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Asymptotic Invariants and Flatness of Local Endomorphisms

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Abstract

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For a local endomorphism of a noetherian local ring we introduce 3 asymptotic invariants one of which we call entropy. We use this notion of entropy to extend numerical conditions in Kunz' regularity criterion to every contracting endomorphism of a noetherian local ring, and to give a characteristic-free interpretation of the definition of Hilbert-Kunz multiplicity. We also show that every finite endomorphism of a complete noetherian local ring of equal characteristic can be lifted to a finite endomorphism of a complete regular local ring. The local ring of an algebraic or analytic variety at a point fixed by a finite self-morphism inherits a local endomorphism whose entropy is well-defined. This situation arises at the vertex of the affine cone over a projective variety with a polarized self-morphism, where we compare entropy with degree.

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Contents

Index of notations

Unless otherwise stated, all rings in this paper are assumed to be noetherian, local, commutative and with identity element 1, while all ring homomorphisms are assumed to be local. The following notation is used:

φ^n	n-fold composition of φ with itself for a self-map $\varphi : X \to X$ of a space.
$\mathrm{length}_R(M)$	length of an R-module of finite length M.
$\lambda(f)$	$\mathrm{length}_S(S/f(\mathfrak{m})S)$ of the closed fiber of a local homomorphism $f : (R, \mathfrak{m}) \to S$.
$H(\varphi)$	local entropy of an endomorphism of finite length $\varphi : R \to R$.
q_φ	$H(\varphi)^{1/\dim R}$ where $\varphi : R \to R$ an endomorphism of finite length.
$f_* N$	R-module obtained from an S-module N via a homomorphism $f : R \to S$.
$f_* S$	R-algebra obtained from a ring S via a homomorphism $f : R \to S$ (see [8]).
$f^* N$	S-module obtained from an R-module N by extending coefficients via $f : R \to S$.
$\deg(f)$	the rank of $f_* S$ where $f : R \to S$ is a finite homomorphism and R is a domain.
$\mu_R(M)$	minimum number of generators of a finitely generated R-module M.
$\mathrm{Specmin}\, R$	set of all minimal prime ideals of a ring R.
${}^a f$	morphism $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$ corresponding to a homomorphism $f : R \to S$.

Introduction

Let $(R, \mathfrak{m})$ be a local ring and $\varphi : R \to R$ an endomorphism. The objective of this study is to investigate the limit of the sequence:

$$H = \lim_{n\to\infty} \left(\operatorname{length} R/\varphi^n(\mathfrak{m})R\right)^{1/n} \tag{1}$$

We say a homomorphisms $\varphi : (R, \mathfrak{m}) \to S$ is *african*[1] if $S/\mathfrak{m}S$ is artinian. Thus african endomoprhisms are exactly the endomorphisms for which the sequence in (1) makes sense. More globally, let $f : X \to Y$ be a morphism of schemes where X is noetherian, $p \in X$, $q = f(p)$. Then p is a generic point of the fiber X_q precisely when

$$f^\sharp : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$$

is african. As we detail in section 2.6, this situation arises, for example, at the vertex of the affine cone over a projective variety with a polarized self-morphism.

[1] This is an acronym for Artinian Fiber RIng at the Closed point.

We prove that the sequence length $R/\varphi^n(\mathfrak{m})R$ is submultiplicative ($\{a_n\}$ is submultiplicative when $a_{n+m} \leqslant a_n a_m$) and hence the limit (1) exists. We christen it *local entropy* and denote by H. We use two more similar invariants along with H.

When R is a domain and φ is finite, we obtain the inequality

$$H \geqslant \frac{\dim_K K \otimes (\varphi_\star R)}{\dim_k k \otimes (\varphi_\star R)}$$

where K and k are the field of fractions and the residue field of R respectively (Proposition 43). We prove that the two sides of this inequality are in fact equal when:

1. R is Cohen-Macaulay (Proposition 43)
2. R is the local ring at the vertex of the cone over a projective variety and φ is induced by a polarized self-morphism of the variety, assuming that polarization is greater than 1 (Proposition 46)
3. R is of prime characteristic and φ is the Frobenius. (Example 44)

In Section 3.2, we use Nagata's flatness criterion to prove that if $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ is african and *contracting* (Definition 9), then $\operatorname{length}(R/\varphi^n(\mathfrak{m})R) = H(\varphi)^n$ for some n implies that φ is flat. This implication allows us to use entropy H to extend the numerical condition of Kunz' regularity criterion from the Frobenius to arbitrary contracting african endomorphisms (Theorem 53).

Along the way we give some properties of H. These properties allow one to "reduce" the task of computing entropy to the case of a better ring. Thus, in Proposition 36 we show how to pass from an endomorphism of a ring with mixed characteristic to an endomorphism of an equi-characteristic ring without changing entropy. Then in Corollary 26 we show that one can pass to an endomorphism of the completion of a local ring without changing entropy. In Proposition 38 we start with any endomorphism and show how to pass to an endomorphism of an *integral domain* which has the same entropy. Finally, we show in Theorem 63 in Section 4 that an african endomoprhism of a complete equi-characteristic local ring comes from an african endomoprhism of a ring of power series, although at this step entropy changes.

In fact, Section 4 has less to do with H. This section is inspired by results of Fakhruddin [15, Corollary 2.2], and Bhatnagar and L. Szpiro [5, Theorem 2.1] on extending a polarized self-morphism of a projective variety over an infinite field to an ambient projective space. Recently in [41] Poonen gave a proof for the main result of [5] over finite fields. Here we consider a similar lifting problem for an african endomorphism of an equicharacteristic complete noetherian local ring, and prove a Cohen-Fakhruddin- type structure theorem (Theorem 63), which says that *any african endomoprhism of a* *complete local ring containing a field comes from an african endomoprhism of a ring of formal power series.* Our proof is an adaptation of Fakhruddin's proof to the case of complete local rings.

One motivation for studying local invariants of rational self-maps of schemes was the equidistribution results for the measure of maximal entropy obtained by Brolin [10], Lyubich [33], L. Szpiro, Ullmo and Zhang [48].

Chapter 1

Examples of rings with endomorphisms

In this chapter we have collected some examples of local rings that admit african endomorphisms. For more examples of such local rings the reader can consult [6, 6.2-6.3] and [16, 2.3-2.5].

There is a recent literature on constructing examples of finite endomorphisms $\varphi_P^\sharp : \mathcal{O}_{X,P} \to \mathcal{O}_{X,P}$ that are not automorphisms, with (X, P) a normal singularity. References include [6, 6.2-6.3] and [16, 2.3-2.5]. Other authors, see, e.g., [11], have addressed the question of whether the existence of a finite surjective self-morphism of degree > 1 of a normal variety imposes restrictions on its local geometry or the nature of its singularities.

Example 1 (Frobenius)**.** If R is a noetherian local ring of positive prime characteristic p, then the Frobenius endomorphism $x \mapsto x^p$ is contracting and african.

Example 2. A power series ring $R := k[\![X_1, \dots, X_n]\!]$ over a field k has many african endomorphisms. If elements $f_1, \dots, f_n$ of R generate an ideal of height n in R, then we obtain an african endomorphism by setting $X_i \mapsto f_i$ for $1 \leqslant i \leqslant n$. Conversely, in Theorem 63 we will show using Cohen's structure theorem that every african endomorphism of a complete equicharacteristic local ring is induced by an endomorphism of a power series ring.

Definition 3. Let $\varphi : R \to R$ be an endomorphism of a noetherian local ring. An ideal $\mathfrak{a}$ of R is called φ-stable, if $\varphi(\mathfrak{a})R \subseteq \mathfrak{a}$.

Example 4. Let $R := k[\![X_1, \dots, X_n]\!]$ be a power series ring over a field k, and let φ be an african endomorphism of R, e.g., as defined in Example 2. Let $z \neq 0$ be an arbitrary element of the maximal ideal of R. Then the ideal $\mathfrak{a}$ generated by $z, \varphi(z), \varphi^2(z), \dots$ (orbit of z under φ) is φ-stable. Thus φ induces an african endomorphism on $R/\mathfrak{a}$. Moreover, if φ is contracting, then so is the induced map. Macaulay 2 can be used to generate examples of this type. We mention three examples here. In these examples k is a field of characteristic zero, R and $\mathfrak{a}$ are as above and $\mu(\,\cdot\,)$ denotes the minimum number of generators of a finitely generated R-module.

a) $n = 4$, $z = X_2^2 + X_3^5$. Define φ as $X_1 \mapsto X_1^3$, $X_i \mapsto X_i^2$ for $i = 2, 3$ and

$X_4 \mapsto X_4^7$. Then $\operatorname{rad}(\mathfrak{a}) = (X_2, X_3)$.

b) $n = 7$, $z = X_1X_2X_3 + X_4^3 + X_5^2X_6 + X_7^3$. Define φ as $X_i \mapsto X_i^2$, for $2 \leqslant i \leqslant 6$ and $X_1 \mapsto X_7^2$, $X_7 \mapsto X_1^2$. Then $\mu(\mathfrak{a}) = 5$ and $\dim R/\mathfrak{a} = 3$. This is completion of the local ring at the vertex of the cone over a projective scheme. The endomorphism is induced by a self-morphism of $\mathbb{P}^6$.

c) $n = 8$, $z = X_1X_4^5X_8^2 + X_3X_5^4 + X_2X_6^3 + X_7$. Define φ as $X_i \mapsto X_i^2$, for $3 \leqslant i \leqslant 8$ and $X_1 \mapsto X_2^2$, $X_2 \mapsto X_1^2$. Then $\mu(\mathfrak{a}) = 5$ and $\dim R/\mathfrak{a} = 4$.

Example 5. Let $R := k[\![X_1, \ldots, X_n]\!]$ be a power series ring over a field k, and let $\mathfrak{a}$ be an ideal of R with generators that can be expressed in the form monomial = monomial. Then the endomorphism of R given by $X_i \mapsto X_i^r$ for an integer $r > 1$, induces a contracting african endomorphism on $R/\mathfrak{a}$. In more geometric terms this example says any singularity of a toric variety admits a contracting self-morphism.

Chapter 2

Local entropy

2.1 Existence and estimates for entropy

In this section we prove the existence of entropy along with two more asymptotic invariants, which we will use to estimate entropy. The proof of existence hinges on the following well-known fact from calculus. For a proof we refer to [50, Theorem 4.9].

Lemma (Fekete). *Let $\{a_n\}_{n\geqslant 1}$ be a sub-multiplicative (super-multiplicative) sequence of positive numbers, that is, $a_{n+m} \leqslant a_n a_m$ (resp. $a_{n+m} \geqslant a_n a_m$) for all $n, m \in \mathbb{N}$. Then the sequence $\{a_n^{1/n}\}$ converges to its infimum (resp. supremum) (In the super-multiplicative case the supremum and hence the limit can be $+\infty$.)*

The first concern in investigating

$$\lim_{n\to\infty} \left(\operatorname{length} R/\varphi^n(\mathfrak{m})R\right)^{1/n}$$

is: when are these lengths finite? This leads us to the following:

Definition 6. We say a local homomorphism $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is *african*[1], if the following equivalent conditions hold:

- **a)** $S/f(\mathfrak{m})S$ is artinian.
- **b)** The only prime ideal of S which contracts to $\mathfrak{m}$ is $\mathfrak{n}$.

Define $\lambda(\varphi) := \operatorname{length} S/f(\mathfrak{m})S$.

Suppose now $f : X \to Y$ is a morphism of schemes where X is noetherian, $p \in X$, $q = f(p)$. Then p is a generic point of the fiber X_q precisely when

$$f^\sharp : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$$

is african.

Remark 7. Note that for a homomorphism of noetherian local rings, finite $\implies$ integral $\implies$ african and finite $\implies$ quasi-finite $\implies$ african. Also the composition of two african homomorphisms is again african.

Here is another nice property of african homomorphisms.

Proposition 8. *Let $f : (R, \mathfrak{m}) \to S$ be african. Let M be an R-module of finite length. Then*

- **a)** *$M \otimes_R S$ is of finite length as an S-module.*

[1]african = Artinian Fiber RIng at the Closed point

b) $\mathrm{length}_S(M \otimes_R S) \leqslant \lambda(f) \cdot \mathrm{length}_R(M)$.

c) *If* f *is* flat*, then* $\mathrm{length}_S(M \otimes_R S) = \lambda(f) \cdot \mathrm{length}_R(M)$.

Proof. By induction on $\mathrm{length}_R(M)$. □

Definition 9. A local endomorphism $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ is called *contracting* if for every $x \in \mathfrak{m}$ the sequence $\{\varphi^n(x)\}_{n \geqslant 1}$ converges to 0 in the $\mathfrak{m}$-adic topology of R.

Remark 10. The terminology of *contracting* endomorphism is due to Avramov, Hochster, Miller and Iyengar, see [3, p. 80] or [2, p. 2]. This terminology is due to the fact that such endomorphisms are contracting in the $\mathfrak{m}$-adic topology of the ring. However, if R is a local ring admitting a contracting endomorphism in the sense of Definition 9, then the closed point of $\operatorname{Spec} R$ corresponds to a *super-attracting* fixed point in the sense of complex (or non-archimedean) dynamics. In [21] for instance, an endomorphism satisfying the condition of Definition 9 is called super-attracting.

Remark 11. It is easy to see that if a local ring has a *contracting* endomorphism, it must be of equal characteristic.

Lemma 12. *Let* $(R, \mathfrak{m})$ *be a noetherian local ring of embedding dimension* δ*. A local endomorphism* $\varphi : R \to R$ *is contracting if and only if* $\varphi^\delta(\mathfrak{m}) \subset \mathfrak{m}^2$.

Proof. See [3, Lemma 12.1.4]. □

The following 2 quantities are needed to estimate entropy:

Definition 13. Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be african. We define

$$\begin{aligned} v(f) &= \max\{r \mid f(\mathfrak{m})S \subset \mathfrak{n}^r\} \\ w(f) &= \min\{r \mid \mathfrak{n}^r \subset f(\mathfrak{m})S\}. \end{aligned}$$

Remark 14. The definition of $v(f)$ and $w(f)$ was inspired by similar definitions of Samuel in [43, p. 11]. Note that $\mathfrak{n}^{w(f)} \subset f(\mathfrak{m})S \subset \mathfrak{n}^{v(f)}$. Thus: $v(f) \leqslant w(f)$.

Lemma 15. *Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ and $g : (S, \mathfrak{n}) \to (T, \mathfrak{p})$ be two african homomorphisms. Then*

- **a)** $\lambda(g) \leqslant \lambda(g \circ f) \leqslant \lambda(g) \cdot \lambda(f)$.
- **b)** *If g is* flat *then $\lambda(g \circ f) = \lambda(g) \cdot \lambda(f)$.*
- **c)** $v(g \circ f) \geqslant v(g) \cdot v(f)$
- **d)** $w(g \circ f) \leqslant w(g) \cdot w(f)$.

Proof. For **a)** and **b)** apply Proposition 8, using the canonical isomorphism of T-modules $(S/f(\mathfrak{m})S) \otimes_S T \cong T/g\,(f(\mathfrak{m})S)\,T$ (see, e.g., [7, Chap. II, § 3.6, Coroll. 2 and 3, pp. 253-254]). For **c)** and **d)** using the fact that for an ideal

$\mathfrak{a}$ of S and $n \in \mathbb{N}$, $g(\mathfrak{a}^n)T = (g(\mathfrak{a})T)^n$, we can write

$$((g \circ f)(\mathfrak{m}))T = g(f(\mathfrak{m})S)T \subset g(\mathfrak{n}^{v(f)})T = (g(\mathfrak{n})T)^{v(f)} R \subset \mathfrak{p}^{v(g)v(f)},$$

and

$$\mathfrak{p}^{w(g)w(f)} \subset (g(\mathfrak{n})T)^{w(f)} = g(\mathfrak{n}^{w(f)})T \subset g(f(\mathfrak{m})S)T = ((g \circ f)(\mathfrak{m}))T.$$

By definition of $v(g \circ f)$ and $w(g \circ f)$ then we obtain $v(g \circ f) \geqslant v(g) \cdot v(f)$ and $w(g \circ f) \leqslant w(g) \cdot w(f)$, respectively. □

Remark 16. This lemma shows that the sequences $\{\lambda(\varphi^n)\}$ and $\{w(\varphi^n)\}$ are sub-multiplicative while $\{v(\varphi^n)\}$ is super-multiplicative and so we can apply Fekete's lemma and introduce 3 asymptotic invariants of an endomoprhism.

Theorem-definition 17. *Let φ be an african endomorphism of a ring $(R, \mathfrak{m})$. Suppose R is of dimension d and embedding dimension δ. Define*

$$\begin{aligned}
H(\varphi) &:= \lim_{n\to\infty} \lambda(\varphi^n)^{1/n} = \lim_{n\to\infty} \text{length}(R/\varphi^n(\mathfrak{m})R)^{1/n} \\
v_h(\varphi) &:= \lim_{n\to\infty} v(\varphi^n)^{1/n} = \lim_{n\to\infty} \max\{r \mid \varphi^n(\mathfrak{m}) \subseteq \mathfrak{m}^r\}^{1/n} \\
w_h(\varphi) &:= \lim_{n\to\infty} w(\varphi^n)^{1/n} = \lim_{n\to\infty} \min\{r \mid \mathfrak{m}^r \subseteq \varphi^n(\mathfrak{m})R\}^{1/n}.
\end{aligned}$$

a) *The invariants $H(\varphi)$, $v_h(\varphi)$ and $w_h(\varphi)$ are finite and not less than* 1.

b) $1 \leqslant v_h(\varphi)^d \leqslant H(\varphi) \leqslant w_h(\varphi)^d$.

c) *If φ is in addition contracting, then $H(\varphi)^\delta \geqslant 2^d$.*

Without pretence, we call the invariant $H(\varphi)$ *local entropy* of φ.

Proof. **a)** Apply Fekete's Lemma in light of Lemma 15. Finiteness of v_h follows from $v(\varphi^n) \leqslant w(\varphi^n)$ for all n (see Remark 7 and Remark 14).

b) From Definition 13 we get $\mathfrak{m}^{w(\varphi^n)} \subset \varphi^n(\mathfrak{m})R \subset \mathfrak{m}^{v(\varphi^n)}$. Thus

$$\operatorname{length}(R/\mathfrak{m}^{v(\varphi^n)}) \leqslant \operatorname{length}(R/\varphi^n(\mathfrak{m})R) \leqslant \operatorname{length}(R/\mathfrak{m}^{w(\varphi^n)}).$$

We consider two cases: $v(\varphi^n) \to \infty$ and $v(\varphi^n) \not\to \infty$. In the first case by Remark 14 $w(\varphi^n) \to \infty$, as well. Then, for large n, $\operatorname{length}(R/\mathfrak{m}^{v(\varphi^n)})$ and $\operatorname{length}(R/\mathfrak{m}^{w(\varphi^n)})$ are polynomials of precise degree d in $v(\varphi^n)$ and $w(\varphi^n)$, respectively, with highest degree terms $e(\mathfrak{m})(v(\varphi^n))^d/d!$ and $e(\mathfrak{m})(w(\varphi^n))^d/d!$. Thus, for large n

$$\frac{1}{2}\frac{e(\mathfrak{m})}{d!}\left(v(\varphi^n)\right)^d \leqslant \operatorname{length}(R/\varphi^n(\mathfrak{m})R) \leqslant 2\frac{e(\mathfrak{m})}{d!}\left(w(\varphi^n)\right)^d,$$

which gives the result after taking n-th root and taking limits. In the second case, when $v(\varphi^n) \not\to \infty$, the sequence $\{v(\varphi^n)\}$ is bounded. Hence, there is a number c such that $1 \leqslant v(\varphi^n) \leqslant c$. Taking n-th root and taking limits, we get $v_h(\varphi) = 1$. Now, if $w(\varphi^n) \to \infty$, then starting with the inequality

$$1 \leqslant \operatorname{length}(R/\varphi^n(\mathfrak{m})R) \leqslant \operatorname{length}(R/\mathfrak{m}^{w(\varphi^n)})$$

and repeating the same argument as before, we arrive at the desired inequality

$$v_h(\varphi) = 1 \leqslant H(\varphi) \leqslant w_h(\varphi)^d.$$

Finally if $w(\varphi^n) \nrightarrow \infty$, then the sequence $\{w(\varphi^n)\}$ is also bounded and there is a number c' such that $1 \leqslant w(\varphi^n) \leqslant c'$. After extracting n-th root and taking limits, get $w_h(\varphi) = 1$. Since $v_h(\varphi) = 1$ as well, the proof will be completed by showing $H(\varphi, R) = 1$. This follows from inequalities

$$1 \leqslant \operatorname{length}(R/\varphi^n(\mathfrak{m})R) \leqslant \operatorname{length}(R/\mathfrak{m}^{w(\varphi^n)}) \leqslant \operatorname{length}(R/\mathfrak{m}^{c'}).$$

c) There is nothing to prove if $\delta = 0$, so assume $\delta > 0$. By Lemma 12 to say that φ is contracting is equivalent to saying $\varphi^\delta(\mathfrak{m})R \subseteq \mathfrak{m}^2$, which by definition gives $v(\varphi^\delta) \geqslant 2$. Hence, using parts **a)** and **b)** we obtain

$$2^{1/\delta} \leqslant (v(\varphi^\delta))^{1/\delta} \leqslant \sup_n \{v(\varphi^n))^{1/n}\} = v_h(\varphi) \leqslant H(\varphi)^{1/d}.$$

$\square$

Example 18 (Frobenius)**.** If R is of characteristic p and φ is its Frobenius endomorphism, then $v_h(\varphi) = p = w_h(\varphi)$ and $H(\varphi) = p^{\dim R}$.

Proof. Let $d = \dim R$. Clearly $v(\varphi^n) \geqslant p^n$ and it is easy to see that $w(\varphi^n) \leqslant p^n\delta$, where δ is the embedding dimension of R. Taking n-th roots and limits, we get $v_h(\varphi) = p = w_h(\varphi)$. Now apply the inequality in part b) of the preceding proposition 17. $\square$

Corollary 19. *Let* $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ *be african. If* $\dim R$ *is zero, then* $H(\varphi) = 1$. *Conversely, if* φ *is contracting and* $H(\varphi) = 1$*, then* $\dim R$ *is zero.*

Proof. If $\dim R = 0$, then R is artinian and the result follows from inequalities $1 \leqslant \text{length}(R/\varphi^n(\mathfrak{m})R) \leqslant \text{length}(R) < \infty$. The converse statement follows from part **c)** of Theorem 17. $\square$

Entropy can be computed using any nonzero module of finite length. Let us recall

Definition 20. Given any (local or not) ring homomorphism: $f : R \to S$, Let R-Mod be the category of R-modules and S-Mod the category of S-modules. The functor $f^\star$: R-Mod $\to$ S-Mod given by

$$f^\star(M) = M \otimes S$$

When f is an iteration of the Frobenius endomorphism, the functor defined in Definition 20 is known as the Frobenius functor. Frobenius functors were first introduced in [40, Definition 1.2]. Properties of Frobenius functors were established in [40] and [25]. When φ is a local endomorphism, one can compare the functors $(\varphi^n)^\star = (\varphi^\star)^n$ to the Frobenius functors.

Proposition 21. *Let $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be african. If M is a nonzero module of finite length, then*

$$H(\varphi) = \lim_{n\to\infty} \operatorname{length}(\varphi^{\star n}(M))^{1/n}.$$

Proof. It is easy to show $\varphi^{\star n}(R/\mathfrak{m}) \cong R/\varphi^n(\mathfrak{m})R$ (see [40, p. 54] or [25, no. 2.6]). Thus,

$$\operatorname{length}(\varphi^{\star n}(R/\mathfrak{m})) = \operatorname{length}(R/\varphi^n(\mathfrak{m})R).$$

Since M is of finite length, there is a surjection $M \to R/\mathfrak{m} \to 0$. Apply the functor $\varphi^{\star n}$ to obtain a surjection $\varphi^{\star n}(M) \to \varphi^{\star n}(R/\mathfrak{m}) \to 0$. Then using Proposition 8-**b**

$$\operatorname{length}(\varphi^{\star n}(R/\mathfrak{m})) \leqslant \operatorname{length}(\varphi^{\star n}(M)) \leqslant \operatorname{length}(R/\varphi^n(\mathfrak{m})R) \cdot \operatorname{length}(M).$$

The result follows after taking n-th root and letting $n \to \infty$. $\square$

In [17] Favre and Jonsson studied a number of local invariants of endomorphisms of local rings of germs of analytic functions over the complex numbers (also see [20]). Their study of these local invariants explains the obstructions encountered in establishing an equidistribution theorem for $(1, 1)$-currents, that extends results of Brolin [10], Lyubich [33], L. Szpiro, Ullmo and Zhang [48]. The work of Favre and Jonsson has also been a key to many subsequent works, including [49], [14], and [39].

In particular, Favre and Jonsson studied the invariant $v_h(\varphi)$ in [19] and [17] in a different guise. In [19, Theorem A] they prove the remarkable result that if k is an arbitrary field and φ is a k-endomorphism of the ring $k[\![X,Y]\!]$, then $v_h(\varphi)$ is a quadratic algebraic integer. A priori it is assumed in [19, Theorem A] that the characteristic of the field is equal to 0, but the method relies on the technique of key polynomials of [18, Appendix E], which is valid in arbitrary characteristic. Bellon and Viallet have conjectured in [4] that their notion of algebraic entropy for dominant rational self-maps of projective space is also always the logarithm of an algebraic integer (see also [46, Conjecture 8]). This conjecture is proved for *monomial* self-maps in [24, Corollary 6.4]. It is, therefore natural to ask a similar question about the invariants $H(\varphi), v_h(\varphi)$ and $w_h(\varphi)$:

Question 22. Let $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be african. Are the invariants $H(\varphi), v_h(\varphi)$ and $w_h(\varphi)$ always algebraic integers?

2.2 Entropy-like properties of local entropy

Roughly speaking, in the theory of dynamical systems, *entropy* is a notion that measures the rate of increase in dynamical complexity as the system evolves with time ([51, p. 313]). Various forms of entropy exist in the literature. For instance, Adler, Konheim, and McAndrew introduced the notion of topological entropy in [1] for continuous maps of compact topological spaces. Measure-theoretic entropy was introduced by Kolmogorov in [28] and later improved by Sinai in [47] for measure-preserving morphisms of probability spaces, and in [4] Bellon and Viallet introduced a notion of algebraic entropy for dominant rational self-maps of projective space.

Our notion of local entropy shares the following basic properties with topological entropy:

1) $H(\varphi^t) = H(\varphi)^t$ for all $t \in \mathbb{N}$ (cf. Proposition 23 below)

2) If $Y \subset X$ is a closed φ-stable subspace, then $H(\varphi\restriction_Y) \leqslant H(\varphi)$. (cf. Proposition 24)

3) If $X = \bigcup Y_i$, $i = 1, \ldots, m$, where the Y_i are closed φ-stable subspaces, then $H(\varphi) = \max\left\{h(\varphi\restriction_{Y_i}) : 1 \leqslant i \leqslant m\right\}$ (cf. Lemma 31)

Proposition 23. *Let $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be african and let $r \in \mathbb{N}$. Then $H(\varphi^r) = H(\varphi)^r$.*

Proof. By definition of entropy

$$\begin{aligned} H(\varphi^r) &= \lim_{n\to\infty} \lambda(\varphi^{rn})^{1/n} \\ &= \left(\lim_{n\to\infty} \lambda(\varphi^{rn})^{1/(rn)}\right)^r \\ &= H(\varphi)^r. \end{aligned}$$

□

Proposition 24. *Given a commutative diagram*

$$\begin{array}{ccc} (R,\mathfrak{m}) & \xrightarrow{\varphi} & (R,\mathfrak{m}) \\ \downarrow f & & \downarrow f \\ (S,\mathfrak{n}) & \xrightarrow{\psi} & (S,\mathfrak{n}) \end{array}$$

where φ, ψ and f are all african.

a) *In general $H(\psi) \leqslant H(\varphi)$.*

b) *If in addition f is flat, then $H(\psi) = H(\varphi)$.*

Proof. **a**) We use Lemma 15-**a)**

$$\begin{aligned} \lambda(\psi^n) &\leqslant \lambda(\psi^n \circ f) = \lambda(f \circ \varphi^n) \\ &\leqslant \lambda(f) \cdot \lambda(\varphi^n). \end{aligned}$$

Take n-th root and limits as $n \to \infty$.

b) If f is flat, using Lemma 15-**a)** and -**b)** we obtain

$$\begin{aligned} \lambda(\varphi^n) &= \lambda(f \circ \varphi^n)/\lambda(f) \\ &= \lambda(\psi^n \circ f)/\lambda(f) \\ &\leqslant \lambda(\psi^n). \end{aligned}$$

Take n-th root and limits as $n \to \infty$. For the inequality in the other direction we use part **a)**. □

With regard to Proposition 24, C. Huneke has asked us the following question:

Question 25. Given a commutative diagram

$$\begin{array}{ccc} (R,\mathfrak{m}) & \xrightarrow{\varphi} & (R,\mathfrak{m}) \\ \downarrow f & & \downarrow f \\ (S,\mathfrak{n}) & \xrightarrow{\psi} & (S,\mathfrak{n}) \end{array}$$

where φ, ψ are african and f any local homomorphism. The ideal $f(\mathfrak{m})S$ is easily seen to be ψ-stable and ψ induces an endomorphism $\overline{\psi} : S/f(\mathfrak{m})S \to S/f(\mathfrak{m})S$ which is african. If f is flat, does the equality $H(\psi) = H(\varphi)H(\overline{\psi})$ hold?

One can see quickly that the inequality $H(\psi) \leqslant H(\varphi)H(\overline{\psi})$ always holds even if f is not flat.

Corollary 26. *Let $\varphi : (R,\mathfrak{m}) \to (R,\mathfrak{m})$ be african. If $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R then $H(\varphi) = H(\widehat{\varphi})$.*

Proof. We have a flat african endomorphism $\widehat{\cdot} : R \to \widehat{R}$ giving a commutative diagram

$$\begin{array}{ccc} (R,\mathfrak{m}) & \xrightarrow{\varphi} & (R,\mathfrak{m}) \\ \downarrow\widehat{\cdot} & & \downarrow\widehat{\cdot} \\ \widehat{R} & \xrightarrow{\widehat{\varphi}} & \widehat{R} \end{array}$$

Apply Proposition 24. □

Corollary 27. *Consider african homomorphisms* $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ *and* $g : (S, \mathfrak{n}) \to (R, \mathfrak{m})$ *of noetherian local rings. Then*

$$H(g \circ f) = H(f \circ g).$$

Proof. we have commutative diagrams:

$$\begin{array}{ccc} R & \overset{g\circ f}{\to} & R \\ \downarrow_f & & \downarrow_f \\ S & \overset{f\circ g}{\to} & S \end{array} \qquad\qquad \begin{array}{ccc} R & \overset{f\circ g}{\to} & R \\ \downarrow_g & & \downarrow_g \\ S & \overset{g\circ f}{\to} & S \end{array}$$

By Proposition 24

$$H(f \circ g) \leqslant H(g \circ f) \quad \text{and} \quad H(g \circ f) \leqslant H(f \circ g).$$

□

Corollary 28 (Invariance)**.** *Let* $(R, \mathfrak{m})$ *and* $(S, \mathfrak{n})$ *be noetherian local rings. Suppose* $f : R \to S$ *is an isomorphism, and let* φ *be an african endomorphism of* R *. Then* $H(f \circ \varphi \circ f^{-1}) = H(\varphi)$*.*

Proof. Apply Corollary 27 to homomorphisms $f \circ \varphi : R \to S$ and $f^{-1} : S \to R$. □

Corollary 29. *Let* $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ *be african and let* $\mathfrak{a}$ *be a* φ*-stable ideal of* R*. Write* $\overline{\varphi}$ *and* $\tilde{\varphi}$ *for endomorphisms induced by* φ *on* $R/\mathfrak{a}$ *and* $R/\varphi(\mathfrak{a})R$*, respectively. Then* $H(\overline{\varphi}) = H(\tilde{\varphi})$*.*

Proof. Let $f : R/\mathfrak{a} \to R/\varphi(\mathfrak{a})R$ and $g : R/\varphi(\mathfrak{a})R \to R/\mathfrak{a}$ be homomorphisms induced by φ and the identity map of R. Apply Corollary 27. □

The next two lemmas will be used in our proof of Proposition 32.

Lemma 30. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers not less than 1 such that $\lim_{n\to\infty} a_n^{1/n} = \alpha$ and $\lim_{n\to\infty} b_n^{1/n} = \beta$ exist. Then*

$$\lim_{n\to\infty}(a_n + b_n)^{1/n} = \max\{\alpha, \beta\}.$$

Proof. See [1, p. 312]. □

Lemma 31. *Let $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be african. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ be a collection of not necessarily distinct φ-stable ideals of R. Let $\overline{\varphi}$ and $\overline{\varphi}_i$ be the endomorphisms induced by φ on $R/\prod_i \mathfrak{a}_i$ and $R/\mathfrak{a}_i$, respectively. Then*

$$H(\overline{\varphi}) = \max\{H(\overline{\varphi}_i) \mid 1 \leqslant i \leqslant s\}.$$

Proof. We proceed by induction on s, the number of ideals, counting possible repetitions. There is nothing to prove if $s = 1$, so suppose $s = 2$. We may assume $\mathfrak{a}_1\mathfrak{a}_2 = 0$; if not, we can replace R with $R/\mathfrak{a}_1\mathfrak{a}_2$ without loss of generality. Then $\mathfrak{a}_2$ is a finitely generated $(R/\mathfrak{a}_1)$-module. Hence for some integer r, we have an exact sequence

$$(R/\mathfrak{a}_1)^r \to R \to R/\mathfrak{a}_2 \to 0.$$

Tensoring by $R/\varphi^n(\mathfrak{m})R$ and taking lengths we get:

$$\lambda(\varphi^n) \leqslant r \cdot \lambda(\overline{\varphi}_1^n) + \lambda(\overline{\varphi}_2^n).$$

Now let $n \to \infty$ and apply Lemma 30 to obtain

$$H(\varphi) \leqslant \max\{H(\overline{\varphi}_1), H(\overline{\varphi}_2)\}.$$

On the other hand by Proposition 24-**a)** we know that $H(\varphi) \geqslant H(\overline{\varphi}_i)$, for each i. This establishes the case $s = 2$. The general case follows easily by induction. □

The following corollary is a special case of the lemma which is worth noting. It says that if all minimal prime ideals of a noetherian local ring R are stable under an endomorphism of the ring, then the entropy is equal to the maximum entropies of the endomorphisms induced on irreducible components of $\operatorname{Spec} R$.

Corollary 32. *Let $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be african. Suppose all minimal prime ideal of R are φ-stable and for each $\mathfrak{p}_i \in \operatorname{Specmin} R$, let ψ_i be the endomorphism induced by φ on $R/\mathfrak{p}_i$. Then*

$$H(\varphi) = \max\{H(\psi_i) \mid \mathfrak{p}_i \in \operatorname{Specmin} R\}. \tag{2.1}$$

Proof. Let $\operatorname{Specmin} R = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ and let $\mathfrak{a} = \prod_i \mathfrak{p}_i$. Then $\mathfrak{a}$ is contained in the nilradical of R, hence $\mathfrak{a}^N = (0)$ for some N, that is, $R = R/\mathfrak{a}^N$. Apply Lemma 31 to obtain

$$H(\varphi) = \max\{H(\psi_i) \mid \mathfrak{p}_i \in \operatorname{Specmin} R\}.$$

$\square$

Remark 33. In Proposition 38 we will generalize Formula 2.1.

2.3 Reduction to equal characteristic

The main result in this section shows that computing local entropy in mixed characteristic can be reduced to the case of equal characteristic $p > 0$.

For a given african endomorphism $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$, we define

$$S := \bigcap_{n=1}^{\infty} \varphi^n(R) \quad \text{and} \quad \mathfrak{n} := \bigcap_{n=1}^{\infty} \varphi^n(\mathfrak{m}). \tag{2.2}$$

Lemma 34. *Let $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be african. Let S and $\mathfrak{n}$ be as defined in Equation 2.2, and let $\mathfrak{a}$ be the ideal generated by $\mathfrak{n}$ in R. Then*

a) *S is a local subring of R with maximal ideal $\mathfrak{n}$.*

b) *$\mathfrak{a}$ is a φ-stable ideal of R.*

c) *If φ is in addition injective, then $\varphi(\mathfrak{a})R = \mathfrak{a}$.*

Proof. **a**) It is immediately clear that S is a subring of R and that $\mathfrak{n}$ is an ideal of S. To show that $\mathfrak{n}$ is the (only) maximal ideal of S, consider an element $s \in S\backslash\mathfrak{n}$. Since $s \notin \mathfrak{n}$, there is an n_0 such that $s \notin \varphi^{n_0}(\mathfrak{m})$. In fact, since for $n \geqslant n_0$, $\varphi^n(\mathfrak{m}) \subseteq \varphi^{n_0}(\mathfrak{m})$, we see that $s \notin \varphi^n(\mathfrak{m})$ for all $n \geqslant n_0$. Hence, there are units $y_n \in R\backslash\mathfrak{m}$ such that $s = \varphi^n(y_n)$ for all $n \geqslant n_0$. Since s is clearly a unit in R, it has a unique multiplicative inverse s^{-1} in R. From uniqueness of multiplicative inverse it immediately follows that we must have $s^{-1} = \varphi^n(y_n^{-1})$, for all $n \geqslant n_0$. Hence, $s^{-1} \in S$, that is, s is also a unit in S.

b) Note that by its definition, $\mathfrak{a}$ has a set of generators $x_1, \ldots, x_g \in \mathfrak{n}$. So $\varphi(\mathfrak{a})R$ can be generated by $\varphi(x_1), \ldots, \varphi(x_g)$ and it suffices to show that each $\varphi(x_i)$ is in $\mathfrak{a}$. Since $x_i \in \mathfrak{n}$, there is a sequence of element $y_{i,n} \in \mathfrak{m}$ such that $x_i = \varphi(y_{i,1}) = \ldots = \varphi^n(y_{i,n}) = \ldots$. Thus, $\varphi(x_i) = \varphi^2(y_{i,1}) = \ldots = \varphi^{n+1}(y_{i,n}) = \ldots$, showing that $\varphi(x_i) \in \mathfrak{n} \subset \mathfrak{a}$.

c) Now suppose φ is injective. To show $\varphi(\mathfrak{a})R = \mathfrak{a}$ it suffices to show that each x_i is in $\varphi(\mathfrak{a})$. Since $x_i \in \mathfrak{n}$, there is a sequence of element $y_{i,n} \in \mathfrak{m}$ such that $x_i = \varphi(y_{i,1}) = \ldots = \varphi^n(y_{i,n}) = \ldots$. Since $x_i = \varphi(y_{i,1})$, we will be done by showing that $y_{i,1} \in \mathfrak{n}$. By injectivity of φ, $y_{i,1} = \varphi(y_{i,2}) = \ldots = \varphi^{n-1}(y_{i,n}) = \ldots$, which means $y_{i,1} \in \mathfrak{n}$. $\square$

Remark 35. Let $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be african and let $\mathfrak{n}$ be as defined in Equation 2.2. If $\mathfrak{n} = (0)$, then by Lemma 34 the ring R contains a field and is of equal characteristic. As noted in [2, Remark 5.9, p. 10], this occurs, for example, if φ is a contracting endomorphism.

Proposition 36. *Let $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be african. Let $\mathfrak{a}$ be the ideal of R defined in Lemma 34, and let $\overline{\varphi}$ be the endomorphism induced by φ on $R/\mathfrak{a}$. Then*

a) $H(\overline{\varphi}) = H(\varphi)$.

b) *If R is of mixed characteristic, then $R/\mathfrak{a}$ is of equal characteristic $p > 0$.*

Proof. **a**) Note that $\varphi^n(\mathfrak{m})R \supset \mathfrak{a}$ for all $n \geqslant 1$. Hence $\varphi^n(\mathfrak{m})R + \mathfrak{a} = \varphi^n(\mathfrak{m})R$, showing that $\mathrm{length}(R/(\varphi^n(\mathfrak{m})R + \mathfrak{a})) = \mathrm{length}(R/\varphi^n(\mathfrak{m})R)$, giving the result.

b) With reference to Lemma 34, the image of the subring S of R in $R/\mathfrak{a}$ is a field, because it's maximal ideal $\mathfrak{n}$ is contained in $\mathfrak{a}$ and is mapped to 0. Hence $R/\mathfrak{a}$ contains a field and must be a local ring of equal characteristic $p > 0$, as its residue field is of characteristic $p > 0$. □

2.4 Integral endomorphisms: entropy and minimal primes

In this section we study the action of integral endomorphisms on minimal primes. An endomorphism $\varphi : R \to R$ induces a self-morphism ${}^a\varphi$ of $\operatorname{Spec} R$. When φ is *integral*, Proposition 38 of this section shows that there exist an irreducible component of $\operatorname{Spec} R$ that is stable under an iteration of φ and which controls entropy of φ.

We begin with a simple observation. Let $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be african. Then for every $n \in \mathbb{N}$,

$$\varphi\left(\ker \varphi^n\right) \subset \ker \varphi^{n-1} \subset \ker \varphi^n.$$

Hence φ induces a local endomorphism of $R/\ker \varphi^n$.

Proposition 37. *Let $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be african. Let $\overline{\varphi}_n$ be the local endomorphism induced by φ on $R/\ker \varphi^n$, $n \in \mathbb{N}$. Then*

a) $H(\varphi) = H(\overline{\varphi}_n)$.

b) *If φ is integral, then so is $\overline{\varphi}_n$.*

c) *If n is large enough, then $\overline{\varphi}_n : R/\ker \varphi^n \to R/\ker \varphi^n$ is injective.*

Proof. **a**) Apply Corollary 29 to the endomorphism φ^n of R, taking $\ker \varphi^n$ as the ideal $\mathfrak{a}$ in that corollary. Since $\varphi^n(\ker \varphi^n)R = (0)$, by that corollary $H(\overline{\varphi}_n^n) = H(\varphi^n)$. Now use Proposition 23.

b) This is clear (see [8, Chapter V, Proposition 2, p. 305]).

c) R is noetherian, so the ascending chain $\ker \varphi \subset \ker \varphi^2 \subset \ker \varphi^3 \subset \dots$ is stationary. Let n_0 be such that $\ker \varphi^n = \ker \varphi^{n+1}$ for $n \geqslant n_0$. We will show that if $n \geqslant n_0$, then $\overline{\varphi}_n : R/\ker \varphi^n \to R/\ker \varphi^n$ is injective. Let $\overline{x} \in R/\ker \varphi^n$. Saying $\overline{\varphi}_n(\overline{x}) = 0$ is equivalent to saying $\varphi(x) \in \ker \varphi^n$, which is equivalent to saying $x \in \ker \varphi^{n+1}$. Since $\ker \varphi^{n+1} = \ker \varphi^n$, we see that $x \in \ker \varphi^n$, or $\overline{x} = 0$ in $R/\ker \varphi^n$. Thus, $\overline{\varphi}_n$ is injective. □

Proposition 38. *Let* $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ *be integral. Let* $\mathfrak{a}$ *be the ideal obtained as the stable limit of the ascending chain of ideals* $\ker \varphi \subset \ker \varphi^2 \subset \ker \varphi^3 \subset \dots$ *Write* ${}^a\varphi$ *for the self-morphism of* $\operatorname{Spec} R$ *induced by* φ*. Then*

- **a)** ${}^a\varphi$ *permutes the minimal prime ideals of* $\mathfrak{a}$*.*
- **b)** *Every minimal prime ideal of* $\mathfrak{a}$ *is in* $\operatorname{Specmin} R$*.*
- **c)** *Assume that the permutation in part* ***a****) is of order* p*. For a minimal prime ideal* $\mathfrak{p}_i$ *of* $\mathfrak{a}$ *let* ψ_i *be the endomorphism induced by* φ^p *on* $R/\mathfrak{p}_i$*. Then*

$$H(\varphi) = \max \{H(\psi_i) \mid \mathfrak{p}_i \text{ is a minimal prime of } \mathfrak{a}\}^{1/p}.$$

Proof. We recall that if a ring S is integral over a subring R, then over every prime ideal $\mathfrak{p}$ of R there lies a prime ideal $\mathfrak{q}$ of S. Moreover, if $\mathfrak{p}$ is a *minimal* prime ideal, then so is $\mathfrak{q}$ (see [35, Theorem 9.3]). Let $\overline{\varphi} : R/\mathfrak{a} \to R/\mathfrak{a}$ be the endomorphism induced by φ.

a) By Proposition 37, the endomorphism $\overline{\varphi} : R/\mathfrak{a} \to R/\mathfrak{a}$ is integral and injective. Apply the fact that we recalled above and note that $\mathfrak{a}$ has only a finite number of minimal prime ideals. (Any surjective map from a finite set to itself is also injective.)

b) Let n be large enough so that $\ker \varphi^n = \mathfrak{a}$. Then φ^n induces an integral injection $\tilde{\varphi}_n : R/\mathfrak{a} \hookrightarrow R$ and we have a commuting diagram

$$\begin{tikzcd} R \arrow[r, "\varphi^n"] \arrow[d] & R \\ R/\mathfrak{a} \arrow[ur, hook, "\tilde{\varphi}_n"'] & \end{tikzcd}$$

If $\mathfrak{q}$ is a minimal prime ideal of $\mathfrak{a}$, then by part **a**) (applied to φ^n) the ideal $\mathfrak{p} := (\varphi^n)^{-1}(\mathfrak{q})$ is a minimal prime of $\mathfrak{a}$. Thus, $(\tilde{\varphi}_n)^{-1}(\mathfrak{q}) = \mathfrak{p}/\mathfrak{a}$. We apply the fact that we recalled at the beginning of the proof to the ring injection $\tilde{\varphi}_n : R/\mathfrak{a} \hookrightarrow R$ and conclude $\mathfrak{q} \in \operatorname{Specmin} R$.

c) Let n be large enough so that $\ker \varphi^{np} = \mathfrak{a}$. By Proposition 37-**a** we know that $H(\varphi^{np}) = H(\overline{\varphi}^{np})$. Since all minimal prime ideals of $R/\mathfrak{a}$ are $\overline{\varphi}^{np}$-stable, by Proposition 32 we obtain

$$H(\overline{\varphi}^{np}) = \max\left\{H(\psi_i^n) \mid \mathfrak{p}_i \text{ is a minimal prime of } \mathfrak{a}\right\}.$$

Now an application of Proposition 23 quickly concludes the proof. $\square$

Corollary 39. *Let $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be integral and* $\operatorname{Spec} R = V(\ker \varphi)$*. Then*

a) *${}^a\varphi$ permutes the minimal prime ideals of R.*

b) *If $\mathfrak{p} \notin \operatorname{Specmin} R$ is a prime ideal of R, then $\varphi^{-1}(\mathfrak{p}) \notin \operatorname{Specmin} R$.*

c) *An element $x \in R$ belongs to a minimal prime ideal of R, if and only if $\varphi(x)$ belongs to a minimal prime ideal of R.*

c) *Assume that the permutation in part **a**) is of order p. For $\mathfrak{p}_i \in \operatorname{Specmin} R$ let ψ_i be the endomorphism induced by φ^p on $R/\mathfrak{p}_i$. Then*

$$H(\varphi) = \max\{H(\psi_i) \mid \mathfrak{p}_i \in \operatorname{Specmin} R\}^{1/p}.$$

Proof. **a**) Since φ is integral, $\operatorname{image}({}^a\varphi) = V(\ker\varphi)$. Thus, from the hypothesis $V(\ker\varphi) = \operatorname{Spec} R$ we see that ${}^a\varphi$ is surjective. Hence, ${}^a\varphi^n$ is also surjective for every $n \in \mathbb{N}$, i.e., $V(\ker\varphi^n) = \operatorname{image}({}^a\varphi^n) = \operatorname{Spec} R$. So for every $n \in \mathbb{N}$, the set of minimal prime ideals of $\ker\varphi^n$ is equal to $\operatorname{Specmin} R$. Take n large enough and apply Proposition 38-**a**.

b) Suppose $\mathfrak{q} := \varphi^{-1}(\mathfrak{p}) \in \operatorname{Specmin} R$. If $\mathfrak{p}$ is not minimal, it contains a minimal prime ideal $\mathfrak{p}'$. Moreover, $\ker\varphi \subset \mathfrak{p}'$. By part **a**), $\varphi^{-1}(\mathfrak{p}') \in \operatorname{Specmin} R$. Since $\mathfrak{q} \supseteq \varphi^{-1}(\mathfrak{p}')$ and $\mathfrak{q}$ is minimal, we must have $\mathfrak{q} = \varphi^{-1}(\mathfrak{p}')$. This is a contradiction, because there can be no inclusion between prime ideals that lie over $\mathfrak{q}$ in the integral ring inclusion $\varphi : (R/\ker\varphi) \hookrightarrow R$.

c) Let x be an element of R. If $\varphi(x) \in \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Specmin} R$, then $x \in \varphi^{-1}(\mathfrak{p})$. By part **a**), $\varphi^{-1}(\mathfrak{p}) \in \operatorname{Specmin} R$. Conversely, suppose $x \in \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Specmin} R$. Then by part **a**) there is a $\mathfrak{p} \in \operatorname{Specmin} R$ such that $\mathfrak{q} = \varphi^{-1}(\mathfrak{p})$. Hence $\varphi(x) \in \mathfrak{p}$.

d) Since all minimal prime ideals of R are φ^p-stable, by Proposition 32, $H(\varphi^p) = \max\{H(\psi_i) \mid \mathfrak{p}_i \in \operatorname{Specmin} R\}$. The result quickly follows from Proposition 23. $\square$

2.5 Finite endomorphisms of local domains: entropy vs degree

The various invariants that measure the complexity of a given system are often related (see, e.g., [32]). We now compare $H(\varphi)$ to *degree* of φ when R is a domain and φ is finite. It turns out that the analogy between local and topological entropies also extends to their relation to *degree*. Misiurewicz and Przytycki showed in [36], that if f is a C^1 self-map of a smooth compact orientable manifold M, then

$$h_{\text{top}}(f) \geqslant \log|\deg(f)|.$$

For a holomorphic self-morphism f of $\mathbb{P}^n(\mathbb{C})$, Gromov showed in [23]

$$h_{\text{top}}(f) = \log|\deg(f)|.$$

Here $\deg(f)$ is the topological degree of f.

In this section we obtain similar formulas for finite endomorphisms of local domains, relating their local entropy to degree. For local Cohen-Macaulay domains we prove an analog of Gromov's formula. Note that we define our local entropy H without taking logarithm typically found in the definition of entropy.

Definition 40. Let $f : R \to S$ be a finite homomorphism of noetherian local rings. Assume that R is a domain. Then by *degree* of f, $\deg(f)$, we mean the rank of the R-module $f_\star S$.

Lemma 41. *Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a finite homomorphism of noetherian local rings with residue fields k_R and k_S, respectively. Assume that R is a domain. Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of R. Then*

$$e_S(f(\mathfrak{q})S) = \frac{e_R(\mathfrak{q}) \cdot \deg(f)}{[f_\star k_S : k_R]}. \tag{2.3}$$

Proof. If $\dim R > \dim S$, both sides are 0. Assume $\dim R = \dim S = d$. By definition of multiplicity and properties of length we quickly obtain

$$\begin{aligned} e_R(\mathfrak{q}, f_\star S) &= \lim_{m\to\infty} \frac{d!}{m^d} \cdot \mathrm{length}_R \big(f_\star \big(\frac{S}{(f(\mathfrak{q})S)^m} \big) \big) \\ &= [f_\star k_S : k_R] \cdot \lim_{m\to\infty} \frac{d!}{m^d} \cdot \mathrm{length}_S \big(\frac{S}{(f(\mathfrak{q})S)^m} \big) \\ &= [f_\star k_S : k_R]^n \cdot e_S(f(\mathfrak{q})S). \end{aligned}$$

On the other hand $e_R(\mathfrak{q}, f_\star S) = e_R(\mathfrak{q}) \cdot \deg(f)$ (see [35, Theorem 14.8]), and Formula 2.3 follows. $\square$

Remark 42. When $f : (R, \mathfrak{m}, k_R) \to (S, \mathfrak{n}, k_S)$ is a *finite* homomorphism of local rings, then $[f_\star k_S : k_R] \cdot \mathrm{length}_S(S/\mathfrak{m}S)$ is the minimal number of generators of $f_\star S$ over R, by Nakayama lemma.

Proposition 43. *Let* $(R, \mathfrak{m})$ *be a domain,* $\varphi : R \to R$ *finite and* $\dim R = d$*. Let* k *be the residue field of* R*. Then*

- **a)** $v_h(\varphi)^d \leqslant \deg(\varphi)/[\varphi_\star k : k] \leqslant H(\varphi)$.
- **b)** *If in addition* R *is Cohen-Macaulay,* $\deg(\varphi)/[\varphi_\star k : k] = H(\varphi)$.

Proof. **a**) By Theorem-definition 17, Remark 16 and Fekete's Lemma, we know $v_h(\varphi) = \sup_n\{(v(\varphi^n))^{1/n}\}$. Hence, to show the first inequality it suffices to show

$$v(\varphi^n)^{d/n} \leqslant \deg(\varphi)/[\varphi_\star k : k].$$

Since $\varphi^n(\mathfrak{m})R \subset \mathfrak{m}^{v(\varphi^n)}$, with the aid of Lemma 41 we obtain

$$e(\mathfrak{m})v(\varphi^n)^d = e(\mathfrak{m}^{v(\varphi^n)}) \leqslant e(\varphi^n(\mathfrak{m})R) = \frac{e(\mathfrak{m}) \cdot \deg(\varphi^n)}{[\varphi^n_\star k : k]}.$$

For the second inequality, let $\mu(\varphi^n_\star R)$, or simply μ, be the minimum number of generators of the R-module $\varphi^n_\star R$. Localizing the surjection $R^\mu \twoheadrightarrow \varphi^n_\star R$ at (0) we see $\mathrm{rank}\, \varphi^n_\star R \leqslant \mu(\varphi^n_\star R)$. On the other hand, as mentioned in Remark 42, we have $\mu(\varphi^n_\star R) = [\varphi_\star k : k]^n \cdot \mathrm{length}(R/\varphi^n(\mathfrak{m})R)$. Since by definition of degree, $\operatorname{rank} \varphi_*^n R = \deg(\varphi^n) = (\deg(\varphi))^n$, we conclude

$$(\deg(\varphi))^n/[\varphi_* k : k]^n \leqslant \operatorname{length}(R/\varphi^n(\mathfrak{m})R).$$

Take n-th root and limits as $n \to \infty$.

b) Let $\mathfrak{q}$ be an arbitrary parameter ideal of R. Then

$$\operatorname{length}(R/\varphi^n(\mathfrak{m})R) \leqslant \operatorname{length}(R/\varphi^n(\mathfrak{q})R).$$

If R is Cohen-Macaulay, then $\operatorname{length}(R/\varphi^n(\mathfrak{q})R) = e(\varphi^n(\mathfrak{q})R)$ (see, e.g., [35, Theorem 17.11]). Thus, using Lemma 41

$$\operatorname{length}(R/\varphi^n(\mathfrak{m})R) \leqslant e(\varphi^n(\mathfrak{q})R) = \frac{e(\mathfrak{q})(\deg(\varphi))^n}{[\varphi_* k : k]^n},$$

Taking n-th root and limits as $n \to \infty$ we obtain

$$H(\varphi) \leqslant \deg(\varphi)/[\varphi_* k : k].$$

This inequality together with the inequality in **a**) give the desired equality.

□

Example 44 (Frobenius)**.** Let $(R, \mathfrak{m})$ be a noetherian local domain of prime characteristic p, and let φ be the Frobenius endomorphism of R. Then by Example 18, $v_h(\varphi) = p$ and $H(\varphi) = p^{\dim R}$, hence by the preceding Proposition 43 **a)**,

$$H(\varphi) = \deg(\varphi)/[\varphi_\star k : k] = p^{\dim R}$$

2.6 Entropy at the vertex of the cone over a projective variety

The following result, which will be used in this section was proved in [5, Theorem 2.1] by Bhatnagar and L. Szpiro over infinite fields and in [41, Theorem 1.4] by Poonen over finite fields.

Theorem. Let X be a projective variety defined over a field k, $\imath : X \hookrightarrow \mathbb{P}^N_k$ a given embedding and $\mathcal{L} := \imath^\star \mathcal{O}_{\mathbb{P}^N_k}(1)$. Let $\varphi : X \to X$ be a self-morphism that is polarized *by $\mathcal{L}$, that is, $\varphi^\star(\mathcal{L}) \cong \mathcal{L}^{\otimes q}$ for an integer $q \geqslant 2$. Then there exists a positive integer r and a finite morphism $\mathbb{P}^N_k \to \mathbb{P}^N_k$ extending φ^r, where $N + 1 = \dim_k \mathrm{H}^0(X, \mathcal{L})$.*

Keeping the notation as above, let $\mathfrak{a}$ be the largest homogeneous ideal in $k[X_0, \ldots, X_N]$ defining $\imath(X)$ and let $R := k[X_0, \ldots, X_N]/\mathfrak{a}$. Also let $R^{(q)} := \bigoplus_{n \geqslant 0} R_{nq}$ be the qth Veronese subring of R. Using the above theorem, we can fix an $r \in \mathbb{N}$ such that φ^r extends to a finite self-morphism $\mathbb{P}^N_k \to \mathbb{P}^N_k$. The proof of this theorem in [5, 41] shows that this self-morphism is given by $N + 1$ forms $F_0, \ldots, F_N$ of degree q^r that have no non-trivial common zeros in the algebraic closure of k. The assignment $X_i \mapsto F_i$ defines a finite endomorphism $\psi : R \to R$ that can be factored as

$$R \to R^{(q^r)} \hookrightarrow R,$$

where $R^{(q^r)} \hookrightarrow R$ is inclusion and $R \to R^{(q^r)}$ is a graded homomorphism that induces φ^r on $X = \operatorname{Proj} R$. We want to calculate the local entropy of the local endomorphism induced by ψ at the vertex of the affine cone $\operatorname{Spec} R$ over X.

Remark 45. While the self-morphism $\mathbb{P}_k^N \to \mathbb{P}_k^N$ obtained from the theorem is not unique, the endomorphism $\psi : R \to R$ is unique up to a scalar multiple.

Proposition 46. *Let X be an integral projective variety of dimension d over a field k with a given embedding $\imath : X \hookrightarrow \mathbb{P}_k^N$ and set $\mathcal{L} := \imath^\star \mathcal{O}_{\mathbb{P}_k^N}(1)$. Let $\varphi : X \to X$ be a self-morphism and assume that $\varphi^\star(\mathcal{L}) \cong \mathcal{L}^{\otimes q}$ for an integer $q \geqslant 2$. Let r, R and ψ be as defined in the previous paragraph and let $H(\psi)$ be the local entropy of ψ at the vertex of the affine cone* $\operatorname{Spec} R$ *over X. Then*

$$H(\psi) = \deg(\psi) = q^{r(d+1)}. \tag{2.4}$$

Proof. By Theorem 17-**b** and Proposition 43-**a** it suffices to show $v_h(\psi) \geqslant q^r$ and $w_h(\psi) \leqslant q^r$. For the first inequality, note that as discussed prior to this proposition, ψ is induced by assignments $X_i \mapsto F_i$ for $N+1$ homogeneous forms $F_0, \ldots, F_N$ of degree q^r in the variables X_i. Thus, ψ^n is given by forms of degree q^{nr}. This shows, with notations of Theorem 17, that $v(\psi^n) \geqslant q^{nr}$. Hence

$$v_h(\psi) = \lim_{n\to\infty} v(\psi^n)^{1/n} \geqslant q^r.$$

To prove the second inequality we use elimination theory: by [31, Corollary to Theorem 1, p. 169] we get $\langle X_0, \dots X_N\rangle^s \subset \langle F_0, \dots, F_N\rangle$, where $s = q^r(N+1) - N$. Thus, $w(\psi^n) \leqslant q^{nr}(N+1) - N$ and we obtain

$$w_h(\psi) = \lim_{n\to\infty} w(\psi^n)^{1/n} \leqslant\ q^r.$$

This concludes the proof. $\square$

Chapter 3

Regularity, flatness and entropy

In this chapter we study the case of regular local rings. Namely, we use entropy to extend numerical conditions of Kunz' regularity criterion to arbitrary contracting african endomorphisms.

3.1 Kunz' Numerical Criterion of Regularity

Let $(R, \mathfrak{m})$ be a noetherian local ring of positive prime characteristic p and dimension d, and let φ be the Frobenius endomorphism of R. In [29] Kunz showed that the following conditions are equivalent:

- **a)** R is regular.
- **b)** $\varphi : R \to R$ is flat.
- **c)** $\lambda(\varphi) = p^d$.
- **d)** $\lambda(\varphi^n) = p^{dn}$ for some $n \in \mathbb{N}$.

Later Rodicio showed in [42], that these conditions are also equivalent to

e) flat $\dim_R \varphi_* R < \infty$.

In [3, Theorem 13.3] Avramov, Iyengar and Miller have extended the equivalence of conditions **a**) and **b**) of Kunz and **e**) of Rodicio to arbitrary contracting local endomorphisms of noetherian local rings.

At first glance, Kunz' conditions **c**) and **d**) may appear to be stated in terms of the characteristic p of the ring. How does one extend, or even state them for an arbitrary endomorphism in arbitrary characteristic? Local entropy H can be used to make sense of Kunz' numerical conditions **c**) and **d**) for all african endomorphisms in any characteristic. Indeed, as shown in the Example 18, in the case of the Frobenius, $H(\varphi) = p^d$. For an arbitrary african endomorphism we have the following

Proposition 47. *Let $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be contracting and african. Then the following conditions are equivalent:*

a) *φ is flat*

b) *$\lambda(\varphi^n) = \lambda(\varphi)^n$ for all $n \geqslant 1$*

c) *$\lambda(\varphi) = H(\varphi)$*

Furthermore, the above conditions imply:

d) If $\mathfrak{a}$ is a non-zero φ-stable ideal of R, and $\bar{\varphi}$ is the endomorphism of $R/\mathfrak{a}$ induced by φ, then $H(\bar{\varphi}) < H(\varphi)$.

Remark 48. For the implications **b)** $\Longleftrightarrow$**c)** and **a)** $\Longrightarrow$ **b)**, φ need not be contracting.

This proposition follows from a flatness criterion that is due to Nagata. A proof can be found in [38, Chap. II, Theorem 19.1]. See also [35, Ex. 22.1, p. 178].

Theorem (Nagata). *Let $f:(R,\mathfrak{m})\to(S,\mathfrak{n})$ be an injective african homomorphism of noetherian local rings. Then S is flat over R, if and only if for every $\mathfrak{m}$-primary ideal $\mathfrak{q}$ of R,*

$$\mathrm{length}_R(R/\mathfrak{q})\cdot\lambda(f)=\mathrm{length}_S(S/f(\mathfrak{q})S). \tag{3.1}$$

We need a stronger version of Nagata's criterion that we state and prove here.

Lemma 49. *Let $f:(R,\mathfrak{m})\to(S,\mathfrak{n})$ be an african homomorphism of noetherian local rings. If Equation 3.1 holds for a family of $\mathfrak{m}$-primary ideals $\{\mathfrak{q}_\alpha\}_{\alpha\in A}$ that define the $\mathfrak{m}$-adic topology, then it holds for all $\mathfrak{m}$-primary ideals.*

Proof. Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal. We will show Equation 3.1 holds for $\mathfrak{q}$. First, using Proposition 8

$$\mathrm{length}_S(S/f(\mathfrak{q})S)=\mathrm{length}_S(S\otimes_R R/\mathfrak{q})\leqslant\lambda(f)\cdot\mathrm{length}_R(R/\mathfrak{q}).$$

To show the reverse inequality, note that by assumption there is a $\mathfrak{q}_\alpha \subseteq \mathfrak{q}$. The exact sequence $0 \to \mathfrak{q}/\mathfrak{q}_\alpha \to R/\mathfrak{q}_\alpha \to R/\mathfrak{q} \to 0$ yields

$$\mathrm{length}_R(R/\mathfrak{q}_\alpha) = \mathrm{length}_R(R/\mathfrak{q}) + \mathrm{length}_R(\mathfrak{q}/\mathfrak{q}_\alpha). \tag{3.2}$$

If we tensor the previous exact sequence with S, we obtain an exact sequence of S-modules $(\mathfrak{q}/\mathfrak{q}_\alpha) \otimes_R S \to S/f(\mathfrak{q}_\alpha)S \to S/f(\mathfrak{q})S \to 0$. Thus

$$\mathrm{length}_S(S/f(\mathfrak{q}_\alpha)S) \leqslant \mathrm{length}_S(S/f(\mathfrak{q})S) + \mathrm{length}_S((\mathfrak{q}/\mathfrak{q}_\alpha) \otimes_R S).$$

Since Equation 3.1 holds for $\mathfrak{q}_\alpha$, and using Proposition 8 we quickly see

$$\begin{aligned} \mathrm{length}_R(R/\mathfrak{q}_\alpha) \cdot \lambda(f) &\leqslant \mathrm{length}_S(S/f(\mathfrak{q})S) + \\ &\quad \mathrm{length}_R(\mathfrak{q}/\mathfrak{q}_\alpha) \cdot \lambda(f). \end{aligned}$$

Now using Equation 3.2 we obtain

$$\mathrm{length}_R(R/\mathfrak{q}) \cdot \lambda(f) \leqslant \mathrm{length}_S(S/f(\mathfrak{q})S).$$

□

Now we can give a

Proof of Proposition 47. **a)** $\Longrightarrow$ **b)**: Immediately from Lemma 15**b)**.

b) $\Longrightarrow$ **c)** Clear from definition of H.

c) $\Longrightarrow$ **b)** Use $H(\varphi)^n \leqslant \lambda(\varphi^n) \leqslant \lambda(\varphi)^n$ where the second inequality follows from sub-multiplicativity of $\{\lambda(\varphi^n)\}$.

b),c) $\Longrightarrow$ **d)**: By our hypothesis, $\lambda(\varphi^n) = \lambda(\varphi)^n = H(\varphi)^n$ for all $n \geqslant 1$. Similar to the previous part, we can write

$$\begin{aligned} H(\overline{\varphi})^n &\leqslant \lambda(\overline{\varphi}^n) = \operatorname{length}(R/(\varphi^n(\mathfrak{m})R + \mathfrak{a})) \\ &\leqslant \operatorname{length}(R/\varphi^n(\mathfrak{m})R) = \lambda(\varphi^n) = H(\varphi)^n. \end{aligned} \tag{3.3}$$

Suppose that $H(\overline{\varphi}) = H(\varphi)$. Then, from Equation 3.3 we can conclude

$$\operatorname{length}(R/(\varphi^n(\mathfrak{m})R + \mathfrak{a})) = \operatorname{length}(R/\varphi^n(\mathfrak{m})R), \quad \forall n \in \mathbb{N}. \tag{3.4}$$

The surjection $R/\varphi^n(\mathfrak{m})R \to R/(\varphi^n(\mathfrak{m})R + \mathfrak{a})) \to 0$ and Equation 3.4 then show

$$R/(\varphi^n(\mathfrak{m})R + \mathfrak{a}) = R/\varphi^n(\mathfrak{m})R, \quad \forall n \in \mathbb{N}.$$

Hence,

$$\mathfrak{a} \subset \textstyle\bigcap_{n\in\mathbb{N}} \varphi^n(\mathfrak{m})R = (0),$$

where the last equality follows from Lemma 12 because φ is by assumption, contracting.

b),c) $\Longrightarrow$ **a)** We will use Nagata's Flatness Criterion to show that φ is flat. We first need to show that φ is injective. Clearly $\ker\varphi$ is φ-stable. Let

$\overline{\varphi}$ be the local endomorphism induced by φ on $R/\ker\varphi$. Then by Proposition 37, $H(\varphi) = H(\overline{\varphi})$. From **d)** it follows that $\ker\varphi = (0)$.

Now since φ is contracting, using Lemma 12 we quickly see that the family $\{\varphi^n(\mathfrak{m})R\}_{n\in\mathbb{N}}$ defines the $\mathfrak{m}$-adic topology of R. By Lemma 49 it suffices to verify Equation 3.1 for this family of $\mathfrak{m}$-primary ideals. We need to show

$$\operatorname{length}_R\left(R/\varphi(\varphi^n(\mathfrak{m}))R\right) = \operatorname{length}_R\left(R/\varphi^n(\mathfrak{m})R\right)\cdot\lambda(\varphi).$$

Using the assumption, we can rewrite this equality as follows:

$$\lambda(\varphi)^{n+1} = \lambda(\varphi)^n \cdot \lambda(\varphi).$$

Since this equality holds trivially, by Nagata's Flatness Theorem φ is flat. □

3.2 The complete Kunz Theorem

Theorem 53 below states that with the new interpretation, all conditions in Kunz' result are equivalent. We include proofs of all implications in this theorem for completeness.

We list two results here that we will need for the proof of Theorem 53.

Lemma 50 ([25, Lemma 3.2])**.** *Let $(R,\mathfrak{m})$ be a noetherian local ring, and let M be a finitely generated R-module. Consider an ideal $\mathfrak{b}\subseteq\mathfrak{m}$ of R. Then there exists an integer $\mu_0 \geqslant 0$ such that* $\operatorname{depth}(\mathfrak{m}, \mathfrak{b}^\mu M) > 0$ *for all $\mu \geqslant \mu_0$.*

Remark 51. In using Lemma 50 we must pay particular attention to the standard convention that the depth of the zero module is ∞ (see, e.g., [26, p. 291]). Otherwise, if M is an R-module of finite length, then for $\mu \gg 0$ we have $\mathfrak{m}^\mu M = (0)$, and this would have been a counter-example to Lemma 50.

The next proposition is taken from [9, Chap. 10, § 1, Proposition 1].

Proposition 52. *Let R be a noetherian ring and let $\mathfrak{a}$ be an ideal of R. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. If we define $d' = \operatorname{depth}(\mathfrak{a}, M')$, $d = \operatorname{depth}(\mathfrak{a}, M)$, and $d'' = \operatorname{depth}(\mathfrak{a}, M'')$, then one of the following mutually exclusive possibilities hold:*

$$d' = d \leqslant d'' \text{ or } d = d'' < d' \text{ or } d'' = d' - 1 < d.$$

Theorem 53. *Let $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be african. Consider the following conditions:*

a) *R is regular.*

b) *$\varphi : R \to R$ is flat.*

c) *$\lambda(\varphi) = H(\varphi)$.*

d) *$\lambda(\varphi^n) = H(\varphi)^n$ for an integer $n \in \mathbb{N}$.*

Then **a**) $\to$ **b**) $\to$ **c**) $\to$ **d**). *If in addition φ is contracting, all these conditions are equivalent.*

Proof. **a**) → **b**): To say that φ is african means $\dim R/\varphi(\mathfrak{m})R = 0$. Hence, the following equation holds:

$$\dim R = \dim R + \dim R/\varphi(\mathfrak{m})R.$$

Since R is regular, the result follows from [35, Theorem 23.1].

b) → **c**): Since φ is flat, by Lemma 15-**b**

$$\lambda(\varphi^n) = \lambda(\varphi)^n, \quad \forall n \in \mathbb{N}.$$

Thus, by definition of local entropy

$$H(\varphi) = \lim_{n\to\infty} \lambda(\varphi^n)^{1/n} = \lim_{n\to\infty} \lambda(\varphi) = \lambda(\varphi).$$

c) → **d**): This is clear.

b) → **a**): We rewrite Herzog's proof for the Frobenius endomorphism from [25, Satz 3.1], for an arbitrary contracting endomorphism here. Burns and Gubeladze have also used this proof in [12, Lemma 3]. We include it here for completeness. To show that R is regular, it suffices to show all finitely generated R-modules have finite projective dimension. So let M be a finitely generated R-module. Suppose M were of infinite projective dimension. Consider a minimal (infinite) free resolution of M

$$L_\bullet \to M \to 0.$$

Let $s := \operatorname{depth}(\mathfrak{m}, R)$, and take an R-regular sequence of elements $\{x_1, \ldots, x_s\}$ in $\mathfrak{m}$. Write $\mathfrak{a}$ for the ideal generated by this regular sequence. (If $s = 0$, take $\mathfrak{a} = (0)$.) Let $\varphi^{\star n}$ be the functor defined in Definition 20. For every $n \in \mathbb{N}$ we set

$$C_\bullet^n := \varphi^{\star n}(L_\bullet) \otimes_R R/\mathfrak{a} \quad \text{and} \quad B_i^n := \operatorname{image}(C_{i+1}^n \to C_i^n).$$

Using properties of $\varphi^{\star n}$, see [40] or [25], it is easy to see that $C_i^n \cong L_i/\mathfrak{a}L_i$ and $B_i^n \subseteq \varphi^n(\mathfrak{m})C_i^n$. Thus, for every i the module C_i^n is independent of n and is nonzero, finitely generated and of depth zero. Applying Lemma 50, let μ_{i_0} be such that $\operatorname{depth}(\mathfrak{m}, \mathfrak{m}^{\mu_{i_0}} C_i^n) > 0$. Since φ is contracting, using Lemma 12 if n is large enough then $\varphi^n(\mathfrak{m})R \subseteq \mathfrak{m}^{\mu_{i_0}}$, hence $B_i^n \subseteq \varphi^n(\mathfrak{m})C_i^n \subseteq \mathfrak{m}^{\mu_{i_0}} C_i^n$. This shows that $\operatorname{depth}(\mathfrak{m}, B_i^n) > 0$ for large n. On the other hand, since φ is flat, $\varphi^{\star n}(L_\bullet)$ is exact. Thus, using properties of $\varphi^{\star n}$ again, we see that

$$\varphi^{\star n}(L_\bullet) \to \varphi^{\star n}(M) \to 0$$

is a minimal (infinite) free resolution of $\varphi^{\star n}(M)$. Hence

$$\mathrm{H}_i(C_\bullet^n) = \operatorname{Tor}_i^R(\varphi^{\star n}(M), R/\mathfrak{a}) = 0, \quad \text{for} \quad i > s.$$

This shows that if $i > s$, then the sequences

$$0 \to B_{i+1}^n \to C_{i+1}^n \to B_i^n \to 0 \tag{3.5}$$

are exact for all $n \in \mathbb{N}$. Take $i = s + 1$ in Sequence 3.5, for instance. By the above argument, if we take n large enough, we will obtain $\mathrm{depth}(\mathfrak{m}, B^n_{s+1}) > 0$ and $\mathrm{depth}(\mathfrak{m}, B^n_{s+2}) > 0$, while $\mathrm{depth}(\mathfrak{m}, C^n_{s+2}) = 0$. By Proposition 52 this is not possible. Hence, the projective dimension of M must be finite.

d) $\to$ **b**): Applying Proposition 47 to φ^n we conclude that φ^n is flat. The implication **b** $\to$ **a**) applied to φ^n then tells us that R is regular, and the implication **a** $\to$ **b**) shows that φ is flat, as well. □

Remark 54. There exist normal singularities (X, P) such that $\mathcal{O}_{X,P}$ admits a finite contracting endomorphism, see [6, Sections 6.2-6.3] or [16, Sections 2.3-2.5]. In this case by Theorem 53 the endomorphism is not flat. This gives examples of finite morphisms of normal varieties which are not flat.

3.3 Generalized Hilbert-Kunz multiplicity

Following ideas of Kunz, Monsky in [37] defined the Hilbert-Kunz multiplicity for the Frobenius endomorphism of noetherian local rings of positive prime characteristic. He then showed that in this case, Hilbert-Kunz multiplicity always exists. Since then, it has become evident through works of various authors, that the Hilbert-Kunz multiplicity provides a reasonable measure of the singularity of the local ring. Thus, a regular local ring of prime characteristic has Hilbert-Kunz multiplicity of 1. Here, inspired by Example 18, we propose a characteristic-free interpretation of the definition of Hilbert-Kunz multiplicity associated with an african *endomorphism*.

Definition 55 (Hilbert-Kunz multiplicity)**.** Let $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be african and set $d := \dim R$. The Hilbert-Kunz multiplicity of R with respect to φ is defined as

$$e_{\mathrm{HK}}^{\varphi}(R) := \lim_{n \to \infty} \frac{\lambda(\varphi^n)}{H(\varphi)^n}, \tag{3.6}$$

provided that the limit exists.

Remark 56. We do not know whether the limit in Equation 3.6 always exists or not. Nevertheless, the next corollary shows that in the case of a regular local ring the Hilbert-Kunz multiplicity is 1, as expected.

Corollary 57. *Let φ be an african endomorphism of a regular local ring R. Then $e_{\mathrm{HK}}^{\varphi}(R) = 1$.*

Proof. This quickly follows from Theorem 53 and Lemma 15-**b**. □

We end this section with a note. Not every homological property of the Frobenius endomorphism can be immediately extended to arbitrary endomorphisms. For example, in [40, Theorem 1.7, p. 58] Peskine and L. Szpiro showed that in positive prime characteristic, a finite free resolution of a module remains exact after applying the Frobenius functor (see Definition 20). This property may fail in general, for an arbitrary endomorphism.

One should note, however, that it is true for resolutions of modules of finite length. (Apply the acyclicity lemma [40, Lemma 1.8].) Away from modules of finite length, this property may fail even in the simple case of a Koszul complex with one element. The image of a non-zerodivisor under an integral endomorphism could be a zerodivisor, as the next example shows.

Example 58. Consider the polynomial ring $k[x, y, z, w]$ over a field k. Let $\mathfrak{a}$ be the ideal (x^2, xy, xz, zw) and let $A = k[x, y, z, w]/\mathfrak{a}$. Then

$$\operatorname{Ass}(A) = \{(x, z), (x, w), (x, y, z)\}.$$

Define an endomorphism φ of $k[x, y, z, w]$ as $x \overset{\varphi}{\mapsto} x^2$; $y \overset{\varphi}{\mapsto} y$; $z \overset{\varphi}{\mapsto} w$; $w \overset{\varphi}{\mapsto} z$. Then $\mathfrak{a}$ is φ-stable. Let $\overline{\varphi}$ be the endomorphism of A induced by φ. The A-module $\overline{\varphi}_* A$ is finitely generated. In fact, it is generated by 1 and x as an A-module. Now, $y + w$ is not a zerodivisor in A because it does not belong to any prime ideal in $\operatorname{Ass}(A)$. But $\overline{\varphi}(y + w) = y + z$ is a zerodivisor in A; it is killed by x, for example. On the other hand, $y + z$ is a zerodivisor but is mapped to $y + w$, a non-zerodivisor.

Nonetheless, in the previous example $\overline{\varphi}^2$ sends any A-regular sequence to an A-regular sequence. This motivates the following

Question 59. Let $\varphi : (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be african. Does there exist a positive integer n such that φ^n will send any R-regular sequence to an R-regular sequence?

Chapter 4

Lifting of endomorphisms of complete equicharacteristic rings

In this chapter we prove Theorem 63, which is inspired by results of Fakhruddin [15, Corollary 2.2], and Bhatnagar and L. Szpiro [5, Theorem 2.1] on extending a polarized self-morphism of a projective variety over an infinite field to an ambient projective space. Recently in [41] Poonen gave a proof for the main result of [5] over finite fields.

Consider a self-morphism φ of a projective variety X over an infinite field k and let $\mathcal{L}$ be an ample line bundle on X with $\varphi^{\star}(\mathcal{L}) \cong \mathcal{L}^{\otimes q}$ for an integer $q \geqslant 1$. In [15] Fakhruddin showed that there exists an embedding $\imath$ of X in $\mathbb{P}_k^N$ given by an appropriate tensor power of $\mathcal{L}$ and a self-morphism ψ of $\mathbb{P}_k^N$ such that $\psi \circ \imath = \imath \circ \varphi$. In [5] Bhatnagar and L. Szpiro relaxed some of Fakhruddin's hypotheses and showed that (assuming $\mathcal{L}$ is very ample) one can keep the same embedding of X given by $\mathcal{L}$ and instead extend an appropriate iteration of φ to the ambient projective space.

Our Theorem 63 is an analogous result about lifting of african endomorphisms of complete noetherian local rings of equal characteristic. In this local version of Fakhruddin's result we do not assume our fields to be infinite. We will begin with a few preparatory results that we will need in the proof of Theorem 63.

Definition 60 ([44, p. 159])**.** In a noetherian local ring R of dimension d and of embedding dimension δ, a system of parameters $\{x_1, \ldots, x_d\}$ is called a *strong system of parameters* if it is part of a minimal set of generators $\{x_1, \ldots, x_d, \ldots, x_\delta\}$ of the maximal ideal.

Lemma 61. *A noetherian local ring* $(R, \mathfrak{m})$ *has strong systems of parameters.*

Proof. The proof is by induction on $\dim R$. If $\dim R = 0$ then the statement is vacuous, since every system of parameters is empty. So assume $\dim R > 0$ and using the Prime Avoidance Lemma [34, p. 2], pick an element $x \in \mathfrak{m}$ that is neither in any minimal prime ideal of R, nor in $\mathfrak{m}^2$. Apply the induction hypothesis to $R/\langle x\rangle$. $\square$

Lemma 62. *Let $(R, \mathfrak{m})$ be a complete local ring of equal characteristic and assume that A is a homomorphic image $\pi : R \to A$ of R. If K is a subfield of A, then there is a subfield L of R such that $\pi|_L : L \to K$ is an isomorphism.*

Proof. Let $B = \pi^{-1}(K)$. Then B is a local subring of R with maximal ideal $\mathfrak{q} = \pi^{-1}(0)$. Note that $\mathfrak{q} = \ker \pi$ as subsets of R. Since $B/\mathfrak{q} \cong K$, B is also of equal characteristic. In general B need not be noetherian. We claim that $B \subseteq R$ is a closed subset in the $\mathfrak{m}$-adic topology of R. To see this, let $\mathfrak{n}$ be the maximal ideal of A and note that the topology induced from the $\mathfrak{n}$-adic topology of A on any subfield of A is the discrete topology. Therefore, any subfield of A is complete with respect to the topology induced from A, and hence is closed in A. Since π is a continuous map and $B = \pi^{-1}(K)$, the claim follows. In particular, B is complete with respect to the topology induced from the $\mathfrak{m}$-adic topology of R.

Denote the $\mathfrak{q}$-adic completion of B by $\widehat{B}$. Since B is a local subring of R and R is complete, we obtain a map $\widehat{i} : \widehat{B} \to R$, where $i : B \hookrightarrow R$ is the inclusion homomorphism. Furthermore, since B is complete with respect to the topology induced from the $\mathfrak{m}$-adic topology of R, we see that $\widehat{i}(\widehat{B}) = B$. Let L' be a coefficient field of $\widehat{B}$ (For the existence of coefficient fields in complete local rings that are not necessarily noetherian, see [38, Theorem 31.1], or [35, Theorem 28.3] or [22, Corollary 2]). Let $L := \widehat{i}(L')$.

Then L is subfield of B that is isomorphic to L'. Furthermore, the following diagram is commutative, and shows that $\pi|_L : L \to K$ is an isomorphism.

$$\begin{tikzcd}
L' \arrow[r, hook] \arrow[rrr, bend left, "\simeq"] \arrow[dr, "\simeq"'] & \widehat{B} \arrow[r, two heads, "\widehat{i}"] \arrow[d] & B \arrow[d, "\pi|_B"'] & L \arrow[l, hook'] \arrow[dl] \\
 & \widehat{B}/\widehat{\mathfrak{q}} \arrow[r, "\simeq"] & K &
\end{tikzcd}$$

$\square$

Theorem 63. *Let $\varphi : (A, \mathfrak{n}) \to (A, \mathfrak{n})$ be african and assume that A is a homomorphic image $\pi : R \twoheadrightarrow A$ of an equicharacteristic complete regular local ring $(R, \mathfrak{m})$. Then φ can be lifted (non uniquely) to an african endomorphism ψ of R such that $\pi \circ \psi = \varphi \circ \pi$.*

Proof. Let K be an arbitrary coefficient field of R. Then $\varphi(\pi(K))$ is a subfield of A, and can be lifted to a subfield L of R, by Lemma 62, in such a way that $\pi|_L : L \to \varphi(\pi(K))$ is an isomorphism. We will use L at the end of our proof to construct an endomorphism ψ of R. Let $d = \dim A$ and let δ be the embedding dimension of A. By Lemma 61 we can choose a strong system of parameters $\{x_1, \ldots, x_d\}$ of A which is part of a minimal set of generators $\{x_1, \ldots, x_d, \ldots, x_\delta\}$ of $\mathfrak{n}$. Choose elements $X_1, \ldots, X_\delta$ in $\mathfrak{m}$ in such a way that $\pi(X_i) = x_i$ for each i. We claim that since the images of

$x_1, \ldots, x_\delta$ in $\mathfrak{n}/\mathfrak{n}^2$ are linearly independent over $A/\mathfrak{n}$, the images $\overline{X}_1, \ldots, \overline{X}_\delta$ of $X_1, \ldots, X_\delta$ in $\mathfrak{m}/\mathfrak{m}^2$ are also linearly independent over $R/\mathfrak{m}$. If not, there will be a dependence relation $\alpha_1\overline{X}_1 + \ldots + \alpha_\delta\overline{X}_\delta = 0$ with $\alpha_i \in R/\mathfrak{m}$ not all zero. This means if we choose $a_i \in R$ such that they map to α_i in $R/\mathfrak{m}$ for $1 \leqslant i \leqslant \delta$, then

$$a_1X_1 + \ldots + a_\delta X_\delta \in \mathfrak{m}^2.$$

If we apply π to this relation, we obtain $\pi(a_1)x_1 + \ldots + \pi(a_\delta)x_\delta \in \mathfrak{n}^2$. But then the image in $\mathfrak{n}/\mathfrak{n}^2$ would provide a nontrivial dependence relation

$$\pi(a_1)\overline{x}_1 + \ldots + \pi(a_\delta)\overline{x}_\delta = 0,$$

contradicting the linear independence of $\overline{x}_1, \ldots, \overline{x}_\delta$ in $\mathfrak{n}/\mathfrak{n}^2$ over $A/\mathfrak{n}$. Our claim follows. Hence, we can extend $\{\overline{X}_1, \ldots, \overline{X}_\delta\}$ to a basis $\{\overline{X}_1, \ldots, \overline{X}_\delta, \ldots, \overline{X_n}\}$ of $\mathfrak{m}/\mathfrak{m}^2$ over $R/\mathfrak{m}$, where $n = \dim R$. If we choose elements $X_i \in \mathfrak{m}$ so that they map to $\overline{X}_i$ in $\mathfrak{m}/\mathfrak{m}^2$ for $\delta + 1 \leqslant i \leqslant n$, then by Nakayama's Lemma $\{X_1, \ldots, X_n\}$ is a minimal set of generators of $\mathfrak{m}$. Furthermore, it follows from the Cohen Structure Theorem that $R = K[\![X_1, \ldots, X_n]\!]$.

Now consider elements $\varphi(\pi(X_i))$ in A and for $1 \leqslant i \leqslant d$ choose $f_i \in \mathfrak{m}$ such that $\pi(f_i) = \varphi(\pi(X_i))$. We claim that the ideal $\langle f_1, \ldots, f_d\rangle$ of R has height d. First, by Krull's Theorem $\operatorname{ht}\langle f_1, \ldots, f_d\rangle \leqslant d$. For the inequality in the other direction, note that the ideal $\mathfrak{b} := \langle \varphi(\pi(X_1)), \dots, \varphi(\pi(X_d)) \rangle$ is $\mathfrak{n}$-primary. Hence, $\pi^{-1}(\mathfrak{b}) = \langle f_1, \dots, f_d \rangle + \ker \pi$ is an $\mathfrak{m}$-primary ideal in R. Since R is regular, by Serre's Intersection Theorem [45, Chap. V, Theorem 1]

$$\dim(R/\ker \pi) + \dim(R/\langle f_1, \dots, f_d \rangle) \leqslant \dim R,$$

or, $d + (\dim R/\langle f_1, \dots, f_d \rangle) \leqslant n$. But $\dim(R/\langle f_1, \dots, f_d \rangle) = n - \operatorname{ht} \langle f_1, \dots, f_d \rangle$ as R is regular. We obtain $\operatorname{ht} \langle f_1, \dots, f_d \rangle \geqslant d$ and our claim follows.

Next, we will choose elements $f_{d+1}, \dots, f_n \in \mathfrak{m}$ inductively, making sure at each step that $\pi(f_t) = \varphi(\pi(X_t))$ and that $\dim R/\langle f_1, \dots, f_t \rangle = n - t$. Assume $d \leqslant t < n$ and that $f_1, \dots f_t$ have been chosen with desired properties. To choose f_{t+1} we use the *coset version* of the Prime Avoidance Lemma due to E. Davis (see [27, Theorem 124] or [35, Exercise 16.8]), that can be stated as follows: *let I be an ideal of a commutative ring R and $x \in R$ be an element. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be prime ideals of R none of which contain I. Then*

$$x + I \nsubseteq \bigcup_{i=1}^{s} \mathfrak{p}_i.$$

Choose an element $u \in \mathfrak{m}$ such that $\pi(u) = \varphi(\pi(X_{t+1}))$. If

$$\dim R/\langle f_1, \dots f_t, u \rangle = n - t - 1,$$

then set $f_{t+1} = u$. If not, let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the set of minimal associated prime ideals of $R/\langle f_1, \ldots, f_t\rangle$ that satisfy

$$\dim R/\mathfrak{p}_i = \dim R/\langle f_1, \ldots, f_t\rangle .$$

Since $\langle f_1, \ldots, f_t\rangle + \ker \pi$ is an $\mathfrak{m}$-primary ideal in R, none of these $\mathfrak{p}_i$'s can contain $\ker \pi$. Therefore by the coset version of the Prime Avoidance Lemma there exists an element $a \in \ker \pi$ such that

$$u + a \notin \bigcup_{i=1}^{s} \mathfrak{p}_i.$$

Setting $f_{t+1} = u + a$ we see that $\dim R/\langle f_1, \ldots, f_{t+1}\rangle = n - t - 1$ and $\pi(f_{t+1}) = \varphi(\pi(X_{t+1}))$, as desired.

After choosing $\{f_1, \ldots, f_n\}$ as described, we define an endomorphism ψ of $R = K[\![X_1, \ldots, X_n]\!]$ as follows. For each $1 \leqslant i \leqslant n$, we define $\psi(X_i)$ to be f_i and for every element α of K we define $\psi(\alpha)$ to be $(\pi|_L)^{-1}(\varphi(\pi(\alpha)))$. Then we extend the definition of ψ to all elements of R by continuity. Since $\psi(\mathfrak{m})R = \langle f_1, \cdots, f_n\rangle$ is $\mathfrak{m}$-primary by construction of the f_i's, ψ is african. Moreover, it is clear from the construction that $\varphi \circ \pi = \pi \circ \psi$. □

Corollary 64. *If φ in Theorem 63 is finite, then so is ψ.*

Proof. This follows from [13, Theorem 8]: a local homomorphism $f : S \to T$ of complete noetherian local rings is finite if and only if f is african, and

$[f_\star k_T : k_S]$ is a finite (algebraic) field extension, where k_S and k_T are residue fields of S and T. $\square$

Question 65. Is it possible in Theorem 63 to take ψ to satisfy $v_h(\psi) = v_h(\varphi)$, $w_h(\psi) = w_h(\varphi)$ and $H(\psi)^{\dim A} = H(\varphi)^{\dim R}$?

In light of the lifting theorem oft his section, we ask the following question: given an african endomoprhism ψ of an equicharacteristic complete local ring R, what can be said about the set of prime ideals $\mathfrak{a}$ of R for which there exist n such that $\psi^n(\mathfrak{a}) \subseteq \mathfrak{a}$. If $\mathfrak{a}$ is such an ideal then ψ descends to give an african endomorphism of $R/\mathfrak{a}$.

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