bar{X}/\mathcal{O}_\kappa}$. Weil proved in [We2] that there is a global p-adic measure μ_ω on the κ-analytic manifold A determined by the κ-rational points of $\bar{X}$ with the property that

$$\int_A d\mu_\omega = \#_m(\bar{X}) p^{-nm} \,. \tag{1.1.9}$$

Here $\mathcal{O}_\kappa$ is the ring of integers of the valued field κ. Thus, we see that one of the central features of counting rational points over a finite field is p-adic integration.

[2] This means that ω generates global sections. Thus, ω generates $\Omega^n_{\bar{X}/\mathcal{O}_\kappa}$ as a trivial line bundle.

One thing that I hope will become clear to the reader in these introductory remarks is how this beautiful equation is related to the notion of a motivic measure.

1.1.3 The beginning of motivic integration

Looking at the factorization of $Z_C(t)$ in Theorem 1.1.1, we see that degree of the two polynomials in the denominator are 1 and the degree of the numerator is $2g$. These are none other than the Betti numbers of the curve (i.e., $b_0(C) = b_2(C) = 1$ and $b_1(C) = 2g$) as a topological manifold over $\mathbb{C}$. In this way, we are beginning to see that p-adic integration can be a tool for proving things about the cohomology groups of a variety. The creation of motivic integration was originally inspired by the power of this approach. In particular, we have the following theorem proved by Batyrev.

1.1.2 Theorem. *Birationally equivalent Calabi-Yau varieties over $\mathbb{C}$ have the same Betti numbers.*

The proof relies on the Weil conjectures for varieties (as stated in [We1]), the p-adic version of Hironaka's resolution of singularities, and p-adic integration (notably the change of variables formula for the p-adic integral). See [B] for reference. At the time of the proof, the situation for the Hodge numbers was unknown. Then, in 1995, Kontsevich introduced (cf. [K]) a theory which has come to be called (geometric) motivic integration in order to prove the following theorem.

1.1.3 Theorem. *Birationally equivalent Calabi-Yau varieties over $\mathbb{C}$ have the same Hodge numbers.*

What is notable about the proof of this fact, which we will present in §1.2.1, is that its geometric viewpoint makes the proof intuitive and relatively easy to understand. This is because the value of a motivic integral takes place in what one may call a poor man's motive: the Grothendieck ring of varieties over a field. We briefly introduce this object now.

1.1.4 Construction. Consider the set S of varieties over a field κ up to isomorphism – i.e., an element of S is the isomorphism class of a variety over κ. We form the free abelian group F with generating set S. Consider the subgroup N of F formed by all elements of the form $\langle X - Y\rangle - \langle X\rangle + \langle Y\rangle$ whenever Y is a locally closed subvariety of X. Here $\langle X\rangle$ denotes the isomorphism class of X. We denote the resulting quotient group by $\mathbf{Gr}(\mathbf{Var}_\kappa)$ and denote the equivalence class of a κ-variety X in $\mathbf{Gr}(\mathbf{Var}_\kappa)$ by $[X]$. There is a natural notion of multiplication in $\mathbf{Gr}(\mathbf{Var}_\kappa)$ defined by $[X]\cdot[Y] := [X \times_\kappa Y]$ which turns this group into a ring. This ring is called *the Grothendieck ring of varieties over the field κ*.

1.1.5 Enter model theory

We mentioned in §1.1.3 how Kontsevich's proof does not rely on the Weil conjectures. However, it strongly relies on Hironaka's resolution of singularities. The basic idea of the proof involves "integrating" the function $f^*\Omega_Y^{\dim Y}$ where $f : X \to Y$ is a proper birational morphism with X smooth. In fact, his construction of the motivic integral principally involves data coming from the exceptional divisor of a proper birational morphism.

Denef and Loeser began to develop the theory of motivic integration much further by introducing concepts from model theory. In particular, the notion of a *semi-algebraic subset* of the jet space $\mathcal{L}(X)$ of a variety X defined as a definable subset in the language L of Denef-Pas. Basically, L is a three sorted language with a sort for the value field $\kappa((t))$, a sort for the residue field κ, and a sort in presburger arithmetic of the value group $\mathbb{Z}$. One important fact about L is that it admits quantifier elimination. This allows one to generalize Kontsevich's construction to semi-algebraic functions from $\alpha : \mathcal{L}(X) \to \mathbb{Z}$, which in turn removes the need for the construction of the motivic integral to only involve data coming from an exceptional divisor. More generally, L has an extremely nice feature which it shares with p-adic subanalytic sets: *cell-decomposition*. The condition for definable sets in a language L to have the property of cell-decomposition can be regarded as strictly weaker than quantifier elimination. Specifically, in [DL1], Denef and Loeser proved the following change of variables formula

$$\int_{\mathcal{L}(Y)} \mathbb{L}^{-\alpha} d\mu = \int_{\mathcal{L}(X)} \mathbb{L}^{-\alpha\circ f - \mathrm{ord} f^{*}\Omega^{n}_{Y}} d\mu \tag{1.1.10}$$

whenever $f : X \to Y$ is a proper birational morphism between varieties of pure dimension n, α is a semi-algebraic function, and the characteristic of the underlying field is 0. The main difference between this change of variables formula and Kontsevich's formula is that the former does not rely on resolution of singularities. Much of the rapid progress in motivic integration has occurred through introducing model-theoretic techniques in this way.

1.1.6 Other versions of motivic integrals

The motivic integration discussed in §1.1.5 is termed geometric motivic integration. It was the first among many different types. We also have p-adic orbital integrals [TH], arithmetic motivic integrals [DL2], and motivic integrals for rigid varieties [NS], to name a few. All of these partly involve a model-theoretic approach to constructing the underlying measure. This is somewhat natural since an element of $\mathbf{Gr}(\mathbf{Var}_{\kappa})$ can be identified with a constructible subset of a variety. There is a downside to this approach however. Namely, the way that things stand currently we are more or less confined to working in pure characteristic. This is because a different set of model theoretic techniques must be established before putting motivic integration over any field on a firm foundation. This is probably the biggest open problem in motivic integration.

Schoutens had the idea to consider a functorial approach to motivic integration. For this, one has to forsake definability and hence forsake much of the properties of semi-algebraic sets. In point of fact, one could consider semi-algebraic sets as an assignment associating a field extension κ' of κ to a semi-algebraic set $A_{\kappa'}$, which is more or less the approach of Cluckers and Loeser in [CL], yet this assignment will fail to be a functor. Thus, taking the functorial point of view, Schoutens employed the analogous idea of a *sieve*. A sieve on a category C is just a subfunctor of a representable functor $\mathrm{Mor}_C(-, c)$, and from this one can define the notion of a Grothendieck topology as a collection J of sieves on C subject to certain conditions. The pair (C, J) is called a *site*. An important property of a Grothendieck topology is that the collection of sieves $J(c)$ of J which are subfunctors of $\mathrm{Mor}_C(-, c)$ form a distributive lattice. In this context, if J is a Grothendieck topology, and E is a class of natural transformations[3] between the functors in J which make it into the small category with objects coming from J and morphisms coming from E (lets denote this category by $J\{E\}$ for lack of better notation), then one can form a Grothendieck group of J with respect to E

[3]Note, that the objects of J are actually objects in $\mathbf{Pre}(C)$, the category of presheaves on C. Thus, formally speaking, we are requiring $J\{E\}$ to be a subcategory of $\mathbf{Pre}(C)$ with objects coming only from J.

in the following way.

1.1.7 Construction. Under the conditions above, we show how to construct the Grothendieck group which we will denote by $\mathbf{Gr}^E(J)$. First, we form the free abelian group F with generating set J. Next, we mod out by the equivalence relation $S \sim S'$ if and only if there is a natural isomorphism in $J\{E\}$ between S and S'. We denote the resulting quotient group by $\langle J\{E\}\rangle$ and the residue class of an object S of J by $\langle S\rangle$. Then, we form the subgroup N of $\langle J\{E\}\rangle$ formed by all elements of the form $\langle S \cup S'\rangle - \langle S\rangle - \langle S'\rangle + \langle S \cap S'\rangle$ where S and S' are elements of $\bar{J}$. We define

$$\mathbf{Gr}^E(J) := \langle J\{E\}\rangle / N \ . \tag{1.1.11}$$

This is called *the Grothendieck group of the site (C, J) with respect E.* If $J\{E\}$ has products, then this categorical product will induce a binary operation on $\mathbf{Gr}^E(J)$. If this categorical product commutes with union and intersection, then it induces a multiplicative structure which is compatible with addition. This ring will be called *the Grothendieck ring of the site (C, J) with respect E.*

One should note at this point that one does not need to assume that J is a Grothendieck topology. In fact, it is enough to assume that $J(c)$ is a distributive lattice with respect to union and intersection for each object c, that finite products exists in J, and that the products commute with union and intersection. This type of object goes under the nomenclature *motivic site* in [Sch1] and [Sch2], which is terminology that we use and extend in Chapter 4. One should also note that $J(c)$ being a distributive semi-lattice, also termed *partial motivic site*, is still enough to define this object. Even still, distributivity is not really necessary here although it is always given. In fact, usually, the motivic sites of interest, in particular, the formal motivic site studied in Chapters 2 and 3, are Grothendieck pre-topologies stable under pull-back. The formal motivic site is a Grothendieck pre-topology on $\mathbb{S}\mathbb{c}\mathbb{h}_\kappa$ which is stable under pull-back. However, we will stick to the nomenclature of motivic sites as they are more general than Grothendieck pre-topologies stable under pull-back and since the terminology is extensively used in the work of Schoutens cited above.

As we will see, the construction of the Grothendieck ring in this context partly resolves the issue of worrying about model theoretic properties of subsets of a jet space. It is perhaps one approach to take if one would like to develop a foundational theory of motivic integration in pure or mixed characteristic.

1.1.8 Finite schemic integration

We now quickly give an introduction to Schoutens' theory of finite schemic integration. Using the notation in §1.1.5, we let $C = \mathbb{S}\mathbb{c}\mathbb{h}_\kappa$ be the collection of separated schemes of finite type over a field κ. We form the Grothendieck pre-topology $\mathbb{Form}_\kappa$ on $\mathbb{Sch}_\kappa$ in the following way. First, we fix the class E for $\mathbb{Sieve}_\kappa$, where $\mathbb{Sieve}_\kappa$ is the collection of all sieves on $\mathbb{Sch}_\kappa$.

1.1.4 Definition. Given two sieves $\mathcal{X}$ and $\mathcal{Y}$, we say that a natural transformation $\nu : \mathcal{Y} \to \mathcal{X}$ is *a morphism of sieves* if given any morphism of schemes $\varphi : Z \to Y$ such that $\mathrm{im}(\varphi^\circ) \subset \mathcal{Y}$, there exists a morphism of schemes $\psi : Z \to X$ with $\mathcal{X} \subset X$ such that the following diagram commutes

$$\begin{tikzcd} Z^\circ \arrow[d, "\varphi^\circ"'] \arrow[drr, "\psi^\circ"] & & \\ \mathcal{Y} \arrow[r, "\nu"'] & \mathcal{X} \arrow[r, "\iota"'] & X^\circ \end{tikzcd}$$

where ι is the natural inclusion defining $\mathcal{X}$ as a subfunctor of $X^\circ := \mathrm{Mor}_{\mathbb{Sch}_\kappa}(-, X)$. This forms a category which we again denote by $\mathbb{Sieve}_\kappa$.

We say that a sieve $\mathcal{Y}$ is *subschemic* if it is of the form $\mathtt{Im}(\varphi^\circ)$ where $\varphi : X \to Y$ is a morphism in $\mathbb{Sch}_\kappa$. The collection of subschemic sieves satisfies the axioms of a Grothendieck pre-topology; however, from the topos-theoretic viewpoint, this collection is of limited interest due to the following theorem proved by Schoutens ([Sch1], Theorem 3.15).

1.1.5 Theorem. *Let $\nu : \mathcal{Y} \to \mathcal{X}$ be a morphism in $\mathbb{Sieve}_\kappa$ and assume that $\mathcal{X}$ and $\mathcal{Y}$ are subschemic sieves with $\mathcal{X}$ affine. Then, ν is rational – i.e., there exists a morphism $\varphi : Y \to X$ in $\mathbb{Sch}_\kappa$ such that*

$$\varphi^\circ \circ \iota = \nu \,, \tag{1.1.12}$$

where $\iota : \mathcal{Y} \hookrightarrow Y^{\circ}$ *is a natural inclusion.*

However, there is a large class of sieves which do not have this property. Recall the construction of a formal scheme. One starts with a closed subscheme Y of X with corresponding ideal sheaf $\mathcal{I}_Y$. For each $n \in \mathbb{N}$, $\mathcal{I}_Y^n$ is a quasi-coherent sheaf of ideals of $\mathcal{O}_X$. Thus, we have the closed subscheme Y_n of X determined by the ideal sheaf $\mathcal{I}_Y^n$. Then, the formal scheme of X along Y is the locally ringed topological space $\widehat{X}_Y$ which is isomorphic to $\varinjlim_{n\in\mathbb{N}} Y_n$ in the category of locally ringed spaces. This leads us to make the following definition.

1.1.6 Definition. ?? We say that a sieve $\mathcal{X}$ is *formal* if for each connected finite κ-scheme $\mathfrak{m}$, there is a subschemic sieve $\mathcal{Y}_{\mathfrak{m}} \subset \mathcal{X}$ such that the sets $\mathcal{Y}_{\mathfrak{m}}(\mathfrak{m})$ and $\mathcal{X}(\mathfrak{m})$ are equal.

In Theorem 7.8 of [Sch1], Schoutens proved that the collection of all formal sieves, denoted by $\mathbb{Form}_{\kappa}$ forms a Grothendieck pre-topology. It can be shown as well that categorical product and coproduct commute in the subcategory $\mathbb{Form}_{\kappa}$ of $\mathbb{Sieve}_{\kappa}$. Thus, we may form the Grothendieck ring a la Construction 1.1.7. We denote the resulting ring by $\mathbf{Gr}(\mathbb{Form}_{\kappa})$ and call it *the Grothendieck ring of the formal motivic site*. In Theorem 2.3.1, we show that there is a surjective ring homomorphism

$$\mathbf{Gr}(\mathbb{Form}_{\kappa}) \twoheadrightarrow \mathbf{Gr}(\mathbf{Var}_{\kappa}) . \tag{1.1.13}$$

In geometric motivic integration, we deal with the arc space $\mathcal{L}(X)$ which is the projective limit of the n-th order arc spaces[4] $\mathcal{L}_n(X)$ defined to be the separated scheme of finite type over κ representing the functor from connected κ-schemes of finite type to the category of sets

$$\mathfrak{m} \mapsto X^\circ(\mathfrak{m} \times_\kappa \operatorname{Spec}(\kappa[t]/(t^n))) \ . \tag{1.1.14}$$

It was noticed by Nash, jr. long before the inception of motivic integration that these arc spaces carry data concerning the structure of singular points on a variety, cf. [Na]. However, we intend to work in a more general context. Let $\mathbb{F}\mathtt{at}_\kappa$ be the full subcategory of $\mathbb{S}\mathtt{ch}_\kappa$ whose objects are connected finite κ-schemes. We call $\mathfrak{m} \in \mathbb{F}\mathtt{at}_\kappa$ a *fat point* over κ. All sieves $\mathcal{X}$ restrict to $\mathbb{F}\mathtt{at}_\kappa$. We will abuse notation and denote the restriction of a sieve $\mathcal{X}$ to $\mathbb{F}\mathtt{at}_\kappa$ as $\mathcal{X}$ as well. Moreover, we will denote the resulting category of all sieves $\mathcal{X}$ restricted to $\mathbb{F}\mathtt{at}_\kappa$ by $\mathbb{S}\mathtt{ieve}_\kappa$. The reason that we may perform this restriction is due to the following fact.

1.1.7 Theorem. *Let X and Y be closed subschemes contained in a separated κ-scheme Z of finite type over κ. Then, X and Y are non-isomorphic over κ if and only if there exists $\mathfrak{m} \in \mathbb{F}\mathtt{at}_\kappa$ such that $X^\circ(\mathfrak{m})$ and $Y^\circ(\mathfrak{m})$ are distinct subsets of $Z^\circ(\mathfrak{m})$.*

Proof. This is a restatement of Lemma 2.2 of [Sch1]. A proof can be found there.

[4] Typically, one only considers the reduced structure of $\mathcal{L}_n(X)$.

□

One of the insights of Schoutens was that the construction of the arc space works just as well when we replace $\mathrm{Spec}(\kappa[t]/(t^n))$ with an arbitrary fat point $\mathfrak{n}$. This leads us to define the generalized arc space of a sieve $\mathcal{X}$ along the fat point $\mathfrak{n}$ by

$$\nabla_{\mathfrak{n}}\mathcal{X}(-) := \mathcal{X}(- \times_{\kappa} \mathfrak{n}) \tag{1.1.15}$$

as a functor from $\mathbb{Fat}_{\kappa}$ to $\mathbf{Set}$. Schoutens proved in §3 of [Sch2] that if $\mathcal{X} = X^{\circ}$ for some $X \in \mathbb{Sch}_{\kappa}$ where κ is algebraically closed, then it follows that $\nabla_{\mathfrak{n}}\mathcal{X}$ is represented by an element of $\mathbb{Sch}_{\kappa}$. Thus, it follows immediately that $\nabla_{\mathfrak{n}}\mathcal{X} \in \mathbb{Sieve}_{\kappa}$ for any $\mathcal{X} \in \mathbb{Sieve}_{\kappa}$ and any $\mathfrak{n} \in \mathbb{Fat}_{\kappa}$. Moreover, Schoutens showed that if $\mathcal{X}$ is formal, then so is $\nabla_{\mathfrak{n}}\mathcal{X}$. Following the work on the geometric motivic measure, it is natural to define the (weightless) finite schemic measure to be

$$\mu_{\mathfrak{n}}(\mathcal{X}) := [\nabla_{\mathfrak{n}}\mathcal{X}]\mathbb{L}^{-\dim \nabla_{\mathfrak{n}}\mathcal{X}} \tag{1.1.16}$$

in the Grothendieck ring $\mathbf{Gr}(\mathbb{Form}_{\kappa})_{\mathbb{L}}$ where $\mathbb{L} = [\mathbb{A}^1_{\kappa}]$.

However, it should be noted that the proofs of these facts do not really depend on κ being algebraically closed. In point of fact, $\nabla_{\mathfrak{m}}X$ is nothing but the internal hom in the topos of quasi-coherent sheaves on X. Grothendieck documented in [G3], Lecture 5, Proposition 2 that $\nabla_{\mathfrak{m}}X$ will be representable in schemes as long as $\mathfrak{m}$ is a flat, projective S-scheme which is locally of finite presentation over S.

Moreover, Artin proved that if we replace the condition of projective with proper, then it will be representable in the category of Artin stacks, cf. [AR]. This also goes by the name of Weil restriction, and for extensive details, the reader may wish to consult section 7.6 of [BLR]. More specifically, It is easy to see that the argument that $\nabla_{\mathfrak{m}}\mathcal{X}$ is a formal sieve when $\mathcal{X}$ is a formal sieve goes through when κ is not algebraically closed.

1.1.9 Issues with the finite schemic measure

Philosophically speaking, one would like for the motivic measure to reduce via the surjective ring homomorphism $\mathbf{Gr}(\mathbb{Form}_{\kappa}) \twoheadrightarrow \mathbf{Gr}(\mathbf{Var}_{\kappa})$ to the geometric motivic measure. The fact that it does not highlights a few of issues with the definition of this version of the finite schemic measure:

1. It is not additive on mutually disjoint sieves.

2. $\dim\nabla_{\mathfrak{n}}\mathcal{X}$ is not well behaved – i.e., it is often the case that $\dim\nabla_{\mathfrak{n}}\mathcal{X} > \ell(\mathfrak{n})\dim\mathcal{X}$.

3. It should be a function on certain projective limits of sieves.

Schoutens side-stepped (1) by fixing the ambient space of a sieve. That is one can only integrate if first we make a choice of both a fat point $\mathfrak{n}$ and a representable sieve of X° containing the sieve in question $\mathcal{X}$. This is somewhat natural and in this case the resulting integral in [Sch2] becomes finite sums of arc spaces of certain types of subsieves X° along $\mathfrak{n}$ which are then multiplied by $\mathbb{L}^{-\ell(\mathfrak{n})\dim X}$.

Issue (3) shows that his construction forsakes any hope with specialization maps to geometric motivic integration. At the very least, one would like to define integration as a potentially infinite summation which collects data from the projective limit of sieves – i.e., to construct an infinite schemic measure. The problem of understanding to what degree one can form a proper notion of a schemic motivic integral which specializes to other types of motivic integrals is the central subject and theme of this dissertation.

1.2 Further background on motivic integration

In this section, we present some details of both geometric motivic integration and constructible motivic integration. This is important as it puts my work on schemic motivic integration into context with these two theories.

1.2.1 A proof of Kontsevich's theorem

Perhaps to both cover some necessary background and for a little extra motivation, it would be helpful to present a proof of Kontsevich's theorem. For this, assume that $f : X_1 \to X_2$ is a birational map between two Calabi-Yau varieties of dimension n. Note that, by definition, a Calabi-Yau variety is proper, so that asking questions about their Hodge numbers makes sense. Moreover, they are always assumed to be smooth. By applying Chow's lemma, we may assume that each X_i is projective (removing the compliment of a dense open set will effect nothing as we will see). Now, applying Resolution of Singularities, there exists a smooth projective variety Z and proper birational morphisms $g_i : Z \to X_i$ for $i = 1,\ 2$ such that $g_2 = f \circ g_1$. Denote by β_i the simple function $\operatorname{ord} g_i^* \Omega^n_{X_i}$. We will give a description of β_i after we finish the proof. Using the change of variables formula (Theorem 1.2.2), we have the following

$$\mu(\mathcal{L}(X_i)) = \int_{\mathcal{L}(X_i)} d\mu = \int_{\mathcal{L}(Z)} \mathbb{L}^{-\beta_i} d\mu\ . \tag{1.2.1}$$

Note that, for each i, $g_i^* \Omega^n_{X_i}$ is a trivial line bundle by assumption. Thus, they are isomorphic on some dense open subset of Z. This is enough to show that $\mu(\mathcal{L}(X_1)) = \mu(\mathcal{L}(X_2))$ because there is a general property of the geometric measure which says that we can remove semi-algebraic sets of the form $\mathcal{L}(S)$ from $\mathcal{L}(Z)$ whenever S is a closed subvariety of positive codimension (this is the same reason we can apply Chow's lemma). In the next section, we will sketch the construction of the geometric motivic measure. For now, we just state that, since X_i are smooth, $\mu(\mathcal{L}(X_i)) = [X_i]\mathbb{L}^{-n}$ for each i.

Now, the final part of the proof is showing how elements of $\mathbf{Gr}(\mathbf{Var}_\kappa)_\mathbb{L}$ relate to Hodge numbers. For this, we define a ring homomorphism P from $\mathbf{Gr}(\mathbf{Var}_\kappa)_\mathbb{L}$

to the ring $\mathbb{Z}[u, v, (uv)^{-1}]$ which sends the class $[X]$ of a variety X to its Hodge polynomial

$$P([X]) = \sum_{p,q} \dim_{\mathbb{C}}(H^q_c(X, \Omega^p_X))u^p v^q \tag{1.2.2}$$

where the subscript c represents that we take complex cohomology with compact support. By some general Hodge theory developed by Deligne (cf. [D1], [D2], and [D3]), this is truly a ring homomorphism – i.e., this map in fact factors through the Grothendieck ring of Hodge structures over $\mathbb{C}$. A nice and concise reference for this fact is Section 1 of [Po]. Note that $\mathbb{L}$ is sent to uv under this ring homomorphism. Thus, $\mathbb{L}^{-1}$ must be sent to $(uv)^{-1}$. We can multiply both sides of the equation $P([X_1]\mathbb{L}^{-n}) = P([X_2]\mathbb{L}^{-n})$ by $(uv)^n$ to get that $P([X_1]) = P([X_2])$. Clearly then the Hodge numbers are the same.

1.2.1 Remark. As the reader may realise this result may be generalised to proving that κ-equivalent varieties have the same Hodge numbers, cf. [Ito] and [Wa]. Also, it is worth mentioning here the connection of this theorem to study of mathematical physics and orbifolds – specifcally, mirror symmetry (see Chapter 7 and more specifically §3 of Chapter 7 of [J]). For example, one may show that for complete varieties, the orbifold euler characteristic and the so-called stringy euler characteristic (the one defined via motivic volumes) agree, cf. [Y]. Perhaps then, it is also worth mentioning in this context that motivic integration may also has been used to prove a general McKay Correspondence [DL4] as well as the Lie Algebra version of the Fundamental Lemma (cf., §4 of [DN] and [CHL]).

1.2.2 Construction. We now explain how β_i is defined. For any κ-variety X, there is a function $\mathrm{ord} : \mathcal{L}(X)(\kappa') \to \mathbb{Z} \cup \{\infty\}$ induced by choosing a uniformization parameter in the discrete valuation ring $\kappa[[t]]$. Here, κ' is a field extension of κ.

Now, assume that X is smooth[5] and of pure dimension n. Let $\mathcal{F}$ be a coherent sheaf equipped with a morphism of sheaves $\iota : \mathcal{F} \to \Omega^n_X$. We define a sheaf of ideals $\mathcal{I}(\mathcal{F})$ on X by choosing for sufficiently small open U of X, a local volume form dx_U on U and defining

$$\mathcal{I}(\mathcal{F})(U) := \{\sum_{i\in I} a_i \iota(s_i)/dx_U \mid s_i \in \mathcal{F}(U), a_i \in \mathcal{O}_X(U), \mathrm{card}(I) < \infty\} \ . \tag{1.2.3}$$

Now, let $f : Z \to X$ be a morphism with X still smooth. Then, there is a morphism of quasi-coherent sheaves $f^*\Omega^n_X \to \Omega^n_Z$. Since Z and X are Noetherian, this is actually a morphism of coherent sheaves. Let $\mathcal{J}$ denote the sheaf of ideals $\mathcal{I}(f^*\Omega^n_X)$ on Z. Let a be a point in $\mathcal{L}(Z)$ and let $\Bbbk_a$ be the residue field. Note that a can be identified with a unique element $\bar{a} \in \mathcal{L}(Z)(\Bbbk_a)$. In other words, there is an open subset U of $\pi_1(a) \in Z$ such that as an element of the open subset

[5] Strictly speaking, smoothness is not necessary, but the definition is notationally simpler when X is smooth.

$\mathcal{L}(U)$ of $\mathcal{L}(Z)$, a can be identified with a injective limit of ring homomorphisms $a_i : \mathcal{O}_Z(U) \to \Bbbk_a[t]/(t^i)$. Note that we may choose U to be affine and thus $\mathcal{O}_Z(U)$ is isomorphic to a finitely generated κ-algebra $\kappa[x_1, \ldots, x_m]/I$. Thus, the injective limit of these ring homomorphism can be identified with choice of substitution $x_i = p_i(t)$ where $p_i(t) \in \Bbbk_a[[t]]$ for all i. For each $g \in \mathcal{J}(U)$, we may apply this substitution to g to get an element of $\Bbbk_a[[t]]$. We denote this element by $g(a)$. Clearly then we can apply the valuation ord of $\Bbbk_a[[t]]$ to $g(a)$. Thus, we define

$$\mathrm{ord} f^*\Omega_X^n(a) := \mathrm{ord}\mathcal{J}(a) := \min_g\{\mathrm{ord}(g(a))\}\ , \tag{1.2.4}$$

where g runs over all elements of $\mathcal{J}(U)$ for small enough open set U of Z containing $\pi_1(a)$. Note that, in the above, π_1 denotes the canonical morphism from $\mathcal{L}(Z)$ to Z induced by sending t to zero at the level of coordinate rings. It is the case that the graph of $\mathrm{ord} f^*\Omega_X^n$ is determined by a semi-algebraic condition – i.e., it is a so-called simple function.

Using this construction, we have the following change of variables formula for geometric motivic integration:

1.2.2 Theorem. *Let κ be a field of characteristic zero. Let $f : X \to Y$ be a proper birational morphism of κ-varieties of pure dimension n where X is smooth. Let $\alpha : A \to \mathbb{N}$ be a simple function (i.e., a function whose fibers form a semi-*

algebraic family of semi-algebraic sets) where A is a semi-algebraic subset of $\mathcal{L}(Y)$. Then,

$$\int_A \mathbb{L}^{-\alpha} d\mu = \int_{f^{-1}(A)} \mathbb{L}^{-\alpha \circ f - \mathrm{ord} f^* \Omega_Y^n} d\mu \tag{1.2.5}$$

in the completed Grothendieck ring along the dimensional filtration[6]whenever both sides converge.

This is Lemma 3.3 of [DL1] and a proof can be found there. Note that $f^{-1}(A)$ is somewhat an abuse of notation. As we have already mentioned, removing subschemes of the arc spaces $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ of infinite codimension does not effect the value of the measure and moreover away from a choice of appropriate subschemes of infinite codimension of both $\mathcal{L}(X)$ and $\mathcal{L}(Y)$, the morphism f induces a set-theoretic bijection, which we again denote by f. Also, note that this lemma holds when X is not smooth and the function $\mathrm{ord} f^* \Omega_Y^n$ can also be defined in this case as well (cf. [DL1]).

1.2.3 Geometric motivic integrals

There is one part of the proof of Kontsevich's theorem which remains unclear – namely, the definition of integration. Let A be a semi-algebraic subset of $\mathcal{L}(X)$ where X is some κ-variety. The general construction is as follows. We can find closed subscheme S of $\mathcal{L}(X)$ of infinite codimension such that the semi-algebraic

[6]We define this in §2.3.

set $A \setminus S$ breaks down into a stratification $\{A_i\}$ where A_i is *stable*. Recall that we can throw away subsets of infinite codimension – i.e., $\mu(A) = \mu(A \setminus S)$. Thus, we define

$$\mu(A) := \sum_i \mu(A_i) \ . \tag{1.2.6}$$

Note that this is an infinite summation taking place in a certain completion of the localized Grothendieck ring $\mathbf{Gr}(\mathbf{Var}_\kappa)_{\mathbb{L}}$, where the filtration defining the completion is induced by dimension. Thus, we just need to define μ for stable semi-algebraic sets. The semi-algebraic sets A_i being stable means that for some m_i the canonical morphism $\pi^j_{m_i} : \mathcal{L}_j(X) \to \mathcal{L}_{m_i}(X)$ restricted to A_i is a piecewise trivial fibration over $\pi^j_{m_i}(A_i)$ with fiber $\mathbb{A}^{n(j-m_i)}_\kappa$ for all $j \geq m_i$ where $n = \dim X$. Here, this basically says that if we shrink X (and thereby A_i), $\pi_j(A_i)$ can be naturally identified with $\pi^j_{m_i}(A_i) \times_\kappa \mathbb{A}^{n(j-m_i)}_\kappa$. The reason this condition is important is that we define

$$\mu(A_i) := [\pi_{m_i}(A_i)]\mathbb{L}^{-nm_i} \ , \tag{1.2.7}$$

and this condition implies that it is independent of m_i. There are some technical lemmas in [DL1] which insure that we can choose the stratification in such a way that the integers m_i do not grow too rapidly, which implies that their summation over i converges. Note also that the language of Denef-Pas admits quantifier elimination. In particular, A_i being a semi-algebraic subset of $\mathcal{L}(X)$ implies that

$\pi_{m_i}(A_i)$ is a constructible subset of the reduced structure on $\mathcal{L}_n(X)$ – i.e., $\pi_{m_i}(A_i)$ corresponds to an element of $\mathbf{Gr}(\mathbf{Var}_\kappa)$ which we denote by $[\pi_{m_i}(A_i)]$.

With this in mind, one defines integration in the following manner. For any κ-variety X and a simple function $\alpha : A \to \mathbb{Z}$, the motivic integral of α is

$$\int_A \mathbb{L}^{-\alpha} d\mu := \sum_{m \in \mathbb{Z}} \mu(\alpha^{-1}(m))\mathbb{L}^{-m} , \tag{1.2.8}$$

whenever the sum on the right hand side converges. The similarity with this definition and the definition of the p-adic integral (Equation 1.1.4) is apparent. This is because they are both defined in analogy with the Lebesgue integral in classical real analysis. The situation is even more striking in that there are "specialization maps" from geometric motivic integration to p-adic integration when κ is a number field (cf. §5 of Chapter 1 of [CNS]).

1.2.4 Constructible motivic integration

As we noted in §1.2.3, the geometric motivic volume takes place in a completion $\hat{\mathcal{G}}_\kappa$ of the ring $\mathcal{G}_\kappa := \mathbf{Gr}(\mathbf{Var}_\kappa)_{\mathbb{L}}$. This is somewhat undesirable as it is currently unknown if the canonical ring homomorphism $\mathcal{G}_\kappa \to \hat{\mathcal{G}}_\kappa$ is injective. Thus, given an additive invariant of a variety, it is not immediate that knowing the motivic volume of a variety will give us information about the invariant in question. In short, there is a ring homomorphism from $\mathcal{G}_\kappa$ to the Grothendieck ring of Chow motives (in characteristic zero), but this ring homomorphism does not necessarily extend[7] to the image of $\mathcal{G}_\kappa$ in $\hat{\mathcal{G}}_\kappa$. In the proof of Kontsevich's theorem, the algebraic realization known as the category of mixed Hodge structures $\mathbf{Hdg}_{\mathbb{C}}$ is sufficiently nice to not cause any problems, but in general, we cannot assume this to be the case. Moreover, dealing with completions of rings which are highly non-Noetherian (such as the Grothendieck ring of varieties) can be difficult. In [CL], the authors found a rather simple and ingenious way to simplify this problem. Here we will outline their general approach to motivic integration known as *constructible motivic integration* as it will give the reader extra background and motivation for understanding the work in this dissertation.

As we mentioned earlier, a definable subassignment is an assignment which sends each field κ' containing κ to a subset of $\mathbb{A}^m_{\kappa'((t))} \times_{\kappa'} \mathbb{A}^n_{\kappa'} \times \mathbb{Z}^r$ for some fixed $m, n,\ r \in \mathbb{N}$ which is definable in the language of Denef-Pas. We form the category of definable subassignments $\mathbf{Def}_\kappa$ by requiring morphisms to be set maps whose graphs are again definable subassignments. A definable S-subassignment X for a definable subassignment S is a morphism $j : X \to S$ in $\mathbf{Def}_\kappa$, called the *structure morphism*, which admits an injective morphism $\iota : X \hookrightarrow S \times h[0, m', 0]$ such that j factors as $p \circ \iota$ where $p : S \times h[0, m, 0] \to S$ is the projection. Note that here, $h[n, m, r]$ denotes the definable subassignment which sends a field extension

[7]There is a well known conjecture concerning the weight structure of Chow motives whose proof would get rid of some of these concerns. One may refer to Remark 1.2.3 of [DL3] and Section 6 of [DL1]

κ' of κ to $\mathbb{A}^m_{\kappa'((t))} \times_{\kappa'} \mathbb{A}^n_{\kappa'} \times \mathbb{Z}^r$ for some fixed $m, n,\ r \in \mathbb{N}$. Note also that fiber products exist in $\mathbf{Def}_\kappa$. We denote the category of definable S-subassignments by $\mathbf{Def}_S$.

Even though definable subassignments are not functors, we can still associate a Grothendieck ring to $\mathbf{Def}_S$. This is done by forming the free abelian group generated by the isomorphism classes of elements of $\mathbf{Def}_S$ and moding out all elements of the form

$$\langle X \cup Y \to S\rangle - \langle X \to S\rangle - \langle Y \to S\rangle + \langle X \cap Y \to S\rangle \tag{1.2.9}$$

whenever X and Y are both subassingments of $S \times h[0, m', 0]$ for some fixed $m' \in \mathbb{N}$. On this group, we define multiplication as usual – i.e., via the fiber product over S. This induces a ring structure on the group. We denote the resulting ring by $\mathbf{Gr}(\mathbf{Def}_S)$ and call it the *Grothendieck ring of definable subassignments relative to* S. The Leftschetz motive relative to S is defined to be the definable S-subassignment $S \times_\kappa h[0, 1, 0]$.

As discussed earlier, our goal is to define an notion of integration without passing to a completion of this ring along some filtration. For this, we need to define the correct notion of a function from a definable subassignment to $\mathbb{Z}$. First, we define the ring

$$\mathbb{A} := \mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}, (\frac{1}{1 - \mathbb{L}^{-i}})_{i>0}]\ . \tag{1.2.10}$$

Now, we do have a notion of a morphism $\alpha : S \to h[0,0,1]$ where S is a definable subassignment. In fact, any such morphism uniquely corresponds to a set-theoretic function $\bar{\alpha} : |S| \to \mathbb{Z}$ where

$$|S| := \{(x, \kappa') \mid x \in S(\kappa'), \kappa' \text{ is a field containing } \kappa\} . \tag{1.2.11}$$

We call any such function $\bar{\alpha}$ a $\mathbb{Z}$-function. They are in one-to-one correspondence with morphisms from S to $h[0,0,1]$. Then, we may form *the ring of constructible presburger functions on* S, denoted by $\mathcal{P}(S)$, as the ring generated by elements of $\mathbb{A}$, all $\mathbb{Z}$-functions, and functions into $\mathbb{A}$ of the form $\mathbb{L}^{\bar{\alpha}}$ where $\bar{\alpha}$ is a $\mathbb{Z}$-function. Addition and multiplication is of course defined pointwise. Now, let $\mathcal{P}^0(S)$ be the ring generated by characteristic functions of definable subassignments contained in S and $\mathbb{L} - 1 \in \mathbb{A}$. Thus, in particular we have that the function $\mathbb{L} - 1 + \mathbb{1}_S$ is equal to the constant constructible presburger function $\mathbb{L}$ on S, where $\mathbb{1}_S$ denotes the characteristic function of S. Clearly, we could have chosen to use $\mathbb{L}$ instead of $\mathbb{L} - 1$, but we do not do this for philosophical reasons[8]. We want to think of $\mathbb{L}$ as being the absolute Leftschetz motive when it is an element of $\mathbb{A}$ and we want to posit that moving from the absolute case to the relative case (or, to formal schemes or even to something else entirely) is constructed in such a way that the relative Leftschetz motive is obtained from the absolute case by removing the origin in $\mathbb{A}^1_\kappa$

[8]Probably, Cluckers and Loeser chose to generate $\mathcal{P}^0(S)$ like this also because they deal with the notion of positive functions and Grothendieck semirings. In that context, their goal was most likely just to make the semiring version of all this work as effectively as possible.

and replacing it with a suitable substitute. For another example, in $\mathbf{Gr}(\mathbb{Form}_\kappa)$, Schoutens proved

$$\mathbb{L} = \mathbb{L}_* + \hat{\mathbb{L}} \tag{1.2.12}$$

where $\hat{\mathbb{L}}$ is the equivalence class of the completion of $\mathbb{A}^1_\kappa$ at the origin, $\mathbb{L}_*$ is the class of the punctured affine line $\mathbb{A}^1_\kappa \setminus \{a\}$ in $\mathbf{Gr}(\mathbb{Form}_\kappa)$, whenever κ is algebraically closed and a a closed point of the affine line. At any rate, there is a natural ring homomorphism from $\mathcal{P}^0(S)$ to $\mathbf{Gr}(\mathbf{Def}_S)$. Thus, we define *the ring of constructible motivic functions on* S to be

$$\mathcal{C}(S) := \mathbf{Gr}(\mathbf{Def}_S) \otimes_{\mathcal{P}^0(S)} \mathcal{P}(S)\ . \tag{1.2.13}$$

Laying out the complete theory involved in performing integration in this context is beyond the scope of this thesis. It will be enough to quickly highlight some important features of constructible motivic integrals.

1.2.5 Construction. The elements involved in forming a constructible motivic integral are as follows.

1. A summability condition in the value group sort.

2. Projection along residue field sort.

3. Cell-decomposition for the value field sort.

With these tools, integration of a constructible presburger function in $\mathcal{P}(X)$ over a definable subassignment S where $X = S \times h[n, m, r]$ is defined by progressively integrating through each of the variables–i.e., integrating each variable in the value group sort, then integrating each variable in the residue field sort, and then integrating each variable in the value field sort (or, vice versa, the order of integration of the variables does not matter). Note that constructible motivic integration specializes to both geometric motivic integration and arithmetic motivic integration (cf. Theorem 16.3.1 and Theorem 16.3.2 of [CL], respectively).

1.3 Summary of the results of this thesis.

In this section, I summarize what work is carried out in this dissertation. The main purpose is to describe the structure, results, unresolved questions, and future possibilities for research vis-à-vis this dissertation.

The summary of the entire thesis could be described as the development of 4 slightly different types of motivic integrals which are aimed at capturing infinitesimal information provided to us by the the language of schemes. The first of these types turns out not to be schemic in nature – in fact, it is best used when a scheme is generically smooth over a field (i.e., the scheme will automatically be reduced from the start). The main difference to this and the classical theory is that we employ Schoutens' notion of the formal motivic site in order to avoid model theoretic concerns. This is important as we also develop the notion of an motivic integral for any complete Noetherian local ring with residue field κ. This technique of constructing motivic integrals over any complete Noetherian ring is critical to capturing the infinitesimal information in the later three types of motivic integrals.

1.3.1 What is done in Chapter 2.

As mentioned in the introduction to this section, Chapter 2 is where we introduce the first of four slightly different schemic motivic measures. This first measure is denoted by $\mu_{\mathfrak{x}}$ where $\mathfrak{x}$ is the one point formal scheme whose coordinate ring could be any complete Noetherian ring with residue field κ. It is in some sense the most natural generalization of motivic integration to schemes from its classical counterpoint, yet, at the same time, it is the most naive. This is because it is quickly seen that the notion of stability, which is used in the classical theory of motivic integration, is too strong to be used when a scheme is non-reduced. Thus, in reference to the particular problem I am aiming to solve, much of this material can be skimmed; however, it should be noted that some of the most basic notions that I employ while developing the schemic theories occurring later will first occur in this chapter. Therefore, it is recommended that the reader not completely skip this chapter immediately even if they are only interested non-reduced schemes. For example, we will see that the notion of lax stability takes an extremely prominent role in our later theories.

So, in summary, we define a motivic measure $\mu_{\mathfrak{x}}$ for any $\mathfrak{x}$-stable scheme (resp. $\mu_{\mathfrak{x}}^{l}$ for any $\mathfrak{x}$-laxly stable scheme) from a boolean algebra of subsieves of the infinite arc space $\nabla_{\mathfrak{x}}X$ where $\mathfrak{x}$ is the formal scheme whose underlying topological space is a point and whose structure sheaf is any complete Noetherian ring with residue field κ. Here, κ can be any field. In particular, if a scheme is generically smooth over κ, one will be able to use the measure $\mu_{\mathfrak{x}}$ to measure the arc space $\nabla_{\mathfrak{x}}X$ without the use of model theory.

The price that is paid for this lack of model theory is that the values take place in the completion of the localized Grothendieck ring of the formal site $\mathbb{Form}_{\kappa}$. Grothendieck rings of motivic sites is an idea originally developed by Schoutens in [Sch1]. This means that a change of variables formula is not immediately forthcoming. Of course, one may easily write down some kind of change of variables formula in the schemic world when the structure sheaf of the formal scheme $\mathfrak{x}$ is $\kappa[[t]]$ so that one arrives at the original change of variables formula via a ring homomorphism $\hat{\mathcal{H}}_{\kappa} \to \hat{\mathcal{G}}_{\kappa}$ where $\hat{\mathcal{H}}_{\kappa}$ is the completion of the localized Grothendieck ring of the formal site and $\hat{\mathcal{G}}_{\kappa}$ is the completion of the localized Grothendieck ring of varieties. Here, completion takes place along the natural dimensional filtration of subgroups (an idea originally developed by Kontsevich in [K]). However, it should be noted that this result explicitly uses model theory – specifically, it relies on the proof of the change of variables formula for the classical geometric motivic integral developed by J. Denef and F. Loeser (which is proved in [DL1] using model theoretic techniques), and moreover, such a formula would really just be a restatement of this aforementioned result. It then is an interesting and open question to ask whether or not a completely functorial change of variables formula (possibly over non-linear arcs) can be proved – i.e., without the use of a model theory or, said differently, within our first theory of schemic motivic integration. We give hints toward a possible positive answer toward this in §**??** and in Chapter 4; however, we do not focus on this question as our focus turns toward developing a motivic integral which will include infinitesimal information. This brings us to our next type of motivic measure.

1.3.2 What is done in Chapter 3.

As mentioned in the previous subsection, we initially define motivic volumes $\mu_{\mathfrak{r}}$ and $\mu_{\mathfrak{r}}^{l}$ when the underlying scheme is generically smooth, and these volumes take values in the completion of the localized Grothendieck ring of the formal site which is denoted by $\hat{\mathcal{H}}_{\kappa}$. However, in order to consider the case where the underlying scheme is non-reduced, we need to adjust $\mu_{\mathfrak{r}}$ (resp., $\mu_{\mathfrak{r}}^{l}$) to obtain measures $\lambda_{\mathfrak{r}}$ (resp., $\lambda_{\mathfrak{r}}^{l}$) by first taking the reduced structure on the projections $\nabla_{\mathfrak{r}/\mathfrak{n}}X$ (where

$\nabla_{\mathfrak{x}} X$ is the infinite arc space with respect to some limit point $\mathfrak{x}$ and $\nabla_{\mathfrak{x}/\mathfrak{n}} X$ is the formal sieve determined by the image of the natural morphism $\nabla_{\mathfrak{x}} X \to \nabla_{\mathfrak{n}} X$) and then sending $(\nabla_{\mathfrak{x}/\mathfrak{n}} X)^{red}$ to its class in $\mathbf{Gr}(\mathbb{Form}_\kappa)$.

It happens often that this type of motivic volume will be equivalent to sending $[\nabla_{\mathfrak{x}/\mathfrak{n}} X]$ to its class in $\mathbf{Gr}(\mathbf{Var}_\kappa)$ via the ring homomorphism $\sigma : \mathbf{Gr}(\mathbb{Form}_\kappa) \to \mathbf{Gr}(\mathbf{Var}_\kappa)$. This leads us to speculate on the immense usefulness of this latter approach in general – i.e., it will prove beneficial in the end to have a measure which takes values in $\hat{\mathcal{G}}_\kappa$ as it will often be the case that $\hat{\sigma} \circ \lambda^l_{\mathfrak{x}}$ will be well-defined (whereas this will occur less often for just $\lambda^l_{\mathfrak{x}}$).

We then turn our attention to the difference between definability and constructibility over a non-algebraically closed field, and, for this, we define a measure $\bar{\mu}^l_{\mathfrak{x}}$ in analogy with $\mu^l_{\mathfrak{x}}$ and $\lambda^l_{\mathfrak{x}}$. We then develop the notion of integration for these measures in analogy with the definition of integration given in Chapter 2. However, the reason for our preference for $\lambda^l_{\mathfrak{x}}$ over $\bar{\mu}^l_{\mathfrak{x}}$ is more or less apparent from the very start, but we feel that it is a good idea to include this material as it might prove useful at a later date.

Let me now take a moment to display why the measure $\lambda^l_{\mathfrak{x}}$ is preferable to $\mu^l_{\mathfrak{x}}$ and $\bar{\mu}^{\mathfrak{l}}_{\mathfrak{x}}$ when constructing a theory of motivic integration for schemes which captures infinitesimal information. Assume for the moment that κ is algebraically closed. Let $\mathfrak{x}$ be an analytic germ[9] of a smooth scheme Y and let us assume that X is smooth over any fat point $\mathfrak{n} \in \mathbb{F}\mathtt{at}_{\kappa}$. Then, it will be the case that the infinite arc space $\nabla_{\mathfrak{x}} X$ has a well-defined motivic measure given by

$$\lambda^l_{\mathfrak{x}}(\nabla_{\mathfrak{x}} X) = [X^{red}]\mathbb{L}^{-\text{dim}(X)}.$$

This appears to me to be the perfect generalization of both $\mu_{\mathfrak{x}}$ (when X is smooth over κ) and of the classical motivic volume specifically.

However, we still notice that this is a little strict as we find that if $\mathfrak{n}$ is the 3rd order infinitesimal neighbourhood of the singular point of the node, $\mathfrak{x}$ is the analytic germ of the singular point of the node, and X is smooth over $\mathfrak{n}$, then one has the formula

$$\hat{\sigma} \circ \lambda^l_{\mathfrak{x}}(\nabla_{\mathfrak{x}} X) = [X^{red}]\mathbb{L}^{-\text{dim}(X)}(\hat{\sigma} \circ \mu_{\mathfrak{l}}(N))^2.$$

in $\hat{\mathcal{G}}_{\kappa}$ where $\hat{\sigma} \circ \mu_{\mathfrak{l}}$ is equivalent to the classical geometric motivic measure and N is the node. We expect this type of behaviour to happen quite often.

Moreover, another issues which is more or less left unexplored is what happens when X is merely flat over a fat point or when X^{red} has singular points. At first glance, the previous situation with the node seems to imply that it would be

[9] We prove at the beginning of Chapter 5 that $\mathfrak{x}$ is the projective limit of $J^n_O \mathbb{A}^m_{\kappa}$ over $n \in \mathbb{N}$ for some fixed $m \in \mathbb{N}$ and where O is the origin. We call such an admissible arc by the name *multilinear arc*. Thus, our theory in Chapter 3 (and in fact, in Chapter 5) works for any multilinear arc (which of course includes the *linear arc* $\mathfrak{l} = \varinjlim \mathfrak{l}_n$ where $\mathfrak{l}_n = \text{Spec}(\kappa[t]/(t^n))$ – i.e., the linear arc recovers the classical case).

reasonable to postulate that

$$\hat{\sigma}(\mu_{\mathfrak{r}}(\nabla_{\mathfrak{r}}(X^{red})) \cdot \lambda^{l}_{\mathfrak{r}}(\nabla_{\mathfrak{r}}\mathfrak{n}))$$

is the appropriate volume when X is flat over $\mathfrak{n}$. However, we include an example in Chapter 3 which seems to imply that such formulas will only work for trivial deformations of generically smooth schemes, and thus a more thorough investigation would have to take place when considering non-trivial deformations of varieties. This leads us to the topic of Chapter 4 and our next way of thinking of motivic measures.

1.3.3 What is done in Chapter 4.

As we noticed at the end of the previous subsection, it became clear that a functorial version of motivic integration as a direct analogue to geometric motivic integration will most likely encounter difficulty when X is not a trivial deformation over a fat point. However, F. Loeser and R. Cluckers have developed (cf. [CL]) a more general theory of motivic integration for varieties, which they term *Constructible Motivic Integration*. Inspired by this more general approach while working with Prof. Loeser in Paris during the academic year of $2012-2013$, I decided to mix some of their ideas, Schoutens' ideas, and some of my own ideas in order to create an extremely general framework. I believe this general framework is the best way forward when dealing with developing further motivic integration for schemes which will properly encode infinitesimal information.

One should note here that we use the notion of sheafification as philosophically speaking it does not cost us much. This process of sheafification along with refinement of covers actually yields a theory that should probably not go by the term motivic integration. However, if one has some special choice of pairwise disjoint open covers (such as in the case of constructible motivic integration where one has the notion of cell-decomposition), then by not applying sheafification (and instead just taking global sections) and refusing to employ the idea of refinement of covers, one will arrive at a theory that looks more like the theory of constructible motivic integration. Thus, one could use the term motivic integration in a reasonable way.

More specifically, there are two central parts of this theory which differ greatly from the approach taken in Chapter 2 and Chapter 3. First, given a point system $\mathbb{I}$ and something which we call a limit sieve $\mathcal{X}$ one may always consider the diagonal ring homomorphism

$$\delta : \mathbf{Gr}(\mathbb{S}\mathbb{i}\mathbb{e}\mathbb{v}\mathbb{e}_{\kappa}) \to \prod_{\sim} \mathbf{Gr}(\mathbb{S}\mathbb{i}\mathbb{e}\mathbb{v}\mathbb{e}_{\kappa})$$

where $\sim$ is some ultrafilter on $\mathbb{I}$. From this diagonal ring homomorphism, one may realize any reasonable kind of motivic measure of $\mathcal{X}$ one may wish to consider. In particular, the types of measures considered in Chapter 2 and Chapter 3 may be realised this way. Also, and perhaps more importantly, it is interesting to ask to what degree one must consider *non-standard motives* when trying to measure any limit scheme with respect to any point system. I believe that this is the best way forward with regards to developing a universel theory.

The second significant difference is our approach to functions on limit sieves and integration. Thus, defining motivic volumes for limit sieves (or specifically schemes) is in some sense orthogonal to defining the notion of a motivic integral. This is because one can realise the motivic integral in terms of pushfowards of presheaves which are locally the tensor product of the Grothendieck ring of a motivic site and a subset (i.e., the integrable functions) of some set of functions whose definition is quite general. In other words, worrying about functorial properties of functions or conic fibers or some other topos-theoretic condition is probably not the best way forward as these conditions may be too restrictive.

The main result of this section is called the main theorem, and it is heavily modelled after the main result of [CL]. We also give hints toward working with higher order integration and with perhaps extending our notion of function. However, after a certain point of working on this, I thought it best not give up the geometric versions displayed in Chapter 2 and Chapter 3 entirely as there should be much more one can say there before thinking about the more abstract theory of Chapter 4. This leads me to my forth and perhaps most interesting motivic measure and the subject of Chapter 5.

1.3.4 What is done in Chapter 5.

In Chapter 5, we outline the development of the 4th and last motivic measure. This measure appears to me to be the most interesting of the 4 measures as it attempts not only to encode the nilpotent information of the scheme but also the behaviour of a scheme as one lifts through nilpotent immersions. Of course then, this also tends to restrict what schemes one may consider (i.e., one hopes that there will not be an obstruction to such lifts). We denote this measure by $\Phi_{\mathfrak{x}}$ where $\mathfrak{x} = \varinjlim \mathbb{J}$ and $\mathbb{J} = \{J_O^n Y \mid n \in \mathbb{N}\}$, where O is a closed point of a variety Y.

The principal types of schemes we wish to apply this measure to are schemes which are affine such that their reduction is smooth over κ. Given such an affine scheme X and using well-known results in EGA IV, one can show that there exists a point system $\mathbb{J}$ where $\mathbb{J}$ is of the form given in the previous section such that for some $\mathfrak{n} \in \mathbb{J}$, X is smooth over $\mathfrak{n}$. Our strategy from there is to continue thinking about lifts (although, of course, we could then use the measure $\lambda_{\mathfrak{x}}^{l}$ for reasonably nice point systems $\mathbb{J}$). Therefore, for each smooth lift $X' \to \mathfrak{m}$ where $\mathfrak{m} \in \mathbb{I}$ where $\ell(\mathfrak{m}) \geq \ell(\mathfrak{n})$, we apply the arc operator $\nabla_{\mathfrak{m}}$ to X'. In a sense then, this could be thought of as a motivic volume of a formal scheme determined by the scheme X and the lifts of X along the point system $\mathbb{J}$. For affine schemes whose reduction is smooth, understanding such a volume comes down to understanding the formal scheme

$$(\nabla_{\mathfrak{x}}\mathfrak{x})^{red}.$$

In this chapter, we show that if Y is a κ-scheme which is smooth at a closed point O and κ is algebraically closed, or, in other words, if $\mathbb{J}$ is determined by the analytic germs of a smooth point of a variety Y over an algebraically closed field, then $(\nabla_{\mathfrak{n}}\mathfrak{n})^{red}$ is isomorphic to $\mathbb{A}_{\kappa}^{l(\mathfrak{n})}$ where $l : \mathbb{J} \to \mathbb{N}$ is some reasonably tame function. We conjecture that the converse is also true. Turning the conjecture into a theorem would be wonderful as it would give an alternative criterion for smoothness of a variety over an algebraically closed field.

One reason I feel confident the above conjecture is true is that also within this chapter I include a sage script that I coded in order to compute the arc space of any affine scheme with respect to any fat point (whose implementation is a central feature of this chapter). This allowed me to run extensive calculations which would otherwise be completely impossible to carry out by hand in any reasonable sense. Interestingly, the calculations that one can do by hand do not tell you much and are a little misleading. Only through this computational approach using Sage was I able to start to have a real feeling for what these spaces look like. In fact, I found some amazing patterns that I would not expect to exist. In particular, we provide formulas for $(\nabla_{\mathfrak{n}}\mathfrak{n})^{red}$ when Y is either the cuspidal curve or the node. In general, we conjecture that if Y is any curve, then infinite auto-arc space $(\nabla_{\mathfrak{x}}\mathfrak{x})^{red}$ is definable in the language of Denef-Pas (i.e., it is a semi-algebraic set). In particular, we conjecture if Y is an irreducible curve and O is its only singular point, then

$$(\nabla_{\mathfrak{n}}\mathfrak{n})^{red} \cong \nabla_{\mathfrak{l}_{P(n)}} Y \times_{\kappa} \mathbb{A}^r_{\kappa}$$

where r is some fixed integer and P is some linear polynomial in $\mathbb{Z}[t]$ where the nilpotency of the maximal ideal of the coordinate ring of $\mathfrak{n}$ is n.

In the future, I hope to prove this conjecture and give a concrete description of the polynomial P in terms of geometric data coming from the curve Y. A complete understanding of this conjecture will give direct formulas for the motivic volume $\Phi_{\mathfrak{x}}$ in terms of the classical motivic volume and could have important applications. In the last section, I include my sage code for computing arc spaces.

Chapter 2

Schemic motivic integration for varieties

In this chapter, we develop the theory of geometric motivic integration for generically smooth schemes over certain infinite arcs. Our construction will specialize to the theory of geometric motivic integration for varieties as developed in [DL1]. See §1.2.1 and 1.2.3 for some background on this theory. The material in this chapter is a reworking of the ideas of Kontsevich found in [K] and the ideas of Denef and Loeser found in [DL1] in lieu of Schoutens' viewpoint via Grothendieck rings of motivic sites found in [Sch1] and [Sch2].

Most of the necessary ideas carry over without issue to Schoutens' theory; however, it is not reasonable to expect a deep understanding of the points of the infinite arc space $\nabla_{\mathfrak{r}}X$ which lie above singular points of X as we cannot apply the language of Denef-Pas when $\mathfrak{r}$ is an arbitrary complete Noetherian local ring. This can be fixed by restricting ones attention to generically smooth schemes X

and removing the infinite arc space of the completion $\hat{X}_S$ of X along the singular locus S of X. This is the analogous construction as the removal of the singular locus in the classical theory.

This latter technical point does not really affect the theory and seems to me to be non-issue. This is because the crucial point of the language of Denef-Pas when it comes to the structure of infinite arc spaces is that it allows one to know that the image of the infinite arc space into the truncated arc space is a constructible set, and, thus, it has a well-defined class in the Grothendieck ring of varieties. However, this issue is fixed exactly by using Schoutens' idea of the Grothendieck ring of the formal site $\mathbf{Gr}(\mathbb{Form}_\kappa)$ which lives above the Grothendieck ring of varieties.

2.1 Admissible arcs.

The classical version of geometric motivic integration takes place over the locally ringed space

$$\mathfrak{l} := (\mathrm{Spec}(\kappa), \kappa[[t]]) ,$$

where κ is a field. Moreover, as we saw in the introduction, we would like to make use of the generalized arc operator. Thus, our first step in constructing a schemic integral is to form certain colimits of fat points, which we will call *admissible arcs* or just *arcs* for short. As a special case, we will recover the arc $\mathfrak{l}$ used in the classical theory.

With this aim in mind, we fix a field κ and let R be a complete Noetherian local ring with maximal ideal $\mathcal{M}$ and with residue field $R/\mathcal{M} = \kappa$. We define the following:

- $R_n := R/\mathcal{M}^n$ for $n \in \mathbb{N} \setminus \{0\}$.
- $\mathfrak{x}_n := (\operatorname{Spec} R_n, \mathcal{O}_{\operatorname{Spec} R_n})$.
- Let $\mathfrak{x}_{n-1} \hookrightarrow \mathfrak{x}_n$ be the closed immersion defined by the surjective ring homomorphism $R_n \twoheadrightarrow R_{n-1}$.
- Let $\mathbb{I}$ denote the resulting directed system of schemes.
- Let $\mathfrak{x} = \varinjlim \mathbb{I}$ denote the direct limit of this directed system in the category of locally ringed spaces.
- Note that $\mathfrak{x}$ is the locally ringed space $(\{x\}, \mathcal{O}_\mathfrak{x})$ where $\mathcal{O}_\mathfrak{x}(\{x\}) = R$.
- (*Working over the field* κ) We have that $\mathfrak{x}_n \in \mathbb{Sch}_\kappa$ where $\mathbb{Sch}_\kappa$ is the category of separated schemes of finite type over the field κ.
- We denote by $\mathbf{Var}_\kappa$ the full sub-category of $\mathbb{Sch}_\kappa$ whose objects are objects X of $\mathbb{Sch}_\kappa$ such that $X = X^{red}$ where X^{red} is the reduction of X. We will call an object in $\mathbf{Var}_\kappa$ a *variety*.

The category $\mathbb{Fat}_\kappa$ of *fat points over the field* κ is defined as the full sub-category of the category of $\mathbb{Sch}_\kappa$ whose objects are connected and zero dimensional.

2.1.1 Theorem. *Let* $\mathfrak{m} \in \mathbb{Sch}_\kappa$ *with* κ *algebraically closed. The following are equivalent:*

1. $\mathfrak{m} \in \mathbb{Fat}_\kappa$.
2. $\mathfrak{m}$ *is the spectrum of a local artinian ring with residue field field* κ.
3. *The underlying topological space of* $\mathfrak{m}$ *is homeomorphic to the one point topological space.*
4. $\mathfrak{m}^{red} = \operatorname{Spec} \kappa$.
5. $\mathfrak{m}$ *is isomorphic to* $J_p^n X$ *for some closed subscheme* $X = \operatorname{Spec} A$ *of* $\mathbb{A}_\kappa^g$*, where* $p \in X$ *is a closed point corresponding to a maximal ideal* $\mathcal{M}$ *of* A *and* $J_p^n X := \operatorname{Spec}(A/\mathcal{M}^n)$.

We let $\mathbf{Arc}_\kappa$ denote the full sub-category of locally ringed spaces whose objects are locally ringed spaces $\mathfrak{x}$ constructed precisely as in 2.1. We call it the *category of admissible arcs over* κ. Sometimes we will also write the objects of $\mathbf{Arc}_\kappa$ as $(\mathfrak{x}, \mathbb{I})$ when $\mathfrak{x}$ and $\mathbb{I}$ are as in 2.1. There is no danger here as $\mathfrak{x}$ uniquely determines $\mathbb{I}$ and vice versa. This is just a notational convenience.

2.1.2 Example. Let $\mathfrak{l}_n := \operatorname{Spec} \kappa[x]/(x^n)$. Then,

$$\mathfrak{l} := \varinjlim_n \mathfrak{l}_n = (\operatorname{Spec} \kappa, \mathcal{O}_{\operatorname{Spec} \kappa[[x]]}) \in \mathbf{Arc}_\kappa \ . \tag{2.1.1}$$

2.1.3 Theorem. *Every element* $(\mathfrak{x}, \mathbb{I})$ *of* $\mathbf{Arc}_\kappa$ *is isomorphic to* $(J^\infty_p X, \{J^n_p X \mid n \in \mathbb{N}\})$ *for some closed subscheme* X *of* $\mathbb{A}^g_\kappa$ *where* p *is a* κ*-rational point of* X*.*

Here $J^n_p X$ denotes the subscheme of X determined by the n-th power of the maximal ideal of p and $J^\infty_p X$ denotes the filtered colimit in locally ringed spaces (i.e., it is a formal scheme). Therefore, an immediate corollary of this theorem is that if $(\mathfrak{x}, \mathbb{I}) \in \mathbf{Arc}_\kappa$, then $\mathbb{I}$ is a collection of fat points such that the embedded dimension of every $\mathfrak{m} \in \mathbb{I}$ is bounded by some natural number g. We use this fact in the next section.

We have a set map $\ell : \mathbb{Fat}_\kappa \to \mathbb{N}$ defined by setting $\ell(\mathfrak{m})$ to be the dimension of $\mathcal{O}_\mathfrak{m}(\mathfrak{m})$ as a vector space over κ – i.e., the cardinality of the basis elements used to generate the global sections of $\mathfrak{m}$ as a vector space over κ. We call $\ell(\mathfrak{m})$ the *length of* $\mathfrak{m}$.

2.2 Arc stability for schemes.

Let $(\mathfrak{x}, \mathbb{I}) \in \mathbf{Arc}_\kappa$ and $X \in \mathbb{Sch}_\kappa$, we define $\nabla_\mathfrak{x} X$ to be the projective limit in the category of locally ringed spaces of the projective system of schemes $\{\nabla_\mathfrak{n} X \mid \mathfrak{n} \in \mathbb{I}\}$. For the definition of the truncated arc space $\nabla_\mathfrak{n} X$, the reader may consult

Equation 1.1.15 of §1.1.8. The reader may also consult §3 and §4 of [Sch2] for further details. The important fact here is that any morphism $\mathfrak{n} \to \mathfrak{m}$ of fat points induces a natural transformation of functors $\nabla_{\mathfrak{m}} \to \nabla_{\mathfrak{n}}$ so that the definition of $\nabla_{\mathfrak{x}} X$ as projective limit of schemes makes sense.

A question one may ask is the following. When will an $X \in \mathbb{S}\mathrm{ch}_\kappa$ be such that there exists a positive integer N such that for all $\mathfrak{n} \in \mathbb{I}$ with $\ell(\mathfrak{n}) \geq N$, the functor

$$(\nabla_{\mathfrak{x}/\mathfrak{n}} X)^\circ : \mathbb{F}\mathrm{at}_\kappa \to \mathbf{Set}$$

is a formal sieve. Here, $\nabla_{\mathfrak{x}/\mathfrak{n}} X$ denotes the image of $\nabla_{\mathfrak{x}} X$ in $\nabla_{\mathfrak{n}} X$ under the natural map $\pi^{\mathfrak{x}}_{\mathfrak{n}}$. Here, as usual, Y° is the functor $\mathbb{F}\mathrm{at}_\kappa \to$ Sets defined by $Y^\circ(\mathfrak{m}) = \mathrm{Mor}_{\mathbb{S}\mathrm{ch}_\kappa}(\mathfrak{m}, Y)$ whenever $Y \in \mathbb{S}\mathrm{ch}_\kappa$.

It is proven in Theorem 8.1 of [Sch2] that every $X \in \mathbb{S}\mathrm{ch}_\kappa$ is such that $(\nabla_{\mathfrak{x}/\mathfrak{n}} X)^\circ$ is a formal sieve provided that $\mathfrak{x} = J^\infty_p X$ and p is some closed point of X when the underlying field κ is algebraically closed. This is the main reason that we restrict our attention to so-called admissible arcs. It is straightforward to extend this result to the case where p is a κ-rational point of X and κ is not necessarily algebraically closed. His proof relies on artin approximation (cf., [BH]).

2.2.1 Definition. Let $X \in \mathbb{S}\mathrm{ch}_\kappa$ and let $(\mathfrak{x}, \mathbb{I}) \in \mathbf{Arc}_\kappa$. We give the following definitions.

- (*Lax Stability*) Suppose that the induced map

  $$\nabla_{\mathfrak{r}/\mathfrak{m}}X \to \nabla_{\mathfrak{r}/\mathfrak{n}}X$$

  given above is a piecewise trivial fibration with fiber $\mathbb{A}^r_\kappa$ whenever $\ell(\mathfrak{m}) \geq \ell(\mathfrak{n}) >> 0$, where $\mathfrak{m}, \mathfrak{n} \in \mathbb{I}$ and where r may depend on $\mathfrak{m}$ and $\mathfrak{n}$. In this case, we say that X is $\mathfrak{r}$-*laxly stable*.

- (*Stability*) Suppose that X is $\mathfrak{r}$-laxly stable so that the r given to us by the previous definition is of the form $r = d(\ell(\mathfrak{m}) - \ell(\mathfrak{n}))$ where $d = \dim X$. Then, we say that X is $\mathfrak{r}$-*stable*.

We may also define a stability function as follows. When X is $\mathfrak{r}$-stable, we will denote by $s_\mathfrak{r}(X)$ the minimum positive integer such that $\forall \mathfrak{m}, \mathfrak{n} \in \mathbb{I}$ with $\ell(\mathfrak{m}) \geq \ell(\mathfrak{n}) \geq s_\mathfrak{r}(X)$, it is the case that the natural map

$$\pi^\mathfrak{m}_\mathfrak{n} : \nabla_\mathfrak{m} X \to \nabla_\mathfrak{n} X \ ,$$

when its range is restricted to $\nabla_{\mathfrak{r}/\mathfrak{n}}X$, is a piecewise trivial fibration with fiber $\mathbb{A}^r_\kappa$ where $r = d(\ell(\mathfrak{m}) - \ell(\mathfrak{n}))$ and $d = \dim X$. In this case, we say that X is $\mathfrak{r}$-*stable at level* $s_\mathfrak{r}(X) - 1$. When X is not $\mathfrak{r}$-stable, we set $s_\mathfrak{r}(X) = +\infty$. Note that s is a function

$$s : \mathbb{S}\mathbb{c}\mathbb{h}_\kappa \times \mathbf{Arc}_\kappa \to \mathbb{N} \cup \{+\infty\}$$

which we call it the *stability function*. We leave it to the reader to define the *lax stability function* ls as it is defined in exactly the same way.

For each $\mathfrak{x} \in \mathbf{Arc}_\kappa$, we define the following subsets of $\mathbb{S}\mathbb{c}\mathbb{h}_\kappa$:

- The collection of all $\mathfrak{x}$-laxly-stable separated κ-schemes of finite type:

$$LStS_{\mathfrak{x}} := ls_{\mathfrak{x}}^{-1}(\mathbb{N}) \ .$$

- The collection of all $\mathfrak{x}$-stable separated κ-schemes of finite type:

$$StS_{\mathfrak{x}} := s_{\mathfrak{x}}^{-1}(\mathbb{N}) \ .$$

2.2.2 Theorem. *Let $\mathbb{S}\mathbb{m}\mathbb{S}\mathbb{c}\mathbb{h}_\kappa$ be the full subcategory of $\mathbb{S}\mathbb{c}\mathbb{h}_\kappa$ formed by smooth separated schemes of finite type over the field κ. For all $\mathfrak{x} \in \mathbf{Arc}_\kappa$ and all $X \in \mathbb{S}\mathbb{m}\mathbb{S}\mathbb{c}\mathbb{h}_\kappa$, X is $\mathfrak{x}$-stable at level 0. Hence, for all $\mathfrak{x} \in \mathbf{Arc}_\kappa$, and all $X \in \mathbb{S}\mathbb{m}\mathbb{S}\mathbb{c}\mathbb{h}_\kappa$*

$$s_{\mathfrak{x}}(X) = ls_{\mathfrak{x}}(X) = 1 \ .$$

Thus, by regarding $\mathbb{S}\mathbb{m}\mathbb{S}\mathbb{c}\mathbb{h}_\kappa$ as a set, we have $\mathbb{S}\mathbb{m}\mathbb{S}\mathbb{c}\mathbb{h}_\kappa \subset StS_{\mathfrak{x}}$ for all $\mathfrak{x} \in \mathbf{Arc}_\kappa$.

Proof. This is proved in Theorem 4.14 of [Sch2]. □

Clearly, for all $\mathfrak{x} \in \mathbf{Arc}_\kappa$,

$$\mathbb{S}\mathbb{m}\mathbb{S}\mathbb{c}\mathbb{h}_\kappa \subset StS_{\mathfrak{x}} \subset LStS_{\mathfrak{x}} \subset \mathbb{S}\mathbb{c}\mathbb{h}_\kappa \ . \tag{2.2.1}$$

2.2.3 Remark. Note that all of these notions of stability are vacuous if the arc $\mathfrak{x}$ has dimension zero. This is because $\mathfrak{x}$ would just be a fat point whose maximal ideal has nilpotency n. More specifically, in the notation of 2.1, we have $R_m = R_n$ for all $m \geq n$.

2.3 Some basic notions for schemic integration

Let $\mathcal{H}_\kappa = \mathbf{Gr}(\mathbb{Form}_\kappa)_{\mathbb{L}}$ be the localization of the Grothendieck ring of the formal motivic site at $\mathbb{L}$ and let $\mathcal{G}_\kappa = \mathbf{Gr}(\mathbf{Var}_\kappa)_{\mathbb{L}}$ be the localization of the Grothendieck ring of varieties over κ at $\mathbb{L}$. One may refer to **??** and 1.1.4, respectively, for the definition of these rings[1] We have a set-theoretic function dim from $\mathcal{H}_\kappa$ to $\mathbb{Z} \cup \{-\infty\}$ defined by sending the element $[\mathcal{X}]\mathbb{L}^{-i}$ to the integer $\dim\mathcal{X} - i$ and extending linearly through $\mathbb{Z}$. Here, $\mathcal{X}$ is of course a formal sieve, and thus has a well-defined dimension:

$$\dim\mathcal{X} := \dim(Z)$$

where Z is the scheme-theoretic image of a morphism of schemes $Y \to X$ such that there is a natural bijection from the κ-rational points of Z to $\mathcal{X}(\kappa)$. The existence of such a Z is given to us by the definition of $\mathcal{X}$ being a formal sieve.

[1]Note that there is a slight abuse of notation as $\mathbb{L}$ stands for both the class of the affine line in the Grothendieck ring of the formal site and the class of the affine line in the Grothendieck ring of varieties. There should be little cause for concern as this never seems to give rise to confusion. However, if the reader wished, they could denote the latter by $\mathbb{L}_{form}$ and reserve the symbol $\mathbb{L}$ purely for the former.

Note that this does make sense because, by Chevalley's Theorem, Z will be a constructible subset of the scheme X and thus will have a well defined dimension. Note also that we set $\dim(0) := \dim([\emptyset]) := \dim(\emptyset) := -\infty$ where $\emptyset \in \mathbb{Sieve}_\kappa$ is the empty sieve.

In an analgous way, we also have a set-theoretic function dim from $\mathcal{G}_\kappa$ to $\mathbb{Z} \cup \{-\infty\}$ albeit the notion of dim here is more straightforward[2]. Let $\hat{\mathcal{H}}_\kappa$ and $\hat{\mathcal{G}}_\kappa$ be the group completion of $\mathcal{H}_\kappa$ and $\mathcal{G}_\kappa$ with respect to the filtration of subgroups given by

$$F^m\mathcal{H}_\kappa = \{X \in \mathcal{H}_\kappa \mid \dim X < m\} \quad \text{and} \quad F^m\mathcal{G}_\kappa = \{X \in \mathcal{G}_\kappa \mid \dim X < m\}\ ,$$

respectively. Multiplication in $\mathcal{H}_\kappa$ (and in $\mathcal{G}_\kappa$) extends to the group completion. This gives both $\hat{\mathcal{H}}_\kappa$ and $\hat{\mathcal{G}}_\kappa$ the structure of a commutative unitial ring.

2.3.1 Theorem. *There is a ring homomorphism*

$$\sigma : \mathbf{Gr}(\mathbb{Form}_\kappa) \to \mathbf{Gr}(\mathbf{Var}_\kappa)\ .$$

Moreover, σ canonically induces ring homomorphisms $\sigma' : \mathcal{H}_\kappa \to \mathcal{G}_\kappa$ and $\hat{\sigma} : \hat{\mathcal{H}}_\kappa \to \hat{\mathcal{G}}_\kappa$. The ring homomorphism $\hat{\sigma}$ is a continuous ring homomorphism of topological rings.

Proof. The case when the underlying field is algebraically closed is done in the proof of Theorem 7.7 of [Sch1]. That argument only depends on the the field being

[2]It is given by the maximum of the dimensions of the irreducible components of the variety.

algebraically closed because Schoutens uses a weak form of Chevalley's theorem to insure that $\mathcal{X}(\kappa) = \text{im}(Y \to X)(\kappa)$ is a constructible subset of $X(\kappa)$. This result was generalized by Grothendieck in [G1] Theorem 1.8.4 to all morphisms of finite presentation between quasi-compact quasi-separated schemes[3]. However, all morphisms in $\mathbb{S}\mathrm{c}\mathbb{h}_\kappa$ satisfy this hypothesis regardless of the ground field. Thus, Schoutens' proof can be adapted to the case where the underlying field κ is not algebraically closed.

In this vein, let $g : Y \to X$ be the morphism of schemes such that $\text{im}(g)(\kappa) = \mathcal{X}(\kappa)$. By the work in Schoutens' proof this is uniquely determined over any field. Moreover, by the generalized Chevalley's theorem above, we have that $\text{im}(g)$ is a constructible subset of the scheme X. Then, we define

$$\sigma([\mathcal{X}]) = [f^{-1}(\text{im}(g))]$$

where f is the reduction map from $X^{red} \to X$. Note that by 1.8.2 of [G1], $f^{-1}(\text{im}(g))$ is indeed a constructible subset of the variety X^{red}.

The rest follows from basic facts concerning localization and completion as functors. The reader could consult [AM] for these facts. □

2.3.2 Remark. It is important to note here that when κ is algebraically closed,

[3] Here, one needs the conditions of quasi-compact and quasi-separated only on X as Y is constructible. See (1.8.1) of [G1] for the details.

$f^{-1}(\mathrm{im}(g))$ can be identified with $\mathrm{im}(g)(\kappa)$. In other words, Schoutens' ring homomorphism and the ring homomorphism agree over an algebraically closed field.

To continue the line of thought in the above proof, one may notice that there are more ring homomorphisms between these two Grothendieck groups. In point of fact, for each fat point $\mathfrak{n} \in \mathbb{Fat}_\kappa$, we have a ring homomorphism

$$\sigma_\mathfrak{n} : \mathbf{Gr}(\mathbb{Form}_\kappa) \to \mathbf{Gr}(\mathbf{Var}_\kappa)$$

defined by sending the class of a formal sieve $[\mathcal{X}]$ to the class of the reduced scheme structure on the image of an associated (in the sense that it comes from the definition of a formal sieve) morphism of schemes $g_\mathfrak{n} : Y \to X$ such that $\mathrm{im}(g_\mathfrak{n})(\mathfrak{n}) = \mathcal{X}(\mathfrak{n})$. Again, by Grothendieck's version of Chevalley's theorem and naturality, this is a well-defined set-map. By the work in Schoutens' proof, it will be a ring homomorphism. Clearly then, these ring homomorphisms induce ring homomorphisms between the localized Grothendieck rings and the completed Grothendieck rings as before. In practice, we will not use these ring homomorphisms, but it should be noted that when considering a non-algebraically closed field, it is helpful to keep in mind that we do in fact need all of these maps because there will be sieves for which $\mathcal{X}(\mathfrak{n})$ is the empty set, yet $\mathcal{X}$ will not be the empty sieve (this cannot happen when κ is algebraically closed). Regardless, for any formal sieve, we do have the following formula,

$$\sigma_{\mathrm{Spec}(\kappa)}([\nabla_{\mathfrak{n}}\mathcal{X}]) = \sigma_{\mathfrak{n}}([\mathcal{X}])$$

where $\mathfrak{n}$ is any fat point. Thus, perhaps, a global view of motivic integration would be as the study of projective limits (or more generally ultra-products) of the ring homomorphisms $\sigma_{\mathfrak{n}}$ where each term in the limit is appropriately renormalized. If one wished, one could put a grading on the respective Grothendieck groups by dimension and make this more precise. However, this chapter and the next focus on more concrete aspects to motivic integration. In Chapter 4, we dabble with pushing the theory of motivic integration further, yet our end results there is a somewhat new theory.

2.3.3 Remark. Another idea that one could embark on here is the fact that $\mathcal{H}_\kappa$ is the minimal localised ring which will work for our purposes. In other words, letting $\mathcal{A}_\kappa := S^{-1}\mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa)$ where $S = \{a\mathbb{L}^i \mid \sigma(a) = 1, i \in \mathbb{N}\}$, we arrive at a factorization $\mathcal{H}_\kappa \to \mathcal{A}_\kappa \to \mathcal{G}_\kappa$ of σ', and moreover, this induces a factorization

$$\hat{\mathcal{H}}_\kappa \to \hat{\mathcal{A}}_\kappa \to \hat{\mathcal{G}}_\kappa$$

of $\hat{\sigma}$ in the category of topological rings. Here, the filtration on $\mathcal{A}_\kappa$ defining $\hat{\mathcal{A}}_\kappa$ is induced by dimension in exactly the same way as for $\hat{\mathcal{H}}_\kappa$. However, we only make use of $\hat{\mathcal{H}}_\kappa$, and in the end, $\hat{\mathcal{G}}_\kappa$ will take prominence.

2.3.4 Definition. We define a map of sets $\mu : StS_{\mathfrak{r}} \times \mathbf{Arc}_{\kappa} \to \hat{\mathcal{H}}_{\kappa}$ by

$$\mu(X, \mathfrak{r}) := \mu_{\mathfrak{r}}(\nabla_{\mathfrak{r}} X) := [(\nabla_{\mathfrak{r}/\mathfrak{n}} X)^{\circ}]\mathbb{L}^{-s_{\mathfrak{r}}(X)\dim X}$$

where $\mathfrak{n} \in \mathbb{I}$ is such that $\ell(\mathfrak{n}) = s_{\mathfrak{r}}(X)$. We call $\mu_{\mathfrak{r}}(\nabla_{\mathfrak{r}} X)$ the *stable motivic* $\mathfrak{r}$*-volume of* X.

By a *function* from $\alpha : (\nabla_{\mathfrak{r}} X)^{\circ} \to \mathbb{N} \cup \{+\infty\}$, we mean an assignment (in general not a functor) which associates to each $\mathfrak{m} \in \mathbb{F}\mathrm{at}_{\kappa}$ a set-theoretic function from $(\nabla_{\mathfrak{r}} X)^{\circ}(\mathfrak{m}) \to \mathbb{N} \cup \{+\infty\}$. As $(\nabla_{\mathfrak{r}} X)^{\circ}$ is itself represented by a scheme, we will often write $\nabla_{\mathfrak{r}} X$ for $(\nabla_{\mathfrak{r}} X)^{\circ}$.

2.3.5 Definition. Let X be an element of $StS_{\mathfrak{r}}$ (resp., and $LStS_{\mathfrak{r}}$). Let $\alpha : \nabla_{\mathfrak{r}} X \to \mathbb{N} \cup \{+\infty\}$ be a function such that for all $\mathfrak{m} \in \mathbb{I}$ with $\ell(\mathfrak{m}) \geq s_{\mathfrak{r}}(X)$ (resp., $\ell(\mathfrak{m}) \geq LStS_{\mathfrak{r}}$) the subsieve $\alpha^{-1}(n)$ of $\nabla_{\mathfrak{r}} X$ is such that $\pi^{\mathfrak{r}}_{\mathfrak{m}}(\alpha^{-1}(n))$ is a formal sieve. In this case, we say that α is a $\mathfrak{r}$*-stable function* (resp., $\mathfrak{r}$*-laxly stable function*).

2.3.6 Definition. Let A be a subsieve of $\nabla_{\mathfrak{r}} X$. Suppose that the characteristic function of A defined by $I_A : \nabla_{\mathfrak{r}} X \to \mathbb{N} \cup \{+\infty\}$ defined by

$$I_A(a) = \begin{cases} 1 & \text{whenever } a \in A \\ 0 & \text{otherwise} \end{cases}$$

is $\mathfrak{r}$-stable (resp., $\mathfrak{r}$-laxly stable). In this case, we say that A is $\mathfrak{r}$*-stable* (resp., $\mathfrak{r}$*-laxly-stable*).

2.3.7 Theorem. *Assume that X is $\mathfrak{x}$-stable where $(\mathfrak{x}, \mathbb{I}) \in \mathbf{Arc}_\kappa$. Let S be a closed subscheme of X. Let $\hat{X}_S$ be the formal completion of X along S. Then, $\hat{X}_S$ is $\mathfrak{x}$-laxly stable, $ls_{\mathfrak{x}}(\hat{X}_S) = s_{\mathfrak{x}}(X)$, and hence $\nabla_{\mathfrak{x}}\hat{X}_S$ is a $\mathfrak{x}$-stable subset of $\nabla_{\mathfrak{x}}X$.*

Proof. Since X is $\mathfrak{x}$-stable, there exists an $\mathfrak{n} \in \mathbb{I}$ such that $\nabla_{\mathfrak{x}/\mathfrak{m}}X \to \nabla_{\mathfrak{x}/\mathfrak{n}}X$ is a piecewise trivial fibration for all $\mathfrak{m} \geq \mathfrak{n}$ in $\mathbb{I}$ with general fiber $\mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-\ell(\mathfrak{n}))}$ where d is the dimension of X. By Theorem 4.4 of [Sch2], we may cover X by a finite collection of opens U_i such that

$$\nabla_{\mathfrak{x}/\mathfrak{m}}U_i \cong \nabla_{\mathfrak{x}/\mathfrak{n}}U_i \times_\kappa \mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-\ell(\mathfrak{n}))} .$$

By Lemma 4.9 of loc. cit., we have that

$$\begin{aligned} \nabla_{\mathfrak{x}/\mathfrak{m}}(U_i)_{U_i \cap S} &\cong (U_i)_{U_i \cap S} \times_\kappa \nabla_{\mathfrak{x}/\mathfrak{n}}U_i \times_\kappa \mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-\ell(\mathfrak{n}))} \\ &\cong \nabla_{\mathfrak{x}/\mathfrak{n}}(U_i)_{U_i \cap S} \times_\kappa \mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-\ell(\mathfrak{n}))} \end{aligned} \tag{2.3.1}$$

Therefore, the natural morphism $\nabla_{\mathfrak{x}/\mathfrak{m}}X_S \to \nabla_{\mathfrak{x}/\mathfrak{n}}X_S$ is a piecewise trivial fibration with general fiber $\mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-\ell(\mathfrak{n}))}$ for all $\mathfrak{m} \geq \mathfrak{n}$. Thus, $\hat{X}_S$ is $\mathfrak{x}$-laxly-stable and $ls_{\mathfrak{x}}(\hat{X}_S) = s_{\mathfrak{x}}(X)$. □

2.4 The induced measure on generalized arc spaces.

In this section, we introduce the motivic measure $\mu_{\mathfrak{x}}$ by first showing how $\mathfrak{x}$-stable schemes have a natural measure and then generalizing the measure to more general notions. I do not know if there are non-smooth schemes which are $\mathfrak{x}$-stable, but it is highly unlikely. The reason that we introduce our motivic measure in this way is so that the reader will see how we arrived at our notion of a motivic measure in more general settings.

2.4.1 Theorem. *Let X be $\mathfrak{r}$-stable and let $\mathbb{B}_{\mathfrak{r}}^X$ be the collection of all subsieves of $\nabla_{\mathfrak{r}} X$ whose truncations under $\pi_{\mathfrak{n}}$ are formal sieves for all $\mathfrak{n} \in \mathbb{I}$ where $\mathfrak{r} = \varinjlim \mathbb{I}$. We have a well-defined set map $\mu_{\mathfrak{r}} : \mathbb{B}_{\mathfrak{r}}^X \to \hat{\mathcal{H}}_\kappa$ which is defined by*

$$\mu_{\mathfrak{r}}(A) := [\pi_{\mathfrak{n}}^{\mathfrak{r}}(A)]\mathbb{L}^{-\ell(\mathfrak{n})dim X} \in \hat{\mathcal{H}}_\kappa$$

for any $\mathfrak{n} \in \mathbb{I}$ such that $\ell(\mathfrak{n}) \geq S_{\mathfrak{r}}(X)$. Moreover, given a countably infinite collection $\{A_i\}$ of mutually disjoint elements of $\mathbb{B}_{\mathfrak{r}}^X$, we may define

$$\mu_{\mathfrak{r}}(\bigsqcup_i A_i) := \sum_i \mu_{\mathfrak{r}}(A_i)$$

provided that the summation converges in $\hat{\mathcal{H}}_\kappa$.

Proof. The first claim simply follows from the definition of $\mathfrak{r}$-stability, and in fact, so does the second claim proved that we show that the definition is independent of representation. The fact that it is independent of representation is extremely basic. Indeed, let $A = \sqcup A_i = \sqcup B_i$ be two mutually disjoint representations defining some functor $A : \mathbb{Fat}_\kappa \to \mathbf{Sets}$ where $\{A_i\}$ and $\{B_i\}$ are two families of mutually disjoint $\mathfrak{x}$-stable subsieves of $\nabla_{\mathfrak{x}}X$, where X is $\mathfrak{x}$-stable. Then,

$$\begin{aligned}\mu_{\mathfrak{x}}(\sqcup_i A_i) &= \sum_j \mu_{\mathfrak{x}}(\sqcup_i (A_i \cap B_j)) \\ &= \sum_{i,j} \mu_{\mathfrak{x}}(A_i \cap B_j) \\ &= \sum_i \mu_{\mathfrak{x}}(\sqcup_j (A_i \cap B_j)) \\ &= \mu_{\mathfrak{x}}(\sqcup_j B_j)\end{aligned}$$

which completes the proof as the summation $\mu_{\mathfrak{x}}(\sqcup_i A_i)$ will converge precisely when the summation $\mu_{\mathfrak{x}}(\sqcup_j B_j)$ converges. □

2.4.2 Remark. The fact that that the measure will be additive through disjoint union is a fact that is need when dealing with the more general situation of generically smooth scheme below.

We have then the following definition of the geometric schemic integral:

2.4.3 Definition. Let $\alpha : \nabla_{\mathfrak{x}}X \to \mathbb{N} \cup \{+\infty\}$ be any $\mathfrak{x}$-stable function on the infinite arc space of a $\mathfrak{x}$-stable scheme, then we define

$$\int_{\nabla_{\mathfrak{x}}X} \mathbb{L}^{-\alpha} d\mu_{\mathfrak{x}} := \sum_{n \in \mathbb{N}} \mu_{\mathfrak{x}}(\alpha^{-1}(n))\mathbb{L}^{-n} ,$$

provided that the summation on the right hand side converges in $\hat{\mathcal{H}}_{\mathfrak{x}}$. We say that α is $\mathfrak{x}$-*integrable* if this summation converges in $\hat{\mathcal{H}}_{\kappa}$.

In general, given a formal sieve $\mathcal{X} \in \mathbb{F}\mathrm{orm}_{\kappa}$ and an admissible arc $\mathfrak{x} \in \mathbf{Arc}_{\kappa}$, we may define a measure

$$\mu^f_{\mathfrak{r}}(\nabla_{\mathfrak{r}}\mathcal{X}) := \varprojlim_{\mathfrak{n}}[\nabla_{\mathfrak{r}/\mathfrak{n}}\mathcal{X}]\mathbb{L}^{-d\ell(\mathfrak{n})-f(\mathfrak{n})} \tag{2.4.1}$$

where $d = \dim(\mathcal{X})$ and f is some function on the point system $\mathbb{I}$ determining $\mathfrak{r}$. This defines an element in $\hat{\mathcal{H}}_\kappa$ provided that f is chosen appropriately and the sequence is Cauchy. For example, it is often natural to let f be the discrepancy of $\mathcal{X}$ at $\mathfrak{r}$ – i.e., $f(\mathfrak{n}) = \dim(\nabla_{\mathfrak{n}}\mathcal{X}) - d\ell(\mathfrak{n})$ – but, this is not the only possibility.

2.4.4 Theorem. *Let X be generically smooth. Then, there exists f such that the limits $\mu^f_{\mathfrak{r}}(A)$ and $\mu^f_{\mathfrak{r}}(A \setminus \nabla_{\mathfrak{r}}\hat{X}_S)$ exist and are equal for any sieve $A \in \mathbb{B}^X_{\mathfrak{r}}$ where S is the singular locus of X. Moreover, we may choose f so that $\mu^f_{\mathfrak{r}}(\nabla_{\mathfrak{r}}X)$ is non-zero.*

Proof. At worst, we can choose f to be the discrepancy. This will guarantee that the sequence is cauchy. The rest of the proof below is a straightforward adaptation of the proof for the analogous statement in [DL1]. Note that I am claiming that for any A,

$$\mu^f_{\mathfrak{r}}(A) = \mu^f_{\mathfrak{r}}(A \setminus \nabla_{\mathfrak{r}}\hat{X}_S)\ . \tag{2.4.2}$$

Now, for ease of notation, we will denote by π_m the canonical morphism from $\nabla_{\mathfrak{r}}X$ to $\nabla_{\mathfrak{m}}X$. We then partition $A \setminus \nabla_{\mathfrak{r}}\hat{X}_S$ in the following way:

$$(A \setminus \pi_m^{-1}(\pi_m(\nabla_{\mathfrak{r}}\hat{X}_S))) \sqcup \bigsqcup_{n \geq m}((\pi_n^{-1}(\pi_n(\nabla_{\mathfrak{r}}\hat{X}_S)) \setminus \pi_{n+1}^{-1}(\pi_{n+1}(\nabla_{\mathfrak{r}}\hat{X}_S))) \cap A)\ ,$$

and we have the following partition of A

$$(A \setminus \pi_m^{-1}(\pi_m(\nabla_{\mathfrak{r}}\hat{X}_S))) \sqcup \pi_m^{-1}(\pi_m(\nabla_{\mathfrak{r}}\hat{X}_S)) \cap A \, .$$

Note these are partitions of sieves into a disjoint union of sieves[4]. Now, the difference between $\mu_{\mathfrak{r}}(A \setminus \nabla_{\mathfrak{r}}\hat{X}_S)$ and $\mu_{\mathfrak{r}}(A)$ is

$$\sum_{n \geq m} \mu_{\mathfrak{r}}^f(\pi_n^{-1}((\pi_n(\nabla_{\mathfrak{r}}\hat{X}_S)) \cap A) - \sum_{n \geq m} \mu_{\mathfrak{r}}^f(\pi_n^{-1}(\pi_n(\nabla_{\mathfrak{r}}\hat{X}_S))) \cap A)) \, ,$$

which is equivalent to 0 in $\hat{\mathcal{H}}$. $\square$

2.4.5 Remark. In the next section, we will choose f in a different way. In that section, the function is denoted by l instead of f.

When X is not generically smooth it is not obvious how to relate $\mu_{\mathfrak{r}}$ and the classical motivic measure. This is one of the main reasons why we find the study of schemes whose reductions are smooth so interesting in this context. In point of fact, any generically smooth scheme over κ will be reduced, so, in this work, what we are really interested in is defining a reasonable notion of motivic integration for schemes (as opposed to the classical version which only works for varieties).

More clearly, it is easy to see that a generically smooth scheme is reduced, since the closure of any reduced open in any scheme X is reduced. If X is gener-

[4] In fact, the is not strictly speaking true. They are inverse images of split formal sieves–i.e., inverse images of formal sieves on a subcategory of $\mathbb{F}\mathtt{at}_\kappa$. However, using split formal sieves is permissible as the Grothendieck groups of the the formal site and the split formal site are equal. This crucial fact is per Proposition 8.6 of [Sch1].

ically smooth over κ, then there is a dense open subscheme U of X such that U is smooth over κ. This implies that U is reduced and hence X is reduced. Thus, whenever X is not reduced it becomes interesting to ask questions about how the schemic measure of $\nabla_{\mathfrak{x}} X$ relates to the classical motivic measure of $\nabla_{\mathfrak{l}} X^{red}$ and the most simple case where this occurs is when X^{red} is smooth, but X itself is not reduced. Much of the third chapter of this thesis is devoted to this problem.

Let us illustrate this with an example. Consider the fat point $\mathfrak{l}_2$ whose coordinate ring is the dual numbers (per our comments above it is not generically smooth) yet its reduction, which is $\mathrm{Spec}(\kappa)$, is smooth. We will show that we can measure $\mathfrak{l}_2$. In terms of the third chapter, $\mathfrak{l}$ is the limit of simple fat points $\mathfrak{l}_n$ and so by adjusting the measure $\mu_{\mathfrak{l}}$, we can define a motivic measure $\lambda^l_{\mathfrak{l}}$ so that $\lambda^l_{\mathfrak{l}}(\nabla_{\mathfrak{l}} \mathfrak{l}_2) = 1$, and therefore we see that this motivic volume is the same as computing the classical motivic volume of $\nabla_{\mathfrak{l}} \mathfrak{l}_2^{red} = \nabla_{\mathfrak{l}} \mathrm{Spec}(\kappa)$ in this case. Having such a formula for any scheme and any point system might be too much to hope for, but we will show that it is often possible to relate the schemic motivic measure with the classical motivic measure under some mild conditions. Adopting a more negative outlook, one also sees that it is not immediately obvious what happens if we work with say a local deformation of a variety over $\mathfrak{l}_2$ (or, for that matter, any other fat point).

2.5 Motivic generating series.

Note that in what follows we index $\mathfrak{n} \in \mathbb{I}$ by the integer n. In other words, $\mathfrak{n}$ is the n-th truncated jet of some scheme Y at κ-rational point O of Y –i.e., $\mathfrak{n} = J_O^n Y$ for some fixed scheme Y. Following §9 of [Sch2], we give the following definition:

2.5.1 Definition. Let $(\mathfrak{x}, \mathbb{I}) \in \mathbf{Arc}_\kappa$ and $X \in \mathbb{S}\mathrm{ch}_\kappa$ where $\dim X = d$, we define the *motivic Igusa-zeta series of X with respect to $\mathfrak{x}$* by

$$\zeta_{\mathfrak{x}}(X)(t) := \sum_{n \in \mathbb{N}} [\nabla_{\mathfrak{n}} X] \mathbb{L}^{-d\ell(\mathfrak{n})} t^n .$$

Moreover, we define *the motivic Poincaré series of X with respect to $\mathfrak{x}$* by

$$P_{\mathfrak{x}}(X)(t) := \sum_{n \in \mathbb{N}} [\nabla_{\mathfrak{x}/\mathfrak{n}} X] \mathbb{L}^{-d\ell(\mathfrak{n})} t^n .$$

2.5.2 Proposition. *When X is $\mathfrak{x}$-stable, then $P_{\mathfrak{x}}(X)$ belongs to $\mathcal{H}_\kappa[t, \frac{1}{1-t}]$.*

Proof. By definition of stability, for sufficiently large length $\ell(\mathfrak{m})$ with $\mathfrak{m} \in \mathbb{I}$, we have the following equalities in $\mathcal{H}_\kappa[[t]]$.

$$\begin{aligned} P_{\mathfrak{x}}(X)(t) &= \sum_{n \in \mathbb{N}} [\nabla_{\mathfrak{x}/\mathfrak{n}} X] \mathbb{L}^{-d\ell(\mathfrak{n})} t^n \\ &= \sum_{n=1}^{k} [\nabla_{\mathfrak{x}/\mathfrak{n}} X] \mathbb{L}^{-d\ell(\mathfrak{n})} t^n + \sum_{n > k} \mu_{\mathfrak{x}}(\nabla_{\mathfrak{x}} X) t^n \\ &= \sum_{n=1}^{k} [\nabla_{\mathfrak{x}/\mathfrak{n}} X] \mathbb{L}^{-d\ell(\mathfrak{n})} t^n + \mu_{\mathfrak{x}}(\nabla_{\mathfrak{x}} X) \frac{t^{k+1}}{1-t} . \end{aligned}$$

∎

When X is $\mathfrak{x}$-laxly stable, the same work above will give us the formula

$$P_{\mathfrak{x}}(X)(t) = \sum_{n=1}^{k} [\nabla_{\mathfrak{x}/\mathfrak{n}} X] \mathbb{L}^{-d\ell(\mathfrak{n})} t^n + \mu^l_{\mathfrak{x}}(\nabla_{\mathfrak{x}} X) \sum_{n>k} \gamma_l(n) t^n \ .$$

Note that γ_l was defined via the function $l(\mathfrak{n}) := l^X_{\mathfrak{x}}(\mathfrak{n})$. When l is linear along n of slope q (meaning that $l(\mathfrak{n}) = qn + b$), we obtain

$$\begin{aligned} P_{\mathfrak{x}}(X)(t) &= \sum_{n=1}^{k} [\nabla_{\mathfrak{x}/\mathfrak{n}} X] \mathbb{L}^{-d\ell(\mathfrak{n})} t^n + \mu^l_{\mathfrak{x}}(\nabla_{\mathfrak{x}} X) \mathbb{L}^b \sum_{n>k} (\mathbb{L}^q t)^n \\ &= \sum_{n=1}^{k} [\nabla_{\mathfrak{x}/\mathfrak{n}} X] \mathbb{L}^{-d\ell(\mathfrak{n})} t^n + \mu^l_{\mathfrak{x}}(\nabla_{\mathfrak{x}} X) \mathbb{L}^b \frac{(\mathbb{L}^q t)^{k+1}}{1 - \mathbb{L}^q t} \ . \end{aligned}$$

Thus, we have the following proposition:

2.5.3 Proposition. *If X is $\mathfrak{x}$-laxly stable and l is linear along n of slope q, then $P_{\mathfrak{x}}(X)$ belongs to $\mathcal{H}_\kappa[t, \frac{1}{1-\mathbb{L}^q t}]$.*

2.5.4 Remark. If X is $\mathfrak{x}$-laxly stable at level 0, then

$$l^X_{\mathfrak{x}}(n) = \dim(\nabla_{\mathfrak{n}} X) - d\ell(\mathfrak{n}) \ ,$$

which is known in [Sch2] as the defect of X at $\mathfrak{n}$.

In the following, we use the notion of a simple point system, which is discussed extensively in Chapter 5. Following §5 of [Sch2], we define the *weightless auto-Igusa zeta series of a limit point* $\mathfrak{x}$ to be

$$\zeta^w_{\mathfrak{x}}(t) = \sum_{n \in \mathbb{N}} \mathbb{L}^{-l(\mathfrak{n})} [\nabla_{\mathfrak{n}} \mathfrak{n}] t^n \ .$$

Assume that $(\mathfrak{x}, \mathbb{I}) \in \mathbf{Arc}_\kappa$ where $\mathbb{I}$ is simple, then we obtain

$$\hat{\sigma}(\zeta^w_{\mathfrak{x}}(t)) = \frac{t}{1-t} \, .$$

Moreover, if $\mathbb{I}$ is eventually simple, we may still conclude that $\hat{\sigma}(\zeta^w_{\mathfrak{x}}(t))$ is rational.

Chapter 3

Motivic Volumes and Deformations

In this chapter, we push the theory further by investigating measures which will include infinitesimal information.

3.1 Local deformations and stability.

First, let us take the definition of smoothness in Chapter III, Section 10 in [H1]. That is, we will assume that $f : X \to Y$ is a smooth morphism in $\mathbb{Sch}_\kappa$ implies that it is of relative dimension d. In particular, if X is smooth over a finite κ-scheme, then all the irreducible components of X have the same dimension d. Thus, X will automatically be of pure dimension d in this case. Although, results easily generalize to the definition of smoothness in EGA, it simplifies the statement of most theorems to include relative dimension as part of the definition of smoothness.

Now, the prototypical example of a stable scheme is any smooth scheme X

in $\mathbb{S}\mathrm{c}\mathbb{h}_\kappa$ of dimension d. One of the goals of this section is to generalize the following theorem.

3.1.1 Theorem. *For any $\mathfrak{x} \in \mathbf{Arc}_\kappa$ and any $X \in \mathbb{S}\mathrm{m}\mathbb{S}\mathrm{c}\mathbb{h}_\kappa$ of dimension d, we have*

$$\mu_{\mathfrak{x}}(\nabla_{\mathfrak{x}} X) = [X]\mathbb{L}^{-d} .$$

Proof. This is proven in Example 3.1.9. □

3.1.2 Definition. We say that a scheme Y is a *local deformation* of a scheme X if there exists a fat point $\mathfrak{n}$ which admits a flat morphism $Y \to \mathfrak{n}$ together with a morphism of schemes from X to Y such that the induced morphism $X \to Y \times_{\mathfrak{n}} \operatorname{Spec}(\kappa)$ is an isomorphism.

If X is smooth and affine, then it is known (cf. [H2] p. 38-39) that every local deformation Y is trivial – i.e., $Y \cong X \times_\kappa \mathfrak{n}$ for any fat point $\mathfrak{n}$. We may generalize even further because this local deformation Y will be smooth over the fat point $\mathfrak{n}$ (just apply base change by $\mathfrak{n}$). Therefore, we have a potential source for a plethora of schemes which admit a well-defined motivic measure – that is, *schemes which are smooth over a fat point*. Of course, here one may ask the same question concerning local deformations of non-smooth schemes, which may be a good topic to take up at a later date.

3.1.3 Lemma. *Let $f : X \to Y$ be an étale morphism of schemes and let $\mathfrak{n}$ be a fat point over κ. Then*

$$\nabla_{\mathfrak{n}} X \cong X \times_Y \nabla_{\mathfrak{n}} Y$$

Proof. Cf. Theorem 4.12 of [Sch2]. □

3.1.4 Lemma. *Let $f : X \to Y$ be an étale morphism of schemes and let $\mathfrak{x}$ be a the limit of a point system (e.g., $\mathfrak{x}$ is an admissible arc over κ), then*

$$\nabla_{\mathfrak{x}} X \cong X \times_Y \nabla_{\mathfrak{x}} Y$$

Proof. This follows directly from the definition of inverse limit, the previous lemma, and Lemma 7.4 of [Sch2]. □

3.1.5 Theorem. *Let $X \to \mathfrak{n}$ be a smooth morphism and let $\mathfrak{m}$ be any fat point. The canonical morphism $\nabla_{\mathfrak{m}} X \to X$ is a piecewise trivial fibration with fiber $\mathbb{A}_{\kappa}^{d(\ell(\mathfrak{m})-1)} \times_{\mathfrak{n}} \nabla_{\mathfrak{m}} \mathfrak{n}$ where $d = dim X$.*

Proof. Using Theorem 4.4 of [Sch2], we may cover $\nabla_{\mathfrak{m}} X$ by opens $\nabla_{\mathfrak{m}} U$ where U is an open in X. Therefore, by shrinking X if necessary, we may assume that there is an étale morphism $X \to \mathbb{A}_{\mathfrak{n}}^d$ (cf., [Liu] Chapter 6, Corollary 2.11).

We apply Lemma 3.1.3 to obtain

$$\nabla_{\mathfrak{m}} X \cong X \times_{\mathbb{A}_{\mathfrak{n}}^d} \nabla_{\mathfrak{m}} \mathbb{A}_{\mathfrak{n}}^d .$$

Note that

$$\nabla_{\mathfrak{m}}\mathbb{A}^d_{\mathfrak{n}} \cong \nabla_{\mathfrak{m}}\mathbb{A}^d_{\kappa} \times_{\kappa} \nabla_{\mathfrak{m}}\mathfrak{n}$$

and that $\nabla_{\mathfrak{m}}\mathbb{A}^d_{\kappa} \cong \mathbb{A}^{d\ell(\mathfrak{m})}_{\kappa}$ by Theorem 4.8 of [Sch2]. Therefore,

$$\nabla_{\mathfrak{m}}X \cong X \times_{\kappa} \mathbb{A}^{d(\ell(\mathfrak{m})-1)}_{\kappa} \times_{\mathfrak{n}} \nabla_{\mathfrak{m}}\mathfrak{n}\ .$$

□

3.1.6 Corollary. *Let $X \to \mathfrak{n}$ be a smooth morphism and let $\mathfrak{m}$ be any fat point. There is an isomorphism*

$$\nabla_{\mathfrak{m}}X \cong X \times_{\kappa} \mathbb{A}^{d(\ell(\mathfrak{m})-1)}_{\kappa} \times_{\mathfrak{n}} \nabla_{\mathfrak{m}}\mathfrak{n}$$

where $d = dimX$.

Proof. This follows because we may cover $\nabla_{\mathfrak{m}}X$ with opens of the form $\nabla_{\mathfrak{m}}U$ as noted at the beginning of the proof of the previous theorem. □

3.1.7 Corollary. *Let $X \to \mathfrak{n}$ be a smooth morphism and let $\mathfrak{m}$ be any fat point. There is an isomorphism*

$$(\nabla_{\mathfrak{m}}X)^{red} \cong X^{red} \times_{\kappa} \mathbb{A}^{d(\ell(\mathfrak{m})-1)}_{\kappa} \times_{\kappa} (\nabla_{\mathfrak{m}}\mathfrak{n})^{red}$$

where $d = dimX$.

3.1.8 Corollary. *Let X be the trivial local deformation with respect to $\mathfrak{n}$ of a smooth variety Y– i.e., $X \cong Y \times_\kappa \mathfrak{n}$. Then,*

$$\nabla_\mathfrak{m} X \cong Y \times_\kappa \mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-1)} \times_\kappa \nabla_\mathfrak{m}\mathfrak{n}$$

where $d = dimX$.

3.1.9 Example. Let us consider the case where $\mathfrak{n} = \operatorname{Spec}(\kappa)$ in the previous corollary. Then, $X = Y$, and we arrive at

$$\nabla_\mathfrak{m} X \cong X \times_\kappa \mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-1)}$$

as $\nabla_\mathfrak{m} \operatorname{Spec}(\kappa) \cong \operatorname{Spec}(\kappa)$ for any fat point $\mathfrak{m}$. This is a particular case of Theorem 4.14 of [Sch2], which we restate in the next corollary.

3.1.10 Corollary. *Let X be a smooth κ-scheme of dimension d. Let $\mathfrak{m}$ and $\mathfrak{v}$ be any two fat points which admit a closed embedding $\mathfrak{m} \hookrightarrow \mathfrak{v}$. Then, the canonical morphism $\nabla_\mathfrak{v} X \to \nabla_\mathfrak{m} X$ is a piecewise trivial fibration with fiber $\mathbb{A}_\kappa^r$ where $r = d(\ell(\mathfrak{v}) - \ell(\mathfrak{m}))$.*

Proof. By the previous corollary, the canonical morphism $\nabla_\mathfrak{v} X \to \nabla_\mathfrak{m} X$ may be rewritten as

$$X \times_\kappa \mathbb{A}_\kappa^{d(\ell(\mathfrak{v})-1)} \to X \times_\kappa \mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-1)}$$

where this morphism is given by the fiber product of the identity morphism on

X and the projection morphism from $\mathbb{A}_\kappa^{d(\ell(\mathfrak{v})-1)}$ to $\mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-1)}$. Thus, the claim is immediate. $\square$

3.1.11 Remark. The previous proof may be regarded as an alternate proof of Theorem 4.14 of [Sch2].

3.1.12 Definition. Let $\mathbb{I}$ be an admissible point system, $\mathfrak{n}$ be any fat point, and V any variety. We say that it is $\mathfrak{n}$-*linear* over V if there is an isomorphism

$$(\nabla_{\mathfrak{m}}\mathfrak{n})^{red} \cong \mathbb{A}_V^r$$

for all $\mathfrak{m} \in \mathbb{I}$ such that $\ell(\mathfrak{m}) >> 0$ where r depends on $\ell(\mathfrak{m})$. Moreover, we say that it is $\mathfrak{n}$-*rational* over V if there

$$\hat{\sigma}([(\nabla_{\mathfrak{m}}\mathfrak{n})]) = [V]\mathbb{L}^r$$

in $\hat{\mathcal{G}}_\kappa$ for all $\mathfrak{m} \in \mathbb{I}$ such that $\ell(\mathfrak{m}) >> 0$ where r depends on $\ell(\mathfrak{m})$.

3.1.13 Remark. Little is known about the structure of the spaces $(\nabla_{\mathfrak{m}}\mathfrak{n})^{red}$ when $\mathfrak{m}$ runs through an admissible point system $\mathbb{I}$. Note however that any point system which is $\mathfrak{n}$-linear over V is $\mathfrak{n}$-rational over V.

3.1.14 Example. Trivially, any point system is κ-linear over κ. The following is a less trivial example Let $\mathbb{I}$ be the admissible point system formed by $J_O^m\mathbb{A}_\kappa^n$–i.e., let $\mathfrak{x} = \varinjlim \mathbb{I}$ be a *multilinear arc*. Then, $\mathbb{I}$ is $J_O^s\mathbb{A}_\kappa^t$-linear over κ for any $s,\ t \in \mathbb{N}$.

3.1.15 Theorem. *Let X be a smooth scheme over $\mathfrak{n}$ and let $\mathbb{I}$ be $\mathfrak{n}$-linear over a variety V. Then, the canonical morphism $(\nabla_{\mathfrak{v}}X)^{red} \to (\nabla_{\mathfrak{m}}X)^{red}$ is a piecewise trivial fibration with general fiber*

$$\mathbb{A}_V^{d(\ell(\mathfrak{v})-\ell(\mathfrak{m}))+r}$$

whenever $\mathfrak{v}$ and $\mathfrak{m}$ are elements of $\mathbb{I}$ equipped with a closed embedding $\mathfrak{m} \hookrightarrow \mathfrak{v}$ such that $\ell(\mathfrak{m}) >> 0$ where $d = dimX$ and r depends on $\ell(\mathfrak{v})$ and $\ell(\mathfrak{m})$.

Proof. Similar to the proof of Theorem 3.1.15, the canonical morphism $(\nabla_{\mathfrak{v}}X)^{red} \to (\nabla_{\mathfrak{m}}X)^{red}$ may be rewritten as

$$X \times_\kappa \mathbb{A}_V^{d(\ell(\mathfrak{v})-1)+r_1} \to X \times_\kappa \mathbb{A}_V^{d(\ell(\mathfrak{m})-1)+r_2}$$

provided that $\ell(\mathfrak{m}) >> 0$. Note then that $r = r_1 - r_2$ where r_1 and r_2 depend on $\ell(\mathfrak{v})$ and $\ell(\mathfrak{m})$, respectively □

3.1.16 Corollary. *Let X be the trivial local deformation of a smooth scheme Y with respect to the fat point $J_O^s\mathbb{A}_\kappa^t$ where $s,\ s \in \mathbb{N}$. Then, $(\nabla_{\mathfrak{m}}X)^{red} \to Y$ is a piecewise trivial fibration with general fiber*

$$\mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-1)+r}$$

whenever $\mathfrak{m}$ is of the form $J_O^m\mathbb{A}_\kappa^n$ where $d = dimX$ and where $r(s,t,m,s)$ is some function depending on s, t, m and n which takes values on in $\mathbb{N}$.

3.1.17 Remark. In point of fact, $r(s,t,m,n)$ will be monotone increasing in m and have polynomial growth in m

3.1.18 Example. Let us again revisit the fat point $\mathfrak{l}_2$ with coordinate ring being the dual numbers. As we mention at the end of 2.4, we may measure $\nabla_{\mathfrak{l}}\mathfrak{l}_2$. First we compute the truncated arc spaces $\nabla_{\mathfrak{l}_n}\mathfrak{l}_2$. This amounts to creating n variables a_i for $i = 0, \ldots, n-1$ and evaluating in $\kappa[t]/(t^n)$ the equation

$$(a_0 + a_1 t + \cdots + a_{n-1}t^{n-1})(a_0 + a_1 t + \cdots + a_{n-1}t^{n-1}) = 0$$

and then computing the reduction of the scheme defined by those equations. For the cases where $n = 1,\ 2,\ 3$ and 4, the reader may quickly check that the space $(\nabla_{\mathfrak{l}_n}\mathfrak{l}_2)^{red}$ is $\mathrm{Spec}(\kappa)$, $\mathbb{A}^1_\kappa$, $\mathbb{A}^1_\kappa$, and $\mathbb{A}^2_\kappa$, respectively. This is part of a general pattern as using the κ-basis for $\kappa[t]/(t^n)$, we see that our first term in the expansion of the left hand side is $a_0^2 = 0$ which in the reduction implies that $a_0 = 0$. Thus, the above equation simplifies to

$$(a_1 + a_2 t + \cdots + a_{n-1}t^{n-2})(a_1 + a_2 t + \cdots + a_{n-1}t^{n-2})t^2 = 0\ .$$

We can iteratively continue this simplification process to obtain

$$(a_s + a_{s+1} t + \cdots + a_{n-1}t^{n-s-1})(a_s + a_{s+1} t + \cdots + a_{n-1}t^{n-s-1})t^{2s} = 0\ .$$

until $2s \geq n$ or, in other words, this equation implies that $a_s = 0$ when $2s < n$ and leaves a_s free when $2s \geq n$. Thus, $(\nabla_{\mathfrak{l}_n}\mathfrak{l}_2)^{red}$ is $\mathbb{A}^{l(n)}_\kappa$ where $l(n) = n - \lceil \frac{n}{2} \rceil$.

In fact, this is part of a general picture in that $(\nabla_{\mathfrak{l}_n}\mathfrak{l}_m)^{red} \cong \mathbb{A}_\kappa^{n-\lceil \frac{n}{m} \rceil}$ for any n and any m. Hence, the point system $\{\mathfrak{l}_n\}$ is our prototypical example of a linear point system. But, let us stick to $\mathfrak{l}_2$ to display the point. Combining our computation for the function l and the previous theorem, we have that if X is a smooth over $\mathfrak{l}_2$ (e.g., the local deformation of a smooth scheme Y with respect to the dual numbers $\mathfrak{l}_2$), then for any natural numbers $m \geq n$, the morphism $(\nabla_{\mathfrak{l}_m} X)^{red} \to (\nabla_{\mathfrak{l}_n} X)^{red}$ is a piece-wise trivial fibration with general fiber

$$\mathbb{A}_\kappa^{d(m-n)+l(m)-l(n)}.$$

In later sections of this chapter, we will construct motivic measures which allow us to measure $\nabla_{\mathfrak{l}} X$ per our results in this section.

It is tempting to think that if $\mathfrak{n}$ is a fat point and $\mathbb{I}$ is a $\mathfrak{n}$-linear point system over κ, then $\mathbb{I}$ must be of the form $\{J^n\mathbb{A}_\kappa^m \mid n \in \mathbb{N}\}$ for some fixed m. The situation is considerably more subtle than this. Using the computational techniques of Chapter 5, I found the following example.

3.1.19 Example. Let $\mathfrak{n} = J_O^3\mathbb{A}_\kappa^2$ and let $\mathbb{J} = \{J_O^n C \mid n \in \mathbb{N}\}$ where $C = \operatorname{Spec}(\kappa[x,y]/(x^2+y^2))$. Then, computationally speaking, it looks extremely unlikely that $\mathbb{J}$ is $\mathfrak{n}$-linear over κ. However, $\mathbb{I} = \{J_O^n N \mid n \in \mathbb{N}\}$ is $\mathfrak{n}$-linear over κ where $N = \operatorname{Spec}(\kappa[x,y]/(xy))$.

3.1.20 Example. Let $\mathfrak{n} = J^3_O C$ where $C = \operatorname{Spec}(\kappa[x,y]/(x^2+y^2))$ and consider the two point systems $\mathbb{I} = \{J^n_O N \mid n \in \mathbb{N}\}$ where $N = \operatorname{Spec}(\kappa[x,y]/(xy))$ and $\mathbb{J} = \{J^n_O C \mid n \in \mathbb{N}\}$. Then, using the computational technique of Chapter 5, it looks extremely unlikely that $\mathbb{J}$ is $\mathfrak{n}$-linear over any variety V. However, $\mathbb{I}$ is $\mathfrak{n}$-linear over κ.

3.1.21 Example. Let $\mathfrak{n} = J^3_O N$ where $N = \operatorname{Spec}(\kappa[x,y]/(xy))$ and let $\mathbb{J} = \{J^n_O C \mid n \in \mathbb{N}\}$. Computationally, it looks extremely unlikely that $\mathbb{J}$ is $\mathfrak{n}$-linear over any variety V. However, we do have the following very interesting example. Consider the point system $\mathbb{I} = \{J^n_O N \mid n \in \mathbb{N}\}$. Then, $\mathbb{I}$ is not $\mathfrak{n}$-linear over any variety V, yet, for $m \geq 5$, we have

$$(\nabla_{J^m_O N} J^3_O N)^{red} \cong \nabla_{\mathfrak{l}_{m-4}} N \times_\kappa \nabla_{\mathfrak{l}_{m-4}} N \times_\kappa \mathbb{A}^r_\kappa$$

where r depends on m. This means that we can still form a motivic measure.

3.1.22 Example. For a more complicated example, let C be the cuspidal cubic and let $\mathfrak{n} = J^3_O C$ and $\mathfrak{m} = J^3_O \mathbb{A}^2_\kappa$. Let $\mathbb{I} = \{J^n_O C \mid n \in \mathbb{N}\}$. Then, $\mathbb{I}$ is not $\mathfrak{n}$-linear over any variety V, but it is $\mathfrak{m}$-linear over κ. However, I believe there is a strong likelihood that it has a similar form to the previous example with the node.

3.1.23 Remark. I have not produced proofs of the above facts, but they are actually very straightforward to prove. Instead, I am relying on my computations using my sage code. There is little cause for concern as once the pattern is noticed, statements of the type above are usually very easy to prove as in our first example where we showed explicitly that the point system $\{\mathfrak{l}_n \mid n \in \mathbb{N}\}$ is $\mathfrak{l}_2$-linear over κ. Using computer algebra, as in Chapter 5, helps tremendously as it allows one to compute a bunch of arc spaces very quickly. Moreover, it is often not feasible to carry out these necessary computations by hand in any reasonable sense.

3.1.24 Example. Let $\mathfrak{v}$ be any fat point and consider the point system $\mathbb{I} = \{J^n_O \mathbb{A}^r_\kappa \mid n \in \mathbb{N}\}$. Then, $\mathbb{I}$ is $\mathfrak{v}$-linear over κ. This is straightforward. This says that every multilinear arc is $\mathfrak{v}$-linear over κ for every fat point $\mathfrak{v} \in \mathbb{F}\mathtt{at}_\kappa$. I highly doubt that there are any points systems whose limit point is not multilinear, yet the point system is $\mathfrak{v}$-linear over κ for every fat point $\mathfrak{v} \in \mathbb{F}\mathtt{at}_\kappa$. This conjecture is similar to our conjecture concerning simple point systems, which can be found in Chapter 5.

The previous example will imply that if X is a scheme which is smooth over a fat point $\mathfrak{v}$, then $\nabla_{\mathfrak{x}} X$ has a well defined motivic measure with respect to $\mathfrak{x} = \varinjlim \mathbb{I}$ where $\mathbb{I}$ is the point system given in the previous example. In particular, we will be able to extend (in the naive way) the classical motivic measure to schemes which are smooth over a fat point.

3.2 Smooth reductions.

Continuing with the theme of the previous section, we investigate here the specific case where X is a scheme such that X^{red} is smooth.

3.2.1 Affine case:

We know that in general smoothness does not descend via a faithfully flat morphism; however, we have the following:

3.2.1 Proposition. *Let $f : X \to Y$ and $h : Y' \to Y$ be two morphisms in $\mathbb{Sch}_\kappa$. Let $X' = X \times_Y Y'$ and let $f' : X' \to Y'$ be the canonical projection. Suppose further that h is quasi-compact and faithfully flat, then f is smooth if and only if f' is smooth.*

Proof. This is a special case of Proposition 6.8.3 of [G2]. $\square$

Let $X \in \mathbb{Sch}_\kappa$ be affine and write $X = \operatorname{Spec} A$. Choose a minimal system of generators $g_1, \ldots, g_s$ of the nilradical $nil(A)$ of A. Let $x_1, \ldots, x_s$ be s variables and let J be the kernel of the map from $\kappa[x_1, \ldots, x_s]$ to A which sends x_i to g_i. We set $R := \kappa[x_1, \ldots, x_s]/J$. Then, $R \hookrightarrow A$. Here, R is nothing other than the *maximum artinian subring of* A. We have the following:

3.2.2 Lemma. *Let* $X = \operatorname{Spec}(A)$ *be a connected affine scheme in* $\mathbb{Sch}_\kappa$*, set* $\mathfrak{n} = \operatorname{Spec} R$ *where* R *is the maximum artinian subring of* A*, and let* l *be any positive integer. Then,* $\mathfrak{n}$ *is fat point over* κ*, and we have the following decompositions:*

(a) $X^{red} \cong X \times_{\mathfrak{n}} \operatorname{Spec} \kappa$

(b) $X \times_{\mathbb{A}^d_{\mathfrak{n}}} \mathbb{A}^{dl}_\kappa \cong X^{red} \times_\kappa \mathbb{A}^{d(l-1)}_\kappa$.

Proof. Write $X = \operatorname{Spec} A$ for some finitely generated κ-algebra. It is basic that $R \hookrightarrow A$. Let $\mathcal{M} = (x_1, \ldots, x_s)R$. Clearly, $\mathcal{M}$ is a maximal ideal of R. Moreover, $\mathcal{M}A \subset nil(A)$ by construction. Therefore, there exists an N such that $\mathcal{M}^N = 0$. Thus, R is artinian ring with residue field κ. We assumed X was connected so that R would be local. Indeed, by injectivity of $R \hookrightarrow A$, any direct sum decomposition of R would immediately imply a direct sum decomposition of A as it would entail that R (and hence A) contains orthogonal idempotents $e_1 \neq e_2$.

Note that the containment $\mathcal{M}A \subset nil(A)$ is actually an equality by construction. Now, use the fact that $\kappa = R/\mathcal{M}$ so that

$$A \otimes_R \kappa \cong A \otimes_R (R/\mathcal{M}) \cong (A/\mathcal{M}A) \otimes_R R \cong A/\mathcal{M}A \cong A/nil(A)$$

where the second isomorphism is a well-known property of tensor products for R-algebras. This proves part (a).

Part (b) is really a restatement of the work done in the preceding paragraph.

One should just note that

$$X \times_{\mathbb{A}^d_{\mathfrak{n}}} \mathbb{A}^{dl}_{\kappa} \cong X \times_{\mathbb{A}^d_{\mathfrak{n}}} \operatorname{Spec}(\kappa) \times_{\kappa} \mathbb{A}^{dl}_{\kappa}$$

so that we can apply (a) to the right hand side to obtain

$$X \times_{\mathbb{A}^d_{\mathfrak{n}}} \kappa \times_{\kappa} \mathbb{A}^{dl}_{\kappa} \cong X^{red} \times_{\mathbb{A}^d_{\kappa}} \mathbb{A}^{dl}_{\kappa} .$$

This proves the result part (b). □

3.2.3 Theorem. *Let $X = \operatorname{Spec}(A)$ be a connected. Then, X^{red} is smooth if and only if there exists a smooth morphism $X \to \mathfrak{n}$ where $\mathfrak{n} = \operatorname{Spec} R$ such that R is the maximal artinian subring of A.*

Proof. This is just a restatement of Proposition 3.2.1 where $Y' = \operatorname{Spec} \kappa$, $Y = \mathfrak{n}$, and $Y' \to Y$ is the canonical morphism. Indeed, by Lemma 3.2.2, $X' := X \times_Y Y' \cong X^{red}$, and the homomorphism of rings $R \to \kappa$ given by modding out by $\mathcal{M}$ is both surjective and flat. □

3.2.4 Remark. Let $X = \operatorname{Spec}(A)$ and let $\mathfrak{n} = \operatorname{Spec}(R)$ where R is the maximal artinian subring of A. Then, a slightly quicker proof of the above theorem can be obtained by noticing that it is necessarily the case that X is an infinitesimal deformation over $\mathfrak{n}$ of X^{red}. Thus, X^{red} being smooth implies that $X \cong X^{red} \times_{\kappa} \mathfrak{n}$. However, one may note that the proof we presented using EGA is more general and could be applied to other situations which may be of interest to motivic integration.

3.2.5 Remark. Let $X = \operatorname{Spec}(A)$ be a connected affine scheme with X^{red} smooth and let $\mathfrak{n} = \operatorname{Spec} R$ be the fat point such that R is the maximal artinian subring of A. Then, per the material in the previous section, if $\mathbb{I}$ is a point system which is $\mathfrak{n}$-rational over a variety V, we may assign to X a motivic volume via $\nabla_{\mathfrak{x}} X$ where $\mathfrak{x} = \varinjlim \mathbb{I}$. The construction of the motivic volume takes place in the proceeding sections of this chapter.

3.2.6 Remark. An immediate corollary of the previous work is that an affine scheme $X \in \mathbb{Sch}_\kappa$ has smooth reduction if and only if X is smooth over some finite κ-scheme. Although we restrict our attention to fat points, the results of this thesis easily generalize to finite κ-schemes.

3.2.2 Non-affine case:

3.2.7 Theorem. *Let $X \in \mathbb{Sch}_\kappa$ be such that X^{red} is smooth. Let d be the dimension of X. Let $\mathfrak{m}$ be any fat point. Then, there exists a finite cover $\{U_i\}$ of X by connected open affines such that for each i*

$$\nabla_{\mathfrak{m}} U_i \cong U_i^{red} \times_\kappa \mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-1)} \times_\kappa \nabla_{\mathfrak{m}} \mathfrak{n}_i \ ,$$

where $\mathfrak{n}_i$ is is the spectrum of the maximum artinian subring of $\mathcal{O}_{U_i}(U_i)$. Moreover, if for each i there exists an isomorphism

$$(\nabla_{\mathfrak{m}}\mathfrak{n}_i)^{red} \cong \mathbb{A}^r_\kappa \ ,$$

then the canonical morphism

$$(\nabla_{\mathfrak{m}}X)^{red} \to X^{red}$$

is a piece-wise trivial fibration over κ with general fiber $\mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-1)+r}$.

Proof. We may cover X by a finite collection of connected open affines $\{U_i\}$. Since the open immersions $U_i^{red} \to X^{red}$ are smooth, each U_i^{red} is a smooth connected affine scheme. By Theorem 3.1.3, for each i, there exists a fat point $\mathfrak{n}_i$ such that $U_i \to \mathfrak{n}_i$ is smooth and, by part (a) of Lemma 3.2.2, such that

$$U_i^{red} \cong U_i \times_{\mathfrak{n}_i} \operatorname{Spec} \kappa \ .$$

We may shrink each U_i (if necessary) so that there is an étale morphism

$$U_i \to \mathbb{A}^d_{\mathfrak{n}_i}$$

and so that the open sets $\nabla_{\mathfrak{m}}U_i$ cover $\nabla_{\mathfrak{m}}X$ (here, $\mathfrak{m}$ is any fat point over κ). Therefore, for each i,

$$\nabla_{\mathfrak{m}}U_i \cong U_i \times_{\mathbb{A}^d_{\mathfrak{n}_i}} \nabla_{\mathfrak{m}}\mathbb{A}^d_{\mathfrak{n}_i} \cong U_i \times_{\mathbb{A}^d_{\mathfrak{n}_i}} \mathbb{A}^{d\ell(\mathfrak{m})}_{\mathfrak{n}_i} \times_\kappa \nabla_{\mathfrak{m}}\mathfrak{n}_i \ ,$$

where the first equation is a consequence of Lemma 3.1.3 and the second equation follows from Proposition 3.3 and Theorem 4.8 of [Sch2]. Using part (b) of Lemma 3.2.2, we obtain

$$\nabla_{\mathfrak{m}} U_i \cong U_i^{red} \times_\kappa \mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-1)} \times_\kappa \nabla_{\mathfrak{m}} \mathfrak{n}_i \ .$$

Because the first two factors in the above fiber product on the right hand side are affine, it is a straightforward verification that

$$(\nabla_{\mathfrak{m}} U_i)^{red} \cong U_i^{red} \times_\kappa \mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-1)} \times_\kappa (\nabla_{\mathfrak{m}} \mathfrak{n}_i)^{red} \ ,$$

from which the theorem follows. □

3.2.8 Theorem. *Let $X \in \mathbb{Sch}_\kappa$ be such that X^{red} is smooth. Let d be the dimension of X. Let $\mathfrak{m}$ be any fat point. Then, there exists a finite collection of fat points $\mathfrak{n}_i$ such that*

(a) $\nabla_{\mathfrak{m}} X \cong X^{red} \times_\kappa \mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-1)} \times_\kappa \nabla_{\mathfrak{m}}(\sqcup_i \mathfrak{n}_i)$

(b) $(\nabla_{\mathfrak{m}} X)^{red} \cong X^{red} \times_\kappa \mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-1)} \times_\kappa (\nabla_{\mathfrak{m}}(\sqcup_i \mathfrak{n}_i))^{red}$.

Proof. Theorem 3.2.7 gives us an open cover $\{U_i\}$ such that $\mathfrak{n}_i$ is the spectrum of the maximum artinian subring of $\mathcal{O}_{U_i}(U_i)$, and it also gives the following formulas:

$$\nabla_{\mathfrak{m}} U_i \cong U_i^{red} \times_\kappa \mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-1)} \times_\kappa \nabla_{\mathfrak{m}} \mathfrak{n}_i \ . \qquad (3.2.1)$$

Using Theorem 4.4 of [Sch2], we may glue the opens $\nabla_{\mathfrak{m}} U_i$ to obtain the expression in (a). Part (b) follows from (a) because the first two factors on the right hand side of (a) are reduced. $\square$

3.2.9 Remark. Let $X \in \mathbb{S}\mathrm{ch}_\kappa$ be such that X^{red} is smooth. Let $\mathfrak{n}_i$ be he spectrum of the maximum artinian subring of $\mathcal{O}_{U_i}(U_i)$ as in the previous theorem. Let $\mathbb{I}_i$ be a point system which is $\mathfrak{n}_i$-rational over some variety V_i for each i and let $\mathfrak{x}$ be the disjoint union of the $\mathfrak{x}_i$ where $\mathfrak{x}_i = \varinjlim \mathbb{I}_i$. Then we may form a motivic volume of X as in the affine case since although $\mathfrak{x}$ is colimit of finite κ-schemes which are not necessarily connected, the definitions easily generalize to this situation.

3.3 The reduced measure.

As mentioned earlier in Chapter 2, Schoutens has shown that $\nabla_{\mathfrak{x}/\mathfrak{n}} X$ defines a formal sieve. We would like to now introduce the notion of the reduction of a formal sieve. In general, if $\mathcal{Y}$ is a formal sieve, then we may consider the sieve $\mathcal{Y}^{red}$ defined as follows. For $\mathfrak{v} \in \mathbb{F}\mathrm{at}_\kappa$, there is a subschemic sieve $\mathcal{S} \subset \mathcal{Y}$ such that $\mathcal{S}(\mathfrak{v}) = \mathcal{Y}(\mathfrak{v})$. Now, $\mathcal{S}$ is actually the sieve associated to the image of a morphism of schemes $\varphi : X \to Y$ and thus $\mathcal{Y}(\mathfrak{v})$ is in fact equal to the $\mathfrak{v}$-rational points of some constructible subset of Y. We may therefore give this constructible set the reduced induced subscheme structure, and we may denote its

$\mathfrak{v}$ rational points by $\mathcal{Y}^{red}(\mathfrak{v})$. It is clear that the resulting assignment $\mathfrak{v} \mapsto \mathcal{Y}^{red}(\mathfrak{v})$ is again a formal sieve, which we term the reduction of $\mathcal{Y}$ and denote it by $\mathcal{Y}^{red}$.

3.3.1 Definition. We say that a scheme X is *reduced* $\mathfrak{x}$*-laxly stable* if the induced map

$$(\nabla_{\mathfrak{x}/\mathfrak{m}}X)^{red} \to (\nabla_{\mathfrak{x}/\mathfrak{n}}X)^{red}$$

is a piecewise trivial fibration with fiber $\mathbb{A}^r_\kappa$ (possibly r depends on the lengths of $\mathfrak{m}$ and $\mathfrak{n}$) whenever $\ell(\mathfrak{n}) >> 0$. If, in addition, $r = d(\ell(\mathfrak{m}) - \ell(\mathfrak{n}))$ with $d = \dim X$, then we say that X is *reduced* $\mathfrak{x}$*-stable*.

As in §2.2, we may define a function

$$rs : \mathbb{S}\mathrm{ch}_\kappa \times \mathbf{Arc}_\kappa \to \mathbb{N} \cup \{+\infty\}$$

by $rs_\mathfrak{x}(X) = \ell(\mathfrak{n})$ where $\mathfrak{n}$ is the minimum fat point in $\mathbb{I}$ which satisfies the definition for rational $\mathfrak{x}$-stability. When X is reduced $\mathfrak{x}$-stable, we say that X is *reduced* $\mathfrak{x}$*-stable at level* $rs_\mathfrak{x}(X) - 1$. We denote the collection of all reduced $\mathfrak{x}$-stable schemes as

$$RS_\mathfrak{x} := rs_\mathfrak{x}^{-1}(\mathbb{N}) \ .$$

For each $\mathfrak{x} \in \mathbf{Arc}_\kappa$, we have a set map

$$\lambda_\mathfrak{x} : RS_\mathfrak{x} \to \hat{\mathcal{H}}_\kappa$$

defined by

$$\lambda_{\mathfrak{r}}(X) := [(\nabla_{\mathfrak{r}/\mathfrak{n}}X)^{red}]\mathbb{L}^{-\ell(\mathfrak{n})d}$$

where $d = \dim X$ and $\ell(\mathfrak{n}) \geq rs_{\mathfrak{r}}(X)$.

3.3.2 Definition. Let X be an element of $RS_{\mathfrak{r}}$, B a subsieve of $\nabla_{\mathfrak{r}}X$, and let $\beta : B \to \mathbb{Z} \cup \{+\infty\}$ be a function such that for all $\mathfrak{m} \in \mathbb{I}$ with $\ell(\mathfrak{m}) \geq rs_{\mathfrak{r}}(X)$, the assignment $\pi^{\mathfrak{r}}_{\mathfrak{m}}(B)$ is a formal sieve. In this case, we say that α is a reduced $\mathfrak{r}$-stable function. We say that a subsieve A of $\nabla_{\mathfrak{r}}X$ is a *reduced* $\mathfrak{r}$*-stable* if its characteristic function is reduced $\mathfrak{r}$-stable.

For any reduced $\mathfrak{r}$-stable subsieve B of $\nabla_{\mathfrak{r}}X$, we define

$$\lambda_{\mathfrak{r}}(B) := [(\pi^{\mathfrak{r}}_{\mathfrak{n}}(B))^{red}]\mathbb{L}^{-d\ell(\mathfrak{n})}$$

where $\ell(\mathfrak{n}) \geq rs_{\mathfrak{r}}(X)$ and $d = \dim X$. Let $\mathbb{B}^X_{\mathfrak{r}}$ denote the collection of all rational $\mathfrak{r}$-stable subsets of $\nabla_{\mathfrak{r}}X$.

3.3.3 Theorem. *Let X be reduced $\mathfrak{r}$-stable and assume that $d = \dim X$. We have a set map*

$$\lambda_{\mathfrak{r}} : \mathbb{B}^X_{\mathfrak{r}} \to \hat{\mathcal{H}}_{\kappa}$$

with the following properties:

(a) For $B \in \mathbb{B}^X_{\mathfrak{r}}$, we may define

$$\lambda_{\mathfrak{r}}(B) := [(\pi^{\mathfrak{r}}_{\mathfrak{n}}(B))^{red}]\mathbb{L}^{-d\ell(\mathfrak{n})} \in \hat{\mathcal{H}}_{\kappa}\ .$$

(b) If X is generically smooth and S is the singular locus of X, then it is the case that $\lambda_{\mathfrak{r}}(\nabla_{\mathfrak{r}} B \setminus \nabla_{\mathfrak{r}} \hat{X}_S) = \lambda_{\mathfrak{r}}(\nabla_{\mathfrak{r}} B)$ where $B \in \mathbb{B}^X_{\mathfrak{r}}$.

(c) When $\{B_i\}$ is a countable collection of mutually disjoint elements of $\mathbb{B}^X_{\mathfrak{r}}$, then we may define

$$\lambda_{\mathfrak{r}}(\cup_i B_i) := \sum_i \lambda_{\mathfrak{r}}(B_i) \ ,$$

whenever the right hand side converges in $\hat{\mathcal{H}}_\kappa$.

Proof. The is exactly the same as the proof of Theorem 2.4.1 and Theorem 2.4.4.

□

3.3.4 Remark. Note here that $\mathcal{X} := (\pi^{\mathfrak{r}}_{\mathfrak{n}}(B))$ will be a formal sieve, and we are using the notation $(\pi^{\mathfrak{r}}_{\mathfrak{n}}(B))^{red}$ to denote the reduction of the constructible subscheme formed by some natural choice of scheme X for which $\mathcal{X}(\kappa)$ will be a constructible subset of $X(\kappa)$. Clearly, when $\mathcal{X}$ is a constructible sieve or even a subschemic sieve, then this notation corresponds to the standard meaning.

Without loss of generality, we can assume Z is irreducible and affine in the following construction. Following §2 of [Sch1], for any $Z \in \mathbb{S}\mathfrak{ch}_\kappa$ and any constructible subset F of $Z(\kappa)$, we define the constructible cone $\mathfrak{C}_Z(F)$ of F over Z to be the sieve

$$\mathfrak{C}_Z(F)(\mathfrak{m}) := \rho_{\mathfrak{m}}^{-1}(F)$$

where $\rho_{\mathfrak{m}} : Z(\mathfrak{m}) \to Z(\kappa)$ is the set map induced by the residue field morphism $\rho_{\mathfrak{m}} : \mathrm{Spec}(\kappa) \to \mathfrak{m}$. More generally, F does not have to be constructible. The functor $\mathfrak{C}_Z$ is well known in deformation theory.

3.3.5 Definition. Let X be a reduced $\mathfrak{x}$-stable scheme and let $B \in \mathbb{B}_{\mathfrak{x}}^X$. We define a set map $\bar{\mu}_{\mathfrak{x}} : \mathbb{B}_{\mathfrak{x}}^X \to \hat{\mathcal{H}}_\kappa$ by

$$\bar{\mu}_{\mathfrak{x}}(B) := [\mathfrak{C}_{(\nabla_{\mathfrak{n}} X)^{red}}((\pi_{\mathfrak{n}}^{\mathfrak{x}}(B)(\kappa))^{red})]\mathbb{L}^{-d\ell(\mathfrak{n})} ,$$

where $d = \mathrm{dim} X$.

3.3.6 Notation. This is notationally heavy. In this chapter, we will just write $F_{\mathfrak{n}}(B)$ in place of $(\pi_{\mathfrak{n}}^{\mathfrak{x}}(B)(\kappa))^{red}$. Let me rephrase to clear up any potential confusion. In the rest of this chapter, we let $F_{\mathfrak{n}}(B)$ denote the constructible subset of $(\nabla_{\mathfrak{n}} X)^{red}$ formed by the inverse image under f of the constructible subset of $\nabla_{\mathfrak{n}} X(\kappa)$ determined by the κ-rational points of the formal sieve $\pi_{\mathfrak{n}}^{\mathfrak{x}}(B)$ where f is the reduction morphism $f : (\nabla_{\mathfrak{n}} X)^{red} \to \nabla_{\mathfrak{n}} X$.

3.3.7 Proposition. *Definition 3.3.5 is independent of choice of $\mathfrak{n} \in \mathbb{I}$ for large enough $\ell(\mathfrak{n})$.*

Proof. In this proof, we find it useful to somewhat abuse notation and denote the representable functor $(\mathbb{A}_\kappa^r)^\circ = \mathrm{Mor}_\kappa(-, \mathbb{A}_\kappa^r)$ by $\mathbb{L}^r(-)$. Also, we will set $m = \ell(\mathfrak{m})$ and $n = \ell(\mathfrak{n})$ for simplicity of notation. At any rate, we may reduce to the case where there exists an isomorphism of constructible sets

$$j : (\nabla_{\mathfrak{x}/\mathfrak{m}} X)^{red} \xrightarrow{\sim} (\nabla_{\mathfrak{x}/\mathfrak{n}} X)^{red} \times_\kappa \mathbb{A}_\kappa^{d(m-n)}$$

This induces a rational isomorphism of sieves

$$j_-^\circ : ((\nabla_{\mathfrak{x}/\mathfrak{m}} X)^{red})^\circ(-) \xrightarrow{\sim} ((\nabla_{\mathfrak{x}/\mathfrak{n}} X)^{red})^\circ(-) \times \mathbb{L}^{d(m-n)}(-)$$

Now, a is a $\mathfrak{v}$-rational point of $\mathfrak{C}_{(\nabla_\mathfrak{m} X)^{red}}(F_\mathfrak{m}(B))(\mathfrak{v})$ if and only if

$$a(\mathfrak{v}) \in F_\mathfrak{m}(B) ,$$

and this is true if and only if

$$j(a(\mathfrak{v})) \in F_\mathfrak{n}(B) \times_\kappa \mathbb{A}_\kappa^{d(m-n)} .$$

Equivalently, $j_\mathfrak{v}^\circ(a) = j \circ a$ is a $\mathfrak{v}$-rational point of $F_\mathfrak{n}(B)^\circ(\mathfrak{v}) \times \mathbb{L}^{d(m-n)}(\mathfrak{v})$ such that

$$(j \circ a)(\mathfrak{v}) = (b \times_\kappa c)(\mathfrak{v}) \in F_\mathfrak{n}(B) \times_\kappa \mathbb{A}_\kappa^{d(m-n)} ,$$

where b is an element of $F_\mathfrak{n}(B)^\circ(\mathfrak{v})$ and c is an element of $\mathbb{L}^{d(m-n)}(\mathfrak{v})$. This is true if and only if $j_\mathfrak{v}^\circ(a)$ is an element of

$$\mathfrak{C}_{(\nabla_\mathfrak{n} X)^{red}}(F_\mathfrak{n}(B))(\mathfrak{v}) \times \mathfrak{C}_{\mathbb{A}_\kappa^{d(m-n)}}(\mathbb{A}_\kappa^{d(m-n)})(\mathfrak{v}) .$$

By Proposition 7.1 of [Sch1], $\mathfrak{C}_{\mathbb{A}_\kappa^{d(m-n)}}(\mathbb{A}_\kappa^{d(m-n)}) \cong \mathbb{L}^{d(m-n)}(-)$. Therefore, j_-° induces a rational isomorphism of sieves

$$\mathfrak{C}_{(\nabla_\mathfrak{m} X)^{red}}(F_\mathfrak{m}(B)) \cong \mathfrak{C}_{(\nabla_\mathfrak{n} X)^{red}}(F_\mathfrak{n}(B)) \times \mathbb{L}^{d(m-n)}(-) .$$

□

3.3.8 Theorem. *Let X be reduced $\mathfrak{r}$-stable and assume that $d = dim X$. We have a set map*

$$\bar{\mu}_{\mathfrak{r}} : \mathbb{B}_{\mathfrak{r}}^{X} \to \hat{\mathcal{H}}_{\kappa}$$

with the following properties:

(a) For $B \in \mathbb{B}_{\mathfrak{r}}^{X}$, we have

$$\bar{\mu}_{\mathfrak{r}}(B) := [\mathfrak{C}_{(\nabla_{\mathfrak{n}} X)^{red}}(F_{\mathfrak{n}}(B))]\mathbb{L}^{-d\ell(\mathfrak{n})} \in \hat{\mathcal{H}}_{\kappa} \ .$$

(b) If X is generically smooth and S is the singular locus of X, then it is the case that $\bar{\mu}_{\mathfrak{r}}(\nabla_{\mathfrak{r}} B \setminus \nabla_{\mathfrak{r}} \hat{X}_S) = \bar{\mu}_{\mathfrak{r}}(\nabla_{\mathfrak{r}} B)$ where $B \in \mathbb{B}_{\mathfrak{r}}^{X}$.

(c) When $\{B_i\}$ is a countable collection of mutually disjoint elements of $\mathbb{B}_{\mathfrak{r}}^{X}$, then we may define

$$\bar{\mu}_{\mathfrak{r}}(\cup_i B_i) := \sum_i \bar{\mu}_{\mathfrak{r}}(B_i) \ ,$$

whenever the sum on right hand side converges

Proof. Again, the proof is the same as Theorem 2.4.1 and Theorem 2.4.4. □

3.4 Lax stability and extension of measures

Let X be a reduced $\mathfrak{r}$-laxly stable and let $\mathfrak{n} \in \mathbb{I}$ be such that $\ell(\mathfrak{n}) = rls_{\mathfrak{r}}(X)$, we define $\mathbb{S}(\mathfrak{n}) := \{\mathfrak{m} \in \mathbb{I} \mid \mathfrak{m} \geq \mathfrak{n}\}$ and a function

$$l : \mathbb{S}(\mathfrak{n}) \to \mathbb{N}$$

by $l(\mathfrak{m}) := l_{\mathfrak{r}}^{X}(\mathfrak{m}) := r - d(\ell(\mathfrak{m}) - \ell(\mathfrak{n}))$ where r is the unique positive number given to us such that the map $(\nabla_{\mathfrak{r}/\mathfrak{m}}X)^{red} \to (\nabla_{\mathfrak{r}/\mathfrak{n}}X)^{red}$ is a piecewise trivial fibration with general fiber $\mathbb{A}_{\kappa}^{r}$ and d is the dimension of X.

When X is a $\mathfrak{r}$-laxly stable scheme, we define $\mathbb{B}_{\mathfrak{r}}^{X}$ to be the collection of all $\mathfrak{r}$-laxly stable subsieves of X. We then have a set map $\mu_{\mathfrak{r}}^{l} : \mathbb{B}_{\mathfrak{r}}^{X} \to \hat{\mathcal{H}}_{\kappa}$ defined by

$$\mu_{\mathfrak{r}}^{l}(A) := [\pi_{\mathfrak{m}}^{\mathfrak{r}}(A)]\mathbb{L}^{-d\ell(\mathfrak{m})-l(\mathfrak{m})} \tag{3.4.1}$$

for large enough $\ell(\mathfrak{m})$.

Moreover, when X is a reduced $\mathfrak{r}$-laxly stable scheme, we define $\mathbb{B}_{\mathfrak{r}}^{X}$ to be the collection of all reduced $\mathfrak{r}$-laxly stable subsieves of X. We then have a set map $\bar{\mu}_{\mathfrak{r}}^{l} : \mathbb{B}_{\mathfrak{r}}^{X} \to \hat{\mathcal{H}}_{\kappa}$ defined by

$$\bar{\mu}_{\mathfrak{r}}^{l}(A) := [\mathfrak{C}_{(\nabla_{\mathfrak{m}}X)^{red}}(F_{\mathfrak{m}}(A))]\mathbb{L}^{-d\ell(\mathfrak{m})-l(\mathfrak{m})} \tag{3.4.2}$$

for large enough $\ell(\mathfrak{m})$.

3.4.1 Remark. If X is (reduced) $\mathfrak{x}$-stable, then the function l is functionally equivalent to 0. Thus,

$$\mu^l_{\mathfrak{x}} \equiv \mu^0_{\mathfrak{x}} \equiv \mu_{\mathfrak{x}} \ , \quad \bar{\mu}^l_{\mathfrak{x}} \equiv \bar{\mu}^0_{\mathfrak{x}} \equiv \bar{\mu}_{\mathfrak{x}} \ , \quad \lambda^l_{\mathfrak{x}} \equiv \lambda^0_{\mathfrak{x}} \equiv \lambda_{\mathfrak{x}}$$

3.4.2 Remark. Note that $\mu^l_{\mathfrak{x}}$, $\bar{\mu}^l_{\mathfrak{x}}$, and $\lambda^l_{\mathfrak{x}}$ are well defined – i.e., they do not depend on choice of fat point $\mathfrak{m}$ when $\ell(\mathfrak{m}) >> 0$. The only work that needs to be done here is to notice that Theorem 3.3.8 does not depend on the dimension of the general fiber being $d(\ell(m) - \ell(n))$.

3.4.3 Theorem. *The analogue of Theorem 2.4.1 (resp., Theorem 3.3.3, Theorem 3.3.8) hold for $\mu^l_{\mathfrak{x}}$, (resp. $\bar{\mu}^l_{\mathfrak{x}}$, $\lambda^l_{\mathfrak{x}}$). Also, the analogue of Theorem 2.4.4 holds for all of these measures as well.*

Proof. The proofs are exactly the same as before. □

3.4.4 Example. Following Example 3.1.18, we have for any n

$$\lambda^l_{\mathfrak{l}}(\nabla_{\mathfrak{l}}\mathfrak{l}_2) = [(\nabla_{\mathfrak{l}_n}\mathfrak{l}_2)^{red}]\mathbb{L}^{-l(n)}$$

where $l(n) = n - \lceil \frac{n}{2} \rceil$. This is equal to 1 in $\hat{\mathcal{H}}_\kappa$.

In some sense then one wants to say that there is a more primitive relation occurring before this example which would basically involve looking at the trivial deformations of $\mathbb{A}^{-l(n)}_\kappa$ for each n over $\mathfrak{l}_2$. However, it is unclear how to make this precise or meaningful as $\nabla_{\mathfrak{l}_n}\mathfrak{l}_2$ is not isomorphic to $\mathbb{A}_{\mathfrak{l}_2}^{-l(n)}$. In fact, in Chapter 5, we investigate the possibility of defining an alternative measure which may shed some light on this.

We may alternatively fix this issue in the following way. We notice that $\hat{\sigma}(\lambda^l_{\mathfrak{l}}(\nabla_{\mathfrak{l}}\mathfrak{l}_2)) = 1$ in $\hat{\mathcal{G}}_\kappa$ and that the sequence $([\nabla_{\mathfrak{l}_n}\mathfrak{l}_2]\mathbb{L}^{-l(\mathfrak{l}_n)})$ can be realized as an element of the ultrapower $\prod_\sim \mathcal{H}_\kappa$ with respect to any ultrafilter $\sim$ on $\mathbb{I}$. Clearly, there is a well defined ring homomorphism

$$\prod_\sim \mathcal{H}_\kappa \to \prod_\sim \mathcal{G}_\kappa.$$

It is also clear that under this ring homomorphism $([\nabla_{\mathfrak{l}_n}\mathfrak{l}_2]\mathbb{L}^{-l(\mathfrak{l}_n)})$ is sent to 1 (or, more generally, to an element of the diagonal in $\prod_\sim \mathcal{G}_\kappa$). Clearly then, if $([\nabla_{\mathfrak{l}_n}\mathfrak{l}_2]\mathbb{L}^{-l(\mathfrak{l}_n)})$ were to stabilize, then we would have the more primitive notion of measurability of $\nabla_{\mathfrak{l}}\mathfrak{l}_2$. This is the perspective taken in Chapter 4. However, it is not the case that $([\nabla_{\mathfrak{l}_n}\mathfrak{l}_2]\mathbb{L}^{-l(\mathfrak{l}_n)})$ is eventually constant, yet, nevertheless, we can still collect this data in the ultraproduct of Grothendieck rings of the formal motivic site and call this a measure of $\nabla_{\mathfrak{l}}\mathfrak{l}_2$. At any rate, let us continue this line of thought with another example.

3.4.5 Example. Let us now consider the trivial deformation X of a smooth affine variety Y over $\mathfrak{l}_2$. Then, we may use $\lambda^l_{\mathfrak{l}}$ to measure X where $l(\mathfrak{l}_n) = n - \lceil \frac{n}{2} \rceil$. As we have already proved Theorem 3.2.7,

$$\nabla_{\mathfrak{l}_n} X \cong Y \times_k \mathbb{A}_\kappa^{d(\ell(\mathfrak{l}_n)-1)} \times_\kappa \nabla_{\mathfrak{l}_n} \mathfrak{l}_2,$$

which in the reduction simplifies to

$$(\nabla_{\mathfrak{l}_n} X)^{red} \cong Y \times_\kappa \mathbb{A}_\kappa^{d(n-1)} \times_\kappa \mathbb{A}_\kappa^{l(n)}$$

where $l(n) = n - \lceil \frac{n}{2} \rceil$. This is a result of our previous example. Then, we may calculate

$$\begin{aligned}
\lambda_{\mathfrak{l}}^l(\nabla_{\mathfrak{l}} X) &= [\nabla_{\mathfrak{l}_n} X]\mathbb{L}^{-d\ell(\mathfrak{l}_n)-l(\mathfrak{l}_n)} \\
&= [Y \times_\kappa \mathbb{A}_\kappa^{d(n-1)} \times_\kappa \mathbb{A}_\kappa^{l(n)}]\mathbb{L}^{-d\ell(\mathfrak{l}_n)-l(\mathfrak{l}_n)} \\
&= [Y]\mathbb{L}^{d(n-1)+l(\mathfrak{l}_n)}\mathbb{L}^{-d\ell(\mathfrak{l}_n)-l(\mathfrak{l}_n)} \\
&= [Y]\mathbb{L}^{-d}
\end{aligned}$$

Here d is the dimension of X. In particular, if $Y = \mathbb{A}_\kappa^d$, then $X = \mathbb{A}_{\mathfrak{l}_2}^d$ and we obtain

$$\lambda_{\mathfrak{l}}^l(\nabla_{\mathfrak{l}} \mathbb{A}_{\mathfrak{l}_2}^d) = 1.$$

The above example works if we replace $\mathfrak{l}_2$ with $\mathfrak{l}_n$, but then note that the function l will need to change accordingly. Going the opposite direction, we have also shown that if X is an affine scheme which is smooth over a fat point $\mathfrak{n}$ and $\mathbb{I}$ is $\mathfrak{n}$-rational over some variety V (whose limit point is $\mathfrak{x}$), then we have a well-defined measure

$$\lambda_{\mathfrak{x}}^l(\nabla_{\mathfrak{x}} X) = [(\nabla_{\mathfrak{m}} X)^{red}][V]\mathbb{L}^{-d\ell(\mathfrak{m})}$$

where $\mathfrak{m}$ is the first simple fat point such that for all $\mathfrak{v} \in \mathbb{I}$ with $\mathfrak{m}$ a closed subscheme of $\mathfrak{v}$, $\mathfrak{v}$ is a $[(\nabla_{\mathfrak{v}}\mathfrak{n})^{red}] = [V]\mathbb{L}^r$ in $\hat{\mathcal{G}}_\kappa$. The previous example is just a specific case of this result.

3.4.6 Example. We continue with the previous two examples, but this time we measure them with respect to $\bar{\mu}^l_{\mathfrak{l}}$. Note first that

$$\mathfrak{C}_{(\nabla_{\mathfrak{l}_n}\mathfrak{l}_2)^{red}}((\nabla_{\mathfrak{l}_n}\mathfrak{l}_2)^{red}(\kappa)) = \mathbb{A}_\kappa^{l(\mathfrak{l}_n)}$$

as the $\mathfrak{C}_X(S)$ behaves like the completion function when X is a scheme and S is a subscheme and in this case X is affine $l(\mathfrak{l}_n)$-space over κ. Thus, it is immediate that $\bar{\mu}^l_{\mathfrak{l}}(\nabla_{\mathfrak{l}}\mathfrak{l}_2) = 1$. Likewise, if X is the trivial deformation of a smooth scheme Y over $\mathfrak{l}_2$ and κ is algebraically closed, then $\bar{\mu}^l_{\mathfrak{l}}(\nabla_{\mathfrak{l}}X) = [Y]\mathbb{L}^{-d}$.

One aspect to schemic motivic integration which has been almost completely ignored until this point is what happens when X^{red} is not smooth. The main reason for this is a matter of limited time and space in writing this dissertation, but there is also the added complexity to studying deformations of non-smooth schemes. Indeed, our main tool in this regard is to hope that X is smooth over some fat point $\mathfrak{n}$ and there is a point system $\mathbb{I}$ which is $\mathfrak{n}$-linear over a variety V, but this approach is quickly seen to fail. This is because if X is a deformation over a fat point $\mathfrak{n}$ of a singular variety, then it will necessarily be the case that the canonical morphism $X \to \mathfrak{n}$ is not smooth because smoothness is stable under base change. However, if X^{red} is generically smooth with singular locus S, then it looks like something can be said if X is a local deformation over an artinian ring. The situation however is significantly more complicated. What can be said immediately is that if X is a trivial deformation of a generically smooth variety Y over a fat point $\mathfrak{n}$ and $\mathbb{I}$ is $\mathfrak{n}$-linear of a variety V , then

$$\nabla_{\mathfrak{m}} X \cong \nabla_{\mathfrak{m}} Y \times_{\kappa} \nabla_{\mathfrak{m}} \mathfrak{n}$$

for any $\mathfrak{m} \in \mathbb{I}$, and we can use a product measure to $\mu_{\mathfrak{r}} \times \bar{\mu}^{l}_{\mathfrak{r}}$ to measure $\nabla_{\mathfrak{r}} X$. Thus, we will have

$$(\mu_{\mathfrak{r}} \times \bar{\mu}^{l}_{\mathfrak{r}})(\nabla_{\mathfrak{r}} X) = [U][V]\mathbb{L}^{-d}$$

where U is the smooth locus of Y and d is the dimension of Y and the measure takes place in $\hat{\mathcal{G}}_{\kappa}$. Note that we are using Theorem 2.4.4 here. What exactly happens when the deformation is not trivial is unclear. Keeping this in mind, we briefly investigate this question a little in the example below.

3.4.7 Example. Let Y be the node–i.e., the spectrum of $\kappa[x, y]/(xy)$. Then, for each $\lambda \in \kappa$, we have a deformation $X_{\lambda} = \operatorname{Spec}(\kappa[x, y, \varepsilon]/(xy - \lambda\varepsilon, \varepsilon^2))$ of Y over $\mathfrak{l}_2$ where $\lambda = 0$ gives the trivial deformation. We have just shown that $\nabla_{\mathfrak{l}} X_0$ has a well defined motivic volume of 2 in $\hat{\mathcal{H}}_{\kappa}$ as the first term in the product is the classical motivic volume of the node and the second volume in the product is 1. In order to understand the case where $\lambda \neq 0$, one needs to investigate the sequence $((\nabla_{\mathfrak{l}_n} X_\lambda)^{red})_n$. With some involved calculation, I obtained X_λ for $n = 1$, $\mathrm{Spec}(\kappa[a_0, a_1, b_0, b_1, c]/I)$ where $I = (a_0 b_0 - \lambda c, a_0 b_1 + a_1 b_0)$ for $n = 2$, and finally $\mathrm{Spec}(\kappa[a_0, a_1, a_2, b_0, b_1, b_2, c]/J)$ where $J = I + (a_0 b_2 + a_2 b_0 + a_1 b_1)$. From the theory, what we do have are formal sieves $\mathcal{X}_n$ such that

$$\nabla_{\mathfrak{l}_n} X_\lambda = (\nabla_{\mathfrak{l}_n} Y) \sqcup \mathcal{X}_n$$

where $\mathcal{X}_n = \nabla_{\mathfrak{l}_n} \mathcal{Y}$ where $\mathcal{Y} = X^\circ \setminus Y^\circ$. Therefore, it would be natural to investigate the sequence $(2 + [\nabla_{\mathfrak{l}_n} \mathcal{Y}]\mathbb{L}^{-n})_n$. Conjecturally, I believe it is possible to understand this sequence. More clearly, for $n = 2$, we obtain $\nabla_{\mathfrak{l}_2} Y \sqcup \mathbb{A}^1_\kappa$, and we also see that $\mathcal{X}_3$ is isomorphic to a parabolic surface. I expect this behaviour to continue – i.e., that $\mathcal{X}_n$ is isomorphic to the functor of points of a rational variety of dimension $n - 1$ for $n \geq 2$. Thus, it is expected that the class of $\mathcal{X}_n$ in $\mathcal{G}_\kappa$ is $\mathbb{A}^{n-1}_\kappa$. Thus, it is expected that this sequence is sent to a convergent sequence under $\hat{\sigma}$ and that this sequence will converge to

$$2 + \mathbb{L}^{-1} = (2\mathbb{L} + 1)/\mathbb{L} = (\mathbb{L} + [\mathbb{P}^1_\kappa])/\mathbb{L} = [\mathbb{P}^1_\kappa]\mathbb{L}^{-1} + 1$$

in $\hat{\mathcal{G}}_\kappa$.

3.4.8 Remark. Motivic integration has been used to form a more general notion of euler characteristic – i.e., the stringy euler characteristic χ^{st}. For example, the reader may wish to briefly consult sections 5, 6, and 7 of [WV] in order to verify that $\chi^{st}(N) = 2$ and $\chi^{st}(C) = \frac{1}{2}$ where N is the node and C is the cuspidal cubic. Keeping the previous example of deformations of the node in mind, one could then use this to define the euler characteristic of a non-trivial deformation as 3 and the euler characteristic of the trivial deformation as 2. With this perspective in mind, we would then expect that the motivic volume of the family of deformations of the node is $2 + \mathbb{L}_*[\mathbb{P}^1_\kappa]\mathbb{L}^{-1} + \mathbb{L}_*$ as this would agree with integrating against the euler characteristic of the bump function defined by the euler characteristic of each fiber along $\mathbb{A}^1_\kappa$. Note then, it becomes interesting to characterize such sequences in $\hat{\mathcal{H}}_\kappa$ with the hope that they will contain even more information about the deformation or the family of deformations in this way. Clearly, much more work needs to be done in this direction. In general, applying our theory of motivic integration to the study of additive invariants appears to be a good direction for further research.

Chapter 4

Generalized Schemic Integration

The goal of this chapter is to present a more general theory of schemic integration. To this aim, we will strongly pull from ideas concerning constructible motivic integration (cf. [CL]) in that we wish to try to avoid the completion of the Grothendieck group if possible. We employ sheaf-theoretic techniques. This is greatly beneficial as it give us the notion of a local integral. Moreover, we account for the possibility that some integrals will not glue and thus construct a notion of higher order integration.

4.1 Measurable Pro-sieves

In §7 of [Sch2], a partial order on $\mathbb{F}\mathtt{at}_\kappa$ was introduced. This is defined as $\mathfrak{m}' \leq \mathfrak{m}$ if $\mathfrak{m}'$ is a closed subscheme[1] of $\mathfrak{m}$. We say that a subset $\mathbb{I}$ of $\mathbb{F}\mathtt{at}_\kappa$ is a *point system* and that the direct limit $\mathfrak{z}$ of elements of $\mathbb{I}$ in the category of locally ringed spaces

[1] In other words, this is the partial order determined by the epimorphisms in the category $\mathbb{F}\mathtt{at}_\kappa$.

is a limit point. We denote the full subcategory of locally ringed spaces formed by fat points over κ by $\widehat{\mathbb{Fat}}_\kappa$. This allows us to define the *limit arc operator* $\nabla_{\mathfrak{z}}$ relative to a limit point $\mathfrak{z} = \varinjlim \mathbb{I}$ as the projective limit of natural transformations $\nabla_{\mathfrak{m}} : \mathbb{Form}_\kappa \to \mathbb{Form}_\kappa$ where $\mathfrak{m} \in \mathbb{I}$.

It is worth noting here that in Chapters 2 and 3, we concerned ourselves with special kinds of limit points termed admissible arcs. We did this to insure that the image of the limit arc space in the truncated arc space is a formal sieve (i.e., measurable); however, in this chapter we work in full generality by introducing the concept of a measurable limit sieve.

4.1.1 Definition. Let $\mathcal{X}_0 \in \mathbb{Sieve}_\kappa$. Let $\mathfrak{z} \in \widehat{\mathbb{Fat}}_\kappa$ and choose a point system $\mathbb{I}$ such that $\mathfrak{z} = \varinjlim \mathbb{I}$. For each $\mathfrak{m} \in \mathbb{I}$, let $\mathcal{X}_{\mathfrak{m}} \in \mathbb{Sieve}_\kappa$ be such that there is natural inclusion

$$\mathcal{X}_{\mathfrak{m}} \hookrightarrow \nabla_{\mathfrak{m}} \mathcal{X}_0. \tag{4.1.1}$$

We define $\mathcal{X} := \varprojlim \mathcal{X}_{\mathfrak{m}}$ and call it *a limit sieve at the point* $\mathfrak{z}$ *with respect to the point system* $\mathbb{I}$. We call $\mathcal{X}_0$ *the base* of $\mathcal{X}$.

Fixing a point system $\mathbb{I}$ with $\mathfrak{z} = \varinjlim \mathbb{I}$, we may form the category of limit sieves at a limit point $\mathfrak{z}$, denoted by $\mathbb{LSieve}_{(\mathfrak{z},\mathbb{I})}$, whose objects are functors

$$\mathcal{X} : \mathbb{Fat}_\kappa \to \mathbf{Set} \tag{4.1.2}$$

where $\mathcal{X}$ is a limit sieve at the point $\mathfrak{z} \in \widehat{\mathbb{Fat}}_\kappa$ with respect to the point system $\mathbb{I}$ and where $\mathcal{X}_0$ is the base of $\mathcal{X}$, and whose morphisms are defined as follows. Let $\mathcal{X}$ and $\mathcal{Y}$ be two objects in $\mathbb{LSieve}_{(\mathfrak{z},\mathbb{I})}$ which have base $\mathcal{X}_0$ and $\mathcal{Y}_0$, respectively. Then, a morphism

$$f : \mathcal{X} \to \mathcal{Y} \tag{4.1.3}$$

in $\mathbb{LSieve}_{(\mathfrak{z},\mathbb{I})}$ will be a projective limit of morphisms of sieves $f_\mathfrak{m} : \mathcal{X}_\mathfrak{m} \to \mathcal{Y}_\mathfrak{m}$ such that for some morphism of sieves $f_0 : \mathcal{X}_0 \to \mathcal{Y}_0$ the following diagram

$$\begin{tikzcd}
\mathcal{X}_\mathfrak{m} \arrow[r, hook, "\subset"'] \arrow[d, "f_\mathfrak{m}"'] & \nabla_\mathfrak{m}\mathcal{X}_0 \arrow[d, "\nabla_\mathfrak{m} f_0"] \\
\mathcal{Y}_\mathfrak{m} \arrow[r, hook, "\subset"'] & \nabla_\mathfrak{m}\mathcal{Y}_0
\end{tikzcd}$$

commutes in the category $\mathbb{Sieve}_\kappa$ for all $\mathfrak{m} \in \mathbb{I}$. Note that the horizontal inclusions are the ones given in Definition 4.1.1.

4.1.2 Example. An important example of a limit sieve is what we call *a limit scheme*. This occurs when the base $\mathcal{X}_0$ of a limit sieve $\mathcal{X} \in \mathbb{LSieve}_{(\mathfrak{z},\mathbb{I})}$ is naturally isomorphic to a scheme $X \in \mathbb{Sch}_\kappa$ and where $\mathcal{X}_\mathfrak{m}$ is naturally isomorphic to $\nabla_\mathfrak{m} X$ for each $\mathfrak{m} \in \mathbb{I}$. This implies that $\mathcal{X}$ is naturally isomorphic to $\nabla_\mathfrak{z} X$ in $\mathbb{LSieve}_{(\mathfrak{z},\mathbb{I})}$ where $\mathfrak{z} = \varinjlim \mathbb{I}$. We denote the full subcategory of $\mathbb{LSieve}_{(\mathfrak{z},\mathbb{I})}$ whose objects are limit schemes by $\mathbb{LSch}_{(\mathfrak{z},\mathbb{I})}$. Thus, by Definition 4.1.1, every limit sieve $\mathcal{X}$ is naturally contained in a limit scheme; we call any such limit scheme *an ambient space* of $\mathcal{X}$.

4.1.3 Proposition. *Every object in the category $\mathbb{LS}\mathrm{ch}_{(\mathfrak{z},\mathbb{I})}$ is represented by a scheme over $\mathfrak{z}$. More clearly, if $X \in \mathbb{S}\mathrm{ch}_\kappa$ and $\mathbb{I}$ is any point system with $\mathfrak{x} = \varinjlim \mathbb{I}$, then $\nabla_\mathfrak{x} X$ is a scheme over $\mathfrak{z}$.*

Proof. By general nonsense, it is enough to prove the second claim. Just note that the transition maps are affine. The claim follows from Lemma 2.2 of [SP]. □

4.1.4 Remark. Note the scheme $\nabla_\mathfrak{x} X$ given to us in Proposition 4.1.3 is in general not separated or of finite type over κ. Also, its underlying topological space is usually not quasi-compact.

Let us now fix an object $\mathcal{S} \in \mathbb{LS}\mathrm{ieve}_{(\mathfrak{z},\mathbb{I})}$. We form the category of limit $\mathcal{S}$-sieves, denoted by $\mathbb{LS}\mathrm{ieve}_\mathcal{S}$, as the full subcategory of the slice category of $\mathbb{LS}\mathrm{ieve}_{(\mathfrak{z},\mathbb{I})}$ over $\mathcal{S}$ defined as follows. Objects are morphisms $j : \mathcal{X} \to \mathcal{S}$ in $\mathbb{LS}\mathrm{ieve}_{(\mathfrak{z},\mathbb{I})}$ such that there exits a non-negative integer n and a natural inclusion $\iota : \mathcal{X} \hookrightarrow \mathcal{S} \times \mathbb{A}^n_\kappa$ such that the following diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\iota"] \arrow[dr, "j"'] & \mathcal{S} \times \mathbb{A}^n_\kappa \arrow[d, "p"] \\ & \mathcal{S} \end{tikzcd}$$

commutes in $\mathbb{LS}\mathrm{ieve}_{(\mathfrak{z},\mathbb{I})}$. Here p is the projection onto the first factor.

4.1.5 Remark. Note that if $\mathcal{S} \in \mathbb{LS}\mathrm{ieve}_{(\mathfrak{z},\mathbb{I})}$, then $\mathcal{S} \times \mathbb{A}^n_\kappa \in \mathbb{LS}\mathrm{ieve}_{(\mathfrak{z},\mathbb{I})}$. Indeed, we may choose as its base $\mathcal{S}_0 \times \mathbb{A}^n_\kappa$ where $\mathcal{S}_0$ is the base of $\mathcal{S}$ as

$$\nabla_\mathfrak{m}(\mathcal{S}_0 \times \mathbb{A}^n_\kappa) \cong \nabla_\mathfrak{m}\mathcal{S}_0 \times \mathbb{A}^{n \cdot \ell(\mathfrak{m})}_\kappa \,. \tag{4.1.4}$$

4.1.6 Remark. Note that $\mathfrak{x}$ is an element of $\mathbb{LSieve}_{(\mathfrak{z},\mathbb{I})}$. Thus, we can form the category $\mathbb{LSieve}_{\mathfrak{z}}$ of limit $\mathfrak{x}$-sieves. Note that $\mathbb{LSieve}_{\mathfrak{z}}$ is a subcategory $\mathbb{LSieve}_{(\mathfrak{z},\mathbb{I})}$, and these two categories are not naturally isomorphic.

4.1.7 Definition. We say that a subcategory $\mathbf{\Box}$ of $\mathbb{LSieve}_{\mathcal{S}}$ is a *partial motivic site* if it is closed under products and for each $X \in \mathbb{LSch}_{\mathcal{S}}$, the set

$$\mathbf{\Box}|_X := \{\mathcal{X} \in \mathbf{\Box} \mid \mathcal{X} \subset X\} \tag{4.1.5}$$

forms a distributive semi-lattice with union as the join operation and intersection as the meet operation. If $\mathbf{\Box}|_X$ is a distributive lattice (not just a semi-lattice) for each $X \in \mathbb{LSch}_{\mathcal{S}}$, then we say that $\mathbf{\Box}$ is a *motivic site*.

4.1.8 Remark. This definition is related to the definition of motivic sites which occur in [Sch1]. However, it is slightly different as we need to make use of the notion of limit sieves. Note that they are always stable under push-forwards of closed immersion; thus, the disjoint union of any two elements in a motivic site $\mathbf{\Box}$ relative to $\mathcal{S}$ is again an element of $\mathbf{\Box}$.

4.1.9 Example. A trivial example of a motivic site is $\mathbb{LSieve}_{\mathcal{S}}$ for any $\mathcal{S} \in \mathbb{LSieve}_{(\mathfrak{z},\mathbb{I})}$.

4.1.10 Definition. Given a morphism $s : \mathcal{Y} \to \mathcal{X}$ in $\mathbb{LSieve}_{\mathcal{S}}$ and a $\mathcal{S}$-subsieve $\mathcal{X}' \subset \mathcal{X}$, we define *the pull-back of $\mathcal{X}'$ along s*, denoted by $s^*\mathcal{X}'$ as the $\mathcal{S}$-subsieve defined by

$$s(\mathfrak{m})^{-1}\mathcal{X}'(\mathfrak{m}) \in \mathbf{Set}. \tag{4.1.6}$$

If $\mathcal{X} \in \mathbb{LSieve}_{\mathcal{S}}$ and X is an ambient space of $\mathcal{X}$, we say that a subsieve of $\mathcal{X}$ is an $\mathcal{S}$-*admissible open* of $\mathcal{X}$ if it is of the form $\mathcal{X} \cap U^\circ$ where U is an open set of X. Note that this definition does not depend on the ambient space of $\mathcal{X}$. We say that a morphism $s : \mathcal{X} \to \mathcal{Y}$ in $\mathbb{LSieve}_{\mathcal{S}}$ is $\mathcal{S}$-*continuous* if the pull-back of an $\mathcal{S}$-admissible open along s is again a $\mathcal{S}$-admissible open. We say that a morphism $s : \mathcal{Y} \to \mathcal{X}$ in a (partial) motivic site $\square$ is a $\square$-*homeomorphism* if it is continuous and bijective whose inverse is also continuous.

4.1.11 Remark. From now on, we will always assume that a morphism of sieves is continuous.

Let $\square$ be a (partial) motivic site. Then, we may form the Grothendieck ring $\mathbf{Gr}(\square)$ in the following way. We denote the isomorphism class (defined by $\square$-homeomorphisms) of $\mathcal{X} \in \square$ by $\langle \mathcal{X} \rangle$. Then, we denote by $\mathbf{Gr}(\square)$ the free abelian group generated by $\langle \mathcal{X} \rangle$ modulo the *scissor relations*

$$\langle \mathcal{X} \cup \mathcal{Y} \rangle + \langle \mathcal{X} \cap \mathcal{Y} \rangle - \langle \mathcal{X} \rangle - \langle \mathcal{Y} \rangle \tag{4.1.7}$$

when $\mathcal{X}$ and $\mathcal{Y}$ share the same ambient space X. We denote by $[\mathcal{X}]$ the residue class of $\langle \mathcal{X} \rangle$ in $\mathbf{Gr}(\square)$ and for simplicity write $[X]$ for $[X^\circ]$. We define multiplication on $\mathbf{Gr}(\square)$ by $[\mathcal{X}]\cdot[\mathcal{Y}] := [\mathcal{X}\times_{\mathcal{S}}\mathcal{Y}]$. We denote this ring by $\mathbf{Gr}(\square)$ and call it the *Grothendieck ring of a motivic site*.

4.1.12 Definition. Let us fix an ultrafilter on the set $\mathbb{I}$ and a base $\mathcal{S}_0$ of $\mathcal{S}$ where $\mathcal{S}_0$ is a scheme. We say that a limit sieve $\mathcal{X}$ is $\mathcal{S}_0$*-measurable with respect to an ultrafilter* $\sim$ or $(\mathcal{S}_0,\sim)$*-measurable* if

$$\mu^{\sim}_{\mathfrak{z},\mathbb{I}}(\mathcal{X}) := \mathrm{ulim}[\mathcal{X}_{\mathfrak{m}}]\mathbb{L}^{-\dim \nabla_{\mathfrak{m}}\mathcal{S}_0} \tag{4.1.8}$$

is an element of the image of the diagonal ring homomorphism

$$\delta : \mathbf{Gr}(\mathbb{S}\mathbb{ieve}_{\kappa})_{\mathbb{L}} \to \prod_{\sim} \mathbf{Gr}(\mathbb{S}\mathbb{ieve}_{\kappa})_{\mathbb{L}}\,. \tag{4.1.9}$$

If $\mathcal{X}$ is $(\mathcal{S}_0,\sim)$-measurable for every ultrafilter $\sim$ on $\mathbb{I}$, then we just say that $\mathcal{X}$ is $\mathcal{S}_0$*-measurable*. We denote the full subcategory of $\mathbb{L}\mathbb{S}\mathbb{ieve}_{(\mathfrak{r},\mathbb{I})}$ whose objects are all measurable limit sieves as $\mathbb{M}\mathbb{S}\mathbb{ieve}_{(\mathfrak{r},\mathbb{I})}$. Moreover, given an object $\mathcal{S}\in\mathbb{L}\mathbb{S}\mathbb{ieve}_{(\mathfrak{r},\mathbb{I})}$, we denote the full subcategory of $\mathbb{L}\mathbb{S}\mathbb{ieve}_{\mathcal{S}}$ whose objects are measurable by $\mathbb{M}\mathbb{S}\mathbb{ieve}_{\mathcal{S}}$. Finally, given a motivic site $\square$, we denote the full subcategory of $\square$ containing all measurable objects of $\square$ by $\Delta(\square)$. We denote the smallest partial motivic site containing $\Delta(\square)\cup\{\mathcal{S}\}$ and contained in $\square$ by $\mathbb{M}\mathbb{es}\square$. We call $\mathbb{M}\mathbb{es}\square$ the *measurable motivic site of* $\square$. If $\square$ is $\mathbb{L}\mathbb{S}\mathbb{ieve}_{\mathcal{S}}$, then we just write $\mathbb{M}_{\mathcal{S}}$ for $\mathbb{M}\mathbb{es}\square$.

4.1.13 Conjecture. *Let us fix a point system* $\mathbb{I}$ *and a base* $\mathcal{S}_0$ *of* $\mathcal{S}$ *where* $\mathcal{S}_0$ *is* *a scheme. Let $\mathcal{X}$ be a limit $\mathcal{S}$-sieve. Then if $\mathcal{X}$ is $(\mathcal{S}_0, \sim)$-measurable for some ultrafilter $\sim$ on $\mathbb{I}$, then $\mathcal{X}$ is $\mathcal{S}_0$-measurable.*

4.1.14 Remark. As it is now understood that measurability depends on the choice of some schemic base $\mathcal{S}_0$, we will often say a limit sieve is measurable in place of $\mathcal{S}_0$-measurable.

4.1.15 Proposition. *The category $\Delta(\square)$ is closed under products, disjoint union and contains the empty sieve. Moreover, if the intersection of two elements in $\Delta(\square)$ are in $\Delta(\square)$, then the union of these two elements are in $\Delta(\square)$ and vice versa.*

Proof. First note that we fix a scheme $\mathcal{S}_0$ as a base of $\mathcal{S}$. Thus, $\nabla_{\mathfrak{m}}\mathcal{S}_0$ is a scheme and $\dim\nabla_{\mathfrak{m}}\mathcal{S}_0$ makes sense. Clearly, the empty sieve is measurable.

We show that $\mathbb{M}\mathrm{es}\square|_X$ is closed under disjoint union. As the diagonal is a subring of the ultraproduct, we may add $\mu^{\sim}_{\mathfrak{z},\mathbb{I}}(\mathcal{X})$ and $\mu^{\sim}_{\mathfrak{z},\mathbb{I}}(\mathcal{Y})$ to again get an element of the diagonal. We have

$$\mu^{\sim}_{\mathfrak{z},\mathbb{I}}(\mathcal{X}) + \mu^{\sim}_{\mathfrak{z},\mathbb{I}}(\mathcal{Y}) = \mathrm{ulim}(([\mathcal{X}_{\mathfrak{m}}] + [\mathcal{Y}_{\mathfrak{m}}]) \cdot \mathbb{L}^{-\dim\nabla_{\mathfrak{m}}\mathcal{S}_0})\,. \tag{4.1.10}$$

Now, applying the scissor relation in the Grothendieck ring, we have that the right hand side is of the form $\mu^{\sim}_{\mathfrak{z},\mathbb{I}}(\mathcal{X} \sqcup \mathcal{Y})$ computed using the arbitrary good base $\mathcal{Z}_0$. The proof for the product follows along the same lines as the proof for the disjoint union. The proof of the last statement follows from the disjoint union case as is readily seen. $\square$

Measurability here is always relative to a schemic base $\mathcal{S}_0$. The reason is that Equation 4.1.8 relative to some fixed schemic base $\mathcal{S}_0$ of $\mathcal{S}$ will give rise to a unique ring homomorphism

$$\mu : \mathbf{Gr}(\mathbb{Mes}\blacksquare) \to \prod_{\sim} \mathbf{Gr}(\mathbb{Sieve}_\kappa) \, . \tag{4.1.11}$$

Thus, we have the following non-exact sequence

$$\mathbf{Gr}(\mathbb{Mes}\blacksquare) \xrightarrow{\mu} \prod_{\sim} \mathbf{Gr}(\mathbb{Sieve}_\kappa) \overset{can}{\twoheadrightarrow} \mathrm{cok}(\delta) \to 0 \tag{4.1.12}$$

in the category of rings. Moreover,

$$\mathrm{gr}(\Delta(\blacksquare)) = \ker(can \circ \mu), \tag{4.1.13}$$

where $\mathrm{gr}(\Delta(\blacksquare))$ denotes the image of the $\blacksquare$-homeomorphism classes of elements of $\Delta(\blacksquare)$ modulo all scissor relations coming from $\mathbb{Mes}\blacksquare$. Here we write gr as to not to confuse the fact that $\Delta(\blacksquare)$ is not a motivic site. Therefore, we have a left exact sequence of rings

$$0 \hookrightarrow \mathrm{gr}(\Delta(\blacksquare)) \hookrightarrow \mathbf{Gr}(\mathbb{Mes}\blacksquare) \xrightarrow{\bar{\mu}} \mathrm{cok}(\delta) \tag{4.1.14}$$

where we denote the composition of μ and can given above as $\bar{\mu}$. In other words, $\bar{\mu}$ obeys the following rule

$$\bar{\mu}([\mathcal{X} \cup \mathcal{Y}]) = \bar{\mu}([\mathcal{X} \cap \mathcal{Y}]) \tag{4.1.15}$$

for all $\mathcal{X}$ and $\mathcal{Y}$ in $\Delta(\Box)$. Of course, it is also invariant under translation by an element of $\Delta(\Box)$ – i.e., for any $a \in \mathrm{gr}(\Delta(\Box))$,

$$\bar{\mu}(X + a) = \bar{\mu}(X) \tag{4.1.16}$$

for all $X \in \mathbf{Gr}(\mathbb{M}\mathrm{es}\Box)$. In particular, if $\mathcal{S}$ itself is measurable, then $\bar{\mu}(\mathbb{L}^n) = 0$ for all n greater than or equal to 0.

4.1.16 Lemma. *Let $\mathcal{S}$ be a limit sieve and $\Box_{\mathcal{S}}$ a motivic site relative to $\mathcal{S}$. For any admissible open $\mathcal{U}$ of $\mathcal{S}$, let $\iota_{\mathcal{U}}$ be the inclusion morphism of $\mathcal{U}$ into $\mathcal{S}$ and let $\iota^*_{\mathcal{U}}$ denote the pullback of motivic sites. Then, $\mathbf{Gr}(\mathbb{M}\mathrm{es}\,\iota^*_{-}\Box_{\mathcal{S}})$ is a flasque sheaf of rings on $\mathcal{S}$.*

Proof. We just need to prove that $G(-) := \mathbf{Gr}(\mathbb{M}\mathrm{es}\,\iota^*_{-}\Box_{\mathcal{S}})$ is in fact a sheaf as the rest of the statement is trivial. Therefore, let $\mathcal{U}$ be an admissible open of $\mathcal{S}$ and let $\mathcal{U}_i$ be an arbitrary cover of $\mathcal{U}$ by admissible opens. For each i, let $s_i \in G(\mathcal{U}_i)$ be such that $s_i|_{\mathcal{U}_j} = s_j|_{\mathcal{U}_i}$. We reduce to the case where $s_i = [\mathcal{X}_i]$ for some sieve $\mathcal{X}_i \in \mathbb{M}\mathrm{es}\,\iota^*_{\mathcal{U}_i}\Box_{\mathcal{S}} \subset \mathbb{M}\mathrm{es}\,\iota^*_{\mathcal{U}}\Box_{\mathcal{S}}$. One may also quickly reduce to assuming that each $\mathcal{X}_i$ is contained in $\mathcal{U}_i$. It is straightforward to show that if $\mathcal{X}$ belongs to $\Delta(\Box)$ then so does $\mathcal{X}_i = \mathcal{X} \cap \mathcal{U}_i$. Now, by definition of $\iota^*_{\mathcal{U}_i}$ there is a $\mathcal{X} \in \mathbb{M}\mathrm{es}\Box$ such that $\iota^*_{\mathcal{U}_i}\mathcal{X} = \mathcal{X}_i$ and note that $\iota^*_{\mathcal{U}_i}\mathcal{X} = \mathcal{X} \cap \mathcal{U}_i$. Now, it is immediate that we have a sheaf of sets assigning to each admissible open $\mathcal{U}$ of $\mathcal{S}$ the motivic site $\mathbb{M}\mathrm{es}\,\iota^*_{\mathcal{U}}\Box$ from which it follows that $G(-)$ is a sheaf. $\square$

4.1.17 Remark. There are alternative possibilities. One could use the constant sheaf of $\mathbf{Gr}(\mathbb{M}\mathrm{es}\square)$ or the constant sheaf of $\mathbf{Gr}(\square)$. Using these sheaves would not effect the material in the rest of the paper.

4.2 Integrable Functions

Consider the ring

$$\mathbb{A} := \mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}, (\frac{1}{1-\mathbb{L}^{-i}})_{i>0}] \ . \tag{4.2.1}$$

along with the ring $\mathbb{B} := \mathbb{Z}[\mathbb{L}][[\mathbb{L}^{-1}]]$. Note that for every positive real number q there is an evaluation map ϑ_q from $\mathbb{B}$ to $\mathbb{R} \cup \{\pm\infty\}$ defined by sending $\mathbb{L}$ to q.

Let $\mathcal{S} \in \mathbb{LS}\mathrm{ieve}_{(\mathfrak{z},\mathbb{I})}$. Let $\mathcal{B}_{\mathcal{S}}$ be the sheaf of $\mathbb{B}$-algebras defined by sending an admissible open $\mathcal{U}$ of $\mathcal{S}$ to

$$\mathcal{B}_{\mathcal{S}}(\mathcal{U}) := \{ f : \mathcal{U} \to \mathbb{B} \mid f \text{ any function } \} \ . \tag{4.2.2}$$

By a function $f : \mathcal{U} \to \mathbb{B}$, we mean that for each $\mathfrak{m} \in \mathbb{F}\mathrm{at}_\kappa$, there is a set-theoretic function $f(\mathfrak{m}) : \mathcal{S}(\mathfrak{m}) \to \mathbb{B}$. In other words, f is an assignment (in general, not a functor), from $\mathbb{F}\mathrm{at}_\kappa$ to $\mathbf{Set}$ which associates to each fat point $\mathfrak{m}$ a set-theoretic function from $\mathcal{S}(\mathfrak{m})$ to $\mathbb{B}$. This is readily seen to be equivalent to f being a function from all pairs $(\mathfrak{m}, a)$ to $\mathbb{B}$ where $\mathfrak{m} \in \mathbb{F}\mathrm{at}_\kappa$ and $a \in \mathcal{S}(\mathfrak{m})$.

4.2.1 Definition. Let $\square$ be a motivic site relative to an object $\mathcal{S}$ in $\mathbb{LS}\mathrm{ieve}_{(\mathfrak{x},\mathbb{I})}$. We call any subsheaf $\mathcal{T}_{\mathcal{S}}$ of $\mathcal{B}_{\mathcal{S}}$ in the Grothendieck topos of sheaves of $\mathbb{Z}[\mathbb{L}]$-algebras on $\mathcal{S}$ a *sheaf of total functions on* $\mathcal{S}$ if for each admissible open $\mathcal{U}$ of $\mathcal{S}$, $\mathcal{T}_{\mathcal{S}}(\mathcal{U})$ contains every characteristic function $\mathbb{1}_{\mathcal{Z}}$ where $\mathcal{Z}$ is measurable subsieve of $\mathcal{U}$ in $\square$ and if there is a non-negative real number $1/R$ such that $\vartheta_q(f)$ is well-defined (i.e., $|\vartheta_q(f(\mathfrak{m})(a))| < \infty$ for all $\mathfrak{m} \in \mathbb{F}\mathtt{at}_\kappa$ and all $a \in \mathcal{S}(\mathfrak{m})$) for every section f of $\mathcal{T}_{\mathcal{S}}$ and for all real q strictly greater than $1/R$. Moreover, we call the minimal such R the *radius* of $\mathcal{T}_{\mathcal{S}}$. If $1/R = 0$, we say that the radius is infinite.

4.2.2 Remark. It is reasonable to let the radius R vary over admissible opens of $\mathcal{S}$. For our purpose, restricting R to be constant over every admissible open is enough. However, results will easily generalize to the case with varying radius.

4.2.3 Example. Let $\mathcal{A}_{\mathcal{S}}$ be the subsheaf of $\mathcal{B}_{\mathcal{S}}$ obtained by restricting the range of sections of $\mathcal{B}_{\mathcal{S}}$ to $\mathbb{A}$. Then, $\mathcal{A}_{\mathcal{S}}$ is a sheaf of total functions. The radius of $\mathcal{A}_{\mathcal{S}}$ is $R = 1$.

4.2.4 Definition. We say that a sequence of elements (b_i) in $\mathbb{B}$ is *q-convergent to* b if $(\vartheta(b_i))$ converges to $(\vartheta_q(b))$ in $\mathbb{R}$ for a fixed real number $q > 0$. Moreover, we say that (b_i) is *convergent to* b *with radius* R if $(\vartheta_q(b_i))$ converges to $\nu_q(b)$ in $\mathbb{R}$ for all real $q > 1/R \geq 0$ and $1/R$ is minimal with respect to this property.

4.2.5 Example. A trivial example is given by a constant sequence determined by an element a of $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}]$. In this case, the sequence converges to a with infinite radius. For good measure, we present a slightly less trivial example. Let (a_i) be a sequence such that $a_i = \mathbb{L}^{n_i}$ for almost all i. Assume also that it is q-convergent to $a \in \mathbb{A}$ for some real $q > 1$. Then,

$$|\vartheta_q(a_i) - \vartheta_q(a_j)| = q^{\min\{n_i,n_j\}}|q^{|n_i-n_j|} - 1|, \quad \forall i, j \ . \tag{4.2.3}$$

Since $|\vartheta_q(a_i) - \vartheta_q(a_j)|$ converges to zero, either n_i diverges to $-\infty$ or (n_i) is a Cauchy sequence. However, a Cauchy sequence of integers is eventually constant. Thus, either n_i diverges to $-\infty$ or $n_i = N$ for all sufficiently large i where N is some fixed integer. Thus, a is either equal to 0 or $\mathbb{L}^N$ for some integer N. Thus, (a_i) converges to either 0 or $\mathbb{L}^N$ with with radius 1. If we assume from the start that (n_i) is a Cauchy sequence, then it follows that (a_i) converges to $\mathbb{L}^N$ for some integer N with infinite radius. Finally, if (a_i) is q-convergent for some q strictly less than 1 and (n_i) is not Cauchy, then it follows that (n_i) diverges to ∞, which means that (a_i) will not be q-convergent for any $q > 1$. In summary, there are only three possible cases:

1. The sequence (a_i) is does not converge.

2. The exponents n_i form a Cauchy sequence in which case (a_i) is eventually a constant sequence determined by an element a in $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}]$, and thus, converges to a with infinite radius.

3. The exponents n_i diverge to $-\infty$ in which case (a_i) converges to 0 with radius 1.

Let $\mathcal{T}_{\mathcal{S}}$ be a sheaf of total functions on $\mathcal{S}$ of radius R. Let $\mathcal{X}_i$ be a countable collection of subsieves of $\mathcal{S}$ and let $\mathcal{T}_{\mathcal{X}_i}$ be the sheaf of total functions on $\mathcal{X}_i$ determined by multiplication of $\mathcal{T}_{\mathcal{S}}(\mathcal{U})$ by $\mathbb{1}_{\mathcal{X}_i}|_{\mathcal{U}}$ on each admissible open $\mathcal{U}$ of $\mathcal{S}$. For an admissible open $\mathcal{U}$ of $\mathcal{S}$, we say that a sequence $f_i \in \mathcal{T}_{\mathcal{X}_i}(\mathcal{U} \cap \mathcal{X}_i)$ is *summable* over $\mathcal{U}$ if the sequence $(\vartheta_q(f_i(\mathfrak{m})(a)))_i$ is summable in $\mathbb{R}$ for all $q > R$ and for all $\mathfrak{m} \in \mathbb{Fat}_\kappa$ and for all $a \in \mathcal{U}(\mathfrak{m})$. Moreover, we say that the sequence (f_i) is *strongly $\mathcal{X}_i$-summable* over $\mathcal{U}$ if there exists a local section $g \in \mathcal{T}_{\mathcal{X}_i}(\mathcal{U} \cap \mathcal{X}_i)$ such that

$$\vartheta_q(g(\mathfrak{m})(a)) = \sum_i \vartheta_q(f_i(\mathfrak{m})(a)) \tag{4.2.4}$$

for all $q > R$ and for all $\mathfrak{m} \in \mathbb{Fat}_\kappa$ and for all $a \in \mathcal{U}(\mathfrak{m})$. Note that here for notational simplicity, we are extending each f_i by zero.

4.2.6 Notation. Now consider an object $\mathcal{S}$ in the category $\mathbb{LSieve}_{(\mathfrak{r},\mathbb{I})}$ along with a sheaf of total functions $\mathcal{T}_{\mathcal{S}}$ on $\mathcal{S}$ relative to a motivic site $\square \subset \mathbb{LSieve}_{\mathcal{S}}$. We let $\square_{\mathcal{X}}$ be the motivic site relative to $\mathcal{X}$ given by $j^*\square$ – i.e., the pullback of all limit sieves in $\square$ along j. For each $\mathcal{X} \in \mathbb{LSieve}_{\mathcal{S}}$ with $\mathcal{X} = \mathcal{S} \times \mathbb{A}^n_\kappa$, we let $\mathcal{T}_{\mathcal{X}}$ to be any sheaf of total functions (relative to $\square_{\mathcal{X}}$) such that $p_*\mathcal{T}_{\mathcal{X}}$ is isomorphic to $\mathcal{T}_{\mathcal{S}}$ in the Grothendieck topos of sheaves of rings on $\mathcal{S}$ where p is the projection morphism onto the first factor. For a general element $\mathcal{X} \in \mathbb{LSieve}_{\mathcal{S}}$, which is a subsieve of $\mathcal{S} \times \mathbb{A}^n_\kappa$, we just restrict the sheaf $\mathcal{T}_{\mathcal{S}\times\mathbb{A}^n_\kappa}$ to $\mathcal{X}$. This, gives us a sheaf of rings

$\mathcal{T}_{\mathcal{X}}$ for each $\mathcal{X} \in \mathbb{LS}\mathbb{ieve}_{\mathcal{S}}$. Thus, whenever we choose a sheaf of total functions on $\mathcal{S}$, this gives rise to sheaves of total functions $\mathcal{T}_{\mathcal{X}}$ for each $\mathcal{X} \in \mathbb{LS}\mathbb{ieve}_{\mathcal{S}}$. We implicitly use this fact throughout the rest of this paper.

4.2.7 Definition. Let $\mathcal{X} \in \mathbb{LS}\mathbb{ieve}_{\mathcal{S}}$. Let $\mathbf{U} = \{\mathcal{U}_i\}$ be a countable admissible open cover of $\mathcal{S}$. Let $\mathcal{T}_{\mathcal{S}}$ be a sheaf of total functions on $\mathcal{S}$. Let $\mathcal{T}_{\mathcal{X}}$ be as in Notation 4.2.6. Let φ be a global section of $\mathcal{T}_{\mathcal{X}}$. We say that φ is *integrable over* $\mathcal{S}$ (or $\mathcal{S}$-*integrable*) *with respect to the cover* $\mathbf{U}$ *of order* k if there exists an automorphism γ of $\mathcal{X}$ such that

1. For all i, there exists a function $\psi_i \in \mathcal{T}_{\mathcal{S}}(\mathcal{U}_i)$ such that $\varphi|_{j^{-1}(\mathcal{U}_i)} = (j \circ \gamma|_{j^{-1}(\mathcal{U}_i)})^{\#}\psi_i$.
2. For this k, $N_{\mathbf{U}}(\psi)_k$ is strongly σ-summable over σ for each $\sigma \in N(\mathbf{U})_k$.

Note that the ψ_i's glue and thus give rise to a global section ψ of $\mathcal{T}_{\mathcal{S}}$. Thus, (2) will make sense once we explain the meaning of N, which we do below. We say that φ is *integrable over* $\mathcal{S}$ *of order* k if it is integrable over $\mathcal{S}$ with respect to any countable admissible open cover $\mathbf{U}$ of $\mathcal{S}$ of order k. If φ is integrable over $\mathcal{S}$ of order 0, then we just say it is *integrable over* $\mathcal{S}$.

The notation in (2) of Definition 4.2.7 needs to be explained. Here, $N(\mathbf{U})$

denotes the Cech nerve of the cover $\mathbf{U}$. Thus, for each k, we have the formula

$$N(\mathbf{U})_k := \bigsqcup_{i_0,\ldots,i_k} \mathcal{U}_{i_0,\ldots,i_k} \tag{4.2.5}$$

where $\mathcal{U}_{i_0,\ldots,i_k}$ denotes the intersection $\cap_{j=0}^{k}\mathcal{U}_{i_j}$ with $\mathcal{U}_{i_j} \in \mathbf{U}$. This operation will extend to an element ψ of $\mathcal{T}_{\mathcal{S}}(\mathcal{S})$ by defining

$$N_{\mathbf{U}}(\psi)_k := \prod_{i_0,\ldots,i_k} \psi|_{\mathcal{U}_{i_0,\ldots,i_k}} . \tag{4.2.6}$$

We may localize the condition by running through Definition 4.2.7 on each admissible open of $\mathcal{X}$. This gives us *the presheaf of $\mathcal{S}$-integrable functions in $\mathcal{T}_{\mathcal{X}}$ of order k*, which we denote by $\mathrm{I}^k_{\mathcal{S}}\mathcal{T}_{\mathcal{X}}$. In general, this presheaf is not a sheaf; however, it is a separated presheaf. At any rate, we sheafify this presheaf to obtain *the sheaf of $\mathcal{S}$-integrable functions in $\mathcal{T}_{\mathcal{X}}$ of order k* which is denoted by $\mathrm{I}^k_{\mathcal{S}}\mathcal{T}^+_{\mathcal{X}}$.

4.2.8 Proposition. *Let $\mathcal{S} \in \mathbb{LS}\mathbb{ieve}_{(\mathfrak{r},\mathbb{I})}$ and let $\mathcal{X} \in \mathbb{LS}\mathbb{ieve}_{\mathcal{S}}$. Then, for all k,*

$$\mathrm{I}^k_{\mathcal{S}}\mathcal{T}^+_{\mathcal{X}} \cong \mathrm{I}^0_{\mathcal{S}}\mathcal{T}^+_{\mathcal{X}} . \tag{4.2.7}$$

Thus, there is only one sheaf of integrable functions which we denote by $\mathrm{I}_{\mathcal{S}}\mathcal{T}_{\mathcal{X}}$.

Proof. The morphism is given by identity on each open set. It is easy to see that one direction of inclusion holds in general – i.e., if a function is integrable of order k, then it is integrable of order $k+1$. The reverse inclusion holds by gluing. $\square$

We define $\mathcal{T}^0_{\mathcal{S}}$ to be the sheaf of rings generated by the characteristic functions of all measurable subsieves $\mathcal{X}$ of $\mathcal{S}$ such that $\mathcal{X} \in \blacksquare$ and by $\mathbb{L}$. Let $\mathcal{T}_{\mathcal{S}}$ be any sheaf of total functions on $\mathcal{S}$. Then, $\mathcal{T}_{\mathcal{S}}$ is a sheaf of $\mathcal{T}^0_{\mathcal{S}}$-modules. Moreover, $\mathbf{Gr}(\mathbb{M}\mathbb{es}\iota^*\_\blacksquare)$ is a sheaf of $\mathcal{T}^0_{\mathcal{S}}$-modules for any motivic site $\blacksquare$ relative to $\mathcal{S}$. Thus, we may form the sheafy tensor products:

$$\begin{aligned} \mathcal{C}_{\mathcal{S}} &:= \mathbf{Gr}(\mathbb{M}\mathbb{es}\iota^*\_\blacksquare) \otimes_{\mathcal{T}^0_{\mathcal{S}}} \mathcal{T}_{\mathcal{S}} \\ \mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}} &:= j^{-1}\mathcal{C}_{\mathcal{S}} \otimes_{j^{-1}\mathcal{T}_{\mathcal{S}}} \mathrm{I}_{\mathcal{S}}\mathcal{T}_{\mathcal{X}} \ . \end{aligned} \tag{4.2.8}$$

We let $\mathrm{I}^k_{\mathcal{S}}\mathcal{C}_{\mathcal{X}}$ denote the presheaf tensor product $j_p^{-1}\mathcal{C}_{\mathcal{S}} \otimes_{j_p^{-1}\mathcal{T}_{\mathcal{S}}} \mathrm{I}^k_{\mathcal{S}}\mathcal{T}_{\mathcal{X}}$, where j is the structure morphism and j_p^{-1} is the presheaf inverse image functor.

4.2.9 Remark. Note that we are suppressing the notation for the motivic site $\blacksquare$ in the above definitions. In practice, this will not lead to confusion as one typically fixes the motivic site at the very start.

4.2.10 Example. Let $\mathcal{S} = \mathrm{Spec}(\kappa)$ where κ is an algebraically closed field. Then, an element of $\mathbb{LS}\mathbb{ieve}_\kappa$ is a subsieve of $\mathbb{A}^n_\kappa$ for some n. One may quickly check that $\mathbb{M}\mathbb{sieves}_\kappa$ contains the motivic site $\mathbf{Var}_\kappa$ of varieties over κ. In addition, $\mathcal{T}^0_{\mathcal{S}}(\mathcal{S})$ can be identified with the ring $\mathbb{Z}[\mathbb{L}]$. Moreover, $\mathcal{A}_{\mathcal{S}}(\mathcal{S}) = \mathbb{A}$. Therefore, for the motivic site $\mathbf{Var}_\kappa$ and for this choice of total functions on $\mathcal{S}$, we have

$$\mathcal{C}_{\mathcal{S}}(\mathcal{S}) \cong \mathbf{Gr}(\mathbf{Var}_\kappa)_{\mathbb{L}}[(\frac{1}{1-\mathbb{L}^{-i}})_{i>0}] \ . \tag{4.2.9}$$

Let $\mathcal{X}$ be any element of $\mathbb{LS}\mathbb{ieve}_\kappa$. Then, using Notation 4.2.6, $\mathcal{T}_{\mathcal{X}} \cong \mathcal{A}_{\mathcal{X}}$. It is easy to see that $\mathrm{I}_{\mathcal{S}}\mathcal{T}_{\mathcal{X}} \cong \mathcal{T}_{\mathcal{X}}$ as the condition for integrability becomes vacuous.

Thus,

$$\mathrm{I}^0_{\mathrm{Spec}(\kappa)}\mathcal{C}_{\mathcal{X}}(\mathcal{U}) \cong \mathbf{Gr}(\mathbf{Var}_\kappa)[\mathcal{A}_{\mathcal{X}}(\mathcal{U})] \tag{4.2.10}$$

on each admissible open $\mathcal{U}$ of $\mathcal{X}$. Thus, at a point $a \in \mathcal{X}(\kappa)$, the stalk is given by

$$(\mathrm{I}^0_{\mathrm{Spec}(\kappa)}\mathcal{C}_{\mathcal{X}})_a \cong \mathbf{Gr}(\mathbf{Var}_\kappa)[\mathcal{A}_{\mathcal{X},a}] \ , \tag{4.2.11}$$

which is in principle enough to define the sheaf $\mathrm{I}_{\mathrm{Spec}(\kappa)}\mathcal{C}_{\mathcal{X}}$. At any rate, next we will define a morphism of rings from $(\mathrm{I}_{\mathrm{Spec}(\kappa)}\mathcal{C}_{\mathcal{X}})_a$ to $\mathbf{Gr}(\mathbf{Var}_\kappa)_{\mathbb{L}}[(\frac{1}{1-\mathbb{L}^{-i}})_{i>0}]$.

Keep the notation of Definition 4.2.7. Let us fix a $k \in \mathbb{N}$ and assume that φ is $\mathcal{S}$-integrable of order k. Then, we have functions $\Sigma_\sigma(N_{\mathbf{U}}(\psi)_k)$ in $\mathcal{T}_{\mathcal{S}}(\sigma)$ for each $\sigma \in N(\mathbf{U})_k$. They agree on the open sets $\mathcal{U}_{i_0,\ldots,i_{k+1}}$. Thus, we may glue to obtain a Cech k-cochain with coefficients in $\mathcal{T}_{\mathcal{S}}$ which we denote by $\mu^k_{\mathcal{S}}(\mathbf{U})(\varphi)$. More specifically, we have the following theorem.

4.2.11 Theorem. *For each k and each cover $\mathbf{U}$ of $\mathcal{S}$, we have a group homomorphism*

$$\mu^k_{\mathcal{S}}(\mathbf{U}) : \mathrm{I}^k_{\mathcal{S}}\mathcal{C}_{\mathcal{X}}(\mathcal{X}) \to \check{\mathrm{C}}^k(\mathbf{U}, \mathcal{C}_{\mathcal{S}}) \ . \tag{4.2.12}$$

Moreover, $d^k \circ \mu^k_{\mathcal{S}}(\mathbf{U}) \equiv 0$.

Proof. This follows from the definition of gluing and the definition of Cech differential. □

Thus, for each k, we obtain group homomorphisms

$$\mu^k_{\mathcal{S}} : \mathrm{I}^k_{\mathcal{S}}\mathcal{C}_{\mathcal{X}}(\mathcal{X}) \to \check{\mathrm{H}}^k(\mathcal{S}, \mathcal{C}_{\mathcal{S}}) \tag{4.2.13}$$

by taking inverse limit of $\mu^k_{\mathcal{S}}(\mathbf{U})$ over the partial order given by refinement of covers. In fact, $\mu^0_{\mathcal{S}}$ is a morphism of presheaves of rings from $\mathrm{I}^0_{\mathcal{S}}\mathcal{C}_{\mathcal{X}}$ to $j_p^{-1}\mathcal{C}_{\mathcal{S}}$ where j_p^{-1} is the presheaf inverse image functor. Applying the sheafification functor to $\mu^0_{\mathcal{S}}$ gives us the morphism of sheaves of rings

$$\mu_{\mathcal{S}} : \mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}} \to j^{-1}\mathcal{C}_{\mathcal{S}}\ . \tag{4.2.14}$$

4.3 The Main Theorem

We present here a schemic analogue to Theorem 10.1.1 of [CL].

4.3.1 Theorem. *Let $\mathcal{S}$ be an element of $\mathbb{L}\mathbb{S}\mathrm{ieve}_{(\mathfrak{r},\mathbb{I})}$. Fix a motivic site $\square$ relative to $\mathcal{S}$ and a sheaf of total functions $\mathcal{T}_{\mathcal{S}}$ on $\mathcal{S}$. Every $\mathcal{X} \in \mathbb{L}\mathbb{S}\mathrm{ieve}_{\mathcal{S}}$ can be endowed with a sheaf of rings $\mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}}$ with the following properties*

1. *Existence of the four functors:*

   (a) *Every continuous morphism $f : \mathcal{X} \to \mathcal{X}'$ in $\mathbb{L}\mathbb{S}\mathrm{ieve}_{\mathcal{S}}$ induces a morphism of sheaves $f_{\#}$ from $\mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}}$ to $f^{-1}\mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}'}$ which is the restriction of the pushforward from $\mathcal{C}_{\mathcal{X}}$ to $f^{-1}\mathcal{C}_{\mathcal{X}}$.*

   (b) *The analogue of (a) is true for $f^{\#}$, $f_!$ and $f^!$ (when it exists).*

2. *Functoriality:*

   (a) *Let* $\lambda : \mathcal{S} \to \mathcal{S}'$ *be a morphism in* $\mathbb{LSieve}_{(\mathfrak{r},\mathbb{I})}$. *This induces a morphism* $\lambda_* : \mathbb{LSieve}_{\mathcal{S}} \to \mathbb{LSieve}_{\mathcal{S}'}$. *There is a natural inclusion of sheaves* $\mathrm{I}_{\mathcal{S}'}\mathcal{C}_{\lambda_*(\mathcal{X})}(\lambda_*(-)) \subset \mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}}(-)$.

   (b) *Given a continuous morphism* $f : \mathcal{X} \to \mathcal{X}'$ *of limit* $\mathcal{S}$*-sieves, we have a commutative diagram*

$$\begin{tikzcd} \mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}} \arrow[r, "f_\#"] \arrow[dr, "\mu_{\mathcal{S}}"'] & f^{-1}\mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}'} \arrow[d, "\mu_{\mathcal{S}}"] \\ & j^{-1}\mathcal{C}_{\mathcal{S}} \end{tikzcd}$$

   in the Grothendieck topos of sheaves of rings on $\mathcal{X}$. *Here* j *is the structure morphism of* $\mathcal{X}$.

3. *Integrability:* $\mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}}$ *is a subsheaf of* $\mathcal{C}_{\mathcal{X}}$.

4. *Additivity: If* $\mathcal{X}$ *is the disjoint union of admissible open subsieves* $\mathcal{X}_i$ *for some countable index* i*, then the isomorphism* $\mathcal{C}_{\mathcal{X}} \cong \prod_i \mathcal{C}_{\mathcal{X}_i}$ *induces an isomorphism* $\mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}} \cong \prod_i \mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}_i}$.

5. *Projection Formula: Given a continuous morphism* $f : \mathcal{X} \to \mathcal{X}'$ *of limit* $\mathcal{S}$*-sieves, an admissible open* $\mathcal{U}$*, an element* $x \in f^{-1}\mathcal{C}_{\mathcal{X}'}(\mathcal{U})$*, and an element* $y \in \mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{S}}(\mathcal{U})$*, then* $xf_\#(y)$ *is an element of* $f^{-1}\mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}'}(\mathcal{U})$ *if and only if*

$f^{\#}(x)y$ is an element of $\mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}}(\mathcal{U})$. If these conditions are satisfied, then $f_{\#}(f^{\#}(x)y) = xf_{\#}(y)$.

6. *Inclusions: Let $f : \mathcal{X} \to \mathcal{X}'$ be injective and let $\mathcal{U}$ be an admissible open of $\mathcal{X}$. Then, φ belongs to $\mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}}(\mathcal{U})$ if and only if $f_{\#}(\varphi)$ belongs to $f^{-1}\mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}'}(\mathcal{U})$.*

7. *Projection along κ-variables: Consider the morphism $p : \mathcal{X} \to \mathcal{X}'$ where $\mathcal{X} = \mathcal{X}' \times \mathbb{A}^m_\kappa$ and p is the projection onto the first factor. Then, for any admissible open $\mathcal{U}$ of $\mathcal{X}$, φ belongs to $\mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}}(\mathcal{U})$ if and only if $p_{\#}(\varphi)$ belongs to $\mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{X}'}(p(\mathcal{U}))$.*

8. *Projection along $\mathbb{N}$-variables: Consider the indexed family of sieves $\mathcal{X}_i = \mathcal{S}$ where $i \in \mathbb{N}$ and set $\mathcal{X} = \sqcup_i \mathcal{X}_i$. For any admissible open $\mathcal{U}$ of the form $\sqcup_i \mathcal{V}$ with $\mathcal{V}$ an admissible open of $\mathcal{S}$, φ belongs to $\mathrm{I}_{\mathcal{Y}}\mathcal{C}_{\mathcal{X}}(\mathcal{U})$ if and only if it is integrable over each copy of $\mathcal{V}$.*

Proof. We list the theorem in this way to make it clear which statements correspond to which in Theorem 10.1.1 of [CL]. We let $\mathcal{C}_{\mathcal{X}}$ be as in Notation 4.2.6 for each $\mathcal{X} \in \mathbb{LSieve}_{\mathcal{S}}$.

1. *Proof of statement 1*: Part (a) follows immediately from properties of push-forwards and using the fact that applying ϑ_q to the sheaves in question in-

duces morphisms in the category of sheaves of topological rings. Note that as $f_!$ is a subfunctor of $f_{\#}$, the analogue for $f_!$ is immediate. The proof of the analogous statement for the pullback $f^{\#}$ follows along the same lines as the proof for part (a). The statement for $f^!$ follows from the adjunction formula.

2. *Proof of statement 2*: For part (a), note that λ_* is the functor induced by inclusion – i.e., if $\mathcal{X}$ is in $\mathbb{LS}\mathrm{ieve}_{\mathcal{S}}$ given by an inclusion $\iota : \mathcal{X} \hookrightarrow \mathcal{S} \times \mathbb{A}^m_\kappa$, then $\lambda_*(\mathcal{X}) = \mathrm{im}((\lambda \times \mathrm{id}_{\mathbb{A}^m_\kappa}) \circ \iota)$. Note that every admissible open subset of $\lambda_*(\mathcal{X})$ is of the form $\lambda_*(\mathcal{U})$ where $\mathcal{U}$ is an admissible open subset of $\mathcal{X}$. The result follows.

3. *Proof of statement 3*: This follows from the fact that any presheaf of integrable functions over $\mathcal{S}$ is separated.

4. *Proof of statement 4*: Follows immediately from the exact sequence defining a sheaf.

5. *Proof of statement 5*: The result is true when f is the projection $\mathcal{X}' \times \mathbb{A}^n_\kappa \to \mathcal{X}'$. Using this together with statement 6 proves the general result.

6. *Proof of statement 6*: One direction is immediate. The other follows from the definition of a sheaf of total functions.

7. *Proof of statement 7*: This is a special case of statement 5.

8. *Proof of statement 8*: This follows immediately by definition.

□

The analogue for statements 1-8 holds for the separated presheaf $\mathrm{I}^0_{\mathcal{S}}\mathcal{C}_{\mathcal{X}}$. To see this, we replace the inverse image functor by its presheaf version and use $\mu^0_{\mathcal{S}}$ instead of $\mu_{\mathcal{S}}$. The proofs are the same except that additivity only holds for finite mutually disjoint covers. Moreover, by only considering global sections, we have analogous statements for $\mathrm{I}^k_{\mathcal{S}}\mathcal{C}_{\mathcal{X}}$. Thus, for the analogues dealing with higher order integration we have to replace $j^{-1}\mathcal{C}_{\mathcal{S}}$ by $\check{\mathrm{H}}^k(\mathcal{S},\mathcal{C}_{\mathcal{S}})$.

4.3.2 Remark. As we do not have the notion of cell-decomposition (instead choosing to work with general open covers), it is unclear if there are analogues to statements A7 and A8 of Theorem 10.1.1 of [CL].

In general, I do not know if $\mathrm{I}_{\mathcal{S}}\mathcal{C}_{\mathcal{S}}$ is isomorphic to $\mathcal{C}_{\mathcal{S}}$. Thus, we give the following definition.

4.3.3 Definition. Let $\mathcal{S} \in \mathbb{LS}\mathbb{ieve}_{(\mathfrak{r},\mathbb{I})}$. We say that a sheaf of total functions $\mathcal{T}_{\mathcal{S}}$ is *integrable* if $\mathrm{I}_{\mathcal{S}}\mathcal{T}_{\mathcal{S}}$ is isomorphic to $\mathcal{T}_{\mathcal{S}}$ in the Grothendieck topos of sheaves of rings on $\mathcal{S}$.

4.3.4 Example. Consider the subsheaf $\mathcal{P}_{\mathcal{S}}$ of $\mathcal{A}_{\mathcal{S}}$ defined as follows. We assign to each admissible open $\mathcal{U}$ the smallest ring generated by elements of $\mathbb{A}$, functions $\alpha : \mathcal{U} \to \mathbb{Z}$, functions $\mathbb{L}^{\beta}$ (with $\beta : \mathcal{U} \to \mathbb{Z}$) subject to the condition that the images of α and β are both presburger sets, and characteristic functions of measurable subsieves of $\mathcal{U}$ in some motivic site $\square_{\mathcal{S}}$. We let $\mathcal{P}_{\mathcal{S}}$ denote the sheafification of the resulting presheaf and call it *the sheaf of presburger functions on* $\mathcal{S}$ (relative to $\square_{\mathcal{S}}$). It is follows from well-known facts about presburger sets that any such sheaf is integrable.

Being a presburger sheaf is sufficient but in general not necessary for a sheaf of total functions to be integrable. For example, if $\mathfrak{z} \in \mathbb{F}\mathtt{at}_{\kappa}$–i.e., we work over finite limit points–then $\mathcal{S}$ will be quasi-compact. In which case, it is easy to see that any sheaf of total functions will be integrable.

Excluding this case, it is interesting to ask what are the necessary conditions on $\mathcal{S}$ and on $\mathcal{T}_{\mathcal{S}}$ for which a sheaf of total functions is integrable. Conjecturally, it seems likely that the only integrable sheaves of total functions (with radius 1) are sheaves of presburger functions on $\mathcal{S}$. At any rate, we want to draw the parallel between $\mathcal{P}_{\mathcal{S}}$ and $\Delta(\square_{\mathcal{S}})$ in that if we allow for functions on $\mathcal{S}$ which involve ultra-limits (in particular, ultra-exponents) in $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}]$, then the notion of integrability still has meaning. Moreover, in this way, one can naturally introduce a much larger class of integrable functions. In point of fact, if there are sheaves of total functions which are not integrable, then there is a strong case for doing this.

Chapter 5

An alternative viewpoint via auto-arcs.

In this chapter, we discuss the possibility of using an alternative measure which keeps track of the infinitesimal lifts of a scheme. Most notably, we use this chapter as an opportunity to introduce computational techniques of arc spaces via the Sage programming language. As we noted in Chapter 2, there is very little known about the *auto-arc spaces*. However, in this chapter, we will see that our Sage script will give much insight into the reduced structure of certain types of auto-arc spaces. In the end, this will lead us to make a rather astonishing conjecture concerning the reduced auto-arcs of truncated jets of curves.

5.1 An alternative measure.

As we noted in the examples of Chapter 3, some of the motivic behavior seems to be lost with the current version of the schemic motivic measure. With this in mind, our approach is to ask questions about lifts to $Y \to \mathfrak{m}$ of a given smooth morphism $X \to \mathfrak{n}$ when $\mathfrak{n} \hookrightarrow \mathfrak{m}$ closed immersion of fat points over κ. Note that a closed immersion $\mathfrak{n} \hookrightarrow \mathfrak{m}$ is only an open immersion of schemes if it is an isomorphism even though it is trivially an open immersion of the underlying topological spaces.

5.1.1 Theorem. *Let $X \in \mathbb{Sch}_\kappa$ and suppose that X is an affine. Let $f : X \to \mathfrak{n}$ be a smooth morphism where $\mathfrak{n} \in \mathbb{Fat}_\kappa$ and let $\iota : \mathfrak{n} \hookrightarrow \mathfrak{m}$ be a closed immersion in $\mathbb{Fat}_\kappa$. Then, there exists a unique smooth morphism $\bar{f} : Y \to \mathfrak{m}$ where $Y \in \mathbb{Sch}_\kappa$ such that $X \cong Y \times_{\mathfrak{m}} \mathfrak{n}$.*

Proof. First, we may reduce to the case where the closed immersion $\mathfrak{n} \hookrightarrow \mathfrak{m}$ is given by a square zero ideal J. Then, it is well-known that the obstruction to lifting smoothly to $Y \to \mathfrak{m}$ is an element of $H^2(X, T_X \otimes \widetilde{J})$ where T_X is the tangent bundle of X. Since $T_X \otimes \widetilde{J}$ is quasi-coherent and X is assumed to be affine, we have that

$$H^2(X, T_X \otimes \widetilde{J}) = 0 \ ,$$

by Theorem 3.5 of Chapter III of [H1]. The uniqueness part quickly follows as the obstruction to uniqueness is an element of $H^1(X, T_X \otimes \widetilde{J})$, which is also trivial since X is affine and $T_X \otimes \widetilde{J}$ is quasi-coherent. □

5.1.2 Remark. Note that X being affine here is important; otherwise, there is a cocycle condition on f that must be satisfied in order to insure that there is such a lift – i.e., to insure that the morphism we would obtain by gluing is smooth.

5.1.3 Remark. Let $X = \mathrm{Spec}(A)$ and let $\mathfrak{n} = \mathrm{Spec}(R)$ where R is the maximum artinian subring of A. Then, a slightly quicker proof of the above theorem can be obtained by noticing that it is necessarily the case that X is an infinitesimal deformation over $\mathfrak{n}$ of X^{red}. Thus, X^{red} being smooth implies that $X \cong X^{red} \times_\kappa \mathfrak{n}$. Then, one may easily see that $Y = X \times_\kappa \mathfrak{m}$ whenever $\mathfrak{n} \hookrightarrow \mathfrak{m}$ and that by base change $Y \to \mathfrak{m}$ is smooth. We present the above proof as its technique is more general and thus could be applied to more situations.

One might be tempted to postulate the following statement. Let $X \in \mathbb{S}\mathrm{ch}_\kappa$ be affine. Assume that $X \to \mathfrak{n}$ and $Y \to \mathfrak{m}$ are smooth morphisms such that $X = Y \times_\mathfrak{m} \mathfrak{n}$ where $\iota : \mathfrak{n} \hookrightarrow \mathfrak{m}$ are elements of the point system $\mathbb{I}$ associated to an admissible arc $\mathfrak{x} \in \mathbf{Arc}_\kappa$. Then, one would hope that there is a canonical morphism

$$\rho_\mathfrak{n}^\mathfrak{m} : \nabla_\mathfrak{m} Y \to \nabla_\mathfrak{n} X,$$

which is a piecewise trivial fibration over κ with general fiber $\mathbb{A}_\kappa^{d(\ell(m)-\ell(n))}$. However, we will see explicitly in this Chapter that this is rarely the case. Moreover, it is unclear exactly when there is a canonical morphism $\rho_\mathfrak{n}^\mathfrak{m} : \nabla_\mathfrak{m} Y \to \nabla_\mathfrak{n} X$ in the first place. However, this does not necessarily limit the theory as we may consider the reduction. To wit, we consider the following question.

5.1.4 Question. Let X be an object of $\mathbb{S}\mathrm{ch}_\kappa$ and let p be a κ-rational point of X. When is there a morphism of varieties $\rho^n_{n-1} : (\nabla_{J^n_p X} J^n_p X)^{red} \to (\nabla_{J^{n-1}_p X} J^{n-1}_p X)^{red}$? Moreover, when does such a morphism arise in a natural way?

Clearly, if $X \in \mathbb{S}\mathrm{ch}_k$ is smooth at p, then, by Theorem 5.1.15, there exists a morphism ρ^n_{n-1} given by projection. Moreover, in the case of the cuspidal cubic C (resp. the node N), the morphism ρ^n_{n-1} is by the truncation $\nabla_{\mathfrak{l}_{2n}} C \to \nabla_{\mathfrak{l}_{2n-2}} C$ (resp. $\nabla_{\mathfrak{l}_n} N^2 \to \nabla_{\mathfrak{l}_{n-1}} N^2$). Thus, a positive proof of Conjecture 5.3.5 will show that, for at least some curves, there is such a natural choice for ρ^n_{n-1}.

5.1.5 Lemma. *Let $X \in \mathbb{S}\mathrm{ch}_\kappa$ and let p be a κ-rational point of X. Then, for all $n \in \mathbb{N}$, there is a natural morphism $\rho^n_{n-1} : (\nabla_{J^n_p X} J^n_p X)^{red} \to (\nabla_{J^{n-1}_p X} J^{n-1}_p X)^{red}$.*

Proof. By the Yoneda lemma, it is enough to show that there is a canonical set map from $(\nabla_{J^n_p X} J^n_p X)^{red}(F) \to (\nabla_{J^{n-1}_p X} J^{n-1}_p X)^{red}(F)$ where F is any field extension of κ. This amounts to showing that there is commutative diagram

$$\begin{array}{ccc} A/\mathfrak{m}^n & \xrightarrow{f} & A/\mathfrak{m}^n \otimes_\kappa F \\ {\scriptstyle c}\big\downarrow & & \big\downarrow{\scriptstyle c\otimes\varphi} \\ A/\mathfrak{m}^{n-1} & \xrightarrow{\bar{f}} & A/\mathfrak{m}^{n-1} \otimes_\kappa F \end{array}$$

where A is a local ring containing κ with maximal ideal $\mathfrak{m}$, φ is an automorphism of F, c is the canonical surjection, and where $\bar{f}$ is induced by f. Indeed, $\bar{f}$ exists since $f(\mathfrak{m}^{n-1}/\mathfrak{m}^n) \subset \mathfrak{m}^{n-1}/\mathfrak{m}^n \cdot F$ for any ring homomorphism $f : A/\mathfrak{m}^n \to A/\mathfrak{m}^n \otimes_\kappa F$. $\square$

5.1.6 Remark. I am therefore in general proposing that a better or alternative type of measure for an affine scheme X could be the following. Let $Z = X^{red}$ and choose some point system $\mathbb{I}$. For each $\mathfrak{n} \in \mathbb{I}$, we find a deformation $Z_\mathfrak{m}$ of X^{red} with the condition that there is some $\mathfrak{v} \in \mathbb{I}$ such that $Z_\mathfrak{v} \cong X$. Then, we wish to understand the morphism $(\nabla_\mathfrak{m} Z_\mathfrak{m})^{red} \to (\nabla_\mathfrak{n} Z_\mathfrak{n})^{red}$ for all $\mathfrak{n},\ \mathfrak{m} \in \mathbb{I}$ such that $\ell(\mathfrak{m}) \geq \ell(\mathfrak{n}) \geq \ell(\mathfrak{v})$. Essentially, understanding this type of measure boils down to understand the reduced structure on the auto-arc spaces as we will see.

Naively, one might proceed to prove the aforementioned statement in the following way. Let $d = \dim X$. We may cover Y by a finite number of opens U, each of which will admit an étale morphism $U \to \mathbb{A}^d_\mathfrak{m}$. As étale morphisms are stable under base change, the restriction $U' \to \mathfrak{n}$ of U also admits an étale morphism $U' \to \mathbb{A}^d_\mathfrak{n}$. Therefore, from the start, we may assume that we have étale morphisms $Y \to \mathbb{A}^d_\mathfrak{m}$ and $X \to \mathbb{A}^d_\mathfrak{n}$. We then have the following isomorphisms:

$$\nabla_\mathfrak{m} Y \cong Y \times_{\mathbb{A}^d_\mathfrak{m}} \nabla_\mathfrak{m} \mathbb{A}^d_\mathfrak{m} \ \text{ and } \ \nabla_\mathfrak{n} X \cong X \times_{\mathbb{A}^d_\mathfrak{n}} \nabla_\mathfrak{n} \mathbb{A}^d_\mathfrak{n} \,.$$

Furthermore, we will have a commutative diagram

$$\begin{array}{ccc} \nabla_\mathfrak{m} Y & \xrightarrow{\cong} & (Y \times_\mathfrak{m} \nabla_\mathfrak{m} \mathfrak{m}) \times_\kappa \mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-1)} \\ {\scriptstyle \pi^\mathfrak{m}_\mathfrak{n}}\Big\downarrow & & \Big\downarrow \\ \nabla_\mathfrak{n} X & \xrightarrow{\cong} & (X \times_\mathfrak{n} \nabla_\mathfrak{n} \mathfrak{n}) \times_\kappa \mathbb{A}_\kappa^{d(\ell(\mathfrak{n})-1)} \end{array}$$

provided that there is a morphism $\nabla_{\mathfrak{m}}\mathfrak{m} \to \nabla_{\mathfrak{n}}\mathfrak{n}$.

It is exactly here that this approach breaks down because even though there exists a natural morphism $(\nabla_{\mathfrak{m}}\mathfrak{m})^{red} \to (\nabla_{\mathfrak{n}}\mathfrak{n})^{red}$, it is not clear exactly when it is a piece-wise trivial fibration. In fact, there can be a complicated scheme structure on the so-called auto-arcs $\nabla_{\mathfrak{n}}\mathfrak{n}$. This is discussed in much more detail in [Sch2] and is ostensibly the focus of this chapter. We will now consider an example to illustrate this behavior. Before that, let us give a definition that we will find useful.

5.1.7 Definition. We say that a fat point $\mathfrak{n}$ is *simple* if $(\nabla_{\mathfrak{n}}\mathfrak{n})^{red} \cong \mathbb{A}^m_\kappa$ for some $m \geq 0$. We say that a point system is *simple* if all of its fat points are simple and that a point system $\mathbb{I}$ is *eventually simple* if $\mathfrak{n} \in \mathbb{I}$ is simple when $\ell(\mathfrak{n}) >> 0$. Moreover, we say that a point system is *eventually simple* over a variety V if $(\nabla_{\mathfrak{n}}\mathfrak{n})^{red} \cong \mathbb{A}^m_V$ when $\ell(\mathfrak{n}) >> 0$.

5.1.8 Remark. Again, very little is known about $(\nabla_{\mathfrak{m}}\mathfrak{n})^{red}$. Thus, in particular, very little is known about the auto-arcs. To investigate the auto-arcs, we take a computational approach in this chapter which seems to be very promising.

5.1.9 Example. In the case where $\mathbb{I} = \{\mathfrak{l}_m\}$, we argued in Chapter 3 that $\mathbb{I}$ is a simple point system. Consider the case where $\mathfrak{m} = \operatorname{Spec} R$ where $R = \kappa[x,y]/(x^2, xy, y^2)$ and $\mathfrak{n} = \operatorname{Spec}(\kappa)$ so that $\nabla_{\mathfrak{n}}\mathfrak{n} = \operatorname{Spec}(\kappa)$. A quick calculation shows that

$$\nabla_{\mathfrak{m}}\mathfrak{m} = \operatorname{Spec}(\kappa[a_1, a_2, b_1, b_2, c_1, c_2]/I)$$

where I is the ideal generated by the elements

$$\{a_1a_2, a_1b_2 + a_2b_1, a_1c_2 + a_2c_1, a_1^2, a_1b_1, a_1c_1, a_2^2, a_2b_2, a_2c_2\}\ .$$

From this it is easy to see that $(\nabla_{\mathfrak{m}}\mathfrak{m})^{red} \cong \mathbb{A}^4_\kappa$. Therefore, $\{\mathfrak{m}, \mathfrak{n}\}$ is a simple point system.

Here is our counter-example to the misguided hope that every fat point is simple. In Example 4.17 of [Sch2], Schoutens found the following counter-example:

$$\mathfrak{m} = \operatorname{Spec} A/\mathcal{M}^4, \quad A := \kappa[x, y]/(y^2 - x^3),\ \mathcal{M} = (\bar{x}, \bar{y})$$

where $\bar{x}$ and $\bar{y}$ are the residue classes of x and y in A. In other words, $\mathfrak{m}$ is the 4th order jet of the cuspidal curve at the origin. This is denoted in loc. cit. as $J^4_O C$ where $C = \operatorname{Spec} A$. It is argued there that $(\nabla_{\mathfrak{m}}\mathfrak{m})^{red}$ is singular. Thus, $\mathfrak{m}$ is not a simple point. It turns out that the truncated jets of the node and the cuspidal cubic are not eventually simple point systems either, yet amazingly, this will not limit the theory. Explicit calculations and proofs are carried out in the following sections.

5.1.10 Question. Let X be a variety over an algebraically closed field and let p be a closed point of X. Under what conditions on X will the point system $\{J^n_p X\}$ be eventually simple?

In this section, we will propose a conjecture for when a point system is simple, yet one should note that many of the results of this chapter will go through when the point system is not simple. However, the general picture is not well understood yet. With all this in mind, it becomes clear that it is necessary to add an additional hypothesis for the moment. We obtain the following theorem:

5.1.11 Theorem. *Let $X = \operatorname{Spec}(A)$ be such that X^{red} is smooth and connected and let $(\mathfrak{x}, \mathbb{I}) \in \mathbf{Arc}_\kappa$ be such that there exists a fat point $\mathfrak{n}$ in $\mathbb{I}$ such that $\mathcal{O}_\mathfrak{n}(\mathfrak{n})$ is the maximal artinian subring of A. Further, suppose that $\mathbb{I}$ is a simple point system. Then, there is a canonical morphism*

$$(\pi_\mathfrak{n}^\mathfrak{m})^{red} : (\nabla_\mathfrak{m} Y)^{red} \to (\nabla_\mathfrak{n} X)^{red}$$

which is a piecewise trivial fibration with general fiber

$$\mathbb{A}_\kappa^{d(\ell(\mathfrak{m})-\ell(\mathfrak{n}))+r(\mathfrak{m},\mathfrak{n})}$$

where $r(\mathfrak{m}, \mathfrak{n})$ is some non-negative integer depending on the lengths of $\mathfrak{m}$ and $\mathfrak{n}$ and where $d = dimX$.

Proof. This reduces to understanding the morphism

$$(\nabla_\mathfrak{m} \mathfrak{m})^{red} \to (\nabla_\mathfrak{n} \mathfrak{n})^{red} ,$$

provided that it exits. Here, we use the hypothesis that the point system is simple to conclude that such a morphism does exist and is just given by some projection

$\mathbb{A}_\kappa^{r_1} \to \mathbb{A}_\kappa^{r_2}$. Clearly, this map is a piecewise trivial fibration with fiber $\mathbb{A}_\kappa^r$ where $r = r(\ell(\mathfrak{m}), \ell(\mathfrak{n}))$ conceivably depends on the lengths of $\mathfrak{m}$ and $\mathfrak{n}$. $\square$

5.1.12 Remark. Note that the connectedness assumption above is not essential if we use finite k-schemes in the place of fat points. Furthermore, later in this chapter, we will see that the hypothesis that the point system is simple can be relaxed considerably.

5.1.13 Remark. One immediate situation in which this theorem may be applied is when $\mathfrak{x} = (\mathrm{Spec}(\kappa), \kappa[[x_1, \ldots, x_m]])$ and X is a trivial deformation of a smooth affine variety over $\mathrm{Spec}(\kappa[x_1, \ldots, x_m]/(x_1, \ldots, x_m)^n)$.

On the topic of simple point systems:

Recall that an eventually simple point system is a sequence of fat points defined by $\mathfrak{n} = \mathrm{Spec}(R/\mathcal{M}^n)$ where $(R, \mathcal{M})$ is a complete Noetherian local ring with residue field κ such that $(\nabla_{\mathfrak{n}}\mathfrak{n})^{red}$ is isomorphic to affine m-space for some integer m whenever $\ell(\mathfrak{n}) >> 0$. Recall that we just showed that if an affine scheme X has as its associated fat point $\mathfrak{n}$ (i.e., if $\mathcal{O}_{\mathfrak{n}}(\mathfrak{n})$ is the maximum artinian subring of $\mathcal{O}_X(X)$), and X^{red} is smooth, then a sufficient condition for there being a well-defined motivic volume which captures not just infinitesimal information but also the ability to lift through admissible point systems is for $\mathfrak{n}$ to belong to an eventually simple point system. After using my Sage script (detailed in later sections of this chapter), I have enough evidence to confidently assert the following conjecture.

5.1.14 Conjecture. *Let κ be algebraically closed. If $\mathbb{I}$ is an eventually simple point system such that the cardinality of $\mathbb{I}$ is infinite, then every $\mathfrak{n} \in \mathbb{I}$ is isomorphic to $J_O^n \mathbb{A}_\kappa^r$ for some fixed $r \in \mathbb{N}$ where O is the origin of $\mathbb{A}_\kappa^r$. In particular, the coordinate ring of the locally ringed space defined by $\varinjlim \mathbb{I}$ is $\kappa[[x_1, \ldots, x_r]]$.*

The conjecture says that the terminology *simple point system* is justified because the conjecture states that a simple point system $\mathbb{I}$ must be the point system defined by the truncated jets of an affine variety at a simple point (i.e., at a smooth point). I am of the opinion that such a result would be interesting in its own right in that it could be included in the long list of necessary and sufficient conditions for a variety to be smooth at a point (see Theorem 5.1.15 below). Currently, I do not have a proof of this conjecture, but through extensive computation, I strongly believe it to be true. The reason I feel that it is necessary to bring this to the attention of the reader is that in this section we will quickly discover that the condition on the point system $\mathbb{I}$ to be simple, although sufficient of course, is far from being necessary with respect to a scheme X having a well defined motivic measure. In fact, perhaps it is even more interesting when we are dealing with point systems which are not simple.

It is straightforward to show that the opposite direction of the conjecture is true Let κ be algebraically closed and assume that $\mathfrak{n} = J^n_O X$ for all $n \in \mathbb{N}$ where X is a variety which is smooth at the origin O. This implies that there is an open subvariety U of X containing the point O and an étale morphism $f : U \to \mathbb{A}^r_\kappa$ where $r = \dim X$. Thus, because f is étale and κ is algebraically closed, $\varinjlim\{\mathfrak{n}\} = \mathfrak{x}$ is such that the coordinate ring of $\mathfrak{x}$ as a locally ringed space is given by $\mathcal{O}_\mathfrak{x} \cong \kappa[[x_1, \ldots, x_r]]$. Thus,

$$\mathfrak{n} \cong \operatorname{Spec}(\kappa[[x_1, \ldots, x_r]]/(x_1, \ldots, x_r)^n) \cong J^n_O \mathbb{A}^r_\kappa$$

for all $n \geq 0$. Thus, we have the following theorem.

5.1.15 Theorem. *Assume that κ is algebraically closed. If $\mathbb{I}$ is the point system determined by the truncated jets of a variety which is smooth at the origin, then $\mathbb{I}$ is a simple point system.*

For the other direction, we will see in the calculations in this chapter that there is ample evidence that it is true. A possible line of attack would be to prove a stronger statement which is related to what is discussed in §5.3.2. In fact, it seems highly likely that $(\nabla_\mathfrak{n}\mathfrak{n})^{red}$ is isomorphic to $V \times_\kappa \mathbb{A}^m_\kappa$ where V is some singular variety (possibly depending on $\mathfrak{n}$) and where m depends on $\ell(\mathfrak{n})$ whenever the $\mathfrak{n}$

are the truncated jets of a scheme of positive dimension which is not smooth at the origin and $\ell(\mathfrak{n}) >> 0$.

5.1.16 Remark. The assumption that κ is algebraically closed can be relaxed in the usual way – i.e., one should expect that $\mathbb{I}$ being a simple point system of infinite cardinality is equivalent to the condition that for all $\mathfrak{n} \in \mathbb{I}$, there is an isomorphism $\mathfrak{n} = J^n_O \mathbb{A}^r_{\kappa'}$ for some fixed $r \in \mathbb{N}$ and some fixed finite field extension κ' of κ.

5.2 Some computations with previous examples.

Naturally, it is important to continue investigating Example 5.1.9 from a computational standpoint. In other words, we wish to understand the reduced structure on auto-arc spaces defined by truncated jets of the cuspidal cubic at the origin and the truncated jets of the node at the origin. We will see that in fact there is a lot that one can say about them. These jets will not be simple points yet any affine scheme whose associated fat point is one of these truncated jets and whose reduction is smooth will have a well-defined motivic volume. Further, we will explicitly describe this motivic volume.

5.2.1 Computation for cuspidal cubic:

First, the code I am going to use will only work in characteristic 0. Thus, we assume κ is of characteristic 0. It is clear to me that the code should work whenever the characteristic is not equal to 2 or 3, and it seems reasonable that one can mod out by p in the equations below to get the equations in any characteristic. However, I have not worked out the details sufficiently yet. Let us consider the cuspidal cubic $C = \mathrm{Spec}(\kappa[x, y]/(y^2 + x^3))$. For each $n \in \mathbb{N}$, let $\mathfrak{n} = J^n_O C$ be the fat point determined by the truncated n-jet of C at the origin. We can run my sage script to compute the auto-arcs $\nabla_{\mathfrak{n}}\mathfrak{n}$ for $n \leq 6$. Note that the complexity of these spaces grows quite rapidly because $\nabla_{\mathfrak{n}}\mathfrak{n}$ is not reduced. This will be made evident in the following. Lets call the coordinate ring of this affine scheme A_n. For $n = 1$, we obtain $A_n = \kappa$ as always. For $n = 2$, the sage output gives the list of equations:

1. $a_0 = 0,$
2. $a_1 = 0,$
3. $a_2 = 0,$
4. $a_3 = 0,$
5. $2a_5a_9 = 0,$
6. $2a_7a_9 = 0,$

7. $a_5a_8 + a_4a_9 = 0,$
8. $a_7a_8 + a_6a_9 = 0,$
9. $2a_4a_8 = 0,$
10. $2a_6a_8 = 0,$
11. $a_9^2 = 0,$
12. $a_8a_9 = 0,$
13. $a_8^2 = 0,$

which take place in $\kappa[a_0, a_1, \ldots, a_9]$. Note that I have manually rendered the sage output, which can either be a python list, a sage ideal, or a singular quotient ring in a tex acceptable way. The 11th and 13th equation will reduce to $a_9 = 0$ and $a_8 = 0$, respectively, in $A_2/\text{nil}(A_2)$, and as either a_8 or a_9 occurs in each term of each equation, we have that the variables a_i are free for $i = 4, 5, 6, 7$. In other words, $A_2/\text{nil}(A_2)$ is isomorphic to $\kappa[a_4, a_5, a_6, a_7]$ and the reduced auto-arc space is $\mathbb{A}^4_\kappa$.

For $n = 3$, our manually rendered sage output is

1. $a_0 = a_1 = a_2 = a_3 = 0$,
2. $2a_7a_{11} + 2a_5a_{13} = 0$,
3. $2a_7a_{13} = 0$,
4. $a_{11}^2 + 2a_9a_{13} = 0$,
5. $2a_{11}a_{13} = 0$,
6. $2a_7a_{10}a_{12} + 2a_6a_{11}a_{12} + a_5a_{12}^2 + 2a_6a_{10}a_{13} + 2a_4a_{12}a_{13} = 0$,
7. $a_7a_{12}^2 + 2a_6a_{12}a_{13} = 2a_{10}a_{11}a_{12} + a_9a_{12}^2 + a_{10}^2a_{13} + 2a_8a_{12}a_{13} = 0$,
8. $a_{11}a_{12}^2 + 2a_{10}a_{12}a_{13} = 0$,
9. $6a_6a_{10}a_{12} + 3a_4a_{12}^2 + 2a_7a_{11} + 2a_5a_{13} = 0$,
10. $3a_6a_{12}^2 + 2a_7a_{13} = 0$,
11. $3a_{10}^2a_{12} + 3a_8a_{12}^2 + a_{11}^2 + 2a_9a_{13} = 0$,
12. $3a_{10}a_{12}^2 + 2a_{11}a_{13} = 0$,
13. $a_{13}^2 = 0$,
14. $a_{12}^2a_{13} = 0$,
15. $a_{12}^3 + a_{13}^2 = 0$,

which takes place in $\kappa[a_0, a_1, \ldots, a_{13}]$. One can see that the list of equations grows rapidly. Here, the first equation which tells us that the first 4 variables are evaluated at zero has to do with the way I wrote the program and tells us nothing substantive mathematically. What we may notice is that equations 11, 13, and 15 all tell us that $a_{11} = 0$, $a_{13} = 0$, and $a_{12} = 0$ in $A_3/\mathrm{nil}(A_3)$, respectively. Then, on may manually check that one of these variables occurs at least once in each term of each equation, as before. Thus, the variable a_i are free for $i = 4, 5, \ldots, 10$, or, in other words, the reduced auto-arc scheme $(\nabla_{\mathfrak{n}}\mathfrak{n})^{red}$ is isomorphic to $\mathbb{A}^7_\kappa$ in this case.

As we stated in Example 5.1.9, at $n = 4$, we should see that $(\nabla_{\mathfrak{n}}\mathfrak{n})^{red}$ is not smooth. In fact, we should verify Schoutens' claim that it is isomorphic to $\nabla_{\mathfrak{l}_2} C \times_\kappa \mathbb{A}^7_\kappa$. Indeed, the sage script will verify this (or, this will ease the sceptical reader into believing that my code works well). For $n = 4$, we have

1. $a_0 = a_1 = a_2 = a_3 = 0$,
2. $-a_{15}^3 + 3a_{11}^2 a_{17} - 6a_{13}a_{15}a_{17} + 3a_5 a_{17}^2 = 0$,
3. $3a_{11}a_{15}^2 + 6a_{11}a_{13}a_{17} + 6a_9 a_{15}a_{17} + 3a_7 a_{17}^2 = 0$,
4. $6a_{11}a_{15}a_{17} + 3a_9 a_{17}^2 = 0$,
5. $3a_{11}a_{17}^2 = 0$,
6. $3a_{15}^2 a_{17} + 3a_{13}a_{17}^2 = 0$,
7. $3a_{15}a_{17}^2 = 0$,

8. $-a_{14}a_{15}^2 + a_{11}^2 a_{16} - 2a_{13}a_{15}a_{16} + 2a_{10}a_{11}a_{17} - 2a_{13}a_{14}a_{17} -$

$- 2a_{12}a_{15}a_{17} + 2a_5 a_{16}a_{17} + a_4 a_{17}^2 = 0,$

9. $2a_{11}a_{14}a_{15} + a_{10}a_{15}^2 + 2a_{11}a_{13}a_{16} + 2a_9 a_{15}a_{16} + 2a_{11}a_{12}a_{17} +$

$+ 2a_{10}a_{13}a_{17} + 2a9a_{14}a_{17} + 2a_8 a_{15}a_{17} + 2a_7 a_{16}a_{17} + a_6 a_{17}^2 = 0,$

10. $2a_{11}a_{15}a_{16} + 2a_{11}a_{14}a_{17} + 2a_{10}a_{15}a_{17} + 2a_9 a_{16}a_{17} + a_8 a_{17}^2 = 0,$

11. $2a_{11}a_{16}a_{17} + a_{10}a_{17}^2 = 0,$

12. $a_{15}^2 a_{16} + 2a_{14}a_{15}a_{17} + 2a_{13}a_{16}a_{17} + a_{12}a_{17}^2 = 0,$

13. $2a_{15}a_{16}a_{17} + a_{14}a_{17}^2 = 0,$

14. $-a_{14}^3 + 3a_{10}^2 a_{16} - 6a_{12}a_{14}a_{16} + 3a_4 a_{16}^2 + a_{11}^2 - 2a_{13}a_{15} + 2a_5 a_{17} = 0,$

15. $3a_{10}a_{14}^2 + 6a_{10}a_{12}a_{16} + 6a_8 a_{14}a_{16} + 3a_6 a_{16}^2 + 2a_{11}a_{13} + 2a_9 a_{15} + 2a_7 a_{17} = 0,$

16. $6a_{10}a_{14}a_{16} + 3a_8 a_{16}^2 + 2a_{11}a_{15} + 2a_9 a_{17} = 0,$

17. $3a_{10}a_{16}^2 + 2a_{11}a_{17} = 0,$

18. $3a_{14}^2 a_{16} + 3a_{12}a_{16}^2 + a_{15}^2 + 2a_{13}a_{17} = 0,$

19. $3a_{14}a_{16}^2 + 2a_{15}a_{17} = 0,$

20. $a_{17}^3 = 0,$

21. $a_{16}a_{17}^2 = 0,$

22. $a_{16}^3 + a_{17}^2 = 0,$

which takes place in $\kappa[a_0, a_1, \ldots, a_{17}]$. It is completely obvious now that putting the full list of equations for this auto-arc space when $n = 5$ and $n = 6$ is untenable. However, given that I am providing the code to my program, it is unnecessary to do so. We are doing it for $n = 4$ so that the reader may see how things work in practice when using my code. At any rate (and as we will see this is a general pattern), we notice that equation 18, 20, and 22 show us that $a_{15} = a_{16} = a_{17} = 0$ in $A_4/\text{nil}(A_4)$. Thus, the list of equations defining $A_4/\text{nil}(A_4)$ will be

1. $a_0 = a_1 = a_2 = a_3 = a_{15} = a_{16} = a_{17} = 0,$
2. $-a_{14}^3 + a_{11}^2 = 0,$
3. $3a_{10}a_{14}^2 + 2a_{11}a_{13} = 0,$

which takes place in $\kappa[a_0, a_1, \ldots, a_{17}]$. We reached these equations by noticing the only equations with terms not involving a_{15}, a_{16}, or a_{17} are equations 14 and 15, and those two equations simplify to equation 2 and 3 above, respectively. So, here, the variables $a_4,\ a_5,\ a_6,\ a_7,\ a_8,\ a_9$ and a_{12} are free so that $A_4/\text{nil}(A_4)$ is the tensor product of a multivariate polynomial ring in 7 variables over κ with S where S is the quotient ring of a multivariate polynomial ring in 4 variables by equations 2 and 3. One may quickly check that S is isomorphic to the coordinate ring of the arc space $\nabla_{\mathfrak{l}_2} C$. Thus, Schoutens' statement is verified – i.e., for $n = 4$, we have an isomorphism

$$(\nabla_{\mathfrak{n}}\mathfrak{n})^{red} \cong \nabla_{\mathfrak{l}_2} C \times_\kappa \mathbb{A}^7_\kappa.$$

For $n = 5$, we will work with the coordinate ring of the reduction $B_5 := A_5/\text{nil}(A_5)$.

I choose to do this by hand as, at least as far as I understand, reduction is not fully implemented in Sage or Singular. As before, we will work in the multivariate polynomial ring $\kappa[a_0, a_1, \ldots, a_{21}]$ and we get a long list of equations which can easily be seen to show that $a_{19} = a_{20} = a_{21} = 0$ in B_5. Then, it is easy to reduce and find that the equations defining B_5 are

1. $a_0 = a_1 = a_2 = a_3 = a_{19} = a_{20} = a_{21} = 0,$
2. $a_{14}^3 - 6a_{14}a_{16}a_{18} - 3a_{12}a_{18}^2 + 2a_9a_{15} - 2a_{13}a_{17} = 0,$
3. $3a_{14}^2a_{18} - 3a_{16}a_{18}^2 + 2a_{13}a_{15} - a_{17}^2 = 0,$
4. $3a_{14}a_{18}^2 + 2a_{15}a_{17} = 0,$
5. $-a_{18}^3 + a_{15}^2 = 0,$

which take place in $\kappa[a_0, a_1, \ldots, a_{21}]$. One quickly notices that $a_4,\ a_5,\ a_6,\ a_7,\ a_8,\ a_{10}$, and a_{11} do not occur in the aforementioned equations. One may check rather quickly (either by running the program or by hand) that equations 2-5 define the coordinate ring of the arc space $\nabla_{\mathfrak{l}_4} C$. Thus, we have shown that

$$(\nabla_{\mathfrak{n}}\mathfrak{n})^{red} \cong \nabla_{\mathfrak{l}_4} C \times_\kappa \mathbb{A}^7_\kappa.$$

Although the complexity increases drastically, the case for $n = 6$ is exactly the same. I personally verified using my sage script that

$$(\nabla_{\mathfrak{n}}\mathfrak{n})^{red} \cong \nabla_{\mathfrak{l}_6} C \times_\kappa \mathbb{A}^7_\kappa,$$

when $n = 6$. It is more complicated, but I am confident that the interested reader could do the same calculation using my code. At any rate, this leads us to conjecture that for $n \geq 4$, we have the following isomorphism

$$(\nabla_{\mathfrak{n}}\mathfrak{n})^{red} \cong \nabla_{\mathfrak{l}_{2(n-3)}} C \times_{\kappa} \mathbb{A}^7_{\kappa}.$$

We will offer a proof of this fact in the next section. The reason I partly carried out the calculation here when I already arrived at the proof is for two reason. First, this is how I arrived at the result, and secondly, it demonstrates how this can be done for other auto-arc spaces. It appears that after computing the first few auto-arc spaces with my program (in general, this will take calculations which cannot be done in any reasonable sense by hand), one will be able to see a general pattern and be able to make a conjecture. Then, at least in my experience so far, a proof can be obtained.

5.2.2 Computation for node:

Perhaps a less interesting yet more manageable computation occurs when $N = \mathrm{Spec}(\kappa[x,y]/(xy))$ and $\mathfrak{n}$ is the truncated n-jet of N. Let A_n be the coordinate ring of $\nabla_{\mathfrak{n}}\mathfrak{n}$ and let B_n be the coordinate ring of the reduction. As usual, we have $A_1 = B_1 = \mathrm{Spec}(\kappa)$. For $n = 2$, the sage script gives the equations

1. $a_0 = a_1 = a_2 = a_3 = 0,$
2. $2a_5a_9 = 0,$
3. $2a_7a_9 = 0,$
4. $a_5a_8 + a_4a_9 = 0,$
5. $a_7a_8 + a_6a_9 = 0,$
6. $2a_4a_8 = 0,$
7. $2a_6a_8 = 0,$
8. $a_9^2 = 0,$
9. $a_8a_9 = 0,$
10. $a_8^2 = 0,$

which take place in $\kappa[a_0, a_1, \ldots, a_9]$ and define A_2. So that the $a_8 = a_9 = 0$ in B_2 so that $B_2 = \kappa[a_4, a_5, a_6, a_7]$ and $(\nabla_{\mathfrak{n}}\mathfrak{n})^{red} \cong \mathbb{A}^4_\kappa$ as we showed in Example 5.1.9.

For $n = 3$, we obtain the equations

1. $a_0 = a_1 = a_2 = a_3 = 0,$
2. $a_6a_7 + a_5a_{12} + a_4a_{13} = 0,$
3. $a_7a_{12} + a_6a_{13} = 0,$
4. $a_{10}a_{11} + a_9a_{12} + a_8a_{13} = 0,$
5. $a_{11}a_{12} + a_{10}a_{13} = 0,$
6. $3a_7^2a_{13} + 3a_5a_{13}^2 = 0,$
7. $3a_{11}^2a_{13} + 3a_9a_{13}^2 = 0,$
8. $3a_{11}a_{13}^2 = 0,$
9. $3a_6^2a_{12} + 3a_4a_{12}^2 = 0,$
10. $3a_6a_{12}^2 = 0,$
11. $3a_{10}^2a_{12} + 3a_8a_{12}^2 = 0,$
12. $3a_{10}a_{12}^2 = 0,$
13. $a_{12}a_{13} = 0,$
14. $a_{12}^3 = a_{13}^3 = 0,$

which take place in $\kappa[a_0, a_1, \ldots, a_{13}]$. In B_3, we may reduce this list to

1. $a_0 = a_1 = a_2 = a_3 = a_{12} = a_{13} = 0,$
2. $a_6a_7 = 0,$
3. $a_{10}a_{11} = 0.$

Thus,

$$(\nabla_{\mathfrak{n}}\mathfrak{n})^{red} \cong N \times_{\kappa} N \times_{\kappa} \mathbb{A}^4_{\kappa}.$$

For $n = 4$, the manually rendered sage output is

1. $a_0 = a_1 = a_2 = a_3 = 0,$
2. $a_7a_8 + a_6a_9 + a_5a_{16} + a_4a_{17} = 0,$
3. $a_8a_9 + a_7a_{16} + a_6a_{17} = 0,$
4. $a_9a_{16} + a_8a_{17} = 0,$
5. $a_{13}a_{14} + a_{12}a_{15} + a_{11}a_{16} + a_{10}a_{17} = 0,$
6. $a_{14}a_{15} + a_{13}a_{16} + a_{12}a_{17} = 0,$
7. $a_{15}a_{16} + a_{14}a_{17} = 0,$
8. $4a_9^3a_{17} + 12a_7a9a_{17}^2 + 4a_5a_{17}^3 = 0,$
9. $6a_9^2a_{17}^2 + 4a_7a_{17}^3 = 0,$
10. $4a_9a_{17}^3 = 0,$
11. $4a_{15}^3a_{17} + 12a_{13}a_{15}a_{17}^2 + 4a_{11}a_{17}^3 = 0,$
12. $6a_{15}^2a_{17}^2 + 4a_{13}a_{17}^3 = 0,$
13. $4a_{15}a_{17}^3 = 0,$
14. $4a_8^3a_{16} + 12a_6a_8a_{16}^2 + 4a_4a_{16}^3 = 0,$
15. $6a_8^2a_{16}^2 + 4a_6a_{16}^3 = 0,$

16. $4a_8a_{16}^3 = 0,$

17. $4a_{14}^3a_{16} + 12a_{12}a_{14}a_{16}^2 + 4a_{10}a_{16}^3 = 0,$

18. $6a_{14}^2a_{16}^2 + 4a_{12}a_{16}^3 = 0,$

19. $4a_{14}a_{16}^3 = 0,$

20. $a_{16}a_{17} = 0,$

21. $a_{17}^4 = 0,$

22. $a_{16}^4 = 0,$

which take place in $\kappa[a_0, a_1, \ldots, a_{17}]$ and describes A_4. From this, one gathers that $a_{16} = a_{17} = 0$ in B_4. Thus, the equations defining B_4 are

1. $a_0 = a_1 = a_2 = a_3 = a_{16} = a_{17} = 0,$

2. $a_7a_8 + a_6a_9 = 0,$

3. $a_8a_9 = 0,$

4. $a_{13}a_{14} + a_{12}a_{15} = 0,$

5. $a_{14}a_{15} = 0,$

From this one sees that $a_4,\ a_5,\ a_{10}$, and a_{11} are free in B_4 and that equations 2 and 3 have no variables in common with equations 4 and 5. In fact, equations 2 and 3 are the same as those which define the arc space $\nabla_{\mathfrak{l}_2}N$, and likewise, equations 4 and 5 are also those which define the arc space $\nabla_{\mathfrak{l}_2}N$. Thus, in the case where

$n = 4$, we arrive at

$$(\nabla_{\mathfrak{n}}\mathfrak{n})^{red} \cong \nabla_{\mathfrak{l}_2}N \times_\kappa \nabla_{\mathfrak{l}_2}N \times_\kappa \mathbb{A}^4_\kappa.$$

In exactly the same way, I used the sage script to find an isomorphism

$$(\nabla_{\mathfrak{n}}\mathfrak{n})^{red} \cong \nabla_{\mathfrak{l}_{n-2}}N \times_\kappa \nabla_{\mathfrak{l}_{n-2}}N \times_\kappa \mathbb{A}^4_\kappa.$$

for $n = 5, \ldots, 8$. The only reason I do not include the calculation here is that it is too lengthy for the uninterested reader and can easily be checked by running my sage script and following my method of reduction for the interested reader. So, in the end, the above isomorphism is expected to hold for all n greater than or equal to 3, which is a fact we will prove in the next section.

5.3 More on reduced auto-arcs of jets of curves.

In the previous section, we found a pattern for the reduced auto-arc of truncated jets of the cuspidal cubic and the node by using my sage script. In this section, we will prove the general formula and discuss their motivic consequences. Finally, we will prove that the reduced auto-arc spaces of truncated jets of a curve are definable in the language of Denef-Pas, which is an extremely hopeful sign for connecting our general schemic approach to motivic integration to the classical motivic integral. In fact, this also partly connects the theory discussed in Chapter

4 with constructible motivic integration as we have cell-decomposition in these cases.

5.3.1 Proofs for the patterns notice in §5.2.

Per our calculations in §5.2, we posit the following theorem.

5.3.1 Theorem. *Let* $C = \operatorname{Spec}(\kappa[x,y]/(y^2 - x^3))$*, and, for* $n \in \mathbb{N}$*, we let*

$$\mathfrak{n} := J_O^n C := \operatorname{Spec}(\kappa[x,y]/((x,y)^n + (y^2 - x^3))).$$

Then, for all $n \geq 4$*,*

$$(\nabla_{\mathfrak{n}} \mathfrak{n})^{red} \cong (\nabla_{\mathfrak{l}_{2(n-3)}} C) \times_\kappa \mathbb{A}_\kappa^7$$

Proof. First, note that

$$(x,y)^n + (y^2 - x^3) = (x^n, x^{n-1}y, y^2 - x^3).$$

as ideals in $\kappa[x,y]$. Thus, we must define two arcs

$$\begin{aligned} \alpha &:= \sum_{i=0}^{n-1} a_i x^i + \sum_{i=0}^{n-2} b_i x^i y \\ \beta &:= \sum_{i=0}^{n-1} c_i x^i + \sum_{i=0}^{n-2} d_i x^i y \end{aligned} \tag{5.3.1}$$

where $a_i,\ b_i,\ c_i$ and d_i are thought of as variables running through κ. We then have the following equations

$$\alpha^n = \alpha^{n-1}\beta = \beta^2 - \alpha^3 = 0 \tag{5.3.2}$$

occurring in

$$R := \kappa[a_i, c_i, b_j, d_j \mid i = 0, \dots n-1, j = 0, \dots, n-2] \otimes_\kappa \kappa[x, y]/(x^n, x^{n-1}y, y^2 - x^3)$$

where we think of R as a finitely generated $\kappa[x, y]/(x^n, x^{n-1}y, y^2 - x^3)$-algebra. Now $0 = \alpha^n = a_0^n$ and in the reduce structure this implies $a_0 = 0$. Likewise, $0 = \alpha^{n-1}\beta$ implies $0 = \beta^n = c_0^n$ since $\alpha^3 = \beta^2$. Thus, $c_0 = 0$ in the reduction. Thus, the equations $\alpha^n = \alpha^{n-1}\beta = 0$ are trivially satisfied in the reduced structure. Now, we consider the following equation:

$$\begin{aligned}
0 = \beta^2 - \alpha^3 &= (\sum_{i=1}^{n-1} a_i x^i + \sum_{i=0}^{n-2} b_i x^i y)^2 - (\sum_{i=1}^{n-1} c_i x^i + \sum_{i=0}^{n-2} d_i x^i y)^3 \\
&= (\sum_{i=1}^{n-1} a_i x^i)^2 + 2(\sum_{i=1}^{n-1} a_i x^i) \cdot (\sum_{i=0}^{n-2} b_i x^i y) + (\sum_{i=0}^{n-2} b_i x^i y)^2 - \\
&- (\sum_{i=1}^{n-1} c_i x^i)^3 - 3(\sum_{i=1}^{n-1} c_i x^i)^2 \cdot (\sum_{i=0}^{n-2} d_i x^i y) - \\
&- 3(\sum_{i=1}^{n-1} c_i x^i) \cdot (\sum_{i=0}^{n-2} d_i x^i y)^2 - (\sum_{i=0}^{n-2} d_i x^i y)^3
\end{aligned}$$

Note the following identities involving each term of the above.

$$\begin{aligned}
(\sum_{i=1}^{n-1} a_i x^i)^2 &= x^2 (\sum_{i=0}^{n-2} a_{i+1} x^i)^2 \\
2(\sum_{i=1}^{n-1} a_i x^i) \cdot (\sum_{i=0}^{n-2} b_i x^i y) &= 2yx(\sum_{i=0}^{n-2} a_{i+1} x^i) \cdot (\sum_{i=0}^{n-2} b_i x^i) \\
(\sum_{i=0}^{n-2} b_i x^i y)^2 &= x^3 (\sum_{i=0}^{n-2} b_i x^i)^2 \\
(\sum_{i=1}^{n-1} c_i x^i)^3 &= x^3 (\sum_{i=0}^{n-2} c_{i+1} x^i)^3
\end{aligned}$$

$$3(\sum_{i=1}^{n-1} c_i x^i)^2 \cdot (\sum_{i=0}^{n-2} d_i x^i y) = 3x^2 y(\sum_{i=0}^{n-2} c_{i+1} x^i)^2 \cdot (\sum_{i=0}^{n-2} d_i x^i)$$
$$3(\sum_{i=1}^{n-1} c_i x^i) \cdot (\sum_{i=0}^{n-2} d_i x^i y)^2 = 3x^4(\sum_{i=0}^{n-2} c_{i+1} x^i) \cdot (\sum_{i=0}^{n-2} d_i x^i)^2$$

Thus, we group the terms involving y and the terms only involving x to obtain the equation

$$\begin{aligned} 0 =& x^2(\sum_{i=0}^{n-2} a_{i+1} x^i)^2 + x^3((\sum_{i=0}^{n-2} b_i x^i)^2 - (\sum_{i=0}^{n-2} c_{i+1} x^i)^3) - 3x^4(\sum_{i=0}^{n-2} c_{i+1} x^i) \cdot (\sum_{i=0}^{n-2} d_i x^i)^2 + \\ &+ 2xy(\sum_{i=0}^{n-2} a_{i+1} x^i) \cdot (\sum_{i=0}^{n-2} b_i x^i) - 3x^2 y(\sum_{i=0}^{n-2} c_{i+1} x^i)^2 \cdot (\sum_{i=0}^{n-2} d_i x^i) \end{aligned}$$

From this we can see that the coefficient of x^2 is a_1^2. This implies $a_1^2 = 0$ and so in the reduction $a_1 = 0$. Thus, we rewrite the previous equation as

$$\begin{aligned} 0 =& x^4(\sum_{i=0}^{n-3} a_{i+2} x^i)^2 + x^3((\sum_{i=0}^{n-2} b_i x^i)^2 - (\sum_{i=0}^{n-2} c_{i+1} x^i)^3) - 3x^4(\sum_{i=0}^{n-2} c_{i+1} x^i) \cdot (\sum_{i=0}^{n-2} d_i x^i)^2 + \\ &+ 2x^2 y(\sum_{i=0}^{n-3} a_{i+2} x^i) \cdot (\sum_{i=0}^{n-2} b_i x^i) - 3x^2 y(\sum_{i=0}^{n-2} c_{i+1} x^i)^2 \cdot (\sum_{i=0}^{n-2} d_i x^i) \end{aligned}$$

Furthermore, from this, we should be able to find exactly 7 free variables. In fact, it is easy to see that $a_{n-1},\ c_{n-1},\ c_{n-2},\ b_{n-2},\ b_{n-3},\ d_{n-2},$ and d_{n-3} are the only free variables. Thus, the space is of the form $S_n \times_\kappa \mathbb{A}^7_\kappa$ where S_n is an algebraic variety in the variables $a_i,\ b_j,\ c_l,\ d_k$ for appropriate indices $i,\ j,\ l,\ k$. Thus, going back to the original equations and setting these free variables to zero, we have

$$0 = (\sum_{i=0}^{n-4} a_{i+2} x^i + \sum_{i=0}^{n-4} b_i x^i y)^2 - (\sum_{i=0}^{n-4} c_{i+1} x^i + \sum_{i=0}^{n-4} d_i x^i y)^3$$

We may perform the substitution $x = s^2$ and $y = s^3$ to obtain an equivalent equation, which defines S is a reduced subscheme of $\nabla_{\mathfrak{l}_{2n-4}} C$. This equation is of the following form:

$$\begin{aligned}
0 =&(a_2 + s^2 \sum_{i=0}^{2n-7} \sigma_i s^i + b_{n-4} s^{2n-5})^2 - (c_1 + \sum_{i=0}^{2n-7} \nu_i s^i + d_{n-4} s^{2n-5})^3 \\
=&a_2^2 + 2a_2 s^2 \sum_{i=0}^{2n-7} \sigma_i s^i + 2a_2 b_{n-4} s^{2n-5} + s^4 (\sum_{i=0}^{2n-7} \sigma_i s^i)^2 - \\
&- c_1^3 - 3c_1^2 s^2 \sum_{i=0}^{2n-7} \nu_i s^i - 3c_1^2 d_{n-4} s^{2n-5} - 3c_1 s^4 (\sum_{i=0}^{2n-7} \nu_i s^i)^2 - s^6 (\sum_{i=1}^{2n-7} \nu_i s^i)^3 \, .
\end{aligned}$$

The above equation is obtained from the previous one by substituting $x = s^2$ and $y = t^3$, defining

$$\sigma_i = \begin{cases} a_{i/2+3} & \text{if } i = 0, 2, \ldots, 2n-6 \\ b_{i/2-1/2} & \text{if } i = 1, 3, \ldots, 2n-7 \end{cases}$$
$$\nu_i = \begin{cases} c_{i/2+2} & \text{if } i = 0, 2, \ldots, 2n-6 \\ d_{i/2-1/2} & \text{if } i = 1, 3, \ldots, 2n-7 \end{cases}$$

and, of course, we also expanded the product.

Now, let $\rho_m : \nabla_{\mathfrak{l}_m} C \to C$ be the natural truncation morphism induced by $\kappa[t]/(t^m) \to \kappa$. The key point to notice is that S contains the subscheme $\rho_{2n-6}^{-1}(O)$ where O is the singularity of the cusp C. In fact, in $\rho_{2n-3}^{-1}(O) \cap S$, is defined by the equation above by setting $a_2 = c_1 = 0$. Thus, it is defined by

$$\begin{aligned}
0 =&s^4 (\sum_{i=0}^{2n-7} \sigma_i s^i)^2 - s^6 (\sum_{i=1}^{2n-7} \nu_i s^i)^3 \\
=&s^2 (s \sum_{i=0}^{2n-7} \sigma_i s^i)^2 - s^3 (s \sum_{i=1}^{2n-7} \nu_i s^i)^3
\end{aligned}$$

which is exactly the equation for $\rho_{2n-6}^{-1}(O)$. Thus, the restriction of the natural truncation morphism gives a morphism f from S to $\nabla_{\mathfrak{l}_{2n-6}} C$ which is surjective and in fact an isomorphism on the inverse images of the singular point of C. However, away from these inverse images of the singular point, the morphism f is a piecewise trivial fibration. Moreover, the most the dimension of the trivial fiber can be is 2 as $\dim(C) = 1$ and $(2n-4)-(2n-6) = 2$. However, S as a subscheme of $\nabla_{\mathfrak{l}_{2n-4}} C$ is cut out by two hyperplanes – i.e., the coefficient in front of the t^{2n-6} term of each arc is zero. This means that this fiber must actually have dimension 0 – i.e., it is an isomorphism away from the singular locus. Thus, on the singular locus f is an isomorphism and away from the singular locus f is an isomorphism. Therefore, f is an isomorphism, which proves the claim. $\square$

5.3.2 Remark. It should be expected to find a similar formula for the reduced auto-arcs of the the truncated jets the curve $C(m,k) = \mathrm{Spec}(\kappa[x,y]/(y^k - x^m))$ where $m > k$. It should be expected that there is an isomorphism

$$(\nabla_{\mathfrak{n}}\mathfrak{n})^{red} \cong (\nabla_{\mathfrak{l}_{k(n-m)}} C(m,k))^{red} \times_\kappa \mathbb{A}^r_\kappa$$

for some fixed $r \in \mathbb{N}$ whenever $n > m$. Perhaps, one may also be able to show that r is equal to $mk+1$. In particular, it is expected that the asymptotic defect of $J^\infty_O C(m,k)$ is given by

$$\delta(J^\infty_O C(m,k)) := \limsup_n \frac{\dim \nabla_{\mathfrak{n}}\mathfrak{n}}{\ell(\mathfrak{n})} = 2$$

Thus, we often expect the asymptotic defect a germ of an irreducible curve at a κ-rational point to be equal to its embedding dimension provided that it is the coordinate ring of a germ of an irreducible curve at a κ-rational point. More general conjectures regarding asymptotic defects can be found in Chapter 5 of [Sch2].

Now, we consider the case of the node $N = \operatorname{Spec}(\kappa[x,y]/(xy))$.

5.3.3 Theorem. *Let $\mathfrak{n}$ be $J^n_O N = \operatorname{Spec}(\kappa[x,y]/(x^n, y^n, xy))$. Then for each $n \geq 3$, we have an isomorphism*

$$(\nabla_{\mathfrak{n}}\mathfrak{n})^{red} \cong \nabla_{\mathfrak{l}_{n-2}} N \times_\kappa \nabla_{\mathfrak{l}_{n-2}} N \times_\kappa \mathbb{A}^4_\kappa.$$

Proof. As in the case of the previous proof, we again define two arcs

$$\alpha := \sum_{i=0}^{n-1} a_i x^i + \sum_{i=1}^{n-1} b_i y^i$$
$$\beta := \sum_{i=0}^{n-1} c_i x^i + \sum_{i=1}^{n-1} d_i y^i$$

and investigate the equations

$$0 = \alpha^n = \beta^n = \alpha\beta \ .$$

Note that

$$0 = \alpha^n = a_0^n \implies 0 = a_0 \text{ in the reduction and}$$

$$0 = \beta^n = c_0^n \implies 0 = c_0 \text{ in the reduction.}$$

Thus, we only have to investigate

$$
\begin{aligned}
0 = \alpha\beta = &(\sum_{i=1}^{n-1} a_i x^i)(\sum_{i=1}^{n-1} c_i x^i) + (\sum_{i=1}^{n-1} a_i x^i)(\sum_{i=1}^{n-1} d_i y^i) + \\
&+ (\sum_{i=1}^{n-1} b_i y^i)(\sum_{i=1}^{n-1} c_i x^i) + (\sum_{i=1}^{n-1} b_i y^i)(\sum_{i=1}^{n-1} d_i y^i) \\
&= x^2(\sum_{i=1}^{n-2} a_{i+1} x^i)(\sum_{i=1}^{n-2} c_{i+1} x^i) + xy(\sum_{i=1}^{n-2} a_{i+1} x^i)(\sum_{i=1}^{n-2} d_{i+1} y^i) + \\
&+ xy(\sum_{i=1}^{n-2} b_{i+1} y^i)(\sum_{i=1}^{n-2} c_{i+1} x^i) + y^2(\sum_{i=1}^{n-2} b_{i+1} y^i)(\sum_{i=1}^{n-2} d_{i+1} y^i) \\
&= x^2(\sum_{i=1}^{n-2} a_{i+1} x^i)(\sum_{i=1}^{n-2} c_{i+1} x^i) + y^2(\sum_{i=1}^{n-2} b_{i+1} y^i)(\sum_{i=1}^{n-2} d_{i+1} y^i),
\end{aligned}
$$

where the terms involving a factor of xy vanish because $xy = 0$. Note that the last equation implies

$$
\begin{aligned}
0 &= (\sum_{i=1}^{n-2} a_{i+1} x^i)(\sum_{i=1}^{n-2} c_{i+1} x^i) \\
0 &= (\sum_{i=1}^{n-2} b_{i+1} y^i)(\sum_{i=1}^{n-2} d_{i+1} y^i) \ .
\end{aligned}
$$

From this it is clear that these equations define $\nabla_{\mathfrak{l}_{n-2}} N \times_\kappa \nabla_{\mathfrak{l}_{n-2}} N$ provided that $n \geq 3$. Note that the variables $a_{n-1},\ b_{n-1},\ c_{n-1}$, and d_{n-1} are free. This gives the result. $\square$

5.3.4 Remark. From this, we may deduced that the asymptotic defect $\delta(J^\infty_O N)$ is 1. Similar results should be possible for the curve $N(k,m) = \mathrm{Spec}(\kappa[x,y]/(x^k y^m))$. Thus, we see that for germs of a reducible curves, there is no reason to expect that, in general, the asymptotic defect will be equal to the embedding dimension of the germ.

5.3.2 Some consequences and conjectures

Given our calculations and subsequent proofs for the cuspidal cubic and node. It can now be reasonably expected that projective limits of reduced auto-arc spaces have nice model-theoretic properties, which as of yet seemed very unlikely. In particular, there should be a firm bridge between schemic motivic integration and the classical version. In particular, we may state the following conjecture.

5.3.5 Conjecture. *Let κ be of characteristic zero and algebraically closed. Let C be a reduced curve and let $\mathfrak{n} = J^n_p C$ be the truncated n-jet of C at κ-rational point p of C for each $n \in \mathbb{N}$. The reduced auto-arc $(\nabla_{\mathfrak{n}}\mathfrak{n})^{red}$ is definable in the language of Denef-Pas $\mathcal{L}$ formed by having a sort in the field $\kappa((t))$, a sort in the residue field κ, and a sort in presburger arithmetic of the value group $\mathbb{Z}$. Moreover, if C is irreducible with singular locus $\{p\}$, we conjecture that*

$$(\nabla_{\mathfrak{n}}\mathfrak{n})^{red} \cong \nabla_{\mathfrak{l}_{L(n)}} C \times_{\kappa} \mathbb{A}^r_{\kappa} \tag{5.3.3}$$

where $L(n)$ is a linear polynomial with integral coefficients in $n \in \mathbb{N}$ and r is some fixed $r \in \mathbb{N}$.

A proof of this fact would be a big step forward in development of the theory of schemic motivic integration. One consequence of this would be the following. Let C and $\mathfrak{n}$ be as in the conjecture. Let $X = \mathrm{Spec}(A)$ be an affine scheme whose reduction is smooth and let $\mathfrak{n} = \mathrm{Spec}(R)$ where R is the maximum artinian subring of A. Let $\mathcal{X}$ be the formal scheme determined by the projective limit of schemes $X_{\mathfrak{m}}$ where $X_{\mathfrak{m}}$ is the lift of the smooth morphism $X \to \mathfrak{n}$ with $\mathfrak{m} \in \mathbb{I}$ where $\mathfrak{x} = \varinjlim \mathbb{I}$ and $\ell(\mathfrak{m}) \geq \ell(\mathfrak{n})$. Then provided that the following conjecture is true, the formal scheme $\nabla_{\mathfrak{x}}\mathcal{X}$ will have a well-defined reduced motivic volume with respect to $\mathfrak{x} = \varinjlim\{J_p^m C \mid m \in \mathbb{N}\}$:

$$\begin{aligned} \Psi_{\mathfrak{x}}(\nabla_{\mathfrak{x}}\mathcal{X}) &:= \mu_{\mathfrak{x}}(\nabla_{\mathfrak{x}}(X^{red})) \cdot \lambda_{\mathfrak{l}}(\nabla_{\mathfrak{l}}C) = [X^{red}]\mathbb{L}^{-d+r} \cdot \lambda_{\mathfrak{l}}(\nabla_{\mathfrak{l}}C) \\ \Psi_{\mathfrak{x}}^r(\nabla_{\mathfrak{x}}\mathcal{X}) &:= \mu_{\mathfrak{x}}^r(\nabla_{\mathfrak{x}}(X^{red})) \cdot \lambda_{\mathfrak{l}}(\nabla_{\mathfrak{l}}C) = [X^{red}]\mathbb{L}^{-d} \cdot \lambda_{\mathfrak{l}}(\nabla_{\mathfrak{l}}C)\ , \end{aligned} \tag{5.3.4}$$

where $d = \dim X$, and we note that $\lambda_{\mathfrak{l}}(\nabla_{\mathfrak{l}}C)$ is the classical motivic volume of the reduced curve C. This is the alternative measure I am proposing in this chapter.

5.3.3 Some remarks pertaining to auto-arcs of jets of higher dimensional varieties.

The situation for auto-arcs of truncated jets of higher dimensional varieties is much more ambiguous. Using my sage script, I investigated first the Whitney umbrella. Briefly, let $W = \mathrm{Spec}(\kappa[x,y,z]/(x^2z - y^2z))$ and $\mathfrak{n} = J_0^n W$. I computed the reduced auto-arc $X_n := (\nabla_{\mathfrak{n}}\mathfrak{n})^{red}$ for $n \leq 4$. The computations are enormous for $n = 3$ and especially for $n = 4$ as the non-reduced structure is very complicated. Thus, here, I will just state my results given that the interested reader may run my sage script to verify my claims. I obtain $X_1 = \mathrm{Spec}(\kappa)$, $X_2 = \mathbb{A}_\kappa^9$, $X_3 = \mathbb{A}_\kappa^{26}$, and $X_4 = Y \times_\kappa \mathbb{A}_\kappa^{44}$ where Y is $\mathrm{Spec}(\kappa[x_1, \ldots, x_9]/I)$ where I is the ideal:

$$I = (x_1^2x_3 - x_2^2x_3, 2x_1x_3x_4 - 2x_2x_3x_5 + x_3^2x_6 - x_1^2x_6, \ldots$$
$$\ldots 2x_1x_3x_7 - 2x_1x_3x_8 + x_1^2x_9 - x_2^2x_9, x_3x_4^2 - x_3x_5^2 + 2x_1x_4x_6 - 2x_2x_5x_6+$$
$$+ x_3x_7^2 - x_3x_8^2 + 2x_1x_7x_9 - 2x_2x_8x_9, 2x_3x_4x_7 + 2x_1x_6x_7 - 2x_3x_5x_8 - 2x_2x_6x_8+$$
$$+ 2x_1x_4x_9 - 2x_2x_4x_9, x_4^2x_6 - x_5^2x_6, 2x_4x_6x_7 - 2x_5x_6x_8 + x_4^2x_9 - x_5^2x_9, \ldots$$
$$\ldots x_6x_7^2 - x_6x_8^2 + 2x_4x_7x_9 - 2x_5x_8x_9, x_7^2x_9 - x_8^2x_9)$$

I could not find any kind of suitable formula as in the case of the previous section for Y. The most I will say is that it looks like it could be a closed subvariety of $(\nabla_{\mathfrak{l}_3}W)^3$. More disturbing is the growth in the affine part from dimension 9 to 26 to 44. Perhaps more computations will yield a better picture in the future although as we saw in the previous section, it would not be surprising if auto-arcs of truncated jets of a reducible variety is complicated.

Continuing this line of thought, I investigated the elliptic surface $S = \mathrm{Spec}(\kappa[x,y,z]/(y^2 - x^3 - zx))$. If $\mathfrak{n} = J_O^nS$ and $X_n = (\nabla_{\mathfrak{n}}\mathfrak{n})^{red}$, I calculated that X_n has a very complicated structure for $n = 2$, 3, and 4. In fact, $X_2 = \mathrm{Spec}(\kappa[x_1, \ldots, x_6]/I)$ where

$$I = (x_6x_1 + x_5x_2 + x_6x_3, x_1x_2 + x_6x_3, x_2x_3, 3x_1^2x_3 + 3x_5x_3^2 + 2x_6x_2, 3x_1x_3^2 + x_2^2).$$

So, we see that even for $n = 2$, the reduced auto-arc is not irreducible. The spaces X_3 and X_4 are much more complicated. I could not find anyway in which they might fit into a general pattern (such as the one noticed for curves). These calculations suggest that any kind generalisation to the conjecture of the previous subsection will have high likelihood of being false. At best, we can say that the case for auto-arcs of truncated jets of irreducible curves seems to be very special.

We also look at the reduced auto-arc of $X_n = (\nabla_{\mathfrak{n}}\mathfrak{n})^{red}$ where $\mathfrak{n} = J^n_O C$ where C is the parabolic cylinder defined by $x^2 + 2zy$. Again, for $n > 2$, X_n is a very complicated space from the looks of it. But, for $n = 3$, we do notice something similar to the Whitney umbrella example. Let I be the ideal of $R = \kappa[x_1, x_2, x_3, x_4]$ generated by the set $\{x_6^2 + 2x_5x_7, 2x_7x_1 + 2x_6x_2 + 2x_5x_3, -2x_2^2 - 4x_1x_3 + 2x_4x_7 + 2x_6x_8 + 2x_5x_9, 2x_3x_7 + 2x_2x_8 + 2x_1x_9, x_8^2 + 2x_7x_9\}$. Then,

$$X_3 = \operatorname{Spec}(R/I) \times_\kappa \mathbb{A}^{14}_\kappa \, .$$

This looks very similar to the reduced auto-arcs of the truncated 4-jets of the Whitney umbrella – i.e., it looks like it could be a closed subvariety of $(\nabla_{\mathfrak{l}_3} C)^2$. However, given these three examples, it looks like either new techniques must be developed to perform motivic integration (e.g., the material in Chapter 4) or a more detailed investigation toward understanding the reduced auto-arcs just in the case of surfaces is needed. The latter appears to me as a difficult question.

5.4 The Sage script for computing affine arc spaces.

In this section, I provide my code, written in Sage 6.2.Beta1 (cf., [S] with needed interface with Singular [DGPS]) and Python 2.7.6 (cf., [P], which will need NumPy [NP] installed), which computes the arc space of an affine scheme X with respect to a fat point $\mathfrak{n}$ in characteristic 0. Note that the running time increases substantially when the length of the fat point $\ell(\mathfrak{n})$ increases even modestly, and it also increases dramatically when the fat point $\mathfrak{n}$ has small length but the affine scheme X is even modestly complicated. I am not sure exactly how to quantify the computational complexity here, but that is an interesting question. It looks like computations of arc spaces are destined to be slow. For example, using the SageMathCloud (available at `https://cloud.sagemath.com/`), it took two hours to compute the auto-arc of the the arc space of $J^8_O N$ where N is the node.

I have decided not to include in the code how to compute the reduced arc space. Thus, this must be done by hand (which can be extremely tedious) or done using Sage at the terminal by the user. Likewise, I have not taken up the matter of computing the arc space in positive characteristic. Although, I am more or less certain that this can be done without issue in Sage. Finally, the output is not great and could be organized in better ways, but this question I leave to the user. It does produce the ideal of definition of the arc space which is enough for my purposes.

```
import sys
import datetime
import operator
from sage.symbolic.expression_conversions import
   PolynomialConverter

## ##########################################################
#
# Sage code for computing arc spaces
#
## ##########################################################

## ##########################################################
## Class to organize methods and storing data variables
## ##########################################################
class Space:

   def __init__(self):
      self.numvars = 0
      self.numeqs = 0
      self.firstequation = 0
      self.fatvars = 0
      self.fateqs = 0
      self.firstfatequation = 0
      return

   def setEquations(self):
      print("Creating functions for your space...")
      return

   def setFatEquations(self):
      print("Creating functions for your fat point...")
      return

   def toString(self):
      msg = "Symbols: " + str(self.numvars) + "\t"
      msg = msg + "Equations: " + str(self.numeqs) + "\n"
      return msg

   def toFatString(self):
```

```
        msg = "Symbols: " + str(self.fatvars) + "\t"
        msg = msg + "Equations: " + str(self.fateqs) + "\n"
        return msg

## ##########################################################
## Helper methods
## ##########################################################
def getInt(msg):
    my_input = raw_input(msg)
    try:
        return int(my_input)

    except:
        print("Input should be an integer, please try again")
        return getInt(msg)
## ##########################################################
def debug(msg):
    now = datetime.datetime.now()
    msg = "[" + str(now) + "] " + str(msg)
    print(msg)
    return

## ##########################################################
## Begin main program
## ##########################################################
if __name__ == '__main__':


    mySpace = Space()
    mySpace.numvars = getInt("How many variables are in
        this space? ")
    mySpace.numeqs = getInt("How many defining equations
        does your space have? ")


    print("Defining ambient space...")
    Poly1=PolynomialRing(QQ,"x",mySpace.numvars)
    print Poly1
    Poly1.inject_variables()
    mySpace.setEquations()
```

```
debug(mySpace.toString())



print('Using the variables above, input the expression
   for your first equation and press return.')
mySpace.firstequation=SR(raw_input())
f=[]
f.append(mySpace.firstequation)

for i in xrange(1, mySpace.numeqs):
   print('Using the variables above, input the
      expression for your next equation and press
      return.')
   mySpace.nextequation=SR(raw_input())
   f.append(mySpace.nextequation)


print('Check that your list of expressions is correct:')
print f



mySpace.fatvars = getInt("How many variables are in
   this fat point? ")
mySpace.fateqs = getInt("How many defining equations
   does your fat point have? ")

print("Defining ambient space...")
Poly2=PolynomialRing(QQ,"y",mySpace.fatvars)
print Poly2
Poly2.inject_variables()
mySpace.setFatEquations()

debug(mySpace.toString())

print('Using the variables above, input the expression
   for your first Equation of your Fat point and press
   return.')
```

```
   mySpace.firstfatequation=SR(raw_input())
   g=[]
   g.append(mySpace.firstfatequation)

   for i in xrange(1, mySpace.fateqs):
      print('Using the variables above, input the
         expression for your next Equation of your Fat
         point and press return.')
      mySpace.nextfatequation=SR(raw_input())
      g.append(mySpace.nextfatequation)
      I=ideal(g)

   debug(mySpace.toFatString())


 ###################################################
 #This code computes a basis for the coordinate ring of the
 #fat point as a vector space over the rationals
 #
 ###################################################

   SingPoly2=singular(Poly2)
   singular.setring(SingPoly2)
   G=[str(g[i]) for i in xrange(mySpace.fateqs)]
   J=singular.ideal(G)
   J=J.groebner()
   B=list(J.kbase())
   length=len(B)
   C=[B[i].sage() for i in xrange(length)]

   arcvars=length*mySpace.numvars
   debug("Defining ambient space for your arc space...")


###################################################
#This block of code defines an ambient space for the arc
   space
#and defines the general symbolic arcs
#
###################################################
```

```
    arcvars=length*mySpace.numvars
    hh=mySpace.numvars+mySpace.fatvars+arcvars
    Poly3=PolynomialRing(QQ,"a",hh)
    Poly3.inject_variables()
    LL=list(Poly3.gens())
    LL1 = [LL[i] for i in xrange(mySpace.numvars)]
    LL2 = [LL[i] for i in
       xrange(mySpace.numvars,mySpace.numvars+mySpace.fatvars)]
    LL3 = [LL[i] for i in
       xrange(mySpace.numvars+mySpace.fatvars,hh)]
    w=Poly2.gens()
##Substitution of variables to force computation that the
    equations for
##the scheme and fat point take place in ambient space
    Dict2={w[i]:LL2[i] for i in xrange(mySpace.fatvars)}
    E=[C[i].subs(Dict2) for i in xrange(length)]
    v=Poly1.gens()
    Dict1={v[i]:LL[i] for i in xrange(mySpace.numvars)}
    F=[f[i].subs(Dict1) for i in xrange(mySpace.numeqs)]

    M=matrix(length,mySpace.numvars,LL3)
    N=matrix(1,length, E)
##Use matrix multiplication to create the general symbolic
    arcs:
    D=N*M

    DD=D.list()
    Dict2={LL1[i]:DD[i] for i in xrange(mySpace.numvars)}
    FF=[F[i].subs(Dict2) for i in xrange(mySpace.numeqs)]
    idealF=ideal(FF)
    debug(idealF)

    tempJ=list(J)
    lll=len(tempJ)
    JJ=[tempJ[i].sage() for i in xrange(lll)]
    w=Poly2.gens()
    Dict2={w[i]:LL2[i] for i in xrange(mySpace.fatvars)}
    tempI= [JJ[i].subs(Dict2) for i in xrange(lll)]
    II=ideal(tempI)
```

```
   debug(II)
##Need the following ring map in order to simplify the
   equations of the arc space
   QR=QuotientRing(Poly3,II)
   QR.inject_variables()
   pi=QR.cover()
##Simplification:
   p=[PolynomialConverter(FF[i],base_ring=QQ) for i in
      xrange(mySpace.numeqs)]
   rr=[p[i].symbol(FF[i]) for i in xrange(mySpace.numeqs)]
   RR=[pi(rr[i]) for i in xrange(mySpace.numeqs)]
   debug("going to factor ring")
   d=[RR[i].lift() for i in xrange(mySpace.numeqs)]
   debug("lifting to the cover")

##The main algorithm. It finds the equations determined by
##the coefficients of the basis elements.

   debug("Computing tempL")
   tempL=[]
   for i in xrange(mySpace.numeqs):
      j=0
      for j in xrange(length-1):
         cc=d[i].quo_rem(E[j])
         #debug("CC: " + str(cc))
         CC=list(cc)
         tempL=tempL+[CC[0]]
         a=simplify(d[i]-CC[0]*E[j])
         if ( d[i] == a ):
            debug("No change")
         #del d[i]
         #debug("d[i] prior to change: " + str(d[i]))
         d[i] = a
         #debug("d[i] after change: " + str(d[i]))
         #d.insert(i,a)
         j=j+1
   bigL=tempL+d

##Simplify again:
   debug("... processing ...")
```

```
    quoL=[pi(bigL[i]) for i in xrange(len(bigL))]
    newL=[quoL[i].lift() for i in xrange(len(bigL))]
##This is not needed but could be useful in the future:
    #runL=[factor(newL[i]) for i in xrange(len(newL))]


##Making sure our list of equations is fully populated:
    ## What is tryL??
    breadth = int(mySpace.numeqs)
    depth = int(length)

    tryL = []
    ## Initialize the list to -1
    for i in xrange(breadth):
        j=0
        for j in xrange(depth):
            tryL.append("NaN")

    debug("... performing division ...")
    ## Populate list with real data
    for i in xrange(breadth):
        j=0
        for j in xrange(depth):
            idx = (i * depth + j)
            tryL[idx] = list( newL[idx].quo_rem( E[j] ))[0]

##Following lists are not needed but could be useful in
    the future:
    #tryL=[list(newL[i].quo_rem(E[i]))[0] for i in
        xrange(len(newL))]
    #finL=[factor(tryL[i]) for i in xrange(len(bigL))]

##Display the length of the fat point
    debug(">>The length of your fat point is:")
    debug(length)
##Display the list of generators for the ideal which
    defines the arc space:
    debug("Create ideal...")
    tempIdeal=Poly3.ideal(LL1+LL2+newL)
    debug(tempIdeal)
```

```
##The following code is an alternate display.
##Singular has a much nicer output possible. However, for
   large spaces, the program hangs when creating a
   quotient ring in sage.
##So, I will comment out this region, but it could be
   useful in the future...
#
#   debug("Quotient ring")
#   finQR=Poly3.quotient_ring(tempIdeal)
#   finQR.inject_variables()
#
#
#
#   debug( ">> Equations for Arc space: " )
#
#   debug("Singular")
#   SingfinQR=singular(finQR)
```

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