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Proofs Methods and Logical Reasoning in Mathematics promote critical thinking, real-life problem-solving, and creativity skills to the new generation

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"It is by logic that we prove, but by intuition that we discover."
— Henri Poincare French Mathematician

Abstract

In mathematics we start, by using discovery method, with small observations and operations, then we look for patterns, algorithms procedures, and furthermore we generalize by writing theorems, formulas, and conclusions. Our intuitive thinking, the needs for solving real-life problems, and curiosity often push us towards research and study more about things that matter to us, especially when we see that problems need to be solved right away. This kind of things have driven people now and back then depended on the era in which they have lived, or they live in. For example: In these days, because of Covid 19 doctors and researchers and other health care organizations, are interested and they are working on collecting data, analyzing, modeling the spread of the virus to track epidemics and make predictions about the progression. While for example, exceedingly early (3400 BC – Mesopotamia) [1] the numerical system was discovered. Everything that is measurable and plays specific role in our lives needs to be well evaluated. For that reason, we need mathematical reasoning and calculations to draw the appropriate conclusions.

Despite what we notice, observe, and we realize by doing some calculations, everything needs proof. We need to make sure that what we state as true statement must be proven as true.

In this article I will discuss some of the most important **Proof Methods in Teaching Mathematics**

Keywords: arbitrary, fixed, variable, proposition, postulate, axiom, premise, inferences, reason, theorem, mathematical proof, lemma, corollary, conjecture etc.

Introduction to Proof Methods in Teaching Mathematics

In early ages students are exposed to many concepts, algorithms and representations in a simple way. They work on order of operations in an arithmetic expression that later helps understanding the equivalent transformations: Ven diagrams, tables, charts, and graphs, and that helps later when they are exposed to functions and different ways of presentations; They learn how to fill in an empty box to make the equality true, and later that helps in understanding the equation and the solution to the equation. They are exposed to patterns and sequences and asked to find the next term(s), and later they are asked to find and write the general term and/or the summation of n terms in the sequence.

But what are some major approaches that we use or apply to teach these concepts. What are some of the crucial methods that are present throughout all mathematics levels and courses?

This is an overly broad issue and never-ending question, but I will focus on only a few of them with the goal of continuing to look and focus on some other ones in the future. The goal for focusing on these issues is because we all know that sometimes students learn steps and procedures to solve a problem, and when it comes to explaining it is hard for them to give a real reason for doing so.

A- The approach to introducing the material to students with the end in mind

- It not only important to know stuff, but it is also just as important to know how to deliver the ideas, the conceptual understanding and of course the procedures for solving a problem. That is what drives me to constantly look for new approaches, pedagogy and methods of teaching. From my experience and observations, I have seen differences and similarities in various faculty in terms of teaching styles.
- Some of the instructors use the approach of discovery and connection between what the students know prior to that moment. By giving the students to solve a problem that involves concepts that they know, but at the same time constructing questions in a way that leads to a new discovery for them is a great approach that encourages learners to think critically and make connections. From analyzing chunks, they synthesize.
- Some other Instructors prefer to present the theorem, statement, or formula and ask students to work toward reasoning and proving it. By having the finalized statement, (Theorem, formula etc.) learners research and break down in small parts, use all the logical reasoning and previous knowledge to get to prove that the statement is true. (Analyzing)
- Some Instructors prefer to use them both and they carefully think of when to use which of the approaches that may be the most effective and easy to understand for learners.

All these methods and approaches require great critical thinking and they match with the top scales in the hierarchy of Bloom’s Taxonomy levels. No wonder why proofs and reasoning are hard for many of the students to perform appropriately.

B- Setting up the communication with the audience by giving some major definitions and terminology.

It is especially important that we set up the bridges of communication to our audience. Giving some vocabulary, definitions and other terminology makes the communication easier and more pleasant. We do this for any kind of communication, with learners, colleagues, and any other audience. A brief discussion to make connections and bring the audience to the necessary level to understand what comes next is also a wonderful way to start.

Since I will be discussing proofs and logical reasoning in this modest written piece, I am providing some definitions and terminology here.

- A **proposition** has the most basic meaning as a statement that can be true or false Examples:
- All even integers are divisible by 2. (True)
- There is no real number between 100 and 101. (False)
- You should go to LA to live. (Not a proposition. Advise, opinion)
- Is it a beautiful song? (Questions are not prop.)

- Let $x, y, z \in Z$. If $x + y = x + z$, then $y = z$
  The proof of this proposition is an example of an axiomatic proof. the proof that refers explicitly to the postulates.

- An **axiom** or **postulate** is a statement that is taken to be true. (A **fact** that is an occurrence in the real world).

  b. Example:
- Three points are always coplanar, and if the points are distinct and non- collinear, the plane they determine is unique
  .
- A **definition** is a precise description of a word used in mathematics.

  c. Example:
- An equilateral triangle is the triangle with three congruent sides.

- A **premise** It is an assumption that something is true. Formal logic uses a set of premises to arrive at a conclusion.

  d. Example:
- P1: x is an even integer
- P2: y is an odd integer
- Conclusion: x +y is an odd integer.

- A **conclusion** is a statement that can be drown as result of applying a set of logical rules known as syllogisms to a set of promises. The process itself is called deduction.

e. Example:

  - P1. All humans are mortal
  - P2: John is a human
  - Conclusion: Therefore, John is mortal

- **Inferences** are steps in reasoning, moving from premises to logical consequences.

  h. Example:
- Consider this argument:
- If Sara goes to College, Sara will study for Medicine.
- Sara goes to College, Therefore, Sara studies for Medicine. (This is a valid argument)

- **Reason** is the ability of consciously making sense of things, applying logic,
- A **theorem** is a non-self-evident statement that has been proven to be true

  a. Example:

Theorem 1 (The sum and Difference Rules of differentiating)

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of the sum f + g (or difference f-g) is the sum (difference) of the derivatives of f and g (It is important to say as under minding that they both are functions of the same variable x).

- A **mathematical proof** is an inferential argument for a mathematical statement, showing that the stated assumptions logically guarantee the conclusion.
- **Lemma** is a theorem that is useful to the proving another theorem.
- **Corollary** is a theorem that follows from another theorem.
- **Conjecture**: a statement that you think is true and can be proven (but has not been proven yet).

C- **Some major proof methods in mathematics associated with examples**

1. **Direct Proofs**

- Usually we use direct proof method when we want to prove a type of implication statement in a form: "If…, then…" or $\forall x \epsilon D, P(x) \rightarrow Q(x)$.(Example). In general, we should be able to prove that the conclusion holds whenever the hypotheses hold.

- We start with assumption that the hypothesis P(a) holds for any arbitrary value of x let say a
- We use the definitions, postulates, other previously proved theorems applying the rules of inference to prove that Q(a) is also true
- Since we proved the implication for an arbitrary a, we use the universal generalization to admit that the statement is true.

- Usually we use direct proof method when we want to prove a type of implication statement in a form: "If…, then…" or $\forall x \epsilon D, P(x) \rightarrow Q(x)$.(Example)

Example 1.1 Direct proof:

Theorem 2: For all integers m and n, if m is odd and n is even, then m+n is odd.

- Assuming that "m is an odd number" is true as given hypothesis, we can write:
- $m = 2k_1 + 1$, also, assuming that n is even, we can write $n = 2k_2$

  So, $m + n = (2k_1 + 1) + 2k_2$

Regrouping using the associative property of addition we will have:

$$m + n = (2k, +2k_2) + 1$$

$$m + n = 2(k_1 + k_2) + 1$$

We can substitute $k_1 + k_2 = k$,

so we get: $m + n = 2k + 1$

So, we proved that for any arbitrary odd m integer, and n even integer
m+n is an odd integer. We use the universal generalization as last step.

Example 1.2 Direct proof

Theorem 3:

Let n ∈ Z. Then $n^2 + n$ is even. We can prove this by writing the expression as $n(n + 1)$.

It is obvious that one of the factors is even and one is odd. So, the product will be even.

By cases: if n is even, then $(n + 1)$ will be odd , as result the product is even.

If n is odd, then n + 1 will be even, still the product will be eve

Example 1.2 Direct proof

I am referring to the theorem for sum and Difference Rules of differentiating.

Theorem 4: The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of the sum f+g (or difference f-g) is the sum (difference) of the derivatives of f and g (It is important to state as under minding that they both are functions of the same variable x).

This Theorem has two parts that can be proven in similar way. It is no necessary to prove them both. Sometimes when the theorem is like this, I let students pick the case that they feel they need the most. I do that for practice examples as well. Giving them a set of practice problems, we discuss the ones that they choose.

Let say students wanted to do the proof for the difference. (Signs uncertainty).

- Start with the assumption that f and g are differentiable.
- By the definition we can write:

$$\frac{d}{dx}\left(f(x)\right) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$\frac{d}{dx}\left(g(x)\right) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

Now we can write the derivative of the sum or difference of functions and use the limit definition to continue with the reasoning to prove the theorem.

$$\frac{d}{dx}\left(f(x) - g(x)\right) = \lim_{\Delta x \to 0} \frac{[f(x+\Delta x) - g(x+\Delta x)] - [f(x) - g(x)]}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - g(x+\Delta x) - f(x) + g(x)}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x) - g(x+\Delta x) + g(x)}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{[f(x+\Delta x) - f(x)] - [g(x+\Delta x) - g(x)]}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} - \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} =$$

$$\frac{d}{dx}\big(f(x)\big) - \frac{d}{dx}\big(g(x)\big)$$

Or alternatively: $(f(x) - g(x))' = f'(x) - g'(x)$

.

One of the direct proofs is based on Mathematical Induction Principle or method.

This method is mainly used to prove formulas especially the summations of n terms in a sequence, or the generalization of n-th term in an arithmetic, geometric or any other sequences.

It is very typical that we can prove the sum of first n terms of the counting numbers sequence, or squares of first n counting numbers etc.

I will focus at another example:

Theorem 5: Prove that the sum of 2k-1 first odd integers equals to the square of k. Where k is a positive integer.

Or: Prove that $1 + 3 + 5 + 7 + 9 + \cdots + (2k - 1) = k^2$ where n is a positive integer.

The First Principle of Mathematical Induction

Let P(n) be defined for all integers n. Let a be an arbitrary but fixed integer. Suppose the following statements are true.

1. P (1) is true. That is, we show that the given property is true for n = 1. **This is the base case**
2. Next, we must assume that this property P(k) is true for an arbitrary integer k where k > 1. **This is the inductive hypothesis**
3. Finally, we must show that this property is true for the next element k + 1 if P(k) is true then P (k + 1) is true.

Let's prove by using the Mathematical Induction Principle that

$1 + 3 + 5 + 7 + 9 + \cdots + (2k - 1) = k^2$ for any arbitrary, but fixed positive integer k.

1) We check if P (1) is true. That is obvious that in that sequence the first term is 1.
2) We assume that this property (formula) holds for k. for any integer k>1.
   That means that we are assuming that: $1 + 3 + 5 + 7 + 9 + \cdots + (2k - 1) = k^2$ is true.
3) Finally, we must show that this property is true for the next element k + 1 if P(k) is true then P (k + 1) is true.

We need to prove that:

$1 + 3 + 5 + 7 + 9 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2$

We can write the left-hand sum above as:

$[\mathbf{1+3+5+7+9}+\cdots+(\mathbf{2k-1})]+(2\mathrm{k}+1)=\mathrm{k}^2$ +(k+1) = $(\mathrm{k}+1)^2$.

We proved that if this formula is true for k terms, then it is true for the k+1 as well.

It is important to highlight that to prove for k+1, you just add on the k+1 term. But also, you need to know what you are expecting by substituting in the original formula the k+1. If they match, that means that the proving is correct. You may also be asked to check if the formula holds. In that case, you need to pay closer attention to procedures. It may be not a true one.

2. **Any other method than the direct proof is considered indirect proofs. It is impossible to tackle all of them, but I will try to discuss some of them**.

2.a. **Contrapositive Proof**

Sometimes the direct proof is very challenging and time consuming, and it incorporates many other theorems to take under consideration to prove it. For that reason, we use different approach of proving it. The contrapositive of an implication $p \to q$ is the implication $p' \to q'$

Example 2.1.

Theorem 3: $\forall x \in R$, if x^3 is irrational, then x is irrational.

Direct proof is not easy to use. The given information is too little to get started to do the proving.

We start with the assumption:

Let x be an arbitrary real number. Then, we establish the contrapositive implication:

If x is not irrational, then x^3 is not irrational. And prove this implication.

This implication can be translated differently as:

If x is rational, then x^3 *is rational as well.*

So, we suppose that x is rational, which means that x can be written as a fraction (ratio). $x=\frac{p}{q}$ where p, q are integers.

We can write: $x^3=\frac{p^3}{q^3}$; Since x^3 *is written as a quotient of integers, we can say that it is rational number.*

So, we proved that if we consider x to be rational, then definitely x^3 *will be rational.* This result came from assuming that x is a rational number.

2.b. Proof by contradiction

In the method of proving by contradiction, we assume that the hypothesis p is true, and that the conclusion q is false.

So, we establish the implication $p \to q$, by proving wrong $p \to q'$ using all the reasoning, postulates and other theorems.

Example 2.b.1:

Theorem 4: $\forall n \in Z, if\ n^2$ is odd, then n is odd.

We want to prove the theorem by contradiction. That means that we assume that n^2 is odd, and the conclusion "n is odd" is false. Meaning that we assume that n is even.

The fact that n is even means that we can write it as $n = 2k$

If $n = 2k$, then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$

This means that n^2 is even. Which contradicts the true hypotheses that n^2 is odd.

This establishes the fact that:$\forall n \in Z, if\ n^2$ is odd, then n is odd. (Theorem above)

This method is largely used in Discrete Mathematics especially in Logic.

2.c. Proofs by cases

This method is used when it is obvious that you must consider different situations and cases.

Example 2.c 1: Let x be an arbitrary real number $x \in R$, and $f(x) = |x|$ be a function of x.

Theorem 5:

Prove that, $\forall x \in R_{,}$ the inequality (1) $x \leq f(x)$ holds.

Since $f(x) = |x|$ the inequality (1) translates into $x \leq |x|$ for $\forall x \in R$.

Obviously since we know that there is no negative value as outcome for this function, we need to consider the inputs in three parts.

1- If $x = 0$ $f(x) = |x|$ so, $f(0) = |0| = 0;\ 0 \leq 0$. $x \leq f(x)$ holds
2- $\forall x \in R^-$ $f(x) = |x|$, Since $x < 0$, $|x| = -x > 0$. But $x < 0$, so $x < |x|\ or\ x < f(x)\ In\ general$ we can say that $x \leq f(x)\ holds$
3- $\forall x \in R^+$ In this case $x = |x|$, so the inequality $x \leq f(x)\ holds$

2.d Equivalence proof

Some of the theorems are in the form "If and only if" or biconditional.

In logic we write that statement as $p \leftrightarrow q$ which is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$.

So, $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$.

This means that we need to prove that "If p, then q" and "If q, then p".

Example 2.d.1:

Theorem 6: The product of two integers a and b is an even integer if and only if at least one of them is an even integer.

Let p be: At least one of the integers a, or b is even

Let q be: The product $a \cdot b$ is even

a. Let's prove the implication $p \rightarrow q$.

Let say only a is an even integer, and b is an odd integer. We can write: $a = 2k_1,\ b = 2k_2 + 1$

Then the product of integers a and b will be: $a \cdot b = (2k_1) \cdot (2k_2 + 1)$

$2 \cdot [k, (2k_2 + 1)]$ so, the product is even number ($k, (2k_2 + 1) =$ k)

So, $a \cdot b = 2$k. Similarly, we can consider a an odd, and b an even integer and prove in the same way. If a and b are both even the, the prof is even easier but pretty like the one shown.

b. Let's prove the implication $q \to p$

So, let the product $a \cdot b$ be an even integer. And let's prove that one of the factors is an even integer. The fact that the product of a and b is even can be written as: $a \cdot b = 2$k . Automatically, we can say that the factor 2 is withing the a, or withing the b (either of them) or in both. Our theorem is proven.

3. Using Counterexamples to prove the importance of the conditions in a true statement, or to prove the falsity of the statements.

In mathematics theorems and any statement must be proven as "True" or "False" conjectures, so that we can use them in solving problems or applications.

One of the ways of teaching students to be able to prove the "True" statements or disprove a "False" statement is using the counterexamples.

In the case of a true statement given to students, they should be able to criticize and find out whether the theorem, or statement given contains satisfied conditions. They should be able to prove it using the definitions and other theorems or statements proven before. Also, they should be able to find the examples that satisfy the statement, as well as some special cases that for one reason or another they fail the statement.

For example, if we consider the **Statement 1**:

Let *f* be a continuous function on (*a; b*) and, without loss of generality,
let *f*(*a*) < *f*(*b*). Then for every value *y*, where *f*(*a*) < *y* < *f*(*b*), there is a
value *c* in [*a; b*] such that *f*(*c*) = *y*.

Students may be asked:

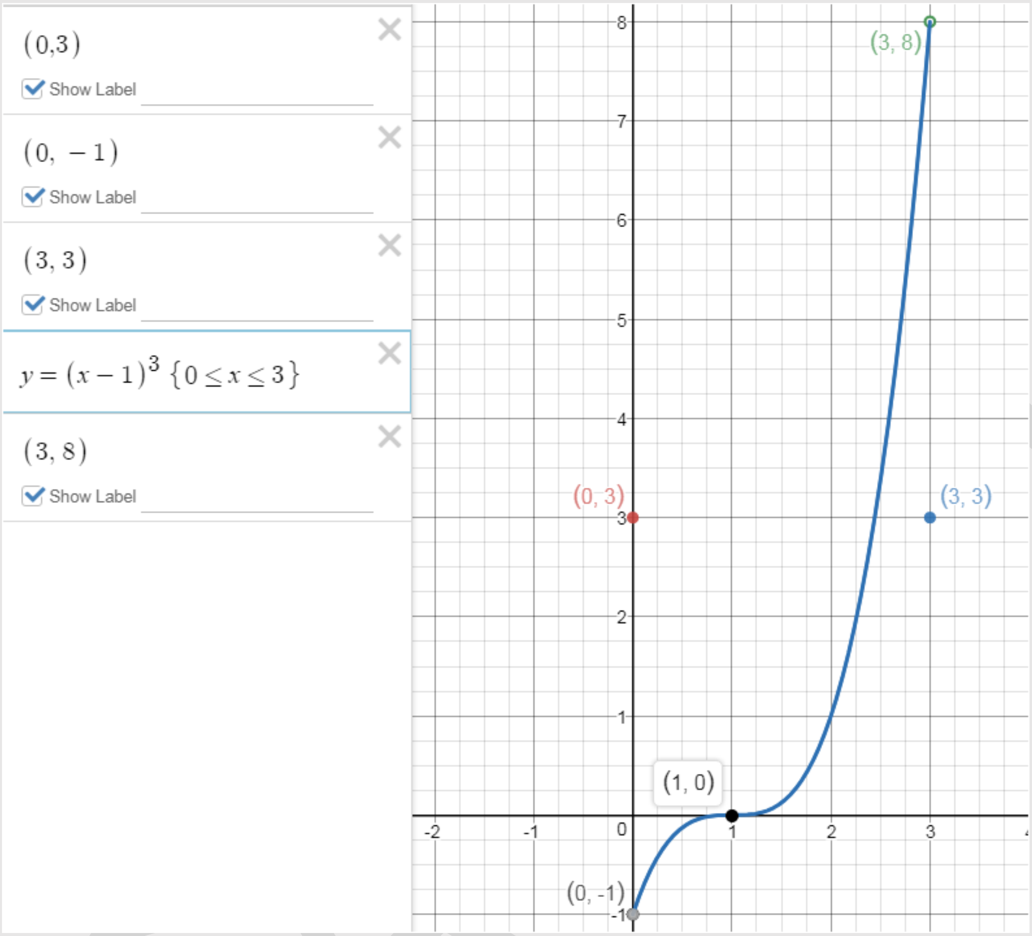
1) Is this statement true?
2) If yes, prove it, if no, find a couple of counterexamples that prove the falsity.

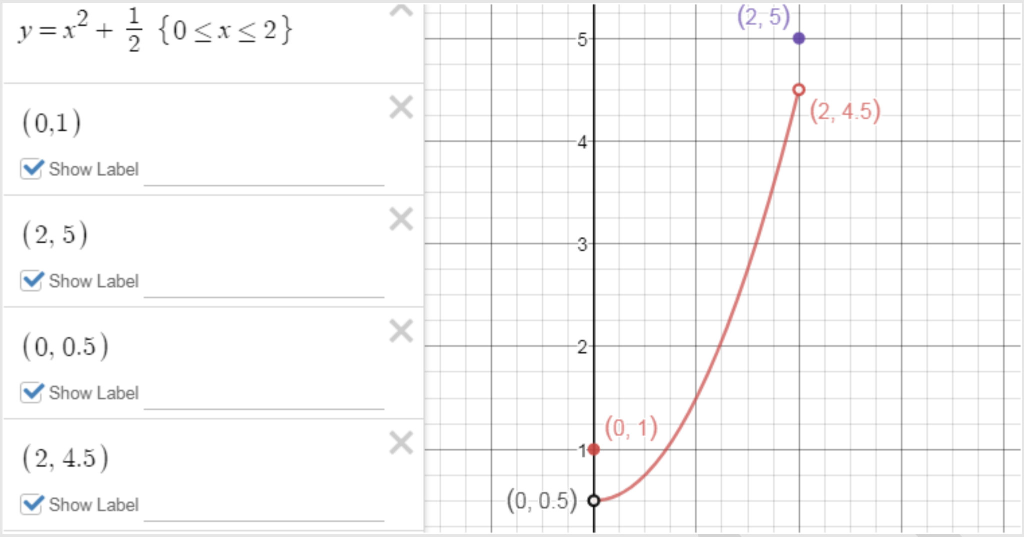
Students can bring the types of counterexamples in the figures below, to prove that the statement is false, and what is missing is that the function must be continued in the closed interval [a, b].

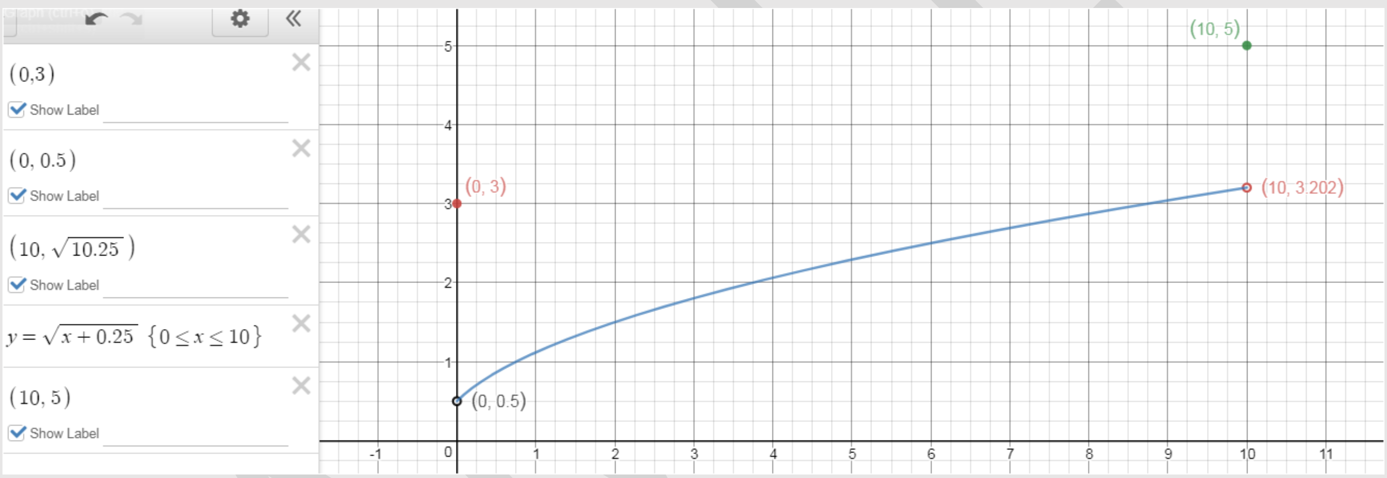
So, the only thing that we need to change to make the statement true is from (a, b) to [a, b].

In the given counterexamples below, we can see the failure of the condition $f(a) < f(b)$ in the extremes of the function, even though the function continues everywhere else without loss of the generality. It is hard for students to drop the "true" value of a statement just because it may not work in one or few cases. (Many times, I ask my students: Is this statement true: Every person in this room is a student at City Tech college? Most of them rush to say: Yes. When I ask them if they are sure, they start thinking more critically. There will be someone in the room that will say: No, because you are not a student, you are a Professor).

Some Counterexamples that make the Statement 1 not true or "False":




$y = x^2 + \frac{1}{2} \; \{0 \le x \le 2\}$
(0,1)
Show Label
(2, 5)
Show Label
(0, 0.5)
Show Label
(2, 4.5)
Show Label
(2, 5)
(2, 4.5)
(0, 1)
(0, 0.5)
1
2
3
4
5


(0,3)
Show Label
(0, 0.5)
Show Label
$(10, \sqrt{10.25})$
Show Label
$y = \sqrt{x + 0.25} \; \{0 \le x \le 10\}$
(10, 5)
Show Label
(10, 5)
(0, 3)
(10, 3.202)
(0, 0.5)
-1
0
1
2
3
4
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8
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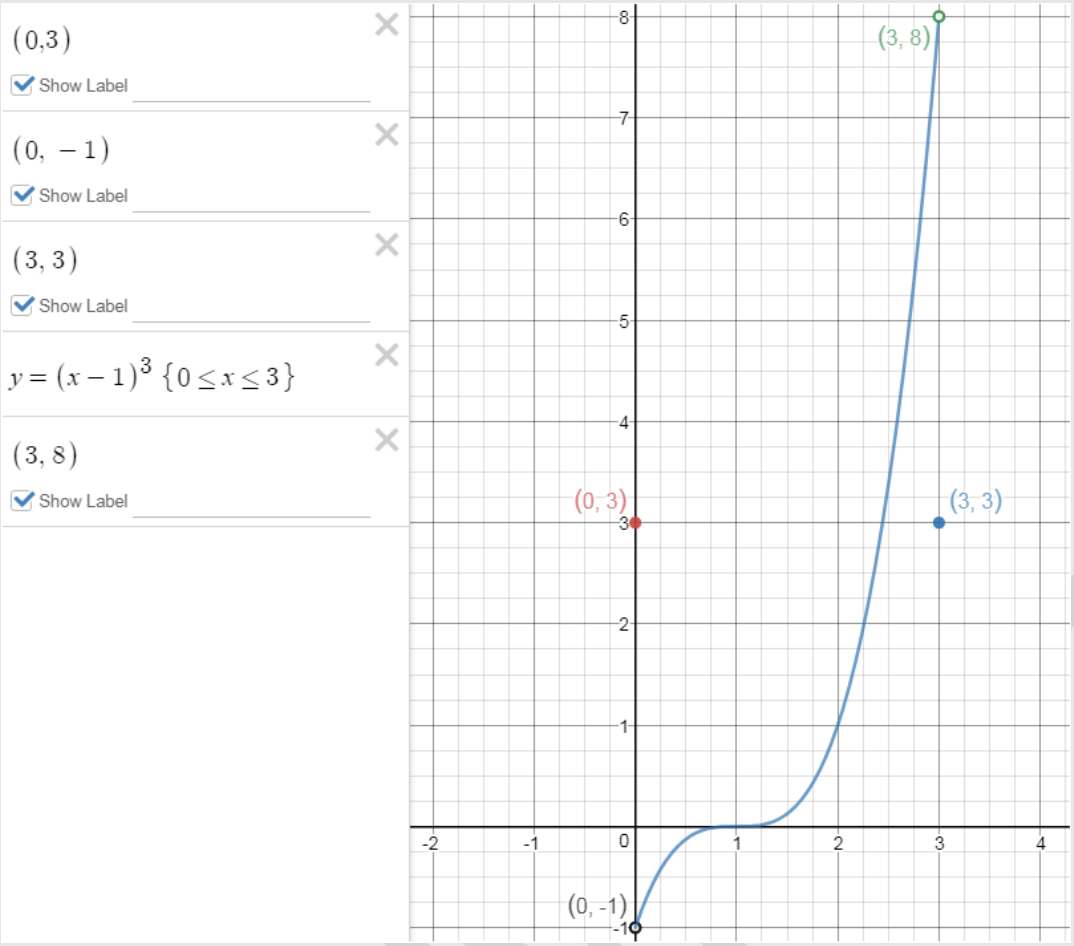
11



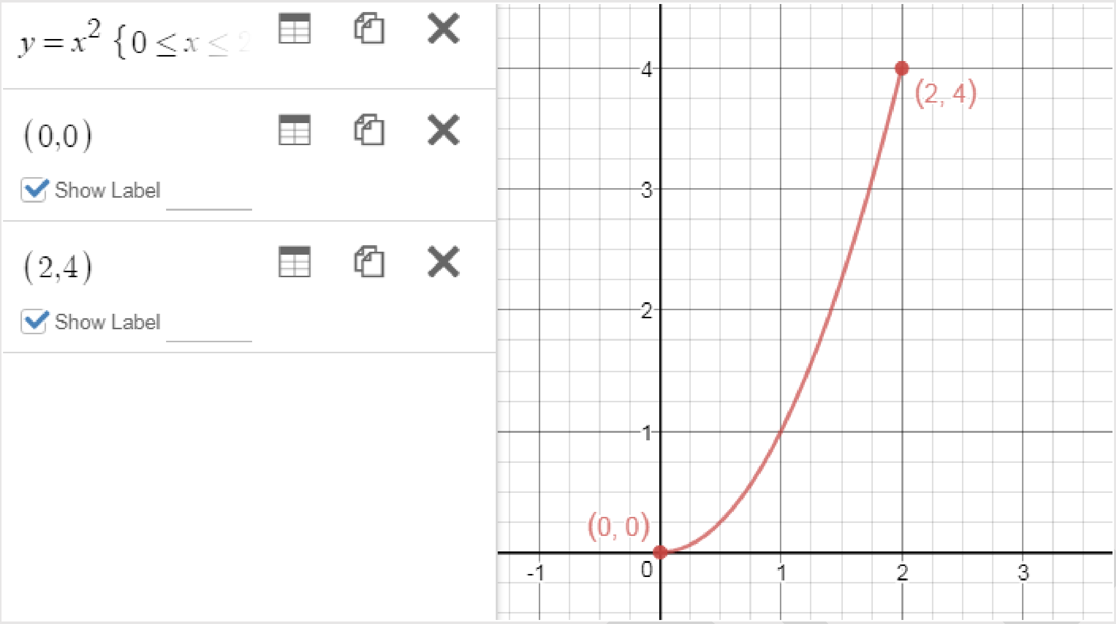
Statement 2: (Intermediate Value Theorem)

Let *f* be a continuous function on [*a; b*] and, without loss of generality, let $f(a) < f(b)$. Then for every value *y*, where $f(a) < y < f(b)$, there is a value *c* in [*a; b*] such that $f(c) = y$.

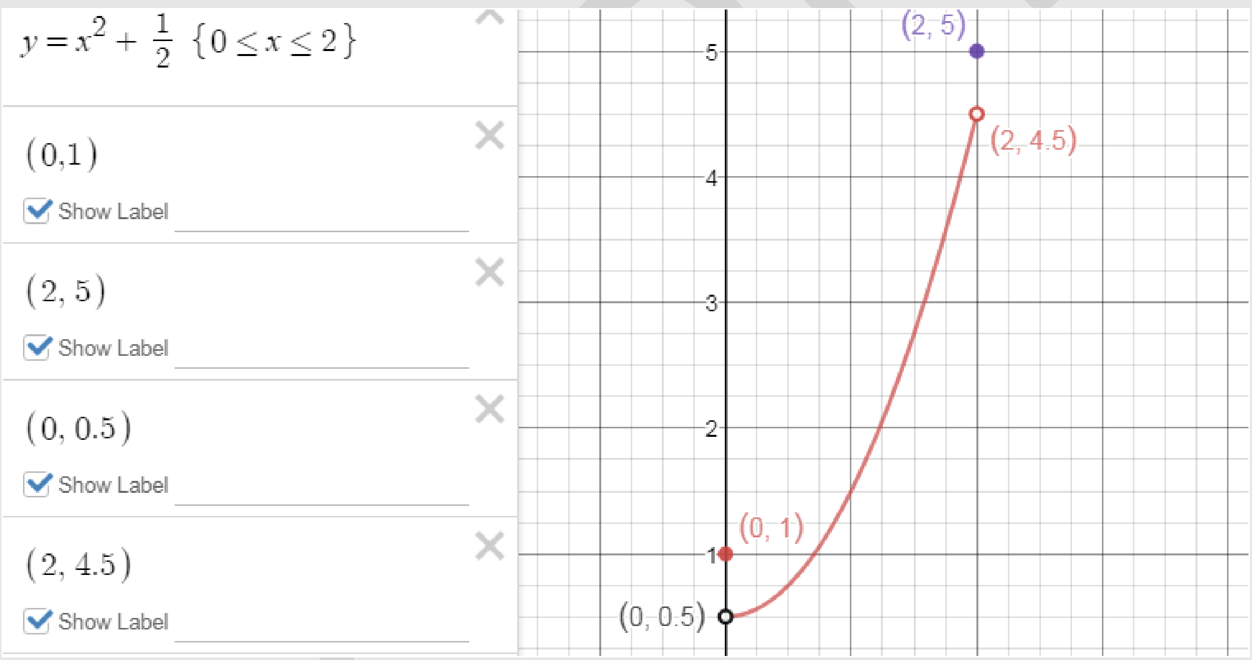
Let say students proved that this statement is true. It is important that students can determine whether a function satisfies or fails the statement. It is important that we ask them to bring examples that satisfy the whole statement regarding every condition and bring some examples (counterexamples) that fail the statement for one or more conditions of the statement.

Students may bring as example that satisfies the statement. Or examples that highlight the failing of the conditions in which the statement is true (Theorem).

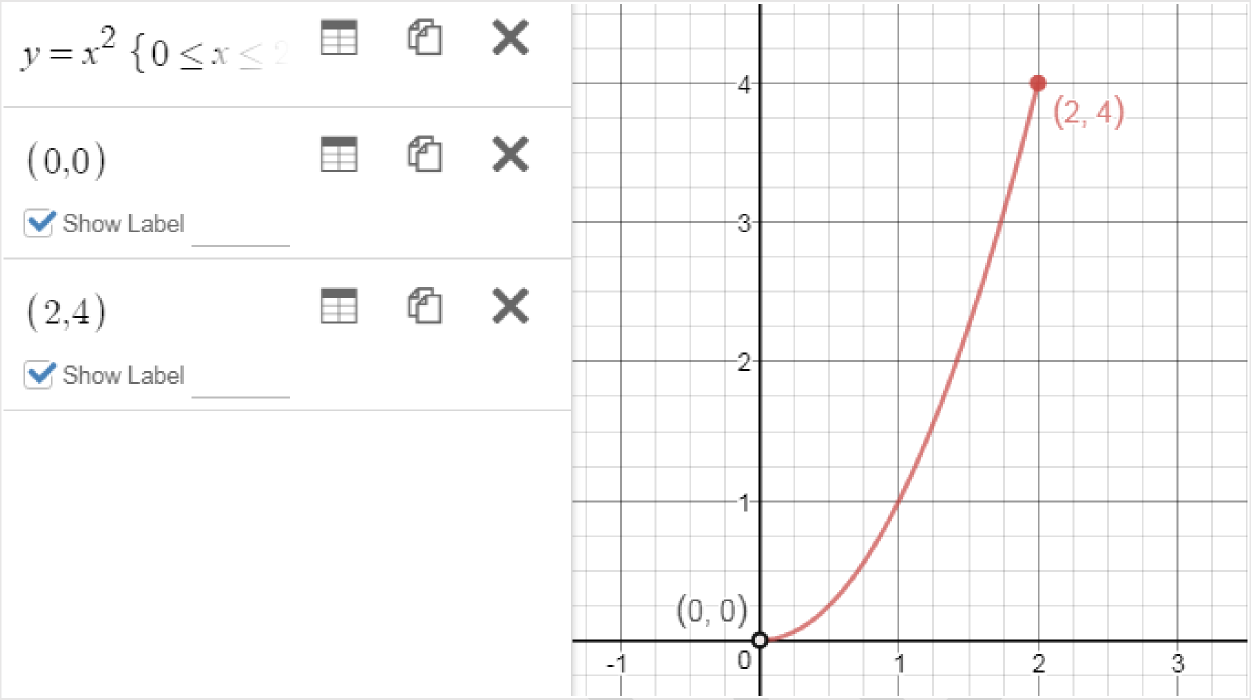
I will let you think about the examples given below.

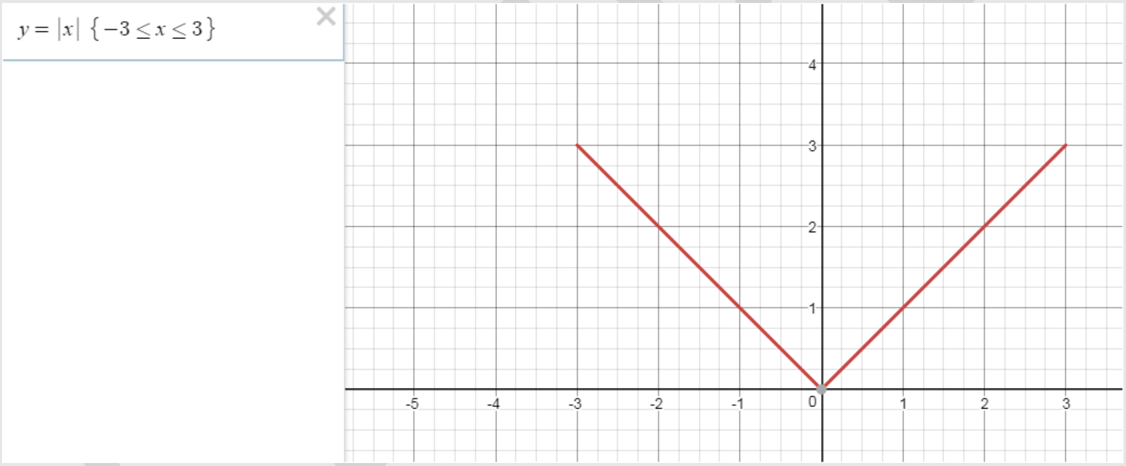


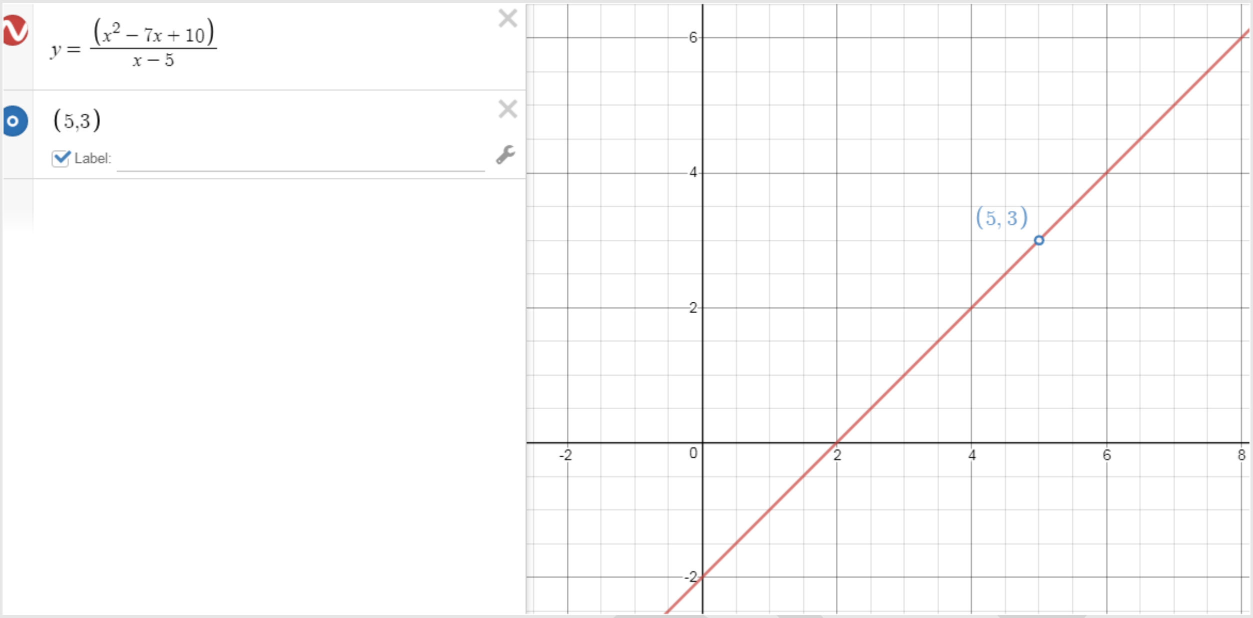
y = x^2 {0 ≤ x ≤
(0,0)
Show Label
(2,4)
Show Label
(2, 4)
(0, 0)
4
3
2
1
-1
0
1
2
3



y = x^2 + 1/2 {0 ≤ x ≤ 2}
(0,1)
Show Label
(2, 5)
Show Label
(0, 0.5)
Show Label
(2, 4.5)
Show Label
(2, 5)
(2, 4.5)
(0, 1)
(0, 0.5)
5
4
3
2
1


$y = x^2 \{0 \le x \le 2$
(0,0)
Show Label
(2,4)
Show Label
(2, 4)
(0, 0)
4
3
2
1
-1
0
1
2
3


$y = |x| \{-3 \le x \le 3\}$
4
3
2
1
-5
-4
-3
-2
-1
0
1
2
3



In my experience and belief, creating their own counterexamples help students think critically, understand the concepts and explore furthermore. (This is a statement that needs to be proven as well……).

References

Larson, Ron; Eduards, Bruce, (2016). *Book Title. Calculus*, Pennsylvania State University, University of Florida: Publisher: Cengage Learning

Johnsonbaugh, Richard, Eighth Edition, *Book Title: Discrete Mathematics, DePaul University, Chicago*

Footnotes

[1][I have used here many of the textbooks such as Calculus, Precalculus, Algebra, Discrete Mathematics etc. I have read a lot of articles and researching papers. I also, have use my long-time experience of teaching different levels of Mathematics.*)*]

Here are only a few links to the resources used:

https://en.wikipedia.org/wiki/Timeline_of_mathematics
https://www.sciencedaily.com/releases/2020/03/200325110855.htm
https://www.cs.sfu.ca/~ggbaker/zju/math/proof.html#disprove

https://mathbitsnotebook.com/Geometry/BasicTerms/BTproofs.html

https://en.wikipedia.org/wiki/Mathematical_proof

http://web.mnstate.edu/jamesju/Fall2015/Content/M311ProofsIntro.pdf https://openstax.org/books/calculus-volume-1/pages/2-4-continuity

https://www.birmingham.ac.uk/Documents/college-eps/college/stem/Student-Summer-Education-Internships/Proof-and-Reasoning.pdf