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D -Structures and their semantics

Rohit Parikh*

1 D -Structures

In these notes we shall be concerned with a semantic object which is a generalization of classical structures, Kripke structures and the regular $*$ -structures of Ehrenfeucht-de Jongh. We shall start by showing how these different cases can be obtained by imposing different regularity conditions on the basic object (D -structures) and the semantics can then be directly interpreted into the semantics of D -structures. We shall then give a game-theoretic explanation of the semantics of the D -structures from which the finite model property of regular $*$ -structures can be easily obtained. We go on to look at the proof theory of these objects.

In the following, μ will be a finite relational type. Constants are permitted but not function symbols.

Definition 1 A D -structure \mathcal{M} of type μ consists of two objects:

1. a family \mathcal{F} of finite relational structures (diagrams), all of type μ and
2. a family \mathcal{H} of homomorphisms between elements of \mathcal{F} . \mathcal{H} includes all the identity maps. \mathcal{H}^t is the closure of \mathcal{H} under composition and clearly $\langle \mathcal{F}, \mathcal{H}^t \rangle$ will be a category.

Remark: Note that homomorphisms preserve atomic formulae but not necessarily their negations. Members of \mathcal{H} will be called *allowable maps*.

Definition 2 A D -structure \mathcal{M} will be said to be *rigid* if all allowable maps are inclusions. It is *directed* if given D_1, D_2 in \mathcal{F} there is a D_3 and allowable maps $p_1 : D_1 \rightarrow D_3$ and $p_2 : D_2 \rightarrow D_3$. \mathcal{M} is *weakly directed* if $\langle \mathcal{F}, \mathcal{H}^t \rangle$ is directed.

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We shall show that a D -structure is a very flexible (but nontrivial) type of object and includes classical structures, intuitionistic structures¹, and the regular $*$ -structures of Ehrenfeucht-de Jongh as special cases (cf. theorems of this section).

Definition 3 Let A be a sentence of the language \mathcal{L}_u augmented by constants from a diagram D (we shall take the elements themselves to be these constants) and modal operators \Box and \Diamond . We recall that \Box means “necessarily” and \Diamond means “possibly”. We define $\mathcal{M}, D \models A$ by induction on the complexity $c(A)$ of A .

1. $c(A) = 0$. Then

$$\mathcal{M}, D \models A \quad \text{iff} \quad A \text{ is true in } D.$$

2. $A = B \wedge C$. Then

$$\mathcal{M}, D \models B \wedge C \quad \text{iff} \quad \mathcal{M}, D \models B \text{ and } \mathcal{M}, D \models C.$$

3. $A = B \vee C$. Then

$$\mathcal{M}, D \models B \vee C \quad \text{iff} \quad \mathcal{M}, D \models B \text{ or } \mathcal{M}, D \models C.$$

4. $A = \neg B$. Then

$$\mathcal{M}, D \models \neg B \quad \text{iff} \quad \mathcal{M}, D \not\models B.$$

5. $A = (\exists x)B(x)$. Then

$$\mathcal{M}, D \models (\exists x)B(x) \quad \text{iff} \quad \text{there exists } a \in |D| \text{ such that } \mathcal{M}, D \models B(a).$$

6. $A = (\forall x)B(x)$. Then

$$\mathcal{M}, D \models (\forall x)B(x) \quad \text{iff} \quad \text{for all } a \in |D|, \mathcal{M}, D \models B(a).$$

7. $A = \Box B(a_1, \dots, a_k)$. Then

$$\mathcal{M}, D \models \Box B(a_1, \dots, a_k) \quad \text{iff} \quad \text{for all allowable } f : D \rightarrow D', \\ \mathcal{M}, D' \models B(f(a_1), \dots, f(a_k)).$$

8. $A = \Diamond B(a_1, \dots, a_k)$. Then

$$\mathcal{M}, D \models \Diamond B(a_1, \dots, a_k) \quad \text{iff} \quad \text{for some allowable } f : D \rightarrow D', \\ \mathcal{M}, D' \models B(f(a_1), \dots, f(a_k)).$$

¹The finiteness requirement on elements of \mathcal{M} has to be dropped in this case, for technical reasons on the diagrams.

In 7, 8 the constants from $|D|$ are explicitly displayed.

Before studying D -structures in general we shall verify the claim made on before Definition 1.

Definition 4 Let A be a formula of the language \mathcal{L}_{μ^*D} , i.e. \mathcal{L}_μ with constants from $|D|$. A^c is the formula obtained from A if we replace \exists everywhere by $\diamond\exists$ and \forall everywhere by $\square\forall$.

Theorem 5 Let \mathcal{A} be a classical μ -structure. $\mathcal{M}^c(\mathcal{A}) = \mathcal{M}$ is the D -structure where \mathcal{F} consists of all finite substructures of \mathcal{A} . \mathcal{H} consists of all inclusion maps. (Thus \mathcal{M} is directed and rigid.) A is any sentence of \mathcal{L}_{μ^*D} . Then

$$\mathcal{A} \models A \quad \text{iff} \quad \mathcal{M}, D \models A^c,$$

where D contains all constants of A .

PROOF. \neg, \vee, \wedge and atomic sentences are trivial. Suppose now that A is $(\exists x)B(x, a_1, \dots, a_k)$ then A^c is $\diamond(\exists x)B^c(x, a_1, \dots, a_k)$.

[left to right] Suppose $\mathcal{A} \models A$. Then there is an $a \in |\mathcal{A}|$ such that

$$\mathcal{A} \models B(a, a_1, \dots, a_k).$$

Let D' be a substructure containing D and a . Then by induction hypothesis, $\mathcal{M}, D' \models B^c(a, a_1, \dots, a_k)$ hence $\mathcal{M}, D' \models (\exists x)B^c(x, a_1, \dots, a_k)$ hence

$$\mathcal{M}, D \models \diamond(\exists x)B^c(x, a_1, \dots, a_k).$$

I.e. $\mathcal{M}, D \models A^c$.

[right to left] Suppose

$$\mathcal{M}, D \models \diamond(\exists x)B^c(x, a_1, \dots, a_k)$$

then there is a D' such that $D \subseteq D'$ and $a \in D'$ such that $\mathcal{M}, D' \models B^c(a, a_1, \dots, a_k)$. But then $\mathcal{A} \models B(a, a_1, a_2, \dots, a_k)$ and hence

$$\mathcal{A} \models (\exists x)B(x, a_1, a_2, \dots, a_k).$$

The \forall case is similar. ■

Theorem 6 Let \mathcal{M} be a directed, rigid D -structure. Let

$$A = \bigcup_{D_a \in \mathcal{F}} D_a.$$

(This union makes sense since \mathcal{M} is directed and rigid.) Then, for sentences A of \mathcal{L}_{u^*A} , we have if D contains all constants of A ,

$$\mathcal{M}, D \models A^c \quad \text{iff} \quad \mathcal{A} \models A.$$

PROOF. Quite similar to above. ■

Definition 7 Let A be a formula of the intuitionistic predicate calculus with symbols from μ and additional constants. We define A^i by induction on $c(A)$.

1. $c(A) = 0, A^i = A$
2. $A = B \wedge C, A^i = B^i \wedge C^i$
3. $A = B \vee C, A^i = B^i \vee C^i$
4. $A = \neg B, A^i = \Box \neg B^i$
5. $A = B \rightarrow C, A^i = \Box(B^i \rightarrow C^i)$
6. $A = (\forall x)B(x), A^i = \Box(\forall x)B^i(x)$
7. $A = (\exists x)B(x), A^i = (\exists x)B^i(x)$.

(In cases 2,3,7, we could take $A^i = \Box(\exists x)B^i(x)$ etc. and the next theorem will still hold.)

Definition 8 Let \mathcal{A} be an intuitionistic structure (as in [Fit69] p.46). Let D_Γ be the structure with base set $P(\Gamma)$, and in which precisely those atomic A hold where $\Gamma \models A$. There is a homomorphism (which comes from set inclusion) from D_Γ to $D_{\Gamma'}$ just in case $R(\Gamma, \Gamma')$. Then, $\mathcal{M} = \mathcal{M}^i(\mathcal{A})$ is $\langle \mathcal{F}, \mathcal{H} \rangle$ where $\mathcal{F} = \{D_\Gamma : \Gamma \in \mathcal{G}\}$ and \mathcal{H} consists of the homomorphisms just mentioned.

Theorem 9 Let A be a sentence in $\hat{P}(\Gamma)$. Then

$$\mathcal{M}, D_\Gamma \models A^i \quad \text{iff} \quad \Gamma \models A.$$

PROOF. The proof is immediate if A is atomic. Also, \wedge, \vee, \exists will work in a parallel way. Suppose $A = \neg B$. Then, $A^i = \Box \neg B^i$. We have:

$$\begin{aligned} \Gamma \models \neg B & \text{ iff for all } \Gamma^*, \quad \Gamma^* \not\models B \\ & \text{ iff for all } D_{\Gamma^*}, \quad \mathcal{M}, D_{\Gamma^*} \not\models B \text{ (ind. hyp)} \\ & \text{ iff for all } D_{\Gamma^*}, \quad \mathcal{M}, D_{\Gamma^*} \models \neg B \\ & \text{ iff } \mathcal{M}, D_\Gamma \models \Box \neg B^i \end{aligned}$$

$A = B \rightarrow C$ and $A = (\forall x)B(x)$ are similar. ■

Suppose now that $\mathcal{M} = \langle \mathcal{F}, \mathcal{H} \rangle$ is a D -structure which is a category. We construct a Kripke structure corresponding to \mathcal{M} . Given $D \in \mathcal{F}$, a *selection* S for D is a set of maps into D such that if there are any maps $D' \rightarrow D$ there is just one such map in S . Take

$$\mathcal{G} = \text{the set of all pairs } \langle D, S \rangle,$$

where $D \in \mathcal{F}$ and S in a selection for D . For $\Gamma = \langle D, S \rangle \in \mathcal{G}$, take $P(\Gamma) = |D|$ and an atomic sentence A in $\mathcal{P}(\Gamma)$ is forced by Γ iff it holds in D . We let $\Gamma R \Gamma'$ iff there is a map $g \in S', g : D \rightarrow D'$ such that for all $f \in S, f \circ g \in S'$. (We point out that given a $g : D \rightarrow D'$ there is always such an S' .)

Theorem 10 For A in the language of IPC with constants from D , with $\Gamma = \langle D, S \rangle$,

$$\Gamma \models A \quad \text{iff} \quad \mathcal{M}, D \models A^i.$$

PROOF. Quite routine. To check one case, suppose $A = \neg B$. Then, $A^i = \Box \neg B^i$. Then,

$$\begin{aligned} \Gamma \models A & \text{ iff } \forall \Gamma^*, \Gamma^* \not\models B \\ & \text{ iff } \forall D' \text{ with allowable } g : D \rightarrow D', \mathcal{M}, D' \not\models B^i \\ & \text{ iff } \forall D' \text{ with allowable } g : D \rightarrow D', \mathcal{M}, D' \models \neg B^i \\ & \text{ iff } \mathcal{M}, D \models \Box \neg B^i \\ & \text{ etc.} \end{aligned}$$

■

Definition 11 Let \mathcal{A} be a (classical) structure of type μ and f a permutation of $|\mathcal{A}|$. Then $f(\mathcal{A})$ is the structure with base set $|\mathcal{A}|$ in which

$$f(\mathcal{A}) \models R(f(a_1), f(a_2), \dots, f(a_n)) \quad \text{iff} \quad \mathcal{A} \models R(a_1, a_2, \dots, a_n),$$

where $R \in u$ and $a_1, a_2, \dots, a_n \in |\mathcal{A}|$. A *regular *-structure* over \mathcal{A} is a family $\{f(\mathcal{A}) \mid f \in G\}$, where G is some group containing all finite permutations of $|\mathcal{A}|$.

Definition 12 Let \mathcal{M} be a family of first order structures all of the same type μ and with the same base set X . If $X_0 \subseteq X$, $M \in \mathcal{M}$ then

$$\mathcal{M}[X_0, M] = \{N \mid N \in \mathcal{M} \text{ and } N|_{X_0} = M|_{X_0}\}$$

Definition 13 (Ehrenfeucht) Let \mathcal{M} be a regular *-structure on \mathcal{A} . $X_0 \subseteq |\mathcal{A}|$, $M \in \mathcal{M}$. A is a sentence of $\mathcal{L}_{\mu * X_0}$. We define $\mathcal{M}[X_0, M] \models A$ by induction on $c(A)$.

1. $c(A) = 0$. Then

$$\mathcal{M}[X_0, M] \models A \quad \text{iff} \quad M \models A.$$

(Note: this depends only on $M|_{X_0}$.)

2. $A = B \wedge C$. Then

$$\mathcal{M}[X_0, M] \models A \quad \text{iff} \quad \mathcal{M}[X_0, M] \models B \text{ and } \mathcal{M}[X_0, M] \models C.$$

3. $A = B \vee C$. Then

$$\mathcal{M}[X_0, M] \models A \quad \text{iff} \quad \mathcal{M}[X_0, M] \models B \text{ or } \mathcal{M}[X_0, M] \models C.$$

4. $A = \neg B$. Then

$$\mathcal{M}[X_0, M] \models A \quad \text{iff} \quad \mathcal{M}[X_0, M] \not\models B.$$

5. $A = (\exists x)B(x)$. Then

$$\mathcal{M}[X_0, M] \models A \quad \text{iff} \quad \text{there exist } a \in X, b \in X_0 \cup \{a\}, N \in \mathcal{M}[X_0, M] \\ \text{such that } \mathcal{M}[X_0 \cup \{a\}, N] \models B(b).$$

6. $A = (\forall x)B(x)$. Then

$$\mathcal{M}[X_0, M] \models A \quad \text{iff} \quad \text{for all } a \in X, b \in X_0 \cup \{a\}, N \in \mathcal{M}[X_0, M], \\ \mathcal{M}[X_0 \cup \{a\}, N] \models B(b).$$

Theorem 14 *Let \mathcal{M} be a regular $*$ -structure on \mathcal{A} . Let $\mathcal{M}_1 = \langle \mathcal{F}, \mathcal{H} \rangle$ be defined as follows*

$$\mathcal{F} = \text{all finite submodels } D_i \text{ of } \mathcal{A}, \\ \mathcal{H} = \text{all monomorphisms } D \rightarrow D' \text{ with } \overline{D'} - \overline{D} \leq 1.$$

Let $X_0 = \{a_1, a_2, \dots, a_n\}$, $A(a_1, a_2, \dots, a_n) \in \mathcal{L}_{u^*X_0}$, $M \in \mathcal{M}$ and

$$b_1, b_2, \dots, b_n \in |\mathcal{A}| \quad \text{such that} \quad \mathcal{A}|_{b_1, b_2, \dots, b_n} = M|_{a_1, a_2, \dots, a_n}.$$

Then

$$\mathcal{M}[X_0, M] \models A(a_1, a_2, \dots, a_n) \quad \text{iff} \quad \mathcal{M}_1, D \models A^c(b_1, b_2, \dots, b_n),$$

where $\{b_1, b_2, \dots, b_n\} \subseteq |D|$.

PROOF. Trivial if A is atomic, a negation, conjunction, or disjunction.

Suppose $A = (\forall x)B(x)$. Then, $\mathcal{M}[X_0, M] \models A(a_1, a_2, \dots, a_n)$ gives, for all N, a, b as provided,

$$\mathcal{M}[X_0 \cup \{a\}, N] \models B(a_1, a_2, \dots, a_n, b).$$

Now, let $g : D \rightarrow D'$ be an allowable map. We need to show that

$$\mathcal{M}, D' \models B^c(g(b_1), \dots, g(b_n), c), \quad \text{for all } c \in |D'|.$$

Now, there is a permutation ϕ such that $\phi(g(b_i)) = a_i$. Take $a = \phi(b)$, where $b \in D' - g[D]$, if $D' \neq g[D]$ and let $a \in \{a_1, \dots, a_n\}$ otherwise. Let $b = \phi(c)$. Let $N = \phi(\mathcal{A})$. Then

$$\begin{aligned} N|_{\{a_1, \dots, a_n\}} &= M|_{\{a_1, \dots, a_n\}} \\ &\simeq \mathcal{A}|_{\{b_1, b_2, \dots, b_n\}} \\ &\simeq \mathcal{A}|_{\{g(b_1), g(b_2), \dots, g(b_n)\}}. \end{aligned}$$

and we get

$$\mathcal{M}[X_0 \cup \{a\}, N] \models B(a_1, a_2, \dots, a_n, b)$$

hence

$$\mathcal{M}, D' \models B^c(g(b_1), \dots, g(b_n), c)$$

Thus

$$\mathcal{M}, D' \models (\forall x)B^c(g(b_1), \dots, g(b_n), x)$$

Hence,

$$\mathcal{M}, D \models \Box(\forall x)B^c(g(b_1), \dots, g(b_n), x)$$

which was to be proved.

The backward argument and the \exists case are quite similar. ■

We now show that a D -structure $\mathcal{M} = \langle \mathcal{F}, \mathcal{H} \rangle$ corresponds to a regular $*$ -structure if

1. \mathcal{M} is weakly directed,
2. $D \in \mathcal{F}$ and $D' \subseteq D \rightarrow D' \in \mathcal{F}$,
3. the allowable maps are those monomorphisms $D \rightarrow D'$ where

$$\overline{D'} = \overline{D} \leq 1.$$

Theorem 15 *Let \mathcal{M} be a D -structure as above. Choose a maximal subfamily $\mathcal{K} \subseteq \mathcal{H}^t$ such that \mathcal{K} is closed under composition and \mathcal{K} contains at most one map from any D to D' . Let \mathcal{A} be the direct limit of \mathcal{F} under \mathcal{K} , and \mathcal{M}_1 a regular $*$ -structure on \mathcal{A} . Suppose $X_0 \subseteq |\mathcal{A}|$, $M \in \mathcal{M}_1$ and $D, a'_1, a'_2, \dots, a'_n$ are such that $D \in \mathcal{F}$ and $D|_{\{a'_1, a'_2, \dots, a'_n\}} \simeq M|_{X_0}$. Then,*

$$\mathcal{M}_1[X_0, M] \models A(a_1, a_2, \dots, a_n) \quad \text{iff} \quad \mathcal{M}, D \models A^c(a'_1, a'_2, \dots, a'_n).$$

PROOF. The proof is straightforward. These notes will continue. ■

2 A Game Theoretic Characterisation

Let μ be a relational type, \mathcal{M} a D -structure of type μ , $D \in \mathcal{M}$, $\mathcal{L} = \mathcal{L}_{\mu^*D}^{\mathcal{M}}$ the language of modal logic (with quantifiers) and nonlogical symbols from μ and $|D|$, A a closed formula of \mathcal{L} . We define a game $\mathcal{G}_{A,D}$ by induction on the complexity of A . (1), (2) are two players.

1. A is atomic. $\mathcal{G}_{A,D}$ is won by (1) iff $D \models A$. Otherwise, it is won by (2).
2. $A = B \wedge C$. Player (2) may choose either game $\mathcal{G}_{B,D}$ or $\mathcal{G}_{C,D}$ which is then played.
3. $A = B \vee C$. Player (1) may choose either game $\mathcal{G}_{B,D}$ or $\mathcal{G}_{C,D}$ which is then played.
4. $A = \neg B$. (1) wins $\mathcal{G}_{A,D}$ iff (s)he loses $\mathcal{G}_{B,D}$.
5. $A = (\forall x)B(x)$. Player (2) chooses an $a \in |D|$. The game $\mathcal{G}_{B(a),D}$ is then played.
6. $A = (\exists x)b(x)$. Player (1) chooses an $a \in |D|$. The game $\mathcal{G}_{B(a),D}$ is then played.
7. $A = \Box B(a_1, a_2, \dots, a_n)$. Player (2) chooses an $f : D \rightarrow D', f \in \mathcal{H}$. The game $\mathcal{G}_{D', B(f(a_1), f(a_2), \dots, f(a_n))}$ is then played.
8. Like (7) except player (1) chooses the f .

(In 7, the elements of $|D|$ are displayed.)

Theorem 16 $\mathcal{M}, D \models A$ iff player (1) has a winning strategy for $\mathcal{G}_{A,D}$.

Corollary 17 *Let $\mathcal{M} = \mathcal{G}_M$ be a regular $*$ -structure where M is classical and \mathcal{G} is a group containing all finite permutations of $|M|$. Let A closed such that, $\mathcal{M} \models A$. There exists a finite $X \subseteq |M|$ such that if $N = M|_X$ and $\mathcal{G}_1 =$ all permutations of X , then $\mathcal{G}, M \models A$. (This can be called the “finite model property”.)*

PROOF. Let $l = c(A)$. There are only finitely many possible diagrams of type μ and size $\leq l$ (upto isomorphism). Choose $X_i \subseteq M$ such that $M|_{X_i}$ is a representative of the i th type occuring inside M . Let $X =$ the union of all the X_i . Let $N = M|_X$.

Let \mathcal{M}_1 be the D -structure consisting of all diagrams in N with allowable maps being monomorphisms $D \rightarrow D'$ with $\overline{D'} - \overline{D} \leq 1$.

\mathcal{M}_2 is the analogous D -structure for M .

Then, clearly, a closed formula of complexity $\leq l$ holds in \mathcal{M}_1, D iff it holds in \mathcal{M}_2, D , where D is the empty diagram. Hence, we get

$$\begin{aligned} \mathcal{G}_M \models A & \text{ iff } \mathcal{M}_2 \models A \\ & \text{ iff } \mathcal{M}_1 \models A \\ & \text{ iff } \mathcal{G}, N \models A \end{aligned}$$

using theorem 14. ■

Theorem 18 (Skolem-Lowenheim theorem for D -structures) *Let $\mathcal{M} = \langle \mathcal{F}, \mathcal{H} \rangle$ be a D -structure. Then there exist countable $\mathcal{F}_1, \mathcal{H}_1, \mathcal{F}_1 \subset \mathcal{F}, \mathcal{H}_1 \subset \mathcal{H}$ such that for all $D \in \mathcal{F}_1$, $A \in \mathcal{L}_{\mu^*D}^M$,*

$$\mathcal{M}_1 = \langle \mathcal{F}_1, \mathcal{H}_1 \rangle \models A \quad \text{iff} \quad \mathcal{M} \models A.$$

Moreover, \mathcal{M}_1 is rigid, directed, weakly directed as a category etc. iff \mathcal{M}_1 is. Thus \mathcal{M}_1 corresponds to an intuitionistic, classical, or regular $$ -structure iff \mathcal{M} does.*

PROOF. Let

$$X = \mathcal{F} \cup \mathcal{H}^t \cup \bigcup \{|D| \mid D \in \mathcal{F}\}.$$

We look at the classical structure with base set and relations, constants corresponding to these in μ plus some others. Thus for a relation $R(x_1, \dots, x_n) \in u$ we have a relation $R'(y, x_1, \dots, x_n)$ which holds iff y is a digram and $R(x_1, \dots, x_n)$ holds in y . We also have monadic predicates corresponding to $\mathcal{F}, \mathcal{H}, \mathcal{H}^t, \bigcup \{|D| \mid D \in \mathcal{F}\}$. In addition we have a function f of two arguments such that

$$\begin{aligned} f(x, y) &= x(y) && \text{whenever } x \in \mathcal{H}^t \text{ and } y \text{ in some } D, \\ & && \text{where } x : D \rightarrow D', \\ &= \text{something not an element} && \text{if the conditions are not fulfilled.} \end{aligned}$$

Then we have the following. For each formula A of $\mathcal{L}_{\mu^*D}^M$, there is a formula A' in the language of M with constants from $|D|$, such that

$$\mathcal{M} \models A \quad \text{iff} \quad M \models A'.$$

Moreover, there are formulae of M expressing various properties of \mathcal{M} mentioned. Now take a countable substructure M_1 of M and take the \mathcal{M}_1 corresponding. ■

Special cases of this theorem include: classical structures, intuitionistic structures, regular *-structures and rigid D -structures. Note that many properties not explicitly mentioned will be elementary in M (possibly after expanding the language) and will be inherited by M_1 .

Game theoretic arguments can be used to give very direct proofs of many results of [EGGdJ] about regular *-structures.

3 The logic of D -structures

We recall the three systems M , M' , M'' for modal quantificational logic.

M consists of

1. the axioms and rules for the predicate calculus,
2. the axioms

$$\begin{aligned} & A \rightarrow \diamond A \\ & \Box A \leftrightarrow \neg \diamond \neg A \\ & \diamond(A \vee B) \leftrightarrow \diamond A \vee \diamond B, \end{aligned}$$

3. the rules

$$\text{if } \vdash A \leftrightarrow B \quad \text{then } \diamond A \leftrightarrow \diamond B$$

and

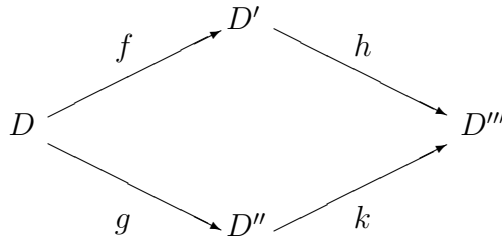
$$\text{if } \vdash A \quad \text{then } \Box A.$$

Theorem 19 *All theorems of M are valid in all D -structures.*

PROOF. It is clear that the axioms are valid and the rules preserve validity. ■

The system M' is **S4** and is obtained by adding the axiom $\Box A \rightarrow \Box \Box A$. The system M'' is **S5** and is obtained by adding, in addition, the axiom $(\diamond \Box A) \rightarrow \Box A$.

Definition 20 $\mathcal{M} = \langle \mathcal{F}, \mathcal{H} \rangle$ is *filtered* if for all allowable maps $f : D \rightarrow D'$, $g : D \rightarrow D''$ there exist D''' , h , k such that the diagram

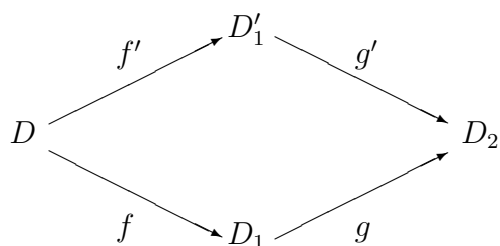


commutes. \mathcal{M} is *weakly filtered* if $\langle \mathcal{F}, \mathcal{H}^t \rangle$ is filtered.

Theorem 21 *If \mathcal{M} is a category then $\mathcal{M} \models \mathbf{S4}$.*

PROOF. Immediate from the definition. ■

The converse is not true. Suppose we have a situation



where $g \circ f$ belongs to \mathcal{H} but $g' \circ f'$ does not. However D'_1 is a copy of D_1 as far as D is concerned. Then the structure given above will act logically like a category. We do not know if there are any nontrivial examples.

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