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**APPROXIMATION BY THE  $K^\lambda$  MEANS OF  
FOURIER SERIES AND CONJUGATE SERIES  
OF FUNCTIONS IN  $H_{\alpha,p}$**

*Dedicated to Academician Professor Gradimir Milovanović  
on the occasion of his 70th birthday.*

*Ben Landon, Holly Carley, and R. N. Mohapatra\**

In this paper, we consider two operators based on the  $K^\lambda$  means of the Fourier series of and conjugate series of functions of class  $L^p$ ,  $p > 1$ . We study the degree of convergence of these operators to the functions whose Fourier series and Allied series are used.

**1. INTRODUCTION**

Various summation methods have been used to make sense of divergent series (see, for example, [5, 6]). These summation methods transform the series into a convergent series and the sum of the divergent series is then defined to be the sum of this associated series. The method is called *regular* if that method agrees with the standard sum on convergent series. Applying a summation method may still be advantageous for a convergent sum as it may speed up the convergence. Here we shall determine the degree of convergence of certain means of the Fourier series and conjugate series of a function  $f$  to itself in  $H_{\alpha,p}$ . We define numbers  $\begin{bmatrix} n \\ m \end{bmatrix}$  by

$$\prod_{\nu=0}^{n-1} (x + \nu) = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} x^m,$$

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where,

$$\prod_{\nu=0}^{n-1} (x + \nu) = \frac{\Gamma(x+n)}{\Gamma(x)}.$$

We will use the conventions  $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$ . The numbers  $\begin{bmatrix} n \\ m \end{bmatrix}$  are known as the absolute values of the Stirling numbers of the first kind. Let  $\lambda > 0$ . The  $K^\lambda$  mean  $K_n^\lambda(t, x)$  of a sequence  $\{t_n(x)\}$  is defined by

$$K_n^\lambda(t, x) := \frac{\gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k t_k(x).$$

If  $K_n^\lambda(t, x) \rightarrow s$  as  $n \rightarrow \infty$ , we say that the sequence  $\{t_n(x)\}$  is summable  $K^\lambda$  to  $s$ . The method  $K^\lambda$  is regular for  $\lambda > 0$  and this case will be supposed in this paper.

Let  $C_{2\pi}$  be the space of all  $2\pi$ -periodic functions defined on  $[0, 2\pi]$  and for  $0 < \alpha \leq 1$  define the Hölder continuous functions

$$H_\alpha := \{f \in C_{2\pi} : \exists M > 0 \forall x, y, |f(x) - f(y)| \leq M|x - y|^\alpha\}.$$

For  $0 < \alpha \leq 1$ ,  $H_\alpha$  is a Banach space [10] under the norm

$$\|f\|_\alpha := \|f\|_\infty + \sup_{x \neq y} \Delta^\alpha f(x, y),$$

where,

$$\Delta^\alpha f(x, y) := \frac{|f(x) - f(y)|}{|x - y|^\alpha}, \quad x \neq y$$

and

$$\Delta^0 f(x, y) := 0.$$

Alextis [1] studied the degree of approximation of the functions in  $H_\alpha$  class by the Cesàro means of their Fourier series in the sup-norm. It was Prössodoorf [11] who initiated the work on the degree of approximation of functions in  $H_\alpha$  by the Féjer mean of the Fourier series in the Hölder metric. This result has been generalized by Chandra [2], Mohapatra and Chandra [9], Singh [13], [14] using different methods. Chandra [2], [3] has studied the degree of approximation problem in the Hölder metric using Borel and Euler means. The degree of approximation of a function  $f$  in  $H_\alpha$  has been studied by Das, Ghosh, and Ray [4] by using (e,c) means of Fourier Series in the Hölder metric.

Let

$$K_n^\lambda(f, x) = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k S_k(f, x).$$

and

$$\tilde{K}_n^\lambda(f, x) = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \tilde{S}_k(f, x)$$

where,  $S_n(f, x)$  and  $\tilde{S}_n(f, x)$ , respectively, denote the  $n$  th partial sums of the series

$$f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

and the conjugate

$$f \sim \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx).$$

The  $K^\lambda$  means were first introduced by Karamata [7]. Lototsky [8] reintroduced the special case  $\lambda = 1$ . Vuckovic [15] first studied the  $K^\lambda$  summability of the Fourier series. Sadangi [12] proved the following two theorems:

**Theorem 1.1 (A).** *Let  $K_n^\lambda(f, x)$  be the  $K^\lambda$  mean of the Fourier series of  $f$  at  $x$ . If*

$$0 \leq \beta < \alpha \leq 1 \text{ and } f \in H_\alpha \text{ then}$$

$$\|K_n^\lambda(f, \cdot) - f\|_\beta = O(1) \frac{\log \log n}{(\log n)^{\alpha-\beta}}$$

**Theorem 1.2 (B).** *Let  $h = \frac{\pi}{l(n)}$ , where  $l(n) = \frac{3}{2} + \lambda \sum_{k=1}^{n-1} \frac{1}{\lambda+k}$ . Let  $\tilde{K}_n(f, x)$  be the  $K^\lambda$  mean of the series conjugate to the Fourier series of  $f$  at  $x$ . If  $0 \leq \beta < \alpha \leq 1$  and  $f \in H_\alpha$ , then*

$$\|\tilde{K}_n^\lambda(f, \cdot) - \tilde{f}(\cdot, h)\|_\beta = O(1) \frac{\log \log n}{(\log n)^{\alpha-\beta}}, 0 \leq \beta < \alpha \leq 1$$

Here

$$\tilde{f}(x, \varepsilon) = -\frac{2}{\pi} \int_\varepsilon^\pi \psi_x(t) \frac{1}{2} \cot \frac{1}{2}t dt$$

$$\tilde{f}(x) = \lim_{\varepsilon \rightarrow 0^+} \tilde{f}(x, \varepsilon),$$

and

$$\psi_x(t) = \frac{1}{2} \{f(x+t) - f(x-t)\}$$

## 2. PRELIMINARY DEFINITIONS

Let  $L_p[0, 2\pi]$  be the space of all  $2\pi$ -periodic integrable functions.

$$H_{\alpha,p} := \left\{ f \in L_p[0, 2\pi] : \left( \int_0^{2\pi} |f(x+t) - f(x)|^p dx \right)^{\frac{1}{p}} \leq K |t|^\alpha \right\},$$

where  $K$  is a positive constant. The space  $H_{\alpha,p}$  ( $p > 1, \alpha \leq 0 < 1$ ) is a Banach space under the norm  $\|\cdot\|_{\alpha,p}$ :

$$\|f\|_{\alpha,p} := \|f\|_p + \sup_{t \neq 0} \frac{\|f(y+t) - f(y)\|_p}{|t|^\alpha}.$$

The metric induced by the norm  $\|\cdot\|_{\alpha,p}$  on  $H_{\alpha,p}$  is called Holder continuous with degree  $p$ . It can be seen that

$$\|f\|_{\beta,p} \leq (2\pi)^{\alpha-\beta} \|f\|_{\alpha,p}.$$

Since  $f \in H_{\alpha,p}$  if and only if  $\|f\|_{\alpha,p} < \infty$ , we have,

$$L_p[0, 2\pi] \supseteq H_{\beta,p} \supseteq H_{\alpha,p}, \quad p > 1, \quad 0 \leq \beta < \alpha \leq 1.$$

### 3. MAIN RESULTS

In this paper we will prove the following two theorems:

**Theorem 3.3.** *Let  $K_n^\lambda(f, x)$  be the  $K^\lambda$  mean of the Fourier series of  $f$  at  $x$ . If  $p > 1, 0 \leq \beta < \alpha \leq 1$ , and  $f \in H_{\alpha,p}$ , then*

$$(1) \quad \|K_n^\lambda(f, \cdot) - f\|_{\beta,p} = O\left(\frac{\log \log n}{(\log n)^{\alpha-\beta}}\right)$$

**Theorem 3.4.** *Let  $h = \frac{\pi}{l(n)}$ , where  $l(n) = \frac{3}{2} + \lambda \sum_{k=1}^{\lambda-1} \frac{1}{\lambda+k}$ . Let  $\tilde{K}_n(f, x)$  be the  $K^\lambda$  mean of the series conjugate to the Fourier series of  $f$  at  $x$ . If  $0 \leq \beta < \alpha \leq 1$  and  $f \in H_{\alpha,p}$ , then*

$$(2) \quad \|\tilde{K}_n(f, x) - \tilde{f}(\cdot, h)\|_{\beta,p} = O\left(\frac{\log \log n}{(\log n)^{\alpha-\beta}}\right)$$

4. ADDITIONAL NOTATIONS AND LEMMAS

We use the following additional notations:

$$\begin{aligned}
 G(t) &= \varphi_{y+u}(t) - \varphi_y(t) \\
 \tilde{G}(t) &= \psi_{y+u}(t) - \psi_y(t) \\
 L_n(y) &= K_n^\lambda(f, y) - f(y) \\
 \tilde{L}_n(y) &= \tilde{K}_n^\lambda(f, y) - \tilde{f}(y, h) \\
 \varphi_x(t) &= \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\} \\
 \tilde{E}_n(t) &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^n \binom{n}{k} \lambda^k \tilde{D}_k(t) \\
 \rho_k(t) &= (\lambda^2 + 2\lambda k \cos t + k^2)^{1/2} \\
 \tan \theta_k &= \frac{\lambda \sin t}{\lambda \cos t + k} \\
 R(n, t) &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \prod_{k=0}^{n-1} \rho_k(t) \\
 l(n) &= \frac{3}{2} + \lambda \sum_0^{n-1} \frac{1}{\lambda + k} \\
 h &= \frac{\pi}{l(n)}
 \end{aligned}$$

We need the following lemmas for the proof of our theorems.

**Lemma 4.5.** *Let  $0 \leq \beta < \alpha \leq 1$ . If  $f \in H_{\alpha,p}$  then for  $0 < t \leq \pi$  and  $u \neq 0$*

$$G(t) = O(1) \begin{cases} t^\alpha \\ |u|^a \\ |u|^\beta t^{\alpha-\beta} \end{cases}, \quad \tilde{G}(t) = O(1) \begin{cases} t^\alpha \\ |u|^a \\ |u|^\beta t^{\alpha-\beta} \end{cases}$$

*Proof.* The first and second estimates of (1) follow from the definition of  $H_{\alpha,p}$  and  $\varphi_x(t)$ . Writing

$$|G(t)| = |G(t)|^{1-\frac{\beta}{\alpha}} |G(t)|^{\frac{\beta}{\alpha}}$$

and using the first two estimates of (1) we can derive the third one. Proof of (2) is similar to that of (1). □

**Lemma 4.6.** *Let  $0 \leq \beta < \alpha \leq 1$ . If  $f \in H_{\alpha,p}$  then for  $x \neq y$*

1.

$$G(t+h) - G(t) = O(1) \begin{cases} h^\alpha \\ |u|^a \\ |u|^\beta h^{\alpha-\beta} \end{cases}$$

2.

$$\tilde{G}(t+h) - \tilde{G}(t) = O(1) \begin{cases} h^\alpha \\ |u|^a \\ |u|^\beta h^{\alpha-\beta} \end{cases}$$

*Proof.* Writing

$$\begin{aligned} G(t+h) - G(t) &= \frac{1}{2} [f(y+u+t+h) - f(y+u+t)] + [f(y+u-t-h) - f(y+u-t)] \\ &\quad + [f(y+t+h) - f(y+t)] + [f(y-t-h) - f(y-t)] \end{aligned}$$

and using the fact that  $f \in H_{\alpha,p}$  we obtain the first estimate of (1). The remaining part of the proof is similar to that of Lemma 4.5 and hence it is omitted.  $\square$

**Lemma 4.7.** *Suppose that  $A$  and  $c$  are both positive constants. Let  $\gamma$  be any real number. Then as  $\mu \rightarrow \infty$ ,*

$$\int_{\frac{\pi}{\mu}}^c t^\gamma e^{-A\mu t^2} dt = O(1) \begin{cases} \mu^{-\gamma-1}, & \gamma < -1 \\ \log \mu, & \gamma = -1 \\ \frac{1}{\mu^k}, & 2k-1 < \gamma \leq 2k, \\ \frac{1}{\mu^{\gamma-k}}, & 2k \leq \gamma \leq 2k+1 \end{cases}, k = 0, 1, 2, 3, \dots$$

**Lemma 4.8.** *We have*

$$\begin{aligned} K_n(t) &= R(n, t) \sin \left( \frac{3}{2}t + \sum_{k=1}^{n-1} \theta_k \right) \\ \tilde{K}_n(t) &= R(n, t) \cos \left( \frac{3}{2}t + \sum_{k=1}^{n-1} \theta_k \right) \end{aligned}$$

*Proof.* By simple computation, we obtain

$$\begin{aligned}
 \tilde{K}_n(t) + iK_n(t) &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^n \binom{n}{k} (\lambda e^{it})^k e^{\frac{1}{2}it} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} e^{\frac{1}{2}it} \prod_{k=0}^{n-1} (\lambda e^{it} + k) \\
 &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} e^{\frac{1}{2}it} \prod_{k=0}^{n-1} [(\lambda \cos t + k) + i\lambda \sin t] \\
 &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} e^{\frac{1}{2}it} \prod_{k=0}^{n-1} \rho_k(t) e^{i\theta_k} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} R(n,t) \exp \left\{ i \left( \frac{3}{2}t + \sum_{k=1}^{n-1} \theta_k \right) \right\}
 \end{aligned}$$

from which the lemma follows. □

**Lemma 4.9.** *Let  $0 < t < \pi$ . Then for some positive constant  $A$ ,*

$$1. R(n,t) = \begin{cases} O(1) \\ O(1)e^{-At^2 \log n} \end{cases}$$

$$2. K_n(t) = \begin{cases} O(1) \\ O(1)e^{-At^2 \log n} \end{cases}$$

$$3. \tilde{K}_n(t) = \begin{cases} O(1) \\ O(1)e^{-At^2 \log n} \end{cases}$$

$$4. \tilde{E}_n(t) = O(t^{-1})$$

*Proof.*  $R(n,t)$  attains its maximum value for  $t = 0$  and it is easy to see that  $R(n,0) = 1$  and this ensures the first estimate of 1. Now



$$\begin{aligned}
 (3) \quad R(n, t) &= \frac{\Gamma(\lambda)}{\Gamma(n + \lambda)} \prod_{k=0}^{n-1} (\lambda^2 + 2\lambda k \cos t + k^2)^{\frac{1}{2}} \\
 (4) \quad &= \frac{\Gamma(\lambda)}{\Gamma(n + \lambda)} \prod_{k=0}^{n-1} (\lambda + k) \left[ 1 - \frac{4k \sin^2 \frac{1}{2}t}{(\lambda + k)^2} \right]^{\frac{1}{2}} \\
 (5) \quad &= \prod_{k=0}^{n-1} \left[ 1 - \frac{4\lambda \sin^2 \frac{1}{2}t}{(\lambda + k)^2} \right]^{1/2} \\
 (6) \quad &= \exp \left[ -\frac{1}{2} \sum_{k=1}^{n-1} \log \left\{ 1 - \frac{4\lambda \sin^2 \frac{1}{2}t}{(\lambda + k)^2} \right\}^{-1} \right]
 \end{aligned}$$

At this stage, we observe that

$$0 < \frac{4\lambda k \sin^2 \frac{1}{2}t}{(\lambda + k)^2} < 1$$

for  $k = 1, 2, 3, \dots$  and  $0 < t < \pi$ . As  $\log(1 - \theta)^{-1} \geq \theta$  for  $0 < \theta < 1$  and  $\sin \geq \frac{2x}{\pi}, 0 \leq x \leq \frac{\pi}{2}$ , we have

$$\begin{aligned}
 (7) \quad \sum_{k=1}^{n-1} \log \left\{ 1 - \frac{4\lambda \sin^2 \frac{1}{2}t}{(\lambda + k)^2} \right\}^{-1} &\geq \sum_{k=1}^{n-1} \frac{4\lambda k \sin^2 \frac{1}{2}t}{(\lambda + k)^2} \\
 &\geq \frac{2\lambda t^2}{\pi^2} \sum_{k=1}^{n-1} \frac{k}{(\lambda + k)^2} \\
 &\geq At^2 \log n
 \end{aligned}$$

where  $A$  is some positive constant. Using (6) and (7) we obtain the second estimate of Lemma 4.9, and 3 follows. As  $\tilde{D}_k(t) = O(t^{-1})$  4 follows at once.  $\square$

**Lemma 4.10.** *Let  $0 < t \leq \frac{\pi}{4}$ . Then*

1.  $\sin \left( \frac{3}{2}t + \sum_{k=1}^{n-1} \theta_k \right) - \sin l(n)t = O(t^3 \log n)$
2.  $\cos \left( \frac{3}{2}t + \sum_{k=1}^{n-1} \theta_k \right) - \cos l(n)t = O(t^3 \log n)$

*Proof.* We have

$$(8) \quad \left| \sin \left( \frac{3}{2}t + \sum_{k=1}^{n-1} \theta_k \right) - \sin l(n)t \right| \leq \left| \left( \frac{3}{2}t + \sum_{k=1}^{n-1} \theta_k \right) - l(n)t \right|.$$

Next, we note that

$$0 < \frac{\lambda \sin t}{\lambda \cos t + k} < 1$$

whenever  $0 < t \leq \frac{\pi}{4}$  and  $k \geq 1$ . Thus for  $0 < t \leq \frac{\pi}{4}, 1 \leq k \leq n-1$

$$\begin{aligned} \theta_k &= \left[ \tan^{-1} \frac{\lambda \sin t}{\lambda \cos t + k} - \frac{\lambda \sin t}{\lambda \cos t + k} \right] + \left[ \frac{\lambda \sin t}{\lambda \cos t + k} - \frac{\lambda t}{\lambda \cos t + k} \right] \\ &\quad + \left[ \frac{\lambda t}{\lambda \cos t + k} - \frac{\lambda t}{\lambda + k} \right] + \frac{\lambda t}{\lambda + k} \\ &= O \left[ \left( \frac{\lambda \sin t}{\lambda \cos t + k} \right)^3 \right] + O \left[ \frac{t^3}{\lambda \cos t + k} \right] + O \left[ \frac{t^3}{(\lambda \cos t + k)(\lambda + k)} \right] + \frac{\lambda t}{\lambda + k} \\ &= O \left( \frac{t^3}{k^3} \right) + O \left( \frac{t^3}{k^2} \right) + O \left( \frac{t^3}{k} \right) + \frac{\lambda t}{\lambda + k} \\ &= \frac{\lambda t}{\lambda + k} + O \left( \frac{t^3}{k} \right) \end{aligned}$$

It follows that

$$(9) \quad \frac{3}{2}t + \sum_{k=1}^{n-1} \theta_k = \frac{3}{2}t + \lambda t \sum_{k=1}^{n-1} \frac{1}{\lambda + k} + O(t^3) \sum_{k=1}^{n-1} \frac{1}{k} = tl(n) + O(t^3 \log n)$$

Using (9) in (8), we obtain Lemma 4.10 part (1). We omit the proof of (2) as it is similar to that of (1).  $\square$

**Lemma 4.11.** *We have*

1.  $R'(n, t) = O(1)t \log n R(n, t), 0 < t \leq \frac{\pi}{2}$
2.  $R(n, t+h) - R(n, t) = O(1)te^{-At^2 \log n}, h \leq t \leq \frac{\pi}{2}$

*Proof.* We have

$$R(n, t) = \lambda \prod_{k=1}^{n-1} \rho_k(t)$$

and so by logarithmic differentiation,

$$\begin{aligned}
R'(n, t) &= R(n, t) \sum_{k=1}^{n-1} \frac{\rho'_k(t)}{\rho_k(t)} \\
&= R(n, t) \sum_{k=1}^{n-1} \frac{(-\lambda k \sin t)}{(\rho_k(t))^2} \\
&= O(1) t R(n, t) \sum_{k=1}^{n-1} \frac{1}{k} \quad (\text{because } \rho_k(t) \geq k, 0 < t \leq \frac{\pi}{2}) \\
&= O(1) t \log n R(n, t)
\end{aligned}$$

and this completes the proof of (1). By the Mean Value Theorem, for some  $\xi$  with  $0 < \xi < 1$ ,

$$\begin{aligned}
R(n, t+h) - R(n, t) &= h R'(n, t + \xi h) \\
&= O(1) h (t + \xi h) R(n, t + \xi h) \log n \\
&= O(1) t e^{-At^2 \log n},
\end{aligned}$$

for  $h \leq t \leq \frac{\pi}{2}$

□

The following lemma is due to Hardy, Littlewood, and Pólya [13].

**Lemma 4.12.** *If  $h(y, t)$  is a function of two variables defined for  $0 \leq t \leq \pi, 0 \leq y \leq 2\pi$ , then*

$$\left\| \int h(y, t) dt \right\|_p \leq \int \|h(y, t)\|_p dt, \quad p > 1$$

Finally, it is known ([16], p.50) that

$$(10) \quad S_n(f, x) - f(x) = \frac{2}{\pi} \int_0^\pi \varphi_x(t) D_n(t) dt$$

and

$$(11) \quad \tilde{S}_n(f, x) = \frac{-2}{\pi} \int_0^\pi \psi_x(t) \tilde{D}_n(t) dt,$$

where

$$D_n(t) = \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t},$$

and

$$\tilde{D}_n(t) = \frac{\cos \frac{1}{2}t - \cos \left(n + \frac{1}{2}\right)t}{2 \sin \frac{1}{2}t}.$$

### 5. PROOF OF THEOREM 3.3

Using (10) we have

$$\begin{aligned} K_n^\lambda(f, x) &= \frac{\Gamma(\lambda)}{\Gamma(n + \lambda)} \sum_{k=0}^n \binom{n}{k} \lambda^k S_k(f, x) \\ &= \frac{\Gamma(\lambda)}{\Gamma(n + \lambda)} \sum_{k=0}^n \binom{n}{k} \lambda^k \left( \frac{2}{\pi} \int_0^\pi \varphi_x(t) D_k(t) dt \right) + f(x) \frac{\Gamma(\lambda)}{(n + \lambda)} \sum_{k=0}^n \binom{n}{k} \lambda^k \\ &= \frac{2}{\pi} \int_0^\pi \frac{\varphi_x(t)}{2 \sin \frac{1}{2}t} K_n(t) dt + f(x), \end{aligned}$$

from which it follows that

$$L_n(x) = K_n^\lambda(f, x) - f(x) = \frac{2}{\pi} \int_0^\pi \frac{\varphi_x(t)}{2 \sin \frac{1}{2}t} K_n(t) dt$$

Now, for fixed  $\delta$  with  $0 < \delta < \frac{\pi}{4}$ , we write for  $u \neq 0$

$$\begin{aligned} L_n(y + u) - L_n(y) &= \frac{2}{\pi} \int_0^\pi \frac{G(t)}{2 \sin \frac{1}{2}t} K_n(t) dt \\ &= \frac{2}{\pi} \left( \int_0^h + \int_h^\delta + \int_\delta^\pi \right) \frac{G(t)}{2 \sin \frac{1}{2}t} K_n dt \\ &= \frac{2}{\pi} (P + Q + R) = I_1 + I_2 + I_3 \end{aligned}$$

By Minkowski's inequality, we have,

$$\|L_n(y + u) - L_n(y)\|_p \leq \|I_1\|_p + \|I_2\|_p + \|I_3\|_p$$

And by Lemma 4.12, we may write,

$$\begin{aligned} (12) \quad \|I_1\|_p &\leq \frac{2}{\pi} \int_0^h \|G(t)\|_p \left| \frac{K_n(t)}{2 \sin \frac{1}{2}t} \right| dt, \\ \|I_2\|_p &\leq \frac{2}{\pi} \int_h^\delta \|G(t)\|_p \left| \frac{K_n(t)}{2 \sin \frac{1}{2}t} \right| dt, \\ \|I_3\|_p &\leq \frac{2}{\pi} \int_\delta^\pi \|G(t)\|_p \left| \frac{K_n(t)}{2 \sin \frac{1}{2}t} \right| dt. \end{aligned}$$

By Lemma 4.5 (1) and Lemma 4.9 (2), we obtain

$$(13) \quad \|I_1\|_p = O\left(|u|^\beta \int_0^h t^{\alpha-\beta-1} dt\right) = O\left(\frac{|u|^\beta}{(\log n)^{\alpha-\beta}}\right).$$

and

$$(14) \quad \begin{aligned} \|I_3\|_p &= O\left(|u|^\beta \int_\delta^\pi t^{\alpha-\beta-1} e^{-At^2 \log n} dt\right) \\ &= O\left(|u|^\beta e^{-A\delta^2 \log n}\right) \\ &= O\left(\frac{|u|^\beta}{(\log n)^\Delta}\right), \Delta \text{ positive however large.} \end{aligned}$$

Using Lemma 4.8, we may write

$$(15) \quad \begin{aligned} \|I_2\|_p &\leq \int_h^\delta \|G(t)\|_p \left| \frac{1}{2 \sin \frac{1}{2}t} - \frac{1}{t} \right| |K_n(t)| dt + \int_h^\delta \frac{\|G(t)\|_p}{t} R(n, t) \sin l(n) dt \\ &\quad + \int_h^\delta \frac{\|G(t)\|_p}{t} R(n, t) \left( \sin\left(\frac{3}{2}t + \sum_{k=1}^{n-1} \theta_k\right) - \sin l(n) t \right) dt \\ &= I + J + K \end{aligned}$$

As  $\left(\frac{1}{2 \sin \frac{1}{2}t} - \frac{1}{t}\right) = O(t)$ , using Lemma 4.5 (1) and Lemma 4.9 (2), we obtain

$$(16) \quad \begin{aligned} I &= O\left(|u|^\beta \int_h^\delta t^{\alpha-\beta+1} e^{-At^2 \log n} dt\right) \\ &= O\left(\frac{|u|^\beta}{\log n}\right) \end{aligned}$$

by Lemma 4.7 (replacing  $\mu$  by  $\log n$ ). Now

$$(17) \quad J = \int_h^\delta \frac{\|G(t)\|_p}{|t|} |R(n, t) \sin l(n) t| dt = \left( \int_h^{2h} + \int_{2h}^{\delta+h} - \int_\delta^{\delta+h} \right) \frac{\|G(t)\|_p}{|t|} |R(n, t) \sin l(n) t| dt$$

Replacing  $t$  by  $t + h$  in the second integral of the above line, we obtain,

$$\begin{aligned}
 J &= \int_h^{2h} \frac{\|G(t)\|_p}{|t|} |R(n,t) \sin l(n) t| dt \\
 &\quad - \int_h^\delta \frac{\|G(t+h)\|_p}{|t+h|} |R(n,t+h) \sin l(n) t| dt \\
 &\quad - \int_\delta^{\delta+h} \frac{\|G(t)\|_p}{|t|} |R(n,t) \sin l(n) t| dt
 \end{aligned}$$

This together with (3.23) gives

$$\begin{aligned}
 (18) \quad 2J &= \int_h^\delta \left[ \frac{\|G(t)\|_p}{|t|} |R(n,t)| - \frac{\|G(t+h)\|_p}{|t+h|} |R(n,t+h)| \right] |\sin l(n) t| dt \\
 &\quad + \int_h^{2h} \frac{\|G(t)\|_p}{|t|} |R(n,t) \sin l(n) t| dt \\
 &\quad - \int_\delta^{\delta+h} \frac{\|G(t)\|_p}{|t|} |R(n,t) \sin l(n) t| dt \\
 &= L + M - N
 \end{aligned}$$

By Lemma 4.5 (1) and Lemma 4.9 (1), we get

$$(19) \quad M = O \left( |u|^\beta \int_h^{2h} t^{\alpha-\beta-1} dt \right) = O \left( \frac{|u|^\beta}{(\log n)^{\alpha-\beta}} \right)$$

and

$$\begin{aligned}
 (20) \quad N &= O \left( |u|^\beta \int_\delta^{\delta+h} t^{\alpha-\beta-1} e^{-At^2 \log n} dt \right) \\
 &= O \left( |u|^\beta e^{-A\delta^2 \log n} \right) \\
 &= O \left( \frac{|u|^\beta}{(\log n)^\Delta} \right), \Delta \text{ positive however large}
 \end{aligned}$$

We rewrite

$$\begin{aligned}
(21) \quad L &= \int_h^\delta \frac{\|G(t)\|_p - \|G(t+h)\|_p}{t} R(n, t) \sin l(n) t dt \\
&+ \int_h^\delta (R(n, t) - R(n, t+h)) \frac{\|G(t+h)\|_p}{t} \sin l(n) t dt \\
&+ \int_h^\delta \left( \frac{1}{t} - \frac{1}{t+h} \right) \|G(t+h)\|_p R(n, t+h) \sin l(n) t dt \\
&= L_1 + L_2 + L_3
\end{aligned}$$

By Lemma 4.5(1) and Lemma 4.6 (1), we obtain

$$(22) \quad L_1 = O \left( |u|^\beta h^{\alpha-\beta} \int_h^\delta \frac{dt}{t} \right) = O \left( |u|^\beta \frac{\log \log n}{(\log n)^{\alpha-\beta}} \right)$$

By Lemma 4.5 (2) and Lemma 4.11 (2), we have

$$\begin{aligned}
(23) \quad L_2 &= O \left( |u|^\beta \int_h^\delta (t+h)^{\alpha-\beta} e^{-At^2 \log n} dt \right) \\
&= O \left( |u|^\beta \int_h^\delta t^{\alpha-\beta} e^{-At^2 \log n} dt \right) \\
&= O \left( \frac{|u|^\beta}{(\log n)^{\alpha-\beta}} \right)
\end{aligned}$$

using Lemma 3 ( $\mu$  is replaced by  $\log n$ ).

Using Lemma 4.5 (2), Lemma 4.9 (1), and Lemma 4.7 (replacing  $\mu$  by  $\log n$ ), we have

$$\begin{aligned}
(24) \quad L_3 &= h \int_h^\delta \frac{\|G(t+h)\|_p}{t(t+h)} R(n, t+h) \sin l(n) t dt \\
&= O \left( h |u|^\beta \int_h^\delta \frac{(t+h)^{\alpha-\beta-1}}{t} e^{-At^2 \log n} dt \right) \\
&= O \left( h |u|^\beta \int_h^\delta t^{\alpha-\beta-2} e^{-At^2 \log n} dt \right) \\
&= O |u|^\beta \begin{cases} \frac{1}{(\log n)^{\alpha-\beta}}, & \alpha - \beta \neq 1 \\ \frac{\log \log n}{\log n}, & \alpha - \beta = 1 \end{cases}
\end{aligned}$$

Collecting the results (18) through (24), we obtain,

$$(25) \quad J = O\left(|u|^\beta \frac{\log \log n}{(\log n)^{\alpha-\beta}}\right), 0 \leq \beta < \alpha \leq 1$$

Now, using Lemma 4.5 (1), Lemma 4.9 (1), and Lemma 4.10 (1), we have

$$(26) \quad \begin{aligned} K &= \int_h^\delta \frac{\|G(t)\|_p}{t} R(n,t) \left( \sin\left(\frac{3}{2}t + \sum_{k=1}^{n-1} \theta_k\right) - \sin l(n)t \right) dt \\ &= O\left(|u|^\beta \log n \int_h^\delta t^{\alpha-\beta+2} e^{-At^2 \log n} dt\right) \\ &= O\left(\frac{|u|^\beta}{(\log n)^{\alpha-\beta}}\right), \end{aligned}$$

by applying Lemma 4.7.

Collecting the results from (12) – (16), (25), and (26), we obtain

$$|L_n(y+u) - L_n(y)| = O\left(|u|^\beta \frac{\log \log n}{(\log n)^{\alpha-\beta}}\right), 0 \leq \beta < \alpha \leq 1$$

which ensures that

$$(27) \quad \sup_{u \neq 0} \frac{|L_n(y+u) - L_n(y)|}{|u|^\beta} = O\left(\frac{\log \log n}{(\log n)^{\alpha-\beta}}\right), 0 \leq \beta < \alpha \leq 1$$

Again  $f \in H_{\alpha,p} \implies \|\varphi_x(t)\|_p = O(t^\alpha)$  and so proceeding as above, we obtain

$$(28) \quad \|L_n(\cdot)\|_p = O\left(\frac{\log \log n}{(\log n)^\alpha}\right), 0 < \alpha \leq 1$$

combining (27) and (28) we obtain (1) and this completes the proof of Theorem 3.3.



## 6. PROOF OF THEOREM 3.4

Using (11) , we obtain

$$\begin{aligned}
\tilde{K}_n^\lambda(f, x) &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \tilde{S}_k(f, x) \\
&= -\frac{2}{\pi} \int_0^\pi \psi_x(t) \left( \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \tilde{D}_k(t) \right) dt \\
&= -\frac{2}{\pi} \int_0^h \psi_x(t) \left( \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \tilde{D}_k(t) \right) dt \\
&\quad - \frac{2}{\pi} \int_h^\pi \psi_x(t) \left( \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \right) \frac{1}{2} \cot \frac{1}{2} t dt \\
&\quad + \frac{2}{\pi} \int_h^\pi \frac{\psi_x(t)}{2 \sin \frac{1}{2} t} \left( \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \cos \left( k + \frac{1}{2} \right) t \right) dt \\
&= -\frac{2}{\pi} \int_0^h \psi_x(t) \tilde{E}_n(t) dt + \tilde{f}(x, h) + \frac{2}{\pi} \int_h^\pi \frac{\psi_x(t)}{2 \sin \frac{1}{2} t} \tilde{K}_n(t) dt
\end{aligned}$$

which further ensures that

$$\begin{aligned}
\tilde{L}_n(x) &= \tilde{K}_n^\lambda(f, x) - \tilde{f}(x, h) \\
(29) \quad &= -\frac{2}{\pi} \int_0^h \psi_x(t) \tilde{E}_n(t) dt + \frac{2}{\pi} \int_h^\pi \frac{\psi_x(t)}{2 \sin \frac{1}{2} t} \tilde{K}_n(t) dt
\end{aligned}$$

Now for fixed  $\delta$  with  $0 < \delta < \frac{\pi}{4}$  and  $x \neq y$ , we write

$$\begin{aligned}
\tilde{L}_n(y+u) - \tilde{L}_n(y) &= \frac{-2}{\pi} \int_0^h \tilde{G}(t) \tilde{E}_n(t) dt + \frac{2}{\pi} \int_h^\delta \frac{\tilde{G}(t)}{2 \sin \frac{1}{2} t} \tilde{K}_n(t) dt \\
(30) \quad &+ \frac{2}{\pi} \int_\delta^\pi \frac{\tilde{G}(t)}{2 \sin \frac{1}{2} t} \tilde{K}_n(t) \\
&= -\tilde{P} + \tilde{Q} + \tilde{R}
\end{aligned}$$

By Minkowski's inequality, we have

$$\|L_n(y+u) - L_n(y)\|_p \leq \|\tilde{P}\|_p + \|\tilde{Q}\|_p + \|\tilde{R}\|_p.$$

By Lemma 4.12,

$$\begin{aligned}
 (31) \quad & \|\tilde{P}\|_p \leq \frac{2}{\pi} \int_0^h \|\tilde{G}(t)\|_p \left| \tilde{E}_n(t) \right| dt \\
 & \|\tilde{Q}\|_p \leq \frac{2}{\pi} \int_h^\delta \|\tilde{G}(t)\|_p \left| \frac{\tilde{K}_n(t)}{2 \sin \frac{1}{2}t} \right| dt \\
 & \|\tilde{R}\|_p \leq \frac{2}{\pi} \int_\delta^\pi \|\tilde{G}(t)\|_p \left| \frac{\tilde{K}_n(t)}{2 \sin \frac{1}{2}t} \right| dt
 \end{aligned}$$

By Lemma 4.5 (2) and Lemma 4.9 (4), we have

$$(32) \quad \|\tilde{P}\|_p = O\left(|u|^\beta \int_0^h t^{\alpha-\beta-1} dt\right) = O\left(\frac{|u|^\beta}{(\log n)^{\alpha-\beta}}\right)$$

Using Lemma 4.5 (2) and Lemma 4.9 (3), we have

$$\begin{aligned}
 (33) \quad & \|\tilde{R}\|_p = O\left(|u|^\beta \int_\delta^\pi t^{\alpha-\beta-1} e^{-At^2 \log n} dt\right) \\
 & = O\left(|u|^\beta e^{-A\delta^2 \log n}\right) \\
 & = O\left(\frac{|u|^\beta}{(\log n)^\Delta}\right), \Delta \text{ positive however large}
 \end{aligned}$$

Now, adopting the lines of arguments similar to those used in estimating  $Q$  in the proof of Theorem 3.3, we can obtain

$$(34) \quad \|\tilde{Q}\|_p = O\left(|u|^\beta \frac{\log \log n}{(\log n)^{\alpha-\beta}}\right), 0 \leq \beta < \alpha \leq 1.$$

Combining the results of (31) – (34), we obtain

$$(35) \quad \sup_{u \neq 0} \frac{\|\tilde{L}_n(y+u) - \tilde{L}_n(y)\|_p}{|u|^\beta} = O\left(\frac{\log \log n}{(\log n)^{\alpha-\beta}}\right), 0 \leq \beta < \alpha \leq 1.$$

Similarly,

$$(36) \quad \|\tilde{L}_n(\cdot)\|_p = O\left(\frac{\log \log n}{(\log n)^\alpha}\right), 0 < \alpha \leq 1.$$

(2) follows from (35) and (36) and this completes the proof of Theorem 3.4.

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