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arXiv:1911.05335v1 [math.AC] 13 Nov 2019

A DIFFERENTIAL-ALGEBRAIC CRITERION FOR OBTAINING A SMALL MAXIMAL COHEN-MACAULAY MODULE

HANS SCHOUTENS

ABSTRACT. We show how for a three-dimensional complete local ring in positive characteristic, the existence of an F-invariant, differentiable derivation implies Hochster's small MCM conjecture. As an application we show that any three-dimensional pseudo-graded ring in positive characteristic satisfies Hochster's small MCM conjecture.

1. INTRODUCTION

Hochster observed that almost all of the homological conjectures over a Noetherian local ring $(R, \mathfrak{m})$ would follow readily from the existence of a *maximal Cohen-Macaulay* module (MCM, for short), that is to say, a module whose depth is equal to the dimension of R (for an overview, see [7]). If R is complete,[1] being an MCM is equivalent with any system of parameters of R becoming a regular sequence on the module. Together with Huneke, he then proved their existence in equal characteristic ([8], with a simplified proof in characteristic zero using ultraproducts by the author in [12]). Recent work of André has now also settled the mixed characteristic case ([1, 2]). Around the same time, he also asked whether in the complete case, we can even get a *small* (=finitely generated) MCM. However, for dimension three and higher, the latter remains largely an open question.[2]

In [14], I gave a new condition (involving local cohomology) for the existence of a small MCM, and deduced the conjecture for three-dimensional F-split complete local rings. In the present paper, I will extend this to some other three-dimensional complete local rings $(R, \mathfrak{m})$. Henceforth, we will in addition assume that R is a domain with algebraically closed residue field k (see footnote 1). I will describe the numerical invariant h from the cited paper and review the argument how the existence of a small MCM follows from the vanishing of h on some unmixed module (Proposition 2.1). For the remainder of this introduction, we now assume that R has moreover positive characteristic, so that we can use Frobenius transforms. Since h is invariant under Frobenius transform and is additive on direct sums, the problem reduces to finding 'enough' *F-decomposable* modules, that is to say, modules whose Frobenius transform is decomposable (see Proposition 2.2 below for a precise statement). For instance, the main result of [14] is an instance of this principle, as F-purity means that R is a direct summand of $\mathbf{F}_*R$. In §3, I then introduce some techniques from differential algebra and deduce the main theorem: if R admits a Hasse-Schmidt derivation $(1, H_1, H_2, \dots)$ with $H_1^p = H_1$, then it admits a small MCM.

Date: November 14, 2019.
2010 *Mathematics Subject Classification.* 13D22,13D45,13A35.

[1] Since all homological conjectures admit faithfully flat descent, there is no loss of generality in proving the existence of MCM's after taking a scalar extension (in the sense of [13, §3]), and so we may assume that R is furthermore complete and has algebraically closed residue field. Moreover, one may always kill a prime ideal of maximal dimension and assume in addition that R is a domain.

[2] Even Hochster has now expressed doubt about the truth of this conjecture.

The last section is then devoted to a special type of rings for which these Hasse-Schmidt derivations always exist, and therefore satisfy Hochster's small MCM conjecture in dimension three: the class of pseudo-graded rings. These include two of the previously known cases of Hochster's conjecture:

- the completion of a three-dimensional graded ring;[3]
- an analytic toric singularity, that is to say, the completion of the coordinate ring of a point on a toric variety.[4]

However, we can now construct new examples from these: e.g., take a hypersurface in a four-dimensional analytic toric singularity with defining equation given by a quadrinomial (=polynomial with four non-zero terms), then each irreducible component is pseudo-graded, whence admits a small MCM (Corollary 4.10). I have also included an appendix, in which I make the connection between differential operators and F-decomposability more explicit, which hopefully gives more credence to this particular approach to solve Hochster's small MCM conjecture in dimension three.

2. A cohomological criterion and F-decomposability

Throughout, fix a complete local domain $(R, \mathfrak{m})$ of dimension $d \geq 2$, with algebraically closed residue field k (see footnote 1), and let M be a finitely generated R-module. Let E be the injective hull of k and denote the Matlis dual of a module Q by $Q^\vee := \operatorname{Hom}_R(Q, E)$.

Local cohomology. We use the following facts

(i) $\mathrm{H}^i_\mathfrak{m}(M)$ is non-zero for i equal to $\operatorname{depth}(M)$ and $\dim(M)$, and some values in between ([4, Theorem 3.5.7]);

(ii) each $\mathrm{H}^i_\mathfrak{m}(M)$ is Artinian ([4, Lemma 3.5.4]), whence $\mathrm{H}^i_\mathfrak{m}(M)^\vee$ is finitely generated, and has dimension at most i ([14, Proposition 2.5]).

Put $\mathbf{K}(M) := \mathbf{K}_R(M) := \mathrm{H}^d_\mathfrak{m}(M)^\vee$. If $S \subseteq R$ is a Noether normalization (i.e., a finite extension with S regular), then $\mathbf{K}(R) = \operatorname{Hom}_S(R, S)$, showing that in general $\mathbf{K}(R)$ is unmixed. In fact, if R is Cohen-Macaulay, then $\mathbf{K}(R)$ is its canonical module. From $\mathrm{H}^d_\mathfrak{m}(M) \cong M \otimes \mathrm{H}^d_\mathfrak{m}(R)$, we get

$$\mathbf{K}(M) = \operatorname{Hom}_R(M, \mathbf{K}(R)) \tag{1}$$

whenever $d = \dim M$, so that the unmixedness of $\mathbf{K}(R)$ implies that of $\mathbf{K}(M)$. The following invariant will play an important role in the sequel:

$$h(M) := \ell(\mathrm{H}^0_\mathfrak{m}(\mathrm{H}^{d-1}_\mathfrak{m}(M)^\vee)) < \infty. \tag{2}$$

2.1. **Proposition.** *Let M be a d-dimensional module, then $\mathbf{K}_R(M)$ has always depth at least two, and even depth at least three whenever $h(M) = 0$.*

Proof. Let $D := \mathrm{H}^0_\mathfrak{m}(M) \subseteq M$. As $\mathrm{H}^d_\mathfrak{m}(D) = \mathrm{H}^{d-1}_\mathfrak{m}(D) = 0$ by (i), the long exact sequence of local cohomology yields $\mathbf{K}(M) = \mathbf{K}(M/D)$, and so upon replacing M by M/D, we may assume that M has positive depth. Choose an M-regular element $a \in \mathfrak{m}$, and let $\bar{R} := R/aR$ and $\bar{M} := M/aM$. The short exact sequence $0 \to M \xrightarrow{a} M \to \bar{M} \to 0$ yields a long exact sequence

$$\dots \to \mathrm{H}^{d-1}_\mathfrak{m}(\bar{M}) \to \mathrm{H}^d_\mathfrak{m}(M) \xrightarrow{a} \mathrm{H}^d_\mathfrak{m}(M) \to \mathrm{H}^d_\mathfrak{m}(\bar{M}) = 0$$

[3] Hochster attributes this case independently to Hartshorne and Peskine-Szpiro, see [7].

[4] The normalization of an analytic toric singularity is a small MCM by [6]; it is pseudo-graded since its ideal of definition can be generated by binomials by [5].

and hence taking Matlis duals, we get an exact seqeunce

$$0 \to \mathbf{K}(M) \xrightarrow{a} \mathbf{K}(M) \to \mathrm{H}^{d-1}_{\mathfrak{m}}(\bar{M})^{\vee} \cong \mathbf{K}_{\bar{R}}(\bar{M}) \tag{3}$$

where the last isomorphism follows since $\bar{M}$ has dimension $d-1$ over $\bar{R}$. Since $\mathbf{K}_{\bar{R}}(\bar{M})$ is unmixed and contains $\mathbf{K}(M)/a\mathbf{K}(M)$ as a submodule, the latter is also unmixed. As a is $\mathbf{K}(M)$-regular, the first assertion follows.

If $h(M) = 0$, then $K' := \mathrm{H}^{d-1}_{\mathfrak{m}}(M)^{\vee}$ has also positive depth, and so we may choose a to be in addition K'-regular (by prime avoidance). The exact sequence (3) extends to

$$0 \to \mathbf{K}(M) \xrightarrow{a} \mathbf{K}(M) \to \mathbf{K}_{\bar{R}}(\bar{M}) \to K' \xrightarrow{a} K' \tag{4}$$

showing that $\mathbf{K}(M)/a\mathbf{K}(M) \cong \mathbf{K}_{\bar{R}}(\bar{M})$. By the first assertion, the latter has depth at least two (as an $\bar{R}$-module, whence as an R-module), and so we are done. □

From now on, we will also assume that R has characteristic $p > 0$. Let $\mathbf{F}_p$ denote the Frobenius map and write $\mathbf{F}_*M$ for the pull-back of M along $\mathbf{F}_p$. That is to say, think of $\mathbf{F}_*M$ as having elements $*x$, for $x \in M$, with scalar multiplication by an element $a \in R$ given by $a{*}x := {*}a^p x$ (and addition as in M). Our assumptions on R imply that $\mathbf{F}_*M$ is again a finitely generated R-module, called the *Frobenius transform* of M. We have

(iii) if D has finite length, then $\ell(D) = \ell(\mathbf{F}_*D)$;
(iv) Frobenius transforms commute with local cohomology and Matlis duality.

For (iii), note that k being algebraically closed implies $\mathbf{F}_*k \cong k$, and the rest now follows by induction on $\ell(D)$ and exactness of $\mathbf{F}_*$. The first statement of (iv) follows from the Čech perspective of local cohomology and the second is proven in [14, Theorem 4.6]).[5] From this, we get

$$h(M) = h(\mathbf{F}_*M). \tag{5}$$

This already yields the main result from [14]: if R is a three-dimensional complete F-pure ring, then it admits a small MCM. Indeed, F-purity implies that $\mathbf{F}_*R \cong R \oplus Q$ for some (finitely generated) Q, and using (5) we get $h(R) = h(\mathbf{F}_*R) = h(R) + h(Q)$, whence $h(Q) = 0$, so that we can apply Proposition 2.1.

F-decomposability. We say that M is *F-decomposable*, if some $\mathbf{F}^n_*M$ is decomposable. Some examples are non-simple modules of finite length, and F-pure rings (see the previous paragraph).[6] Any direct summand of some $\mathbf{F}^n_*M$ will be a called an *F-component*.

2.2. **Proposition.** *If there exists a nonempty class $\mathcal{H}$ of F-decomposable unmixed modules which is closed under F-components (i.e., closed under Frobenius transforms and direct summands), then there is some $M \in \mathcal{H}$ with $h(M) = 0$.*

Proof. Choose $M \in \mathcal{H}$ with $h(M)$ minimal. By assumption, there is some n such that $\mathbf{F}^n_*M \cong P \oplus Q$, with $P, Q \in \mathcal{H}$. Since h is additive on direct sums, (5) yields

$$h(P) + h(Q) = h(\mathbf{F}^n_*M) = h(M),$$

[5] The following shorter argument using derived categories was proposed by an anonymous reviewer. Let $\underline{\omega}_R$ be the normalized dualizing complex and write $\underline{\mathrm{D}}(-) := \underline{\mathrm{RHom}}(-, \underline{\omega}_R)$ for the Grothendieck dual, so that in particular $\underline{\mathrm{R\Gamma}}_{\mathfrak{m}}(\underline{\omega}_R) \cong E$. Since $\mathbf{F}_* = \mathbf{F}_!$ commutes with $\underline{\mathrm{R\Gamma}}_{\mathfrak{m}}$, it also commutes with $\underline{\mathrm{D}}(-) \cong \underline{\mathrm{RHom}}(\underline{\mathrm{R\Gamma}}_{\mathfrak{m}}(-), E)$ by Grothendieck duality, and finally also with Matlis duality, since

$$\underline{\mathrm{R\Gamma}}_{\mathfrak{m}}(\underline{\mathrm{D}}(-)) \cong \underline{\mathrm{RHom}}(-, \underline{\mathrm{R\Gamma}}_{\mathfrak{m}}(\underline{\omega}_R)) \cong \underline{\mathrm{RHom}}(-, E) = -^{\vee}.$$

[6] In fact, I suspect that k is the only F-indecomposable module—note that if this were true, then Proposition 2.2 below would apply to the class of all unmixed modules, proving Hochster's conjecture in dimension three.

so that by minimality, we must have $h(M) = 0$. □

3. HASSE-SCHMIDT DERIVATIONS AND F-DECOMPOSABILITY

Recall that a derivation D on R is a k-linear map satisfying the Leibniz rule $D(ab) = aD(b) + D(a)b$, for $a, b \in R$. Repeating this rule, we get

$$D^p(ab) = \sum_{i=0}^{p} \binom{p}{i} D^i(a)D^{p-i}(b) = aD^p(b) + D^p(a)b \tag{6}$$

proving that D^p is again a derivation; if $D^p = D$ then we say that D is *F-invariant*.

Recall that a k-linear endomorphism f of M is said to have *order at most* n, if, by recursion, $[f, a]$ has order at most $n-1$, for all $a \in R$, where we declare the elements of R, identified with the left multiplication maps on M, as having order zero. Endomorphisms of finite order are then called *differential operators*. Let f be a linear differential operator on M. This means that for each $a \in R$, there exists $d_a \in R$, such that $d_a x = [f, a](x) = (fa - af)(x) = f(ax) - af(x)$. Since the map $a \mapsto d_a$ corresponds to the restriction of $[f, -]$ on R, it is in fact a derivation $\delta_M(f) \in \mathrm{Der}_k(R)$. In other words, $\delta_M(f) = D$ means that $f(ax) = af(x) + D(a)x$, for all $a \in R$ and $x \in M$ (one says that f is a *D-skew* derivation). Let $\mathfrak{K}(M) \subseteq \mathrm{Der}(R)$ be the image of δ_M (it is called the *Kodaira-Spencer kernel* of M as it can be realized as a kernel between Hochschild homology [15, §9]). We have

$$\mathfrak{K}(P \oplus Q) = \mathfrak{K}(P) \cap \mathfrak{K}(Q). \tag{7}$$

Indeed, let π denote the projection $M := P \oplus Q \to P$. Given $D \in \mathfrak{K}(M)$, choose a linear differential operator f on M with $\delta_M(f) = D$, and let $p\colon P \to P\colon x \mapsto \pi(f(x))$. One easily verifies that $[p, a] = D(a)$, for $a \in R$, showing that $\delta_P(p) = D \in \mathfrak{K}(P)$. Conversely, if $D \in \mathfrak{K}(P) \cap \mathfrak{K}(Q)$, then we can find linear differential operators f and g on P and Q respectively with $\delta_P(f) = D = \delta_Q(g)$. It is now easy to see that $f \oplus g$ is a linear differential operator on M with $\delta_M(f \oplus g) = D$.

3.1. **Proposition.** *If there is an F-invariant $D \in \mathfrak{K}(M)$, then $\mathbf{F}_*M$ is decomposable.*

Proof. Suppose not, so that $\mathcal{E} := \mathrm{End}_R(\mathbf{F}_*M)$ is hereditary strongly local with residue field k by Theorem 5.1. Choose a linear differential operator f on M with $\delta_M(f) = D$. As in (6), we have

$$f^p(ax) = \sum_{i=0}^{p} \binom{p}{i} D^i(a)f^{p-i}(x) = af^p(x) + D^p(a)b \quad \text{for all } a \in R \text{ and } x \in M,$$

so that f^p has order one and $\delta_M(f^p) = D^p = D$. Put $\phi := f^p - f$. Since $\delta_M(f^p - f) = 0$, we get $[\phi, a] = 0$, for all $a \in R$, which means that ϕ is R-linear. Let S be the subalgebra of $\mathrm{End}_R(M) \subseteq \mathcal{E}$ generated by ϕ, so that S is commutative, local and complete with residue field k (by Theorem 5.1). The Artin-Schreier polynomial $P(T) := T^p - T - \phi \in S[T]$ has a solution in k, since $P(f) = 0$. Since it is an etale equation, it therefore has already a solution $\tau \in S$ (by Hensel's lemma). Hence $(f-\tau)^p = f^p - \tau^p = f + \phi - (\tau + \phi) = (f - \tau)$, so that $(f - \tau)^{p-1}$ is a (non-trivial) idempotent in $\mathcal{E}$, contradicting (vi). □

Hasse-Schmidt derivations. Recall that a *Hasse-Schmidt derivation* on R is a sequence $\mathbf{H}$ of k-linear endomorphisms H_l, for $l \in \mathbb{N}$, such that $H_0 = 1$ and

$$H_l(ab) = \sum_{i=0}^{l} H_i(a)H_{l-i}(b) \quad \text{for all } a, b \in R \text{ and } l \geq 0. \tag{8}$$

Putting $\Phi_{\mathbf{H}} := \sum H_l t^l$, viewed as a $k[[t]]$-linear endomorphism on $R[[t]]$, then $\Phi_{\mathbf{H}}$ is multiplicative and reduces to the identity modulo t, whence is an automorphism of $R[[t]]$, and conversely any such automorphism induces a unique Hasse-Schmidt derivation (see, for instance, [11, §27]). In particular, H_1 is a derivation on R, and more generally, H_l is a differential operator of order at most l. If an ideal $I \subseteq R$ is $\mathbf{H}$-invariant (i.e., $H_l(I) \subseteq I$, for all l), then $\mathbf{H}$ induces a Hasse-Schmidt derivation on R/I. We call an arbitrary derivation D *integrable*, if there exists some Hasse-Schmidt derivation $\mathbf{H}$ with $H_1 = D$.

3.2. **Corollary.** *Given a prime ideal $\mathfrak{p} \subseteq R$, then a Hasse-Schmidt derivation $\mathbf{H}$ on R induces one on $R/\mathfrak{p}$ in the following two cases:*

(i) *$\mathfrak{p}$ is an associated prime of R;*
(ii) *$\mathfrak{p}$ is a minimal prime of the singular (respectively, non-Cohen-Macaulay, non-Gorenstein, non-normal) locus of R.*

Proof. We need to show that in either case $\mathfrak{p}$ is $\mathbf{H}$-invariant, which amounts to showing that the $k[[t]]$-algebra automorphism $\Phi_{\mathbf{H}}$ of $R[[t]]$ defined by $\mathbf{H} = (H_i)_i$ preserves $\mathfrak{p}R[[t]]$. In case (i), since $\mathfrak{p}R[[t]]$ is then an associated prime of $R[[t]]$, so must its image $\Phi_{\mathbf{H}}(\mathfrak{p}R[[t]])$ be. But any associated prime of $R[[t]]$ is extended from R, that is to say, of the form $\mathfrak{q}R[[t]]$, for some associated prime $\mathfrak{q}$ of R. Since $\Phi_{\mathbf{H}}(a) = a + tf$, for some $f \in R[[t]]$, we see that $a \in \mathfrak{p}$ implies $a \in \mathfrak{q}$, that is to say, $\mathfrak{p} \subseteq \mathfrak{q}$. Reasoning instead with the inverse of $\Phi_{\mathbf{H}}$, we get the other inclusion, showing that $\Phi_{\mathbf{H}}(\mathfrak{p}R[[t]]) = \mathfrak{p}R[[t]]$.

In case (ii), let $\mathfrak{a}$ be the radical ideal defining the singular (respectively, non-Cohen-Macaulay, non-Gorenstein, non-normal) locus of R, and let $J := \Phi_{\mathbf{H}}(\mathfrak{a}R[[t]])$. Let $\mathfrak{Q} \subseteq R[[t]]$ be a prime ideal and set $\mathfrak{P} := \Phi_{\mathbf{H}}^{-1}(\mathfrak{Q})$. If $\mathfrak{Q}$ does not contain J, then $\mathfrak{P}$ does not contain $\mathfrak{a}R[[t]]$. As the latter ideal defines the singular (respectively, non-Cohen-Macaulay, non-Gorenstein, non-normal) locus of $R[[t]]$, we see that $R[[t]]_{\mathfrak{P}}$ is regular (respectively, Cohen-Macaulay, Gorenstein, normal), whence so is $R[[t]]_{\mathfrak{Q}}$ under the isomorphism $\Phi_{\mathbf{H}}$. This proves that $\mathfrak{a}R[[t]] \subseteq J$. Conversely, if $R[[t]]_{\mathfrak{Q}}$ is regular (respectively, Cohen-Macaulay, Gorenstein, normal), then so is $R[[t]]_{\mathfrak{P}}$ and hence $\mathfrak{P}$ does not contain $\mathfrak{a}R[[t]]$, whence neither does $\mathfrak{Q}$ contain J, showing that $J \subseteq \mathfrak{a}R[[t]]$, so that $\Phi_{\mathbf{H}}$ preserves the ideal $\mathfrak{a}R[[t]]$. As $\mathfrak{p}$ is a minimal prime of $\mathfrak{a}$, the result follows from (i). □

3.3. **Proposition.** *If $\mathbf{H}$ is a Hasse-Schmidt derivation, then $H_1 \in \mathfrak{K}(\mathbf{F}_*^n R)$, for all n.*

Proof. Replacing p by some power, we easily reduce to the case that $n = 1$, and so we are done once we show that $f := \mathbf{F}_*(H_p)$ is a linear differential operator on $\mathbf{F}_*R$ with $\delta(f) = H_1$. For $a \in R$ and $x := *b \in \mathbf{F}_*R$, we have $f(ax) = f(*a^pb) = *H_p(a^pb)$. To calculate $H_p(a^pb)$, we need, with Φ the automorphism given by $\mathbf{H}$, the coefficient of t^p in

$$\Phi(a^pb) = \Phi(a)^p\Phi(b) = (a + H_1(a)t + \dots)^p(b + H_1(b)t + \dots + H_p(b)t^p + \dots)$$

which is $a^pH_p(b) + H_1(a)^pb$, so that

$$f(ax) = *a^pH_p(b) + *H_1(a)^pb = a*H_p(b) + H_1(a)*b = af(x) + H_1(a)x$$

showing that $\delta(f) = H_1$. □

3.4. **Theorem.** *If a three-dimensional complete local domain of characteristic p admits an F-invariant, integrable derivation, then it has a small MCM.*

Proof. By assumption, there exists a Hasse-Schmidt derivation $\mathbf{H}$ with $H_1 = H_1^p$. Let $\mathcal{H}$ be the collection of all F-components of R (i.e., all summands of $\mathbf{F}_*^n R$, for all n). If we want to apply Proposition 2.2 to $\mathcal{H}$, so that we get a small MCM in view of Proposition 2.1, then we must show that any $Q \in \mathcal{H}$ is F-decomposable. By assumption, Q is a summand of some $\mathbf{F}^n_* R$. By Proposition 3.3, we have $H_1 \in \mathfrak{K}(\mathbf{F}^n_* R)$ and hence $H_1 \in \mathfrak{K}(Q)$ by (7), so that Q is decomposable by Proposition 3.1. $\square$

S_2-ification and pseudo-perfect modules. Since $\mathbf{K}(R)$ is indecomposable (as R is a domain), its endomorphism ring $S := \mathrm{End}_R(\mathbf{K}(R))$ is a local ring (by Theorem 5.1) satisfying Serre's condition (S_2), called the S_2-*ification* of R (see, for instance [3, Theorem 3.2] or [9]). Since S is a finite R-module (contained in the field of fractions of R), any small MCM over S, is then also a MCM over R. In short, if we want to do so, we may moreover assume that R is an S_2-*domain*, i.e., satisfies Serre's condition (S_2).

Identifying $\mathbf{K}(\mathbf{K}(M))$ with $\mathrm{Hom}_R(\mathrm{Hom}_R(M, \mathbf{K}(R)), \mathbf{K}(R))$ via (1), for a module M, we have a canonical map $M \to \mathbf{K}(\mathbf{K}(M))$, given by sending $x \in M$ to the homomorphism $\mathrm{Hom}_R(M, \mathbf{K}(R)) \to \mathbf{K}(R)\colon \varphi \mapsto \varphi(x)$. If this map is an isomorphism, then we will call M *pseudo-perfect*. In view of (1), we therefore showed that a complete local domain satisfies property (S_2) if and only if it is itself pseudo-perfect. It is not hard to show using (iv), that any F-component of a pseudo-perfect module is again pseudo-perfect. Therefore, analyzing the above proofs, we actually showed

3.5. **Corollary.** *If a three-dimensional complete local S_2-domain R admits an F-invariant, integrable derivation, then some F-component of R is a small MCM.* $\square$

4. Application: pseudo-graded rings

Throughout this section, let $S := k[[x_1, \dots, x_r]]$ and let $A := S[x_1^{-1}, \dots, x_r^{-1}]$ be the *ring of Laurent series*. Any element $f \in A$ can be written as $f = \sum_{\mathbf{a}} u_{\mathbf{a}} x^{\mathbf{a}}$, with $u_{\mathbf{a}} \in k$ and $\mathbf{a} \in \mathbb{Z}^r$ (with the usual convention that $x^{\mathbf{a}} := x_1^{a_1} \cdots x_r^{a_r}$), so that in addition, its *support* $\mathrm{supp}(f)$, that is to say, the set of all $\mathbf{a} \in \mathbb{Z}^r$ such that $u_{\mathbf{a}} \neq 0$, is contained in some translate $\mathbf{c} + \mathbb{N}^r$, for some $\mathbf{c} \in \mathbb{Z}^r$.

Fix a non-zero linear form $\lambda \in \mathrm{Hom}(\mathbb{Z}^r, \mathbb{Z})$ and let $Z(\lambda) \subset \mathbb{Z}^r$ be its kernel. This form is represented by an r-tuple $\mathbf{l} := (l_1, \dots, l_r)$, so that $\lambda(a_1, \dots, a_r) = l_1 a_1 + \dots + l_r a_r$ for any $(a_1, \dots, a_r) \in \mathbb{Z}^r$. Put differently, if $\mathbf{e}_i$ is the i-th standard basis element of $\mathbb{Z}^r$, then $l_i = \lambda(\mathbf{e}_i)$. Let us say that an element $f \in A$ is λ-*homogeneous*, if λ is constant on its support $\mathrm{supp}(f)$, or, equivalently, if $\mathrm{supp}(f) \subseteq \mathbf{a} + Z(\lambda)$, for some $\mathbf{a} \in \mathbb{Z}^r$, called the *weight vector* of f. An ideal $I \subseteq S$ will be called a λ-*ideal*, if it is generated by λ-homogeneous elements. Any complete local ring R that can be realized as a quotient $R := S/I$ with I a λ-ideal for some non-zero linear form λ, will be called a *pseudo-graded ring*. An example is the completion of a standard graded ring at its irrelevant maximal ideal (with all $l_i = 1$). To λ, we also associate the derivation

$$\Delta_\lambda := l_1 x_1 \partial_1 + \dots + l_r x_r \partial_r$$

where $\partial_i := \partial/\partial x_i$ is the i-th partial derivative on S, whence on A. A quick calculation yields

$$\Delta_\lambda(x^{\mathbf{a}}) = \lambda(\mathbf{a}) x^{\mathbf{a}} \tag{9}$$

for all $\mathbf{a} \in \mathbb{Z}^r$. In particular, $\Delta_\lambda(f) = \lambda(\mathbf{a}) f$, for any λ-homogeneous element f with weight vector $\mathbf{a}$, showing that any λ-ideal is invariant under Δ_λ. In particular, Δ_λ induces a derivation on the pseudo-graded quotient R. A p-fold iteration of (9) then yields

$$\Delta_\lambda^p(x^{\mathbf{a}}) = \lambda(\mathbf{a})^p x^{\mathbf{a}} = \lambda(\mathbf{a}) x^{\mathbf{a}} = \Delta_\lambda(x^{\mathbf{a}})$$

and since this holds for any monomial, we get $\Delta_\lambda^p = \Delta_\lambda$, that is to say, Δ_λ is F-invariant.

Our next goal is to show that Δ_λ is integrable, and to this end we will use generalized binomials. For $m, d \in \mathbb{N}$, we have the so-called *negation rule*

$$\binom{-m}{d} := (-1)^d \binom{m+d-1}{d},$$

and the usual additive rule (Pascal identity)

$$\binom{z}{d} = \binom{z-1}{d} + \binom{z-1}{d-1} \qquad \text{for all } z \in \mathbb{Z} \text{ and } d > 0. \tag{10}$$

4.1. **Proposition.** *There exists a Hasse-Schmidt derivation $\mathbf{H}_\lambda = (1, \Delta_\lambda, H_2, \dots)$ on S, which leaves every λ-ideal invariant. In particular, any pseudo-graded local ring admits an F-invariant, integrable derivation.*

Proof. We will be more precise and show that the k-linear maps given by

$$H_n\big(\sum u_{\mathbf{a}} x^{\mathbf{a}}\big) := \sum_{\mathbf{a}} u_{\mathbf{a}} \binom{\lambda(\mathbf{a})}{n} x^{\mathbf{a}} \tag{11}$$

for $\sum_{\mathbf{a}} u_{\mathbf{a}} x^{\mathbf{a}} \in A$, yield a Hasse-Schmidt derivation on the ring of Laurent series A. Note that $H_1 = \Delta_\lambda$ does satisfy (11) in view of (9). To verify (8), let $f = \sum u_{\mathbf{a}} x^{\mathbf{a}}$ and $g = \sum_{\mathbf{b}} v_{\mathbf{b}} x^{\mathbf{b}}$ be in A. We get

$$\begin{aligned} H_n(fg) &= H_n\big(\sum_{\mathbf{a},\mathbf{b}} u_{\mathbf{a}} v_{\mathbf{b}} x^{\mathbf{a}+\mathbf{b}}\big) = \sum_{\mathbf{a},\mathbf{b}} u_{\mathbf{a}} v_{\mathbf{b}} H_n(x^{\mathbf{a}+\mathbf{b}}) \\ &= \sum_{\mathbf{a},\mathbf{b}} u_{\mathbf{a}} v_{\mathbf{b}} \binom{\lambda(\mathbf{a}+\mathbf{b})}{n} x^{\mathbf{a}+\mathbf{b}} \overset{(CH)}{=} \sum_{i,\mathbf{a},\mathbf{b}} u_{\mathbf{a}} v_{\mathbf{b}} \binom{\lambda(\mathbf{a})}{i} \binom{\lambda(\mathbf{b})}{n-i} x^{\mathbf{a}} x^{\mathbf{b}} \\ &= \sum_{i,\mathbf{a},\mathbf{b}} u_{\mathbf{a}} v_{\mathbf{b}} H_i(x^{\mathbf{a}}) H_{n-i}(x^{\mathbf{b}}) = \sum_i H_i\big(\sum u_{\mathbf{a}} x^{\mathbf{a}}\big) H_{n-i}\big(\sum_{\mathbf{b}} v_{\mathbf{b}} x^{\mathbf{b}}\big) \\ &= \sum_{i=0}^n H_i(f) H_{n-i}(g) \end{aligned}$$

where we used the Chu-Vandermonde identity for binomial coefficients

$$\binom{v+w}{n} = \sum_{i=0}^n \binom{v}{i}\binom{w}{n-i} \tag{CH}$$

for $v, w \in \mathbb{Z}$, since $\lambda(\mathbf{a}+\mathbf{b}) = \lambda(\mathbf{a}) + \lambda(\mathbf{b})$.

It follows from (11) that $\mathbf{H}_\lambda$ leaves S invariant, so that it is a Hasse-Schmidt derivation on S. To show that $\mathbf{H}_\lambda$ leaves any λ-ideal of S invariant, let $A_\lambda \subseteq A$ be the subring of all Laurent series whose support lies in the kernel $Z(f)$ of λ. It follows from (11) that each H_n is identically zero on A_λ. Therefore, each H_n is A_λ-linear, since for $f \in A$ and $g \in A_\lambda$, we have $H_n(fg) = fH_n(g) + H_1(f)H_{n-1}(g) + \cdots + H_n(f)g$ in which all but the last term are zero, so that $H_n(fg) = gH_n(f)$. Let $f \in S$ be λ-homogeneous. By definition, it is of the form $f = x^{\mathbf{a}} g$, with $g \in A_\lambda$. Hence, using (9), we get

$$H_n(f) = H_n(x^{\mathbf{a}} g) = gH_n(x^{\mathbf{a}}) = g\binom{\lambda(\mathbf{a})}{n} x^{\mathbf{a}} = \binom{\lambda(\mathbf{a})}{n} f.$$

Hence any λ-ideal is invariant under H_n, for all n. To prove the last assertion, let R be pseudo-graded, say, $R = S/I$ for some linear form $\lambda \colon \mathbb{Z}^r \to \mathbb{Z}$ and some λ-ideal $I \subseteq S$. Since I is invariant under $\mathbf{H}_\lambda$, the latter induces a Hasse-Schmidt derivation on R. $\square$

By Corollary 3.2, any component of a pseudo-graded ring, i.e., any quotient by an associated prime, also admits an F-invariant integrable derivation and Theorem 3.4 yields:

4.2. **Theorem.** *Any three-dimensional component of a pseudo-graded local ring in positive characteristic admits a small MCM.* □

Examples of pseudo-graded rings. To an element f in S or in $k[x]$, we associate its lattice $\Lambda(f) \subseteq \mathbb{Z}^r$, as the subgroup generated by all differences of elements in $\operatorname{supp}(f)$ (with the convention that $\Lambda(f) = 0$ when f is a monomial). We have inclusions

$$\Lambda(fg), \Lambda(f+g) \subseteq \Lambda(f) + \Lambda(g), \tag{12}$$

for all f, g. The *rank* $\operatorname{rk}(f)$ of f is defined to be the rank of $\Lambda(f)$, that is to say, the vector space dimension of $\Lambda(f) \otimes \mathbb{Q}$; in particular, $\operatorname{rk}(f) \leq r$. A *binomial* is an element whose support consists of two elements, and so binomials have rank one. We may extend the above to ideals I in S or $k[x]$: define $\Lambda(I) := \Lambda(f_1) + \cdots + \Lambda(f_s)$ for $I = (f_1, \ldots, f_s)$. By (12), this does not depend on the choice of generators, and we have in particular that $\Lambda(I + J) = \Lambda(I) + \Lambda(J)$ for any two ideals I, J. We then define $\operatorname{rk}(I)$ as the rank of $\Lambda(I)$. More generally, if R is the quotient of S or $k[x]$ by some ideal, then we define the rank of an ideal $I \subseteq R$ as the smallest rank of a lifting of I to S or $k[x]$.

4.3. **Lemma.** *If $R = S/I$ with $\operatorname{rk}(I) < r$, then R is pseudo-graded.*

Proof. By assumption, there is some $(l_1, \ldots, l_r) \in \mathbb{Z}^r$ which is orthogonal to $\Lambda(I)$, and it is now easy to see that I is a λ-ideal for the linear form $\lambda := l_1 z_1 + \cdots + l_r z_r$. □

By the theory of toric varieties, or using [5, Theorem 2.1] for the more general case, we have (recall that r is the length of the tuple of variables x):

4.4. **Theorem.** *If $I \subseteq k[x]$ is generated by binomials, then $k[x]/I$ has dimension $r - \operatorname{rk}(I)$. Moreover, if I is also prime, the quotient $k[x]/I$ is the coordinate ring of a toric variety in $\mathbb{A}^r_k$, and any coordinate ring of a toric variety is obtained this way.* □

By a *toric singularity* (of embedding dimension r), we mean the local ring of a (singular) point on a toric variety (in $\mathbb{A}^r_k$), and the completion of a such a ring is then called an *analytic toric singularity*. Recall that the normalization of a toric variety is again a toric variety which is in addition Cohen-Macaulay ([6]). Since the normalization is a finite extension, any analytic toric singularity therefore admits a small MCM. Immediately from Theorem 4.4 and Lemma 4.3 we see that an analytic toric singularity is pseudo-graded.

4.5. **Theorem.** *Let T be analytic toric singularity, or more generally, a pseudo-graded complete local domain, and let $V \subseteq \operatorname{Spec} T$ be its singular (respectively, non-Cohen-Macaulay, non-Gorenstein, non-normal) locus. If $\mathfrak{p}$ defines a three-dimensional irreducible component of V, then $T/\mathfrak{p}$ admits a small MCM.*

Proof. By Proposition 4.1, there exists a Hasse-Schmidt derivation $\mathbf{H} = (H_i)_i$ on T with $H_1^p = H_1$. By Corollary 3.2(ii), this $\mathbf{H}$ descends to a Hasse-Schmidt derivation on $T/\mathfrak{p}$ and so we are done by Theorem 3.4. □

4.6. **Proposition.** *Let T be a d-dimensional analytic toric singularity. If $I \subseteq T$ is an ideal of rank strictly less than d, then T/I is pseudo-graded.*

Proof. Writing again I for a lift to $S := k[[x]]$ of minimal rank, we must show that $I + J$ is a λ-ideal for some linear form λ, where J is the ideal generated by binomials such that $T = k[[x]]/J$. By Theorem 4.4, the rank of J is $r - d$. Since $\Lambda(I + J) = \Lambda(I) + \Lambda(J)$ and $\Lambda(I)$ has rank at most $d - 1$ by assumption, the rank of $\Lambda(I + J)$ is at most $r - 1$, so that we are done by Lemma 4.3. □

4.7. **Corollary.** *Let T be a d-dimensional analytic toric singularity, and let $C \subseteq \operatorname{Spec}(S)$ be a cylinder with base inside a $(d-1)$-dimensional coordinate hyperplane. Then the coordinate ring of any three-dimensional irreducible component of $C \cap \operatorname{Spec}(T)$ admits a small MCM.*

Proof. By assumption, there exists a subset of the x-variables of size $d-1$ defining the ideal $I \subseteq T$ of C. In particular, $\operatorname{rk}(I) \leq d-1$ and so T/I, the coordinate ring of $C \cap \operatorname{Spec}(T)$ is pseudo-graded by Proposition 4.6, and the result now follows from Theorem 4.2. □

By an *m-nomial* in a quotient of S or $k[x]$, we mean the image f of an element whose support has cardinality m. Note that then $\operatorname{rk}(f) < m$, and any element satisfying the latter inequality is called a *pseudo-m-nomial*. For instance, the trinomial

$$f := u_0 x^2 z^4 + u_1 x y^2 z^2 + u_2 y^4$$

is in fact a pseudo-binomial as $\Lambda(f)$ is generated by $(1, -2, 2)$. Let us call a homomorphism of complete local rings with residue field k *(pseudo-)m-nomial*, if it is given by (pseudo-)m-nomials, that is to say, induced by a homomorphism $k[[y]] \to k[[z]] \colon y_i \mapsto f_i$, where each f_i is a (pseudo-)m-nomial.

4.8. **Corollary.** *Let T be a d-dimensional analytic toric singularity, $I \subseteq S$ a monomial ideal, and $S \to T$ a pseudo-m-nomial homomorphism. If $r(m-1) < d$, then T/IT is pseudo-graded. In particular, if $\mathfrak{p}$ is a three-dimensional associated prime of IT, then $T/\mathfrak{p}$ admits a small MCM.*

Proof. Write T as a quotient of some $k[[z]]$, with $z = (z_1, \dots, z_s)$, and let $S \to T$ be given by sending x_i to the pseudo-m-nomial $f_i \in k[[z]]$, for $i = 1, \dots, r$. By assumption, $\Lambda_i := \Lambda(f_i) \subseteq \mathbb{Z}^s$ has rank at most $m-1$. Let $B := k[[z]][z_1^{-1}, \dots, z_s^{-1}]$ and write $f_i = z^{\mathbf{a}_i} g_i$, for some $\mathbf{a}_i \in \mathbb{Z}^s$ and some $g_i \in B$ with support in Λ_i. Let $\sigma \colon \mathbb{Z}^r \to \mathbb{Z}^s$ be the linear map given by $(b_1, \dots, b_r) \mapsto b_1\mathbf{a}_1 + \dots + b_r\mathbf{a}_r$. It follows that the image of a monomial $f^{\mathbf{b}}$ in T, with $\mathbf{b} \in \mathbb{Z}^r$, is equal to $z^{\sigma(\mathbf{b})} g_{\mathbf{b}}$, for some $g_{\mathbf{b}} \in B$ with $\operatorname{Supp}(g_{\mathbf{b}}) \subseteq \Lambda := \Lambda_1 + \dots + \Lambda_r$. This shows that $\Lambda(IT) \subseteq \Lambda$ and since the latter has rank at most $r(m-1)$, we are done by Proposition 4.6 and Theorem 4.2. □

4.9. *Remark.* More generally, the result still holds if $I \subseteq S$ is an arbitrary ideal such that $\operatorname{rk}(I) < d - r(m-1)$. Indeed, I is then generated by elements of the form $h = x^{\mathbf{b}} \sum u_{\mathbf{c}} x^{\mathbf{c}}$, with $\mathbf{c} \in \Theta := \Lambda(I)$, $u_{\mathbf{c}} \in k$ and $\mathbf{b} \in \mathbb{Z}^r$. The image of h in T is equal to $z^{\sigma(\mathbf{b})} g_{\mathbf{b}} \sum z^{\sigma(\mathbf{c})} g_{\mathbf{c}}$ and so its support is inside $\sigma(\mathbf{b}) + \sigma(\Theta) + \Lambda$, showing that $\Lambda(IT) \subseteq \sigma(\Theta) + \Lambda$. As $\operatorname{rk}(\sigma(\Theta)) \leq \operatorname{rk}(\Theta) = \operatorname{rk}(I)$, we get $\operatorname{rk}(IT) < d$.

4.10. **Corollary.** *Let T be a d-dimensional analytic toric singularity. If $\mathfrak{p} \subseteq T$ is a three-dimensional prime, then $T/\mathfrak{p}$ admits a small MCM in the two following cases:*

(i) *$d = 4$ and $\mathfrak{p}$ contains a non-zero (pseudo-)quadrinomial;*
(ii) *$d = 5$ and $\mathfrak{p}$ contains a height two ideal generated by two (pseudo-)trinomials.*

Proof. Apply Corollary 4.8, where I is the ideal generated by the variables, and $S \to T$ is given, in the case $d = 4$, by the quadrinomial with $r = 1$, and in case $d = 5$, by the two trinomials with $r = 2$. □

4.11. *Remark.* More generally, if there exists an ideal $I \subseteq T$ of height $d-3$ generated either by $d-4$ pseudo-binomials and one pseudo-quadrinomial or by $d-5$ pseudo-binomials and two pseudo-trinomials, then any $T/\mathfrak{p}$ admits a small MCM for any minimal prime $\mathfrak{p}$ of I. The same is true if instead in the above, the number of pseudo-binomials is arbitrary and we also allow monomial generators instead (as long as the height of the ideal is $d-3$).

5. Appendix: hereditary strongly local algebras

Recall that a non-commutative ring A is called *local* if it has a unique maximal left ideal, which is then also the unique maximal right ideal, and this is then also the Jacobson radical $\operatorname{rad}(A)$ (see, for instance, [10]). It follows that A is local if and only if $A/\operatorname{rad}(A)$ is a division ring. Therefore, A is local if and only if, for any $f \in A$, either f or $1-f$ is a unit. We call A *strongly local* if $A/\operatorname{rad}(A)$ is a field, called its *residue field*. Finally, if R is a commutative ring and A an R-algebra, then we call A *hereditary (strongly) local* over R, if any R-subalgebra of A is (strongly) local. The following is folklore:

5.1. **Theorem.** *Let $\mathcal{E} := \operatorname{End}_R(Q)$ be the endomorphism ring of a finitely generated R-module Q over a Henselian local ring R with algebraically closed residue field k. The following are equivalent*

(v) *Q is indecomposable;*
(vi) *$\mathcal{E}$ has no non-trivial idempotents;*
(vii) *$\mathcal{E}$ is local;*
(viii) *$\mathcal{E}$ is strongly local with residue field k;*
(ix) *$\mathcal{E}$ is hereditary strongly local over R.*

Proof. We only need to show that the first condition implies the last. Let $A \subseteq \mathcal{E}$ be an R-subalgebra and take some $f \in A$. Let $S \subseteq A$ be the R-subalgebra generated by f. Since S is commutative and R is Henselian, S must be a direct sum $S_1 \oplus \cdots \oplus S_m$ of local rings, and since k is algebraically closed, they all have residue field k. But then m must be equal to 1 lest we violate (vi). There is a unique $u \in k$ such that $f - u$ lies in the maximal ideal of S, whence in the radical of A, and so A is strongly local with residue field k. □

In the sequel, let R be a complete local ring of characteristic p with algebraic residue field k and let M be a finitely generated R-module. Although we did not use it explicitly, the following is the underlying reason for introducing differential algebra into the problem: let $\mathcal{D}(M) \subseteq \operatorname{End}_k(M)$ denote the subring of differential operators (=the endomorphisms of finite order) on M (see §3); Grothendieck showed (see [16]) that it consists precisely of the $R^{(n)}$-linear endomorphisms, for some n, where $R^{(n)} \subseteq R$ is the subring of p^n-th powers of elements of R.

5.2. **Corollary.** *The ring of differential operators $\mathcal{D}(M)$ is hereditary strongly local if and only if M is F-indecomposable.*

Proof. Note that $\operatorname{End}_{R^{(n)}}(M) \cong \operatorname{End}_R(\mathbf{F}^n_* M)$, and so if $\mathcal{D}(M)$ is local, whence has no non-trivial idempotents, neither therefore does the subring $\operatorname{End}_R(\mathbf{F}^n_* M)$, proving that $\mathbf{F}^n_* M$ is indecomposable. Conversely, suppose M is F-indecomposable, and let $S \subseteq \mathcal{D}(M)$ be an arbitrary R-subalgebra. Since $\mathbf{F}^n_* M$ is by assumption indecomposable, the subalgebra $S_n := S \cap \operatorname{End}_{R^{(n)}}(M)$ is strongly local by Theorem 5.1, for all n. It is not hard to see that $S = \bigcup_n S_n$ is then also strongly local. □

Let us say that a submodule $H \subseteq \mathcal{D}$ is *F-closed* if $h^p \in H$, for all $h \in H$.

5.3. **Proposition.** *If there exists a finitely generated F-closed R-submodule $H \subseteq \mathcal{D}$ containing R, some $f \notin H$, and an l such that $f^{p^l} - f \in H$, then M is F-decomposable.*

Proof. Put $q := p^l$. Towards a contradiction, suppose M is F-indecomposable. Since $\mathcal{D}$ is then hereditary strongly local by Corollary 5.2, we can find a (unique) $u \in k$ such that $g := f - u$ is not a unit, and since then also $g^q - g \in H$, we may assume form the start that f is not a unit. By Corollary 5.2, we may choose d large enough so that

$f \in A := \operatorname{End}_{R^{(d)}}(M) \subseteq \mathcal{D}$ and $H \subseteq A$. Since A is then a local R-algebra, f lies in its Jacobson radical $\mathfrak{n} := \operatorname{rad} A$. Since A is finite over R, so too is $A/\mathfrak{m}A$ over k, and hence some power of $\mathfrak{n}$ lies in $\mathfrak{m}A$, say, $\mathfrak{n}^l \subseteq \mathfrak{m}A$. It follows that $f^{ln} \in \mathfrak{m}^n A$, for all n. Since H is F-closed and $f^q - f$ lies in H, so does $(f^q - f)^q = f^{2q} - f^q$, whence also $f^{2q} - f$. Continuing this way, we get $f^{lnq} - f \in H$, whence $f \in H + \mathfrak{m}^{nq}A$, for all n. Since H, being finitely generated, is $\mathfrak{m}$-adically closed in A, we actually get $f \in H$, contradiction. $\square$

Proposition 3.1 is now an immediate consequence of Proposition 5.3: take f as in the former's proof and let $H := \operatorname{End}_R(M)$, so that $f^p - f \in H$.

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