9-30-2015

Propagation of Non-Stationary Noise in Waveguides

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PROPAGATION OF NON-STATIONARY NOISE IN WAVEGUIDES

By

Jonathan Samuel Ben-Benjamin

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

at

THE CITY UNIVERSITY OF NEW YORK

2015
This manuscript has been read and accepted for the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

PROPAGATION OF NON-STATIONARY NOISE IN WAVEGUIDES

by

Jonathan Samuel Ben-Benjamin

Adviser: Prof. Leon Cohen

This thesis consists of research regarding pulse and noise propagation in dispersive media. The research consists of three parts. In Part I we develop an approach for the propagation of non-stationary noise in waveguides, and in particular, we focus on the two-plate waveguide, which is a standard model for the ocean. The fundamental aim is to obtain the propagation of the space-time autocorrelation function. In our formulation, the noise is described by a Wigner spectrum, from which the autocorrelation function can be obtained. We discuss how to obtain the Wigner spectrum of a noise field from the Wigner spectrum of the noise that created it. We show that the Wigner spectrum of the field can be expressed in terms of the Wigner spectrum of the driving noise function and the Wigner distribution of the Green’s function. We calculate the Green’s function and its Wigner distribution for the two-plate waveguide. A number of special cases are considered, and we show how our result reduces to a known special case.

In Part II, we study the evolution of non-stationary noise in dispersive media in terms of modes and show how modes evolve and how they are effected by sources. Each mode satisfies
a Schrödinger-type equation where the “Hamiltonian” may not be Hermitian. The Hamiltonian operator corresponds to the dispersion relation where the wavenumber is replaced by the wavenumber operator. A complex dispersion relation corresponds to a non-Hermitian operator and indicates that we have attenuation. The case of arbitrary initial conditions is developed. The results expand on a previous approximation in terms of the Wigner distribution of a single mode. This solution requires one to obtain the initial modal functions from the given initial wave, and the cross-Wigner approximation between different modal functions. The approximation is obtained for general linear wave equations by transforming the equations to phase space, and then solving in the new domain. A number of examples are given.

In Part III we consider the motion of a “particle” in a medium of variable index of refraction whose motion is governed by Snell’s law. A stratified medium is considered. We give a derivation of the Newtonian forces that govern the motion and show that a position-dependent variable mass is necessary. Explicit expressions are given for the velocity and acceleration components of the particle. These are derived directly from Snell’s law. It is further shown that momentum is conserved along the interface, and that Snell’s law follows from this conservation law. We apply the equations of motion and find the conditions for a SOFAR channel, a phenomenon where rays are trapped in a channel.
This work is dedicated to the memory of

Martin Ingerman Z"L and ראובן בן-بنيימן ז"ל

who could not see this day.
Acknowledgements

I want to begin by fully expressing my gratitude to my adviser, Professor Leon Cohen; but I know that I lack the skill to adequately do so, and I will unfortunately have to be satisfied with a vastly incomplete expression of thanks. I will also try to resist the urge to embarrass him with kind words, and I assure the reader that any praise has been reduced one-thousand fold.

Professor Leon Cohen is the perfect adviser. He is extremely generous with his time and resources. He also does not immediately accept my speculations and sees through my logical blunders. Upon reading through my early correspondences with him, I am terribly embarrassed at some of my statements and questions. I can only wish to approach his level so that one day I could also be embarrassed about the statements I have made in our more recent correspondence.

When we started working together, Professor Cohen told me that I could research anything I liked, and that he was willing to work on anything as long as it was interesting. He certainly kept his promise. I learned that he is not only a master of many fields, but he is also an expert at finding interesting projects.

Our formulation of the particle view of Snell’s law grew out of a small simulation project that I started for fun. I think that he sensed that it might become interesting and challenged me to find an analytical expression for the simulated results, and he pushed
me further to formulate the dynamics. As shown in this thesis, we were able to formulate the particle dynamics under Snell’s law in terms of variable mass.

Also, our work about multi-mode wave propagation was initiated by an intellectual argument on the night prior to my thesis proposal. Professor Cohen helped me practice my thesis proposal presentation numerous times, until late at night. He also stayed late with me on numerous days, practicing the presentation for my thesis defense.

Professor Cohen and I regularly spent long hours working together, which were always enjoyable. I must have received more hours of guidance than any of my colleagues. I am extremely fortunate for that. He is indeed a very dedicated and invested mentor. My friends were often envious of our frequent and lengthy meetings.

I am incredibly grateful to Professor Leon Cohen, it has been a true pleasure working and learning from him, and even with that, I am sure that I am not appreciative enough.

I want to thank Professor Sultan Catto for serving on my committee, and also for recommending to me, while I was searching for an adviser, that I work with Professor Leon Cohen.

I want to thank Professor Pat Loughlin for serving on my committee, and also for our joint work, the results of which are important for this thesis and for my future work as well.
I would also like to thank my undergraduate professor, Professor Brett Fadem from Muhlenberg College, as he is the main reason that I majored in physics and am in the field of physics today. We would spend hours discussing physics, and he also took me to Brookhaven National Laboratory two summers to do research. He taught me how to think like a physicist, and I thank him for developing and nurturing my joy in science and my interest in fundamental questions in physics. I decided to continue learning about physics so I could find the answers.

I would also like to thank my high school chemistry teacher, Rebecca Isseroff, who invested and developed in me a profound interest in science in general and in chemistry in particular. I am still deeply interested in chemistry, but am enormously glad to study physics.

I would like to further thank the truly excellent professors at the CUNY Graduate Center and at Muhlenberg College.

I would like to thank my handler Daniel Moy who helped me stay on top of things while at The Graduate Center, and for encouraging me to attend The Graduate Center.

I would also like to thank Renee Sasaki, who besides helping Daniel Moy conquer the department’s bureaucratic and administrative toll, has also made my time at the Graduate Center enjoyable.
I would like to thank the Graduate Center security guards who greet me each day, play jokes, and let me in even when the school is closed. I thoroughly enjoyed joking with all of you.

I also want to thank the fourth floor cleaning crew, Hesed and Salman, especially for not throwing away my stuff!

I would like to thank George L. Poppe and Antonios Papaioannou, with whom I am writing a paper, and without whom I would not have survived my first year of graduate school.

I would also like to thank the fundamental physics crew, Arthur Parzygnat, STEVE VAYL, Marcelo Nomura, Ryan Abrahams, Josiah Sugarman, and Thomas C. Proctor for many interesting discussions.

I would like to thank the Sunday philosophy of physics crew, Jamie Bryan Lennox, Billy Mayer, and Amol Deshmukh for many interesting philosophical discussions.

I want to thank Anura Abewickrama, Xing Su, and Aradhana Kumari, from whom I learned alot.

I would also like to further thank my Science Center and fourth-floor colleagues, Franklin Lee, Tayfun Pay, Tal Galfsky, Kelly Greenland, Sean Byrnes, Miguel J. Bentata, Rob J.
Collison, Cesar Andrade, Dara Pir, Francesca Areselucini, Byungdo Park, and Zack McGuirk, for making the time in GC both productive and (mostly) interesting.

I also want to thank my brave allies that gave me peace of mind: Yuval Abrams, Cosim Sayid, David Topel, Asaf Shamis, Erin McKinney-Prupis, and others.

I would like to also thank my siblings and friends.

Finally, I want to wholeheartedly thank my parents, משה בן ראובן ומלכה בן-بنيיני and Shari R. Ingerman, and my proud grandmothers, Helen Ingerman and מלכה בן-בנייני, who allowed me to find what I truly love.
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Part I

Propagation of Non-Stationary Noise in a Two-Plate Waveguide
Chapter 1

Introduction: Noise

Suppose that we know the statistics of noise that is generated in a specific region and at specific times. What would be the statistics of the effect of this noise in a different region and/or at a future time? We develop a general method to deal with propagation of non-stationary noise in a waveguide that contains noise sources. For example, the noise generated by a moving ship is non-stationary, since the noise statistics at each point in space and time change as the ship moves. In particular we will explicitly obtain the propagation of noise in a two-plate waveguide, which is a standard model for the ocean. The way we describe non-stationary noise is by way of the Wigner spectrum which we briefly review.

For a deterministic time function, \( f(t) \), the Wigner distribution is given by\(^1\) [1, 2]

\[
W(t, \omega) = \frac{1}{2\pi} \int f^*(t - \tau/2)f(t + \tau/2)e^{-i\tau\omega}d\tau
\]

(1.1)

Upon ensemble-averaging, (1.1) becomes

\[
\overline{W}(t, \omega) = \frac{1}{2\pi} \int \langle f^*(t - \tau/2)f(t + \tau/2) \rangle e^{-i\tau\omega}d\tau
\]

(1.2)

\(^1\)All integrals are evaluated from \(-\infty\) to \(\infty\), unless otherwise noted. Also, multiple integrals are indicated by the number of differentials.
$\mathcal{W}(t, \omega)$ is called the ‘Wigner spectrum’ [3, 4, 5, 6]. In the Wigner spectrum, the term $\langle f^*(t - \tau/2) f(t + \tau/2) \rangle$ is the non-stationary autocorrelation function and can be obtained by way of

$$\langle f^*(t - \tau/2) f(t + \tau/2) \rangle = \int \mathcal{W}(t, \omega) e^{i\tau\omega} d\omega \quad (1.3)$$

One can also define the position-wavenumber Wigner distribution. For a space function $f(x)$, analogously to Eq. (1.1)

$$W(x, k) = \frac{1}{2\pi} \int f^*(x - \theta/2) f(x + \theta/2) e^{-i\theta k} d\theta \quad (1.4)$$

The Wigner spectrum for this case is then

$$\mathcal{W}(x, k) = \frac{1}{2\pi} \int \langle f^*(x - \theta/2) f(x + \theta/2) \rangle e^{-i\theta k} d\theta \quad (1.5)$$

where now $\langle f^*(x - \theta/2) f(x + \theta/2) \rangle$ is the spatial autocorrelation function.

In this thesis we will study the more general case of combined space-time autocorrelation functions. If we consider noise that is a function of both space and time, $u(x, t)$, then we use the four-dimensional Wigner distribution

$$W(x, k, t, \omega) = \frac{1}{(2\pi)^2} \int u^*(x - \theta/2, t - \tau/2) u(x + \theta/2, t + \tau/2) e^{-i\tau\omega} e^{-i\theta k} d\tau d\theta \quad (1.6)$$

and the four-dimensional Wigner spectrum is obtained by ensemble averaging

$$\mathcal{W}(x, k, t, \omega) = \frac{1}{(2\pi)^2} \int \langle u^*(x - \theta/2, t - \tau/2) u(x + \theta/2, t + \tau/2) \rangle e^{-i\tau\omega} e^{-i\theta k} d\tau d\theta \quad (1.7)$$

where now $\langle u^*(x - \theta/2, t - \tau/2) u(x + \theta/2, t + \tau/2) \rangle$ is the space-time autocorrelation function.
1.1 The Autocorrelation Function from the Wigner Spectrum

In Chap. 5 we show that the Wigner distribution of the response could be obtained from the Wigner distribution of the source via the Wigner distribution of the Green’s function. Taking the ensemble average of both sides, we obtain an expression for the Wigner spectrum of the response in terms of the Wigner spectrum of the source.

\[
W_{u_f}(r, k, t, \omega) = (2\pi)^4 \int W_f(r', -k', t', -\omega') W_G(r, k; x', k'; t, \omega; t', \omega') \, dr' \, dt' \, dk' \, d\omega' \quad (1.8)
\]

We now show how the autocorrelation function could be obtained from the Wigner spectrum. Since the Wigner spectrum of the source response is defined as

\[
W_{u_f}(x, k, t, \omega) = \frac{1}{(2\pi)^4} \int \langle u_f^*(r - \theta/2, t - \tau/2)u_f(r + \theta/2, t + \tau/2) \rangle e^{-i\theta \cdot k} e^{-i\tau \omega} \, d\theta \, d\tau \quad (1.9)
\]

then a double inverse-Fourier-transform yields the ensemble average

\[
\langle u_f^*(r - \theta/2, t - \tau/2)u_f(r + \theta/2, t + \tau/2) \rangle = \int \tilde{W}_{u_f}(r, k, t, \omega) e^{i\theta \cdot k} e^{i\tau \omega} \, dk \, d\omega \quad (1.10)
\]

which is the non-stationary autocorrelation function of the response.

In the general case above,

\[
\langle u_f^*(x - \theta/2, t - \tau/2)u_f(x + \theta/2, t + \tau/2) \rangle = (2\pi)^4 \int \tilde{W}_f(r', -k', t', -\omega') \left( \int W_G(r, k; r', k'; t, \omega; t', \omega') e^{i\theta \cdot k} e^{i\tau \omega} \, dk \, d\omega \right) \, dr' \, dt' \, dk' \, d\omega' \quad (1.11)
\]
Chapter 2

Outline of Approach

1. The fundamental wave equation we consider is

\[
\left( \nabla^2 - \frac{1}{c^2(z)} \frac{\partial^2}{\partial t^2} \right) u(r, t) = f(r, t) \tag{2.1}
\]

where \( u(r, t) \) is the wave and \( f(r, t) \) is a source function and \( c(z) \) is the speed which depends only on depth.

2. The solution is

\[
u(r, t) = u_H(r, t) + u_f(r, t) \tag{2.2}
\]

where \( u_H(r, t) \) is the ‘homogeneous’ part, and satisfies

\[
\left( \nabla^2 - \frac{1}{c^2(z)} \frac{\partial^2}{\partial t^2} \right) u_H(r, t) = 0 \tag{2.3}
\]

and where \( u_f(r, t) \) satisfies

\[
\left( \nabla^2 - \frac{1}{c^2(z)} \frac{\partial^2}{\partial t^2} \right) u_f(r, t) = f(r, t) \tag{2.4}
\]
3. We consider the case where

\[ u_H(\mathbf{r}, t) = 0 \]  

(2.5)

4. The response to the source, \( u_f(\mathbf{r}, t) \), can be expressed in terms of a Green’s function

\[ u_f(\mathbf{r}, t) = \int G(\mathbf{r}, \mathbf{r}'; t, t')f(\mathbf{r}', t')d\mathbf{r}'dt' \]  

(2.6)

where the Green’s function \( G(\mathbf{r}, \mathbf{r}'; t, t') \) satisfies

\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r}, \mathbf{r}'; t, t) = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t') \]  

(2.7)

5. We calculate the Wigner distribution in space and time

\[ W_{u_f}(\mathbf{r}, k, t, \omega) = \frac{1}{(2\pi)^4} \int u_f^*(\mathbf{r} - \mathbf{r}_k/2; t - \tau/2)u_f(\mathbf{r} + \mathbf{r}_k/2; t + \tau/2)e^{-i\tau\omega}e^{-i\mathbf{r}_k \cdot \mathbf{k}}d\tau d\mathbf{k} \]  

(2.8)

and we ensemble average to obtain the Wigner spectrum

\[ \langle W_{u_f}(\mathbf{r}, k, t, \omega) \rangle = \frac{1}{(2\pi)^4} \int \langle u_f^*(\mathbf{r} - \mathbf{r}_k/2; t - \tau/2)u_f(\mathbf{r} + \mathbf{r}_k/2; t + \tau/2) \rangle e^{-i\tau\omega}e^{-i\mathbf{r}_k \cdot \mathbf{k}}d\tau d\mathbf{k} \]  

(2.9)

This is called the Wigner spectrum, which is also denoted by

\[ \langle W_{u_f}(\mathbf{r}, k, t, \omega) \rangle = \overline{W}_{u_f}(\mathbf{r}, k, t, \omega) \]  

(2.10)
From the Wigner spectrum one can obtain the autocorrelation function

\[
\langle u^*(r - \tau k/2 ; t - \tau/2) u_f(r + \tau k/2 ; t + \tau/2) \rangle = \int \langle W_{u_f}(r, k, t, \omega) \rangle e^{i\tau \omega} e^{i\tau k \cdot k} dk d\omega.
\] (2.11)

6. To calculate \( W_{u_f}(r, k, t, \omega) \) we show that

\[
W_{u_f}(r, k, t, \omega) = (2\pi)^4 \int W_f(r', -k', -\omega') W_G(r, k; r', k', t, \omega; t', \omega') dr' dt' dk' d\omega'
\] (2.12)

where \( W_f(r', -k', -\omega') \) is the Wigner distribution of the source term and \( W_G \) is the Wigner distribution of the Green’s function.

\[
W_G(r, k; r', k'; t, \omega; t', \omega') = \int G^*(r - \theta/2, r' - \theta'/2; t - \tau/2, t' - \tau'/2)
\times G(r + \theta/2, r' + \theta'/2; t + \tau/2, t' + \tau'/2) e^{-i\theta \cdot k} e^{-i\theta' \cdot k'} e^{-i\tau \omega} e^{-i\tau' \omega'} d\theta d\tau d\theta' d\tau'
\] (2.13)

7. Since the Green’s function is deterministic, ensemble averaging gives

\[
\langle W_{u_f}(r, k, t, \omega) \rangle = (2\pi)^4 \int \langle W_f(r', -k', -\omega') \rangle W_G(r, k; r', k'; t, \omega; t', \omega') dr' dt' dk' d\omega'
\] (2.14)

8. Our basic aim is to calculate \( W_G \) for the two-plate waveguide, which in turn will allow us to calculate \( \langle W_{u_f}(r, k, t, \omega) \rangle \)

9. The two-plate waveguides we consider are the pressure-release top and bottom waveguide,
which is defined by the boundary conditions [7]

\[ u(x, y, z = 0, t) = 0 \quad \text{(pressure-release surface)} \]  
\[ u(x, y, z = L, t) = 0 \quad \text{(pressure-release bottom)} \]

and the pressure-release top and hard bottom two-plate waveguide, which is defined by the following boundary conditions

\[ u(x, y, z = 0, t) = 0 \quad \text{(pressure-release surface)} \]  
\[ \frac{\partial}{\partial z} u(x, y, z = L, t) = 0 \quad \text{(hard bottom)} \]

where \( L \) is the depth of the waveguide.
Chapter 3

Two-Plate Waveguide

A two-plate waveguide is an infinite three-dimensional region bounded above and below by a pair of parallel planes which have certain properties. To be called a waveguide, each of the planes should enforce some boundary condition, that is, there are properties that any wave must satisfy at the two planes. Each plane is called a plate, and hence the name ‘two-plate waveguide’. Two-plate waveguides are often used to model the ocean, where the top plate stands for the ocean surface, and the bottom plate represents the ocean bottom. Usually the $z$-direction is taken to be the depth direction, and here the two plates are perpendicular to the $z$-direction. Let $z$ point downward, in the direction of increasing depth. Common boundary conditions are the pressure-release boundary condition, which forces the magnitude of any wave to be zero (at a plate), and the hard boundary condition, which forces any wave’s derivative in the direction normal to the plate (at the plate) to be zero. The simplest model for the ocean is the pressure-release top and hard bottom two-plate waveguide [7]. They are defined by the following boundary conditions; for a wave $u(x, y, z, t)$

$$u(x, y, z = 0, t) = 0 \quad \text{(pressure-release surface)} \quad (3.1)$$

$$\frac{\partial}{\partial z} u(x, y, z = L, t) = 0 \quad \text{(hard bottom)} \quad (3.2)$$

9
where \( L \) is the depth of the ocean bottom.

\[ \text{3.1 Noise Propagation} \]

Suppose that we know the statistics of noise that is generated in a specific region and at specific times. What would be the statistics of the effect of this noise in a different region and/or at a future time? In this thesis we aim to develop a general method that can address such questions for both stationary and non-stationary noise situations.

To crystalize our aim, suppose the propagation of a wave is governed by the wave equations

\[
\left( \nabla^2 - \frac{1}{c^2(z)} \frac{\partial^2}{\partial t^2} \right) u(r,t) = f(r,t) \quad (3.3)
\]

where \( u(r,t) \) is the wave and \( f(r,t) \) is the source term. The solution is

\[
u(r,t) = u_H(r,t) + u_f(r,t) \quad (3.4)
\]

where \( u_H(r,t) \) satisfies the ‘homogeneous’ equation

\[
\left( \nabla^2 - \frac{1}{c^2(z)} \frac{\partial^2}{\partial t^2} \right) u_H(r,t) = 0 \quad (3.5)
\]
and $u_f(r, t)$ satisfies
\[
\left( \nabla^2 - \frac{1}{c^2(z)} \frac{\partial^2}{\partial t^2} \right) u_f(r, t) = f(r, t) \tag{3.6}
\]

We will take the homogeneous term to be zero
\[
u_H(r, t) = 0 \tag{3.7}
\]

The response to the source, $u_f(r, t)$, can be expressed in terms of a Green's function
\[
u_f(r, t) = \int G(r, r'; t, t') f(r', t') dr' dt' \tag{3.8}
\]

where the Green's function $G(r, r'; t, t')$ satisfies
\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(r, r'; t, t) = \delta(r - r') \delta(t - t') \tag{3.9}
\]

Given an autocorrelation function for the noise source, we develop a method for finding the Wigner spectrum of the response, $\bar{W}_{u_f}$,
\[
\bar{W}_{f}(r, k, t, \omega) = \frac{1}{(2\pi)^2} \int \langle f^*(r - \theta/2, t - \tau/2) f(r + \theta/2, t + \tau/2) \rangle e^{-i\theta k} e^{-i\tau \omega} d\theta d\tau \tag{3.10}
\]

We show in Chap. 5 that the Wigner spectrum of the source response is
\[
\bar{W}_{u_f}(r, k, t, \omega) = (2\pi)^4 \int W_G(r, k; r', -k'; t, \omega, t'; -\omega') \bar{W}_{f}(r', k', t', \omega') dr' dt' dk' d\omega' \tag{3.11}
\]
where $W_G$ is the Wigner distribution of the Green’s function

$$
W_G(r, k; r', k'; t, \omega, t', \omega')
= \frac{1}{(2\pi)^4} \int G^*(r - \theta/2, r' - \theta'/2; t - \tau/2, t' - \tau'/2)
\times G(r + \theta/2, r' + \theta'/2; t + \tau/2, t' + \tau'/2)e^{-i\theta \cdot k}e^{-i\theta' \cdot k'}e^{-i\tau \omega}e^{-i\tau' \omega'}d\theta d\theta' d\tau d\tau'
$$

(3.12)

In Chap. 4 we derive the Green’s function for the waveguide, and in Chap. 7 we derive the Wigner distribution of the Green’s function from Chap. 4. We derive Eq. (3.11) in Chap. 5.
Chapter 4

Green’s Function

In this chapter we review the standard Green’s function for the waveguides discussed in Chap. 3. However some aspects of our derivation are new. We first consider the general case where the speed is a function of $z$

$$c = c(z)$$  \hfill (4.1)

and then specialize to the case where

$$c = \text{constant}$$  \hfill (4.2)

and to the two types of boundary conditions previously mentioned, namely, for a wave $u(x, y, z, t)$

$$u(x, y, 0, t) = 0 \quad u(x, y, L, t) = 0 \quad \text{(pressure-release top and bottom)} \hfill (4.3)$$

$$u(x, y, 0, t) = 0 \quad \frac{\partial}{\partial z}u(x, y, L, t) = 0 \quad \text{(pressure-release top, hard bottom)} \hfill (4.4)$$
In the variable speed case, the Green’s function will be expressed in terms of general functions, as to allow the speed to be an arbitrary function. In the constant speed case, these will be sine functions.

We consider waves which obey the three-dimensional scalar wave equation

\[ \left( \nabla^2 - \frac{1}{c^2(z)} \frac{\partial^2}{\partial t^2} \right) u(x, y, z, t) = f(x, y, z, t) \]  

(4.5)

Since the speed \( c(z) \) in Eq. (4.5) is only \( z \)-dependent, there is a distinct difference between the \( z \) coordinate and the \( x, y \) coordinates in the differential equation governing the wave, Eq. (9.1). We rewrite it as

\[ \left( \nabla^2 \rho + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2(z)} \frac{\partial^2}{\partial t^2} \right) u_f(\rho, z, t) = f(\rho, z, t) \]  

(4.6)

where we use \( \rho = (x, y) = (r, \theta) \) to designate a two-dimensional position within a horizontal plane, and use \( z \) as the coordinate designating depth.

Assume that we have a Green’s function \( G \) for the solution of Eq. (4.6), then

\[ u_f(\rho, z, t) = \int G(\rho, \rho'; z, z', t, t') f(\rho', z', t') d\rho' dz' dt' \]  

(4.7)

where \( G \) satisfies

\[ \left( \nabla^2 \rho + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2(z)} \frac{\partial^2}{\partial t^2} \right) G(\rho, \rho'; z, z'; t, t') = \delta^{(2)}(\rho - \rho') \delta(z - z') \delta(t - t') \]  

(4.8)

Eq. (4.6) is usually solved in the frequency domain by using a Green’s function \( H \) in the frequency domain.

Going to frequency space, we need to find the source as well as the wave equation in the new space. To accomplish this, we start by writing the Fourier transforms of the wave and
source

\[ u(\rho, z, t) = \frac{1}{\sqrt{2\pi}} \int \hat{u}(\rho, z, \omega) e^{i\omega t} d\omega \]  
(4.9)

\[ f(\rho, z, t) = \frac{1}{\sqrt{2\pi}} \int \hat{f}(\rho, z, \omega) e^{i\omega t} d\omega \]  
(4.10)

with inverses

\[ \hat{u}(\rho, z, \omega) = \frac{1}{\sqrt{2\pi}} \int u(\rho, z, t) e^{-i\omega t} dt \]  
(4.11)

\[ \hat{f}(\rho, z, \omega) = \frac{1}{\sqrt{2\pi}} \int f(\rho, z, t) e^{-i\omega t} dt \]  
(4.12)

To find the appropriate differential equation in the frequency domain, we use Eqs. (4.6), (4.9), and (4.10) to obtain

\[
\left( \nabla_{\rho}^2 + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2(z)} \frac{\partial^2}{\partial t^2} \right) u(\rho, z, t) = \frac{1}{\sqrt{2\pi}} \int \left( \nabla_{\rho}^2 + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2(z)} \right) \hat{u}(\rho, z, \omega) e^{i\omega t} d\omega 
\]
(4.13)

\[
= f(\rho, z, t) = \frac{1}{\sqrt{2\pi}} \int \hat{f}(\rho, z, \omega) e^{i\omega t} d\omega 
\]
(4.14)

which leads to

\[
\frac{1}{\sqrt{2\pi}} \int \left[ \left( \nabla_{\rho}^2 + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2(z)} \right) \hat{u}(\rho, z, \omega) - \hat{f}(\rho, z, \omega) \right] e^{i\omega t} d\omega = 0 
\]
(4.15)

Hence,

\[
\left( \nabla_{\rho}^2 + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2(z)} \right) \hat{u}(\rho, z, \omega) = \hat{f}(\rho, z, \omega) 
\]
(4.16)

We now define the Green’s function in the frequency domain \( H \)

\[
G(\rho, \rho'; z, z', t, t') = \frac{1}{\sqrt{2\pi}} \int H(\rho, \rho'; z, z'; \omega, t') e^{i\omega t} d\omega 
\]
(4.17)
From Eq. (4.8) we see that the Green’s function $H$ must satisfy

$$
\delta^{(2)}(\rho - \rho')\delta(z - z')\delta(t - t')
= \frac{1}{\sqrt{2\pi}} \int \left( \nabla_{\rho}^2 + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2(z)} \right) H(\rho, \rho'; z, z'; \omega, t') e^{i\omega t} d\omega
$$

(4.18)

To satisfy Eq. (4.18), we must have

$$
H(\rho, \rho'; z, z'; \omega, t') = \frac{1}{\sqrt{2\pi}} e^{-i\omega t'} h(\rho, \rho'; z, z'; \omega)
$$

(4.19)

and

$$
\left( \nabla_{\rho}^2 + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2(z)} \right) h(\rho, \rho'; z, z') = \delta^{(2)}(\rho - \rho')\delta(z - z')
$$

(4.20)

From Eqs. (4.17) and (4.19) we see that $G$ is a function of $t - t'$

$$
G(\rho, \rho'; z, z', t, t') = \frac{1}{2\pi} \int h(\rho, \rho'; z, z'; \omega) e^{i\omega(t - t')} d\omega = G(\rho, \rho'; z, z', t - t')
$$

(4.21)

In App. A we find that in frequency space, the Green’s function equation analogous to Eq. (4.7) requires one less integration. This is because the $t$ and $t'$ dependence of the Green’s function $G$ is of the form $t - t'$

$$
\tilde{u}(\rho, z, \omega) = \sqrt{2\pi} \int h(\rho, \rho'; z, z'; \omega) \tilde{f}(\rho', z', \omega) d\rho' dz'
$$

(4.22)

We now further define the function $g'$, which is the Fourier transform of the $\rho$ coordinates in $h$

$$
h(\rho, \rho'; z, z'; \omega) = \frac{1}{2\pi} \int g'(\kappa, \rho'; z, z'; \omega) e^{i\kappa \cdot \rho} d\kappa
$$

(4.23)

where $\kappa$ is a two-dimensional vector, which represents the Fourier conjugate variables to $\rho$ in the wavenumber domain.
Plugging Eq. (4.23) into Eq. (4.20), we find requirements on $g'$

\[
\delta^{(2)}(\rho - \rho')\delta(z - z') = \left( \nabla^2 + \frac{\omega^2}{c^2(z)} \right) h(\rho, \rho', z, z'; \omega) = \int \left( \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2(z)} - \kappa^2 \right) g'(\kappa, \rho'; z, z'; \omega) e^{i\kappa \cdot \rho} d\kappa
\]

(4.24)

To satisfy the above equation, we must have

\[
g'(\kappa, \rho'; z, z'; \omega) = e^{-i\kappa \cdot \rho'} g(\kappa; z, z'; \omega)
\]

(4.26)

and

\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2(z)} - \kappa^2 \right) g(\kappa; z, z'; \omega) = \delta(z - z')
\]

(4.27)

Using Eq. (4.26), the Green's function in Eq. (4.23) is, in terms of $g$

\[
h(\rho, \rho'; z, z'; \omega) = \frac{1}{2\pi} \int g(\kappa; z, z'; \omega) e^{i\kappa \cdot (\rho - \rho')} d\kappa
\]

(4.28)

which shows that $h$ is of the form

\[
h(\rho, \rho'; z, z'; \omega) = h(\rho - \rho'; z, z'; \omega)
\]

(4.29)

and from Eq. (4.21) we also have that $G$ is of the form

\[
G(\rho, \rho'; z, z'; t, t') = G(\rho - \rho'; z, z'; t - t')
\]

(4.30)
4.1 Depth-Dependent Speed

In this section, we present the usual way in which the three-dimensional scalar wave equation with sources and a general depth-dependent speed is addressed in waveguides. The Green’s function is found for waves in the frequency domain in terms of a complete set of functions, \( \phi_n(z, \omega) \), which satisfy the boundary conditions of the waveguide. The \( \phi_n \)'s depend on the speed \( c(z) \), and generally are not trivial to determine.

To obtain the Green’s function, we start by noting that we can express \( g(\kappa; z, z'; \omega, \omega') \) from Eq. (4.23) using an eigenfunction expansion. Consider the eigenfunctions \( \phi_n(z, \omega) \) with eigenvalues \( \lambda_n \)

\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2(z)} \right) \phi_n(z, \omega) = \lambda_n \phi_n(z, \omega) \tag{4.31}
\]

Since they will carry the depth dependence of the Green’s function, and by extension, of the source-response, we demand that the \( \phi \)'s satisfy the boundary conditions of the waveguide.

Now, because the operator \( \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2(z)} \) in Eq. (4.31) is of the form satisfying Sturm-Liouville theory, the eigenfunctions form a complete set \[7\]

\[
\sum_n \phi_n^*(z', \omega) \phi_n(z, \omega) = \delta(z - z') \tag{4.32}
\]

We try to express \( g \) as a sum of the eigenfunctions \( \phi_n \)'s

\[
g(\kappa; z, z'; \omega) = \sum_n A_n(\kappa, z'; \omega) \phi_n(z, \omega) \tag{4.33}
\]

The coefficients, \( A_n(\kappa, z'; \omega) \), could be thought of as the \( \phi \)-transform of \( g \).
Using Eqs. (4.27), (4.32), and (4.33), we have

\[
\delta(z - z') = \sum_n \phi_n^*(z', \omega) \phi_n(z, \omega) \quad (4.34)
\]

\[
= \left( \frac{\partial^2}{\partial z'^2} + \frac{\omega^2}{c^2(z)} - \kappa^2 \right) g(\kappa; z, z'; \omega) \quad (4.35)
\]

\[
= \left( \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2(z)} - \kappa^2 \right) \sum_n A_n(\kappa, z'; \omega) \phi_n(z, \omega) \quad (4.36)
\]

\[
= \sum_n (\lambda_n - \kappa^2) A_n(\kappa, z'; \omega) \phi_n(z, \omega) \quad (4.37)
\]

Since the \(\phi_n\)'s satisfy a Sturm-Liouville equation, they are orthogonal to one another. Using this orthogonality of the \(\phi_n\)'s [7]

\[
\int \phi_n^*(z, \omega) \phi_n(z, \omega) dz = \delta_{nn} \quad (4.38)
\]

we find the coefficients \(A_n(\kappa, z'; \omega)\) to be

\[
A_n(\kappa, z'; \omega) = \frac{\phi_n^*(z', \omega)}{\lambda_n - \kappa^2} \quad (4.39)
\]

Finally, substituting Eq. (4.39) into Eq. (4.33), we find that \(g\) in terms of the \(\phi_n\)'s is

\[
g(\kappa; z, z'; \omega) = \sum_n \frac{\phi_n^*(z', \omega) \phi_n(z, \omega)}{\lambda_n - \kappa^2} \quad (4.40)
\]

From Eq. (4.40) we see that \(g\) only depends on the magnitude of \(\kappa\).

By definition, from Eq. (4.28), \(h\) is

\[
h(\rho - \rho'; z, z'; \omega) = \frac{1}{2\pi} \int \sum_n \frac{\phi_n^*(z', \omega) \phi_n(z, \omega)}{\lambda_n - \kappa^2} e^{i\kappa \cdot (\rho - \rho')} d\kappa \quad (4.41)
\]

and \(G\) could be obtained using Eq. (4.21).
4.2 Constant Speed

Here we consider the case where the speed $d$ is a constant

$$c(z) = c$$ \hspace{1cm} (4.42)

The constant speed case is a special case of the depth-dependent speed. In App. M, we derive directly the Green’s function for the constant speed case. Here we provide the constant speed Green’s functions for two sets of boundary conditions which we discussed, namely, for the pressure-release top and bottom, and for pressure-release top and hard bottom waveguides.

4.2.1 Pressure-Release Surface and Bottom

Here we consider the pressure-release top and bottom waveguide, where for any wave $u(x, y, z, t)$, we must have that

$$u(x, y, z = 0, t) = 0$$ \hspace{1cm} (4.43)

$$u(x, y, z = L, t) = 0$$ \hspace{1cm} (4.44)

In order to find $h$, we begin by considering a complete set of functions compatible with our waveguide (with the boundary conditions). For the constant speed case, with our boundary conditions, the $\phi_n$’s of Eq. (4.31) are sinusoidal functions. In particular, to satisfy the waveguide boundary conditions, the $\phi_n$’s are

$$\phi_n(z, \omega) = \frac{2}{L} \sin(\eta_n z)$$ \hspace{1cm} (4.45)
where

\[ \eta_n = \frac{n}{L} \pi \]  

where the \( n \)'s are positive integers.

The \( \phi_n \)'s satisfy Eq. (4.31) with

\[ \lambda_n = \frac{\omega^2}{c^2} - \eta_n^2 \]  

\[ \left( \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} \right) \phi_n(z, \omega) = \left( \frac{\omega^2}{c^2} - \eta_n^2 \right) \phi_n(z, \omega) \]  

Then, from Eq. (4.40), \( g \) is

\[ g(\kappa; z, z'; \omega) = \frac{4}{L^2} \sum_n \sin(\eta_n z') \sin(\eta_n z) \left( \frac{\omega^2}{c^2} - \eta_n^2 - \kappa^2 \right) \]  

and from Eq. (4.28), \( h \) is

\[ h(\rho - \rho'; z, z'; \omega) = \frac{1}{2\pi} \int \frac{4}{L^2} \sum_n \sin(\eta_n z') \sin(\eta_n z) e^{i\kappa \cdot (\rho - \rho')} d\kappa \]  

### 4.2.2 Pressure-Release surface and Hard Bottom

Here we consider waveguides with pressure-release surface and hard bottom, which restrict any wave inside to satisfy

\[ u(x, y, z = 0, t) = 0 \]  

\[ \frac{\partial}{\partial z} u(x, y, z = L, t) = 0 \]  

In this case the \( \phi_n \)'s are also sinusoids. The difference between this case and the previous case is the form of the \( \eta_n \)'s.
In this case
\[ \eta_n = \frac{2n - 1}{2L} \pi \] (4.53)
where the \( m \)'s are positive integers.

Similarly to the previous subsection, the \( \phi_n \)'s satisfy
\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} \right) \phi(z, \omega) = \left( \frac{\omega^2}{c^2} - \eta_n^2 \right) \phi(z, \omega)
\] (4.54)
and \( g \) and \( G \) are
\[
g(\kappa; z, z'; \omega) = \frac{4}{L^2} \sum_n \frac{\sin(\eta_n z') \sin(\eta_n z)}{\omega^2/c^2 - \eta_n^2 - \kappa^2}
\] (4.55)
\[
G(\rho - \rho'; z, z'; \omega) = \frac{1}{2\pi} \int \frac{4}{L^2} \sum_n \frac{\sin(\eta_n z') \sin(\eta_n z)}{\omega^2/c^2 - \eta_n^2 - \kappa^2} e^{i\kappa(\rho - \rho')} d\kappa
\] (4.56)

### 4.2.3 Further Materials

Many authors express the Green’s function for the above cases in cylindrical coordinates using a Hankel function of the first kind. They claim that the zeroth order Hankel function of the first kind satisfies
\[
\left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + k^2 \right) H_0^{(1)}(kr) \sim i \frac{\delta(r)}{r}
\] (4.57)

We have never seen a direct proof of this, and we present our own proof in App. I.

Also, notice that in our presentation of the Green’s function for depth-dependent speeds, knowledge of the \( \phi_n \)'s was assumed. In practice, the \( \phi_n \)'s may be difficult to determine. In App. J we derive general characteristics of the \( \phi_n \)'s for the case of uniformly increasing or decreasing speed as a function of depth.
Chapter 5

The Wigner Spectrum of the Response

We are interested in deriving a general expression that gives us the relationship between the autocorrelation function of a source and the autocorrelation function of the response to the source. We also want to be able to do this for non-stationary noise. We chose to represent this relationship and all of the terms in phase space using Wigner distributions and spectra, since they naturally lend themselves to studying autocorrelation functions.

Normally, we could use a Green’s function to obtain the response, and then obtain its Wigner spectrum after ensemble averaging. Notice that in this process we treat the source as a deterministic function, and ensemble average at the end. Here we begin with the Wigner spectrum of the noise, and directly obtain the Wigner spectrum of the response.
5.1 Relationship between the Wigner Distribution of the Source, Green’s Function, and Response

For the sake of simplicity, here we consider the one-dimensional case; however, we note that our results generalize in a straightforward manner to higher dimensions, and we present the three-dimensional case in App. K. We now prove that

\[ W_{uf}(x, k, t, \omega) = (2\pi)^2 \int W_G(x, k; x', -k'; t, \omega; t'; -\omega') W_f(x, k, t, \omega) dx' dt' dk' d\omega' \]  

(5.1)

We start by noting that the Wigner distribution of a wave \( u(r, t) \) is given by

\[ W_{uu}(x, k, t, \omega) = \frac{1}{(2\pi)^2} \int u^*(x - \theta/2, t - \tau/2) u(x + \theta/2, t + \tau/2) e^{-ik\theta} e^{-i\omega\tau} d\theta d\tau \]  

(5.2)

Since the full wave is the superposition of a source response and a homogeneous wave, \( u = u_f + u_H \), the Wigner distribution of the full wave is

\[ W_{uu} = W_{u_fu_f} + W_{u_fu_H} + W_{u_Hu_f} + W_{u_Hu_H} \]  

(5.3)

We now focus on the \( W_{u_fu_f} \) term. We could get rid of the other terms by considering the homogeneous wave \( u_H \) to be initially zero, in which case

\[ W_{uu} = W_{u_fu_f} \]  

(5.4)

We now derive the general relationship between the Wigner spectrum of the source response and the Wigner spectrum of the noise source via the Wigner distribution of the Green’s function.

Recall that given a deterministic source, the deterministic response is given by Eq. (M.2),
which we repeat here

\[ u_f(x, t) = \int G(x, x'; t, t') f(x', t') dx'dt' \]  

(5.5)

The Wigner distribution of the response \( u_f(x, t) \) in Eq. (5.5) is

\[
W_{u_f}(x, k, t, \omega) = \frac{1}{(2\pi)^2} \int d\theta dt e^{-i\theta k} e^{-i\omega \tau} u_f^*(x - \theta/2, t - \tau/2) u_f(x + \theta/2, t + \tau/2)
\]  

(5.6)

\[
= \frac{1}{(2\pi)^2} \int dx_2 dx_1 dt_1 f^*(x_2, t_2) f(x_1, t_1)
\]

\[
\times \int d\theta dt e^{-i\theta k} e^{-i\omega \tau} G^*(x - \theta/2, x_2; t - \tau/2, t_2) G(x + \theta/2, x_1; t + \tau/2, t_1)
\]  

(5.7)

Changing variables

\[
x_2 = x' - \phi/2 \quad t_2 = t' - \psi/2
\]  

(5.8)

\[
x_1 = x' + \phi/2 \quad t_1 = t' + \psi/2
\]  

(5.9)

we obtain

\[
W_{u_f}(x, k, t, \omega) = \frac{1}{(2\pi)^2} \int f^*(x' - \phi/2, t' - \psi/2) f(x' + \phi/2, t' + \psi/2)
\]

\[
\times \int G^*(x - \theta/2, x' - \phi/2; t - \tau/2, t' - \psi/2)
\]

\[
\times G(x + \theta/2, x' + \phi/2; t + \tau/2, t' + \psi/2) e^{-i\theta k} e^{-i\omega \tau} d\theta dt dx'dt'
\]  

(5.10)

Using the definition of the Wigner distribution of the source, \( W_f \), and double inverse-Fourier-transforming, we have

\[
f^*(x' - \phi/2, t' - \psi/2) f(x' + \phi/2, t' + \psi/2) = \int W_f(x, k, t, \omega) e^{i\omega\delta'} e^{i\omega\omega'} dk'd\omega'
\]  

(5.11)

Inserting Eq. (5.11) into Eq. (5.10), we have that the Wigner distribution of the source
response is

\[ W_u(x, k, t, \omega) = (2\pi)^2 \int W_f(x', -k', t', -\omega') W_G(x, k; x', k'; t, \omega; t', \omega') dx' dt' dk' d\omega' \quad (5.12) \]

where the Wigner distribution of the Green’s function is defined to be

\[ W_G(x, k; x', k'; t, \omega; t', \omega') = \frac{1}{(2\pi)^4} \int G^*(x - \theta/2, x' - \theta'/2; t - \tau/2, t' - \tau'/2) \times G(x + \theta/2, x' + \theta'/2; t + \tau/2, t' + \tau'/2) e^{-i\tau \omega} e^{-i\theta k} e^{-i\tau' \omega'} e^{-i\theta' k'} d\theta d\theta' d\tau d\tau' \quad (5.13) \]

Eq. (5.12) could be rewritten as

\[ W_u(x, k, t, \omega) = (2\pi)^2 \int W_f(x', k', t', \omega') W_G(x, k; x', -k'; t, \omega; t', -\omega') dx' dt' dk' d\omega' \quad (5.14) \]

For the three dimensional case, we have\(^1\)

\[ W_u(r, k, t, \omega) = (2\pi)^4 \int W_f(r', k', t', \omega') \times W_G(r, k; r', -k'; t, \omega; t', -\omega') dr' dt' dk' d\omega' \quad (5.15) \]

### 5.2 The Case of \( G(x, x'; t, t') = G(x, x'; t - t') \)

In some cases (like the case of the waveguide in Chap. 3), the Green’s function is of a functional form that depends on the difference of coordinates\(^2\)

\[ G(x, x'; t) = G(x, x'; t - t') \quad (5.16) \]

---

\(^1\)See App. K

\(^2\)where it is understood that \( G \) is not the same function on the two sides.
In this case, $W_{uf}$ in Eq. (5.14) becomes

$$W_{uf}(x, k, t, \omega) = (2\pi)^2 \int W_f(x', -k', t', \omega)W_G(x, k; x', k'; t - t', \omega)dx'dt'dk' \quad (5.17)$$

which is equivalent to

$$W_{uf}(x, k, t, \omega) = (2\pi)^2 \int W_f(x', -k', t - t', \omega)W_G(x, k; x', k'; t', \omega)dx'dt'dk' \quad (5.18)$$

Notice that we no longer have an integration over the conjugate variable to $t'$, which here appears in the Green’s function as part of a difference.

### 5.3 The Case of $G(x, x'; t, t') = G(x - x'; t - t')$

If the Green’s function for the situation is of the functional form

$$G(x, x'; t, t') = G(x - x'; t - t') \quad (5.19)$$

then $W_{uf}$ becomes

$$W_{uf}(x, k, t, \omega) = (2\pi)^2 \int W_f(x', k, t', \omega)W_G(x - x', k; t - t', \omega)dx'dt' \quad (5.20)$$

which is Eq. (62) in [8]. This is equivalent to

$$W_{uf}(x, k, t, \omega) = (2\pi)^2 \int W_f(x - x', k, t - t', \omega)W_G(x', k; t', \omega)dx'dt' \quad (5.21)$$

Notice that an effect similar to the previous case occurred and the $k'$ integration disappeared.

Our main results, Eqs. (5.12), (5.17), and (5.20), generalize to arbitrary dimension in a

---

3These could be obtained either directly from Eq. (5.6) or from Eq. (5.12) by using Eq. (L.6) in App. L.
straightforward way. For the case of the waveguides considered in earlier chapters, we utilize Green’s functions which are of the functional form

\[ G(x, x'; y, y'; z, z'|\omega) = G(x - x'; y - y'; z, z'|\omega) \]  

(5.22)

In this case, the equation for \( W_{uf} \) is\(^4\)

\[
W_{uf}(x, k_x, y, k_y, z, k_z|\omega)
= 2\pi \int W_G(x', k_x, y', k_y, z, k_z, z', k_z'|\omega) \\
\times W_f(x - x', k_x, y - y', k_y, z', -k_z'|\omega) dx' dy' dz' dk_z'
\]  

(5.23)

\[
= 2\pi \int W_G(x - x', k_x, y - y', k_y, z, k_z, z', k_z'|\omega) \\
\times W_f(x', k_x, y', k_y, z', -k_z'|\omega) dx' dy' dz' dk_z'
\]  

(5.24)

\(^4\)See App. K.
Chapter 6

Wigner Distribution of the Green’s Function: Variable Speed

Here we calculate the Wigner distribution of the Green’s function Eq. (4.41), found in Chap. 4. In App. E, we show how to directly obtain the single-frequency Wigner distribution of the Green’s function. We also show how to find the single-frequency Wigner distribution of the Green’s function from the full Wigner distribution of the Green’s function. The single-frequency Wigner distribution corresponds to the intensity in the frequency coordinate.

Here we find the Wigner distribution for the full Green’s function, Eq. (4.41) (repeated here for convenience) Consider Eqs. (4.21), and (4.28)

\[
G(\rho, \rho'; z, z', t - t') = \frac{1}{2\pi} \int h(\rho, \rho'; z, z'; \omega) e^{i\omega(t-t')} d\omega 
\]

(6.1)

\[
h(\rho, \rho'; z, z'; \omega) = \frac{1}{2\pi} \int d\kappa e^{i\kappa \cdot (\rho - \rho')} g(\kappa; z, z'; \omega) 
\]

(6.2)

We know that

\[
W_G = W_h 
\]

(6.3)
Calculating Wigner distribution of $h$, 

$$W_h(\rho, \kappa; z, k_z; z', k'_z; t, \omega) = \frac{1}{(2\pi)^5} \frac{1}{(2\pi)^2} \int d\theta_z d\theta'_z d\tau e^{-i\theta_z k_z} e^{-i\theta'_z k'_z} e^{-i\tau t} \times \int d\kappa_1 e^{-i\kappa_1 (\rho - \theta_z/2)} \sum_n \frac{\phi_n(z' - \theta_z/2, \omega - \tau/2) \phi^*_n(z - \theta_z/2, \omega - \tau/2)}{\lambda_n^* - (\kappa_1)^2} \times \int d\kappa_2 e^{i\kappa_2 (\rho + \theta_z/2)} \sum_m \frac{\phi_m(z' + \theta_z/2, \omega + \tau/2) \phi^*_m(z + \theta_z/2, \omega + \tau/2)}{\lambda_m - (\kappa_2)^2}$$ (6.4)

Here the effective limits of integration for $\theta_z$ and $\theta'_z$ do not go from $-\infty$ to $\infty$ because of the finite size of the waveguide. This is discussed in App. G.

The Wigner distribution of $h$ becomes

$$W_h(\rho, \kappa; z, k_z; z', k'_z; \omega, t) = \frac{4}{(2\pi)^5} \sum_n \sum_m \int d\kappa_1 \frac{1}{\lambda_n^* - (\kappa_1)^2} \frac{1}{\lambda_m - (2\kappa - \kappa_1)^2} \int d\tau e^{-i\tau t} \times \int d\theta_z e^{-i\theta_z k_z} \phi^*_n(z - \theta_z/2, \omega - \tau/2) \phi_m(z + \theta_z/2, \omega + \tau/2) \times \int d\theta'_z e^{-i\theta'_z k'_z} \phi_n(z' - \theta'_z/2, \omega - \tau/2) \phi^*_m(z' + \theta'_z/2, \omega + \tau/2)$$ (6.5)

This could alternatively be obtained from the double Fourier transform of the Green’s function, $g$, in Eq. (4.40). This is done in App. D.
Chapter 7

The Wigner Distribution of the Green’s Function: Constant Speed

We now seek to find the Wigner distribution of the Green’s function of the wave equation, \( G \). Here we obtain the Wigner distribution of \( G \) directly from its Fourier-transform, \( H \). We use the fact that

\[ W_G = W_h \quad (7.1) \]

Note that \( G \) in Eq. (4.50) has the same form for both sets of boundary conditions which we considered in Chap. 3, except that the Green’s functions satisfying the two sets of boundary conditions require different \( \eta_m \).

\[ \eta_m = \frac{m}{L} \pi \quad \text{(pressure-release surface and bottom)} \quad (7.2) \]

\[ \eta_m = \frac{2m - 1}{2L} \pi \quad \text{(pressure-release surface and hard bottom)} \quad (7.3) \]

where \( m \) is a positive integer.
We begin with the Green’s function in the frequency domain, \( h \), which could be obtained using Eqs. (4.49) and (4.28)

\[
h(r, r'; \omega) = \frac{1}{2\pi^2 L} \int \sum_{\eta_z} \frac{e^{i\eta_z (x-x')}}{\epsilon^2 - \eta_x^2 - \eta_y^2 - \eta_z^2} \sin(\eta_z z') \sin(\eta_z z) d\eta_x d\eta_y \tag{7.4}
\]

The Wigner distribution of the Green’s function will be the sum of the product of different parts

\[
W_h(\Delta x, k_x; \Delta y, k_y; z, z', k_z, k'_z; \Delta t, \omega) = \frac{1}{(2\pi^2 L)^2} \int \sum_{\eta_z} \sum_{\eta'_z} W_h^{(\Delta x)} W_h^{(\Delta y)} W_h^{(z)} W_h^{(z')} W_h^{(\omega)} d\eta_x d\eta_y d\eta'_x d\eta'_y \tag{7.5}
\]

### 7.1 Pressure-Release Surface and Bottom

To obtain the pressure-release case, we replace the \( \eta_z \)s and \( \eta'_z \)s by \( \frac{m}{L} \pi \) and \( \frac{n}{L} \pi \) respectively, and the summations become summations over \( m \) and \( n \). Therefore Eq. (7.5) becomes

\[
W_h(\Delta x, k_x; \Delta y, k_y; z, z', k_z, k'_z; \Delta t, \omega) = \frac{1}{(2\pi^2 L)^2} \int \sum_{m} \sum_{n} W_h^{(\Delta x)} W_h^{(\Delta y)} W_h^{(z)} W_h^{(z')} W_h^{(\omega)} d\eta_x d\eta_y d\eta'_x d\eta'_y \tag{7.6}
\]

The parts \( W_h^{(\Delta x)} \) through \( W_h^{(\omega)} \) are derived in App. G, and the results are listed below.

In calculating \( W_h \), we notice that we can separate some of the integrations and perform them independently. We list here the following pieces which have to be evaluated so that we
can obtain $W_G$. We provide the derivations in App. G.

$$W_h^{(x,x')} = e^{i2(k_x - \eta_x)(x-x')\delta} \left( k_x - \frac{\eta_x + \eta'_x}{2} \right) \delta \left( k'_x + k_x \right) \tag{7.7}$$

$$W_h^{(y,y')} = e^{i2(k_y - \eta_y)(y-y')\delta} \left( k_y - \frac{\eta_y + \eta'_y}{2} \right) \delta \left( k'_y + k_y \right) \tag{7.8}$$

$$W_h^{(\Delta x)} = 2e^{i2\Delta x(k_x - \eta_x)}\delta(\eta'_x + \eta_x - 2k_x) \tag{7.9}$$

$$W_h^{(\Delta y)} = 2e^{i2\Delta y(k_y - \eta_y)}\delta(\eta'_y + \eta_y - 2k_y) \tag{7.10}$$

$$W_h^{(z)} = -\frac{1}{\pi L} \cos[(n - m)\pi] \left[ \left( e^{+i(n-m)\pi z/L} \frac{\sin[(2k_z + (n + m)\pi/L)z - L/2 - 2Lk_z]}{2k_z + (n + m)\pi/L} - e^{-i(n-m)\pi z/L} \frac{\sin[(2k_z - (n + m)\pi/L)z - L/2 - 2Lk_z]}{2k_z - (n + m)\pi/L} \right) \right] \tag{7.11}$$

$$W_h^{(\omega)} = \frac{1}{2\pi |\Delta t|} \left[ \left( \frac{e^{-i2\omega \Delta t}}{\eta} \frac{e^{-i2c\eta \Delta t}}{(2w - c\eta)^2 - c^2\eta^2} - \frac{e^{-i2c\eta' \Delta t}}{(2w' - c\eta')^2 - c^2\eta'^2} \right) \right] \tag{7.12}$$

where $\eta^2 = \eta_x^2 + \eta_y^2 + \eta_n^2$, and $\eta' = \sqrt{\eta_x - 2k_x)^2 + (\eta_y - 2k_y)^2 + (\eta_m)^2}$. This comes from the Dirac delta functions in $W_h^{(x)}$, $W_h^{(\Delta x)}$, $W_h^{(y)}$, and $W_h^{(\Delta y)}$. 
Combining all of the integrals as in Eq. (7.5), we get

\[ W_h(x, x', k_x, k_x'; y, y', k_y, k_y' ; z, k_z ; z', k'_z ; \Delta t, \omega) = \frac{1}{(2\pi^2 L)^2} \int d\eta_x d\eta_y \sum_{\eta_x} \int d\eta'_x d\eta'_y \sum_{\eta'_y} W_h^{(x,x')} W_h^{(y,y')} W_h^{(z)} W_h^{(z')} \]

\[ = \frac{1}{(2\pi^2 L)^2} \int d\eta_x d\eta_y \sum_{\eta_x} \sum_{\eta'_y} 4e^{i2(k_x-\eta_x)(x-x')} \delta(k'_x + k_x) e^{i2(k_y-\eta_y)(y-y')} \delta(k'_y + k_y) \]

\[ \cos[(n-m)\pi] \left[ \left( e^{+i(n-m)\pi z/L} \sin\left(2k_x + (n+m)\pi/L\right) 2|z - L/2| - 2Lk_z \right) \right] \]

\[ + e^{-i(n-m)\pi z/L} \sin\left(2k_x - (n+m)\pi/L\right) 2|z - L/2| - 2Lk_z \right) \] \]

\[ - \left( e^{+i(n+m)\pi z/L} \sin\left(2k_x + (n-m)\pi/L\right) 2|z - L/2| - 2Lk_z \right) \]

\[ + e^{-i(n+m)\pi z/L} \sin\left(2k_x - (n-m)\pi/L\right) 2|z - L/2| - 2Lk_z \right) \]

\[ \frac{1}{2\pi |\Delta t|} \frac{c^2}{\eta} \left\{ \frac{e^{-2\omega \Delta t / \eta}}{(2\omega + \eta c)^2 - (c\eta')^2} - \frac{e^{-i2\varphi \Delta t c}}{(2\omega - c\eta)^2 - (c\eta')^2} \right\} \]

\[ - \frac{e^{+i2\varphi \Delta t c}}{\eta'} \left\{ \frac{e^{-2\omega \Delta t / \eta'}(2\omega - c\eta)^2 - c^2 \eta'^2}{(2\omega + c\eta'^2 - c^2)} - \frac{e^{+i2\varphi \Delta t c}}{2\omega + c\eta'^2 - c^2} \eta'^2 \right\} \] \]

and if we represent our \( x \) and \( x' \) coordinates to the difference

\[ \Delta x = x - x' \]  

and likewise, our \( y \) and \( y' \) coordinates to

\[ \Delta y = y - y' \]
and appropriately drop the $k'_x$ and $k'_y$ coordinates,

$$W_h(\Delta x, k_x; \Delta y, k_y; z, z', k_z, k'_z; \Delta t, \omega)$$

$$= \frac{1}{(2\pi^2 L)^2} \int d\eta_x d\eta_y \sum_{\eta_z} \int d\eta'_x d\eta'_y \sum_{\eta'_z} W_h(\Delta x) W_h(\Delta y) W_h(z) W_h(z') W_h(\omega)$$

$$= \frac{1}{(2\pi^2 L)^2} \int d\eta_x d\eta_y \sum_{\eta_z} \sum_{\eta'_z} 4e^{i2\Delta x(\Delta k_x - \eta_x)} e^{i2\Delta y(\Delta k_y - \eta_y)}$$

$$\cos[(n - m)\pi] \left[ e^{+i(n-m)\pi z/L} \sin[(2k_z + (n + m)\pi/L)2|z - L/2| - 2Lk_z] \right.$$

$$+ e^{-i(n-m)\pi z/L} \sin[(2k_z - (n + m)\pi/L)2|z - L/2| - 2Lk_z] \right]$$

$$- \left( e^{+i(n+m)\pi z/L} \sin[(2k_z + (n - m)\pi/L)2|z - L/2| + 2Lk_z] \right.$$

$$+ e^{-i(n+m)\pi z/L} \sin[(2k_z - (n - m)\pi/L)2|z - L/2| + 2Lk_z] \right)$$

$$\times \frac{1}{2\pi} \frac{\Delta t}{|\Delta t|} \frac{e^{-i2\omega\Delta t}}{\eta} \left\{ \frac{e^{-2\eta\Delta t}}{\eta} \left[ (2\omega + c\eta)^2 - (c\eta')^2 - (2\omega - c\eta)^2 - (c\eta')^2 \right] \right.$$

$$- \frac{e^{-i2\eta'\Delta t}}{\eta'} \left[ (2\omega - c\eta')^2 - c^2\eta'^2 - (2\omega + c\eta')^2 - c^2\eta^2 \right] \right\}$$

$$= \frac{1}{2\pi^2 L^2} \int d\eta_x d\eta_y \sum_{\eta_z} \sum_{\eta'_z} 4e^{i2\Delta x(\Delta k_x - \eta_x)} e^{i2\Delta y(\Delta k_y - \eta_y)}$$

$$\cos[(n - m)\pi] \left[ e^{+i(n-m)\pi z/L} \sin[(2k_z + (n + m)\pi/L)2|z - L/2| - 2Lk_z] \right.$$

$$+ e^{-i(n-m)\pi z/L} \sin[(2k_z - (n + m)\pi/L)2|z - L/2| - 2Lk_z] \right]$$

$$- \left( e^{+i(n+m)\pi z/L} \sin[(2k_z + (n - m)\pi/L)2|z - L/2| + 2Lk_z] \right.$$

$$+ e^{-i(n+m)\pi z/L} \sin[(2k_z - (n - m)\pi/L)2|z - L/2| + 2Lk_z] \right)$$

$$\times \frac{1}{2\pi} \frac{\Delta t}{|\Delta t|} \frac{e^{-i2\omega\Delta t}}{\eta} \left\{ \frac{e^{-2\eta\Delta t}}{\eta} \left[ (2\omega + c\eta)^2 - (c\eta')^2 - (2\omega - c\eta)^2 - (c\eta')^2 \right] \right.$$

$$- \frac{e^{-i2\eta'\Delta t}}{\eta'} \left[ (2\omega - c\eta')^2 - c^2\eta'^2 - (2\omega + c\eta')^2 - c^2\eta^2 \right] \right\}$$

(7.17)
Chapter 8

The Wigner Spectrum of the Noise Source

8.1 Spatial and Temporal White Noise Source

Consider the important case in which the autocorrelation function of the noise source is white noise in both space and time.

\[ \langle f^*(r_1, t_1) f(r_2, t_2) \rangle = a \delta^3(r_2 - r_1) \delta(t_2 - t_1) \]  

(8.1)

where \( a \) is a constant. Then

\[ \langle f^*(r - \theta/2, t - \tau/2) f(r + \theta/2, t + \tau/2) \rangle = a \delta^3(\theta) \delta(\tau) \]  

(8.2)

Therefore, the Wigner spectrum of the noise source is

\[ \mathcal{W}_f(r, k, t, \omega) = \frac{1}{(2\pi)^4} \int a \delta^3(\theta) \delta(\tau) e^{-i\theta \cdot k} e^{-i\tau \omega} d\theta d\tau = \frac{a}{(2\pi)^4} \]  

(8.3)
In App. K we derive that the Wigner spectrum of the response for this case is given by

\[
W_{u_f}(\rho, \kappa, z, k_z, t, \omega) = (2\pi)^4 \int W_f(\rho - \rho', \kappa, z' - k'_z, t - t', -\omega') \\
\times W_G(\rho', \kappa; z, k_z; z', k'_z; t', \omega) d\rho' d\omega' dt' dk'_z
\]  

(8.4)

8.1.1 Depth-Dependent Speed

In Chap. 6 we found that the Wigner distribution of the Green’s function is Eq. (6.5)

\[
W_h(\rho, \kappa; z, k_z; z', k'_z; \omega, t) = \frac{1}{(2\pi)^5} \sum_n \sum_m \int d\kappa_1 \frac{e^{i2\rho(\kappa - \kappa_1)}}{\lambda^2_n - (\kappa_1)^2} \int d\tau e^{-i\tau t} \\
\times \int \phi^*_n(z - \theta z/2, \omega - \tau/2) \phi_m(z + \theta z/2, \omega + \tau/2) e^{-i\theta z k_z} d\theta z \\
\times \int \phi_n(z' - \theta' z/2, \omega - \tau/2) \phi^*_m(z' + \theta' z/2, \omega + \tau/2) e^{-i\theta' k'_z} d\theta' z
\]

(8.5)

Using Eq. (8.3), we have that the source response is

\[
W_{u_f}(\rho, \kappa, z, k_z, t, \omega) = a \sum_n W_{\phi n}(z, k_z|\omega) \frac{\phi^*_n(z - \theta z/2, \omega) \phi_m(z + \theta z/2, \omega) e^{-i\theta z k_z} d\theta z}{|\lambda^2_n - \kappa^2|^2}
\]

(8.6)

where \(W_{\phi n}^\phi(z, k_z|\omega)\) is defined as

\[
W_{\phi n}^\phi(z, k_z|\omega) = \frac{1}{2\pi} \int \phi^*_n(z - \theta z/2, \omega) \phi_m(z + \theta z/2, \omega) e^{-i\theta z k_z} d\theta z
\]

(8.7)

8.1.2 Constant Speed

To specialize to the constant speed case, we begin by noticing that the \(W_{\phi n}^\phi\) term in Eq. (8.6) is exactly \(W_h^z\) in Chap. 7, whose derivation is in App. G. Also, since we have from
Eq. (4.47) in Chap. 4 that in the constant speed case

\[ \lambda_n = \frac{\omega^2}{c^2} - \frac{\eta_n^2}{c^2} \]  \hspace{1cm} (8.8)

the denominator of Eq. (8.6) is

\[ |\lambda_n - \kappa^2|^2 = |\frac{\omega^2}{c^2} - \kappa^2 - \frac{\eta_n^2}{c^2}|^2 \]  \hspace{1cm} (8.9)

So in this case the Wigner spectrum of the response is

\[ \overline{W}_{uf}(\rho, \kappa, z, k_z, t, \omega) = a \sum_n \frac{W_h^{(z)}}{|\frac{\omega^2}{c^2} - \kappa^2 - \frac{\eta_n^2}{c^2}|^2} \]  \hspace{1cm} (8.10)

From Eq. (G.20), \( W_h^{(z)} \) is, for the \( nn \) case

\[ W_h^{(z)} = -\frac{1}{2\pi L} \left[ \left( \frac{\sin[(k_z + n\pi/L)4|z - L/2| - 2Lk_z]}{k_z + n\pi/L} + \frac{\sin[(k_z - n\pi/L)4|z - L/2| - 2Lk_z]}{k_z - n\pi/L} \right) \right. \\
- 2 \cos[2n\pi z/L] \frac{\sin[4|z - L/2|k_z - 2Lk_z]}{k_z} \right] \]  \hspace{1cm} (8.11)
Chapter 9

Kuperman-Ingenito Result as a Special Case

Kuperman and Ingenito sought to find an expression for noise in the ocean that results from wind or rain striking the ocean surface [9]. They model the ocean as a two-plate waveguide with a pressure-release top and a hard bottom. In their model, all of the noise sources emanate from within a horizontal plane at an arbitrary depth $z'$ (as in fig. 9.1).

\[ z = 0 \quad z = z' \quad z = L \]

Figure 9.1: Kuperman-Ingenito: Noise Sources within a Plane of Depth $z'$
The Helmholtz equation with depth-dependent sound speed

\[
\left( \nabla^2 + \frac{\omega^2}{c^2(z)} \right) \hat{u}(\mathbf{r}, \omega) = \hat{f}(\mathbf{r}, \omega)
\]  

(9.1)

describes wave dynamics in the waveguide, and the noise and sources are considered as a function of frequency \( \omega \).

Since the speed \( c(z) \) is depth-dependent, there is a distinct difference between the \( z \) coordinate and the \( x, y \) coordinates in the differential equation governing the wave, so we rewrite Eq. (9.1) as

\[
\left( \nabla^2_{\rho} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2(z)} \right) \hat{u}(\mathbf{\rho}, z, \omega) = \hat{f}(\mathbf{\rho}, z, \omega)
\]  

(9.2)

where we use \( \mathbf{\rho} = (x, y) = (r, \theta) \) to designate a two-dimensional position within a horizontal plane, and use \( z \) as the coordinate designating depth.

As in our approach, sources and waves are treated deterministically until the end of the calculation, at which point they are ensemble averaged. The disturbance resulting from the noise in the rest of the waveguide is described by the single-frequency autocorrelation function of the source-response in the frequency domain.

\[
R(\mathbf{\rho}_1, \mathbf{\rho}_2, z_1, z_2|\omega) = \langle \hat{u}^*(\mathbf{\rho}_1, z_1, \omega) \hat{u}(\mathbf{\rho}_2, z_2, \omega) \rangle
\]  

(9.3)

Their main result is that the autocorrelation function of the response to a noise source is approximately

\[
R(\mathbf{\rho}, z_1, z_2|\omega) \sim \frac{q^2 \rho(z')}{8} \sum_n \frac{|\phi_n(z', \omega)|^2 |\phi_n(z_1, \omega)|^2 |\phi_n^*(z_2, \omega)|}{\kappa_n(\omega) \alpha_n(\omega)} \int N(\mathbf{\rho}', \omega) J_0(\kappa_n|\mathbf{\rho} - \mathbf{\rho}'|) \, d\mathbf{\rho}'
\]  

(9.4)
for a single frequency $\omega$, where $N(\rho, \omega)$ is the autocorrelation function of the noise source. The $\phi_n$’s are the eigenfunctions of $\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2(z)}$ with eigenvalues $\lambda_n = \alpha_n + i\beta_n$, and $\rho$ is the position of interest in the $xy$ plane.

9.1 Deriving the Kuperman-Ingenito Result

In Chap. 4 we found a general expression for the Green function for the three-dimensional scalar wave equation with arbitrary speed $c(z)$ in frequency space. In App. H we found the autocorrelation function of the source response in general, and also for specific types of noise source autocorrelation functions. In this chapter, we input the noise source autocorrelation function from the paper Kuperman-Ingenito [9] into our Eq. (5.15).

Since the Kuperman-Ingenito source is of the form $\hat{f}(\rho, z, \omega) = \hat{F}(\rho, \omega)\delta(z - z')$, its same-frequency autocorrelation function is

$$R(\rho - \tau/2, \rho + \tau/2, z - \tau_z/2, z + \tau_z/2|\omega) = R(\rho - \tau/2, \rho + \tau/2|\omega)\delta(z - z_z)$$ \hspace{1cm} (9.5)

Further, in the Kuperman-Ingenito case, the autocorrelation of the noise source is stationary, that is, there is no $\rho$ dependence, and the autocorrelation function depends only on the distance $\tau$ within the plane

$$R(\rho - \tau/2, \rho + \tau/2|\omega) = R(\tau|\omega)$$ \hspace{1cm} (9.6)

\textsuperscript{1}where it is understood that the two $R$’s (on either side of the equation) are different functions.
Using Eqs. (9.5) and (9.6), the autocorrelation function of the response becomes

\[
\langle \hat{u}_j^*(\rho - \theta/2, z - \theta z/2, \omega) \hat{u}_j(\rho + \theta/2, z + \theta z/2, \omega) \rangle \\
= -\frac{\pi^2}{2} \sum_n |\phi_n(z', \omega)|^2 \phi_n^*(z - \theta z/2, \omega) \phi_n(z + \theta z/2, \omega) \int d\tau \frac{R(\tau)}{\beta_n(\alpha_n)^2} J_0(|\theta - \tau| \alpha_n) \tag{9.7}
\]

If we further consider a noise source autocorrelation function which depends only on the absolute distance \(\tau\) within the \(xy\) plane, that is, that \(R(\tau) = R(\tau)\)

\[
\langle \hat{u}_j^*(\rho - \theta/2, z - \theta z/2, \omega) \hat{u}_j(\rho + \theta/2, z + \theta z/2, \omega) \rangle \\
= -\frac{\pi^2}{2} \sum_n |\phi_n(z', \omega)|^2 \phi_n^*(z - \theta z/2, \omega) \phi_n(z + \theta z/2, \omega) \int d\tau \frac{R(\tau)}{\beta_n(\alpha_n)^2} J_0(|\theta - \tau| \alpha_n) d\tau \tag{9.8}
\]

which is the main result of Kuperman and Ingenito [9], as per their Eq. (29).
Part II

Waves, Modes, Sources, and Green’s Functions
Chapter 10

Wave Equations, Modes and Sources

We consider waves that evolve according to differential equations with constant coefficients and present the concept of modes. In the absence of a source term, the differential equation is called homogeneous, and the wave is the sum of modal functions. Each modal function evolves in a different manner, which could be obtained from the differential equation for the wave in a straightforward way.

In this chapter, we present a comprehensive theory of modes and modal functions. This is adopted largely from our paper [8].

In part I, we considered the source response exclusively. In the presence of a source, the wave is the superposition of a source-response and a homogeneous wave. Starting at Chap. 12, we describe properties of the source-response in the mode formulation. Examples are given to illustrate the ideas applying to both homogeneous wave equations and wave equations with sources.

In Chaps. 15 and 16, we focus exclusively on the homogeneous wave, and study its Wigner distribution and Spectrogram. The formulation in terms of phase space lends itself to approximations, and we develop approximations to the single mode and multimode wave

\[\text{[1]}\text{These chapters are adopted largely on our papers [12, 13].}\]
We consider wave equations with constant coefficients with a source term

\[
\left( \sum_{n=0}^{N_x} b_n \frac{\partial^n}{\partial x^n} - \sum_{n=0}^{N_t} a_n \frac{\partial^n}{\partial t^n} \right) u(x, t) = f(x, t) \tag{10.1}
\]

where \( u(x, t) \) is the wave and \( f(x, t) \) is a source term that depends on space and time. Under some circumstances we may have wave equations of the more general form

\[
\sum_{m=0}^{N_x} \sum_{n=0}^{N_t} A_{n,m} \frac{\partial^{n+m}}{\partial x^n \partial t^m} u(x, t) = f(x, t) \tag{10.2}
\]

and such an example is considered in section 13.3. We write Eq. (10.1) as

\[
\mathcal{L} u(x, t) = f(x, t) \tag{10.3}
\]

where the operator \( \mathcal{L} \) is

\[
\mathcal{L} = \sum_{n=0}^{N_x} b_n \frac{\partial^n}{\partial x^n} - \sum_{n=0}^{N_t} a_n \frac{\partial^n}{\partial t^n} \tag{10.4}
\]

The solution to Eq. (10.1) or (10.2) is given by

\[
u(x, t) = u_H(x, t) + u_f(x, t) \tag{10.5}
\]

where \( u_H(x, t) \) is the solution of the homogeneous equation

\[
\mathcal{L} u_H(x, t) = \left( \sum_{n=0}^{N_x} b_n \frac{\partial^n}{\partial x^n} - \sum_{n=0}^{N_t} a_n \frac{\partial^n}{\partial t^n} \right) u_H(x, t) = 0 \tag{10.6}
\]

and \( u_f(x, t) \) is a called the source response of Eq. (10.3). We start discussing \( u_H(x, t) \) in the following section, and we start discussing the \( u_f(x, t) \) term in Chap. 12.
10.1 Modes and the Homogenous Equation

The concept of modes arises from the homogenous equation. One substitutes $e^{ikx-i\omega t}$ into Eq. (10.6) to obtain the algebraic equation \[14, 15, 16, 17, 18\]

\[
\sum_{n=0}^{N_x} b_n(ik)^n - \sum_{n=0}^{N_t} a_n(-i\omega)^n = 0 \tag{10.7}
\]

and solves for $\omega$ in terms of $k$. Eq. (10.7) is called the dispersion relation. Generally there are many solutions to the dispersion relation and each solution is called a mode. Let us assume there are $M$ modes

\[
\omega = \omega_\ell(k) \quad \ell = 1, 2, \ldots, M \tag{10.8}
\]

Furthermore a particular mode may have a complex dispersion relation; the complex part of the dispersion relation for a particular mode is responsible for attenuation. Each mode has a corresponding modal function $u_\ell(x,t)$ whose solution is \[14, 15, 16, 17, 18\]

\[
u_\ell(x,t) = \frac{1}{\sqrt{2\pi}} \int S_\ell(k,0) e^{ikx-i\omega_\ell(k)t} dk \tag{10.9}
\]

where $S_\ell(k,0)$ is the initial spatial spectrum which is obtained from $u_\ell(x,0)$ by way of

\[
S_\ell(k,0) = \frac{1}{\sqrt{2\pi}} \int u_\ell(x,0) e^{-ikx} dx \tag{10.10}
\]

The superposition of the source response and the modal functions comprise the full wave.

\[
u(x,t) = u_f(x,t) + \sum_\ell u_\ell(x,t) \tag{10.11}
\]
Generally speaking, one is given initial conditions for the wave \( u(x, t) \) and not for each modal function. In section 10.2 we discuss how the modal functions are obtained from the initial conditions.

If one defines the time dependent spatial spectrum by [19, 20, 22, 23]

\[
S_\ell(k, t) = S_\ell(k, 0) e^{-i\omega_\ell(k)t} \tag{10.12}
\]

then \( u_\ell(x, t) \) and \( S_\ell(k, t) \) form Fourier transform pairs between \( x \) and \( k \) for all time

\[
\begin{align*}
  u_\ell(x, t) &= \frac{1}{\sqrt{2\pi}} \int S_\ell(k, t) e^{ikx} dk \\
  S_\ell(k, t) &= \frac{1}{\sqrt{2\pi}} \int u_\ell(x, t) e^{-ikx} dx
\end{align*} \tag{10.13, 10.14}
\]

The general solution for the homogenous equation is the sum of modal functions

\[
  u_H(x, t) = \sum_{\ell=1}^{M} u_\ell(x, t) \tag{10.15}
\]

Before proceeding, we establish the equation of motion for each modal function [24]. Differentiating Eq. (10.9) with respect to time

\[
\frac{\partial u_\ell(x, t)}{\partial t} = -i \frac{1}{\sqrt{2\pi}} \int \omega_\ell(k) S_\ell(k, 0) e^{ikx-\iota\omega_\ell(k)t} dk
\]

\[
= -i \frac{1}{\sqrt{2\pi}} \omega_\ell \left( \frac{1}{i} \frac{\partial}{\partial x} \right) \int S_\ell(k, 0) e^{ikx-\iota\omega_\ell(k)t} dk \tag{10.16, 10.17}
\]

and hence each modal function satisfies the differential equation

\[
i \frac{\partial u_\ell(x, t)}{\partial t} = \omega_\ell \left( \frac{1}{i} \frac{\partial}{\partial x} \right) u_\ell(x, t) \tag{10.18}
\]
Defining the wavenumber operator, $\mathcal{K}$, by

$$\mathcal{K} = \begin{cases} \frac{1}{i} \frac{d}{dx} & \text{in the position representation} \\ k & \text{in the wavenumber representation} \end{cases} \quad (10.19)$$

we write Eq. (10.18) as

$$i \frac{\partial u_\ell(x, t)}{\partial t} = \omega_\ell(\mathcal{K}) u_\ell(x, t) \quad (10.20)$$

This is a Schrödinger-like equation where the Hamiltonian is replaced by some function of $\mathcal{K}$. Since $\omega_\ell$ may be complex, the operator $\omega_\ell(\mathcal{K})$ may be non-Hermitian. Since this is a differential equation that is first order in time, we can write

$$u_\ell(x, t; t_0) = \int K_\ell(x, x') u_\ell(x', t_0) dx' \quad (10.21)$$

The function $K_\ell(x, x'; t, t_0)$ is called the propagator. The propagator satisfies

$$\left[ i \frac{\partial}{\partial t} - \omega_\ell(\mathcal{K}) \right] K_\ell(x, x'; t, t_0) = 0 \quad (10.22)$$

with the constraint that at $t = t_0$

$$K_\ell(x, x'; t, t) = \delta(x - x') \quad (10.23)$$

Each mode has its own propagator, which could be found directly from the mode, $\omega_\ell$, itself. The propagator for Eq. (10.21) is given by [19, 20]

$$K_\ell(x, x'; t, t_0) = K_\ell(x - x', t - t_0) = \frac{1}{2\pi} \int e^{ik(x-x')-i\omega_\ell(k)(t-t_0)} dk \quad (10.24)$$
Hence for this case the propagator is a function of only two variables, and we write

\[ u_\ell(x, t) = \int K_\ell(x - x', t - t_0) u_\ell(x', t_0) \, dx' \]  \hspace{1cm} (10.25)

10.2 Determination of the Modes From the Initial Conditions for the Wave

In general we are given the initial conditions for the wave and for its time derivatives. We now show how one can obtain the initial modal functions from these initial conditions. We assume full knowledge of the sources \( f(x, t) \). Once the initial modal functions are obtained, each evolves according to Eq. (10.21). For simplicity, we first consider the two-mode case, and subsequently give the results for an arbitrary number of modes.

Suppose that our wave equation has exactly two modes \( \omega_1(k) \) and \( \omega_2(k) \) with corresponding modal functions \( u_1(x, t) \) and \( u_2(x, t) \). The homogeneous part is

\[ u_H(x, t) = u_1(x, t) + u_2(x, t) \]  \hspace{1cm} (10.26)

and the full wave is

\[ u(x, t) = u_f(x, t) + u_1(x, t) + u_2(x, t) \]  \hspace{1cm} (10.27)

Now suppose we are given initially the full wave and its first time derivative

\[ u(x, 0) \]  \hspace{1cm} (10.28)

\[ \frac{\partial}{\partial t} u(x, 0) \]  \hspace{1cm} (10.29)
Then

\[ u(x, 0) = u_f(x, 0) + u_1(x, 0) + u_2(x, 0) \]  

(10.30)

\[ \frac{\partial}{\partial t} u(x, 0) = \frac{\partial}{\partial t} u_f(x, 0) + \frac{\partial}{\partial t} u_1(x, 0) + \frac{\partial}{\partial t} u_2(x, 0) \]  

(10.31)

Notice that the wave and source response term always enter together in the same way from the point of view of initial conditions

\[ \psi(x, t) = u(x, t) - u_f(x, t) = u_H(x, t) \]  

(10.32)

and therefore we write

\[ \psi(x, 0) = u_1(x, 0) + u_2(x, 0) \]  

(10.33)

\[ \frac{\partial}{\partial t} \psi(x, 0) = \frac{\partial}{\partial t} u_1(x, 0) + \frac{\partial}{\partial t} u_2(x, 0) \]  

(10.34)

But using Eq. (10.20) for each mode, Eq. (10.34) can be written as

\[ \frac{\partial}{\partial t} \psi(x, 0) = \frac{1}{i} \omega_1 (\mathcal{K}) u_1(x, 0) + \frac{1}{i} \omega_2 (\mathcal{K}) u_2(x, 0) \]  

(10.35)

Operating on Eq. (10.33) with \( \frac{1}{i} \omega_1 \left( \frac{\partial}{\partial x} \right) \) we have

\[ \frac{1}{i} \omega_1 (\mathcal{K}) \psi(x, 0) = \frac{1}{i} \omega_1 (\mathcal{K}) u_1(x, 0) + \frac{1}{i} \omega_1 (\mathcal{K}) u_2(x, 0) \]  

(10.36)
and then subtracting Eq. (10.35) from (10.36) we obtain

$$\omega_1(K)\psi(x,0) + \frac{1}{i}\frac{∂}{∂t}\psi(x,0) = [\omega_1(K) - \omega_2(K)]u_2(x,0) \quad (10.37)$$

One can think of this as a differential equation for $u_2(x,0)$. Symbolically, one can solve for $u_2(x,0)$

$$u_2(x,0) = \frac{\omega_1(K) + \frac{1}{i}\frac{∂}{∂t}\psi(x,0)}{\omega_1(K) - \omega_2(K)} \quad (10.38)$$

Similarly, for $u_1(x,0)$ we have

$$u_1(x,0) = \frac{\omega_2(K) + \frac{1}{i}\frac{∂}{∂t}\psi(x,0)}{\omega_2(K) - \omega_1(K)} \quad (10.39)$$

### 10.3 The Multi-Mode Case

Suppose that our wave equation admits $M$ modes and that we are given $M$ initial conditions of the form

$$u^{(j)}(x,0) = \frac{∂^j}{∂t^j}u(x,0) \quad j = 0, 1, \ldots, M - 1 \quad (10.40)$$

and assume that we know the source $f(x,t)$ at all times. Then at time $t = 0$ we have that

$$\psi(x,0) = \sum_{ℓ=1}^{M} u_ℓ(x,0) \quad (10.41)$$

$$\psi^{(j)}(x,0) = \sum_{ℓ=1}^{M} \frac{∂^j}{∂t^j}u_ℓ(x,0) \quad j = 1, \ldots, M - 1 \quad (10.42)$$

and rewriting these as per Eq. (10.20), we have

$$\psi^{(j)}(x,0) = \sum_{ℓ=1}^{M} \left[ \frac{1}{iω_ℓ(K)} \right]^j u_ℓ(x,0) \quad j = 0, \ldots, M - 1 \quad (10.43)$$
which in matrix notation is written as

\[
\begin{pmatrix}
\psi(x, 0) \\
\psi(1)(x, 0) \\
\vdots \\
\psi(M-1)(x, 0)
\end{pmatrix}
= 
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
\frac{1}{i} \omega_1(K) & \frac{1}{i} \omega_2(K) & \cdots & \frac{1}{i} \omega_M(K) \\
\vdots & \vdots & \ddots & \vdots \\
\left[\frac{1}{i} \omega_1(K)\right]^{M-1} & \left[\frac{1}{i} \omega_2(K)\right]^{M-1} & \cdots & \left[\frac{1}{i} \omega_M(K)\right]^{M-1}
\end{pmatrix}
\begin{pmatrix}
u_1(x, 0) \\
u_2(x, 0) \\
\vdots \\
u_M(x, 0)
\end{pmatrix}
\]

(10.44)

The modes are then obtained by the inversion of the matrix

\[
\begin{pmatrix}
u_1(x, 0) \\
u_2(x, 0) \\
\vdots \\
u_M(x, 0)
\end{pmatrix}
= 
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
\frac{1}{i} \omega_1(K) & \frac{1}{i} \omega_2(K) & \cdots & \frac{1}{i} \omega_M(K) \\
\vdots & \vdots & \ddots & \vdots \\
\left[\frac{1}{i} \omega_1(K)\right]^{M-1} & \left[\frac{1}{i} \omega_2(K)\right]^{M-1} & \cdots & \left[\frac{1}{i} \omega_M(K)\right]^{M-1}
\end{pmatrix}^{-1}
\begin{pmatrix}
\psi(x, 0) \\
\psi(1)(x, 0) \\
\vdots \\
\psi(M-1)(x, 0)
\end{pmatrix}
\]

(10.45)

Notice that in general, each modal function depends on all \(\psi^{(j)}\).
Chapter 11

Wavenumber Formulation

While the concept of modes is usually formulated in the space variable, we now show that a simpler formulation exists in wavenumber space. We define the appropriate wavenumber functions by

\[ S(k, t) = \frac{1}{\sqrt{2\pi}} \int u(x, t) e^{-ikx} \, dx \]

For the full wave \hspace{1cm} (11.1)

\[ S_f(k, t) = \frac{1}{\sqrt{2\pi}} \int u_f(x, t) e^{-ikx} \, dx \]

For the source response \hspace{1cm} (11.2)

\[ F(k, t) = \frac{1}{\sqrt{2\pi}} \int f(x, t) e^{-ikx} \, dx \]

For the source \hspace{1cm} (11.3)

\[ \varphi(k, t) = \frac{1}{\sqrt{2\pi}} \int \psi(x, t) e^{-ikx} \, dx \]

For the homogeneous wave \hspace{1cm} (11.4)

\[ S_\ell(k, t) = \frac{1}{\sqrt{2\pi}} \int u_\ell(x, t) e^{-ikx} \, dx \]

For each mode \hspace{1cm} (11.5)
and the corresponding inverses are

\[ u(x, t) = \frac{1}{\sqrt{2\pi}} \int S(k, t) e^{ikx} dk \] (11.6)

\[ u_f(x, t) = \frac{1}{\sqrt{2\pi}} \int S_f(k, t) e^{ikx} dk \] (11.7)

\[ f(x, t) = \frac{1}{\sqrt{2\pi}} \int F(k, t) e^{ikx} dk \] (11.8)

\[ \psi(x, t) = \frac{1}{\sqrt{2\pi}} \int \varphi(k, t) e^{ikx} dk \] (11.9)

\[ u_\ell(x, t) = \frac{1}{\sqrt{2\pi}} \int S_\ell(k, t) e^{ikx} dk \] (11.10)

The propagation equation, Eq. (10.1), written in terms of the operator \( \mathcal{K} \) is then

\[
\left( \sum_{n=0}^{N_x} b_n i^n \mathcal{K}^n - \sum_{n=0}^{N_t} a_n \frac{\partial^n}{\partial t^n} \right) u(x, t) = f(x, t)
\] (11.11)

and we can immediately write the equation of evolution for \( S(k, t) \)

\[
\left( \sum_{n=0}^{N_x} b_n i^n k^n - \sum_{n=0}^{N_t} a_n \frac{\partial^n}{\partial t^n} \right) S(k, t) = F(k, t)
\] (11.12)

Also, corresponding to Eq. (10.3) we write

\[ \mathcal{L} S(k, t) = F(k, t) \] (11.13)

where now \( \mathcal{L} \) is expressed in the wavenumber representation

\[ \mathcal{L} = \sum_{n=0}^{N_x} b_n i^n k^n - \sum_{n=0}^{N_t} a_n \frac{\partial^n}{\partial t^n} \] (11.14)
The solution to Eq. (11.13) is given by

\[ S(k, t) = S_F(k, t) + S_H(k, t) \]  

(11.15)

where \( S_H(k, t) \) is the solution of the homogeneous equation

\[
\left( \sum_{n=0}^{M} b_n i^n k^n - \sum_{n=0}^{N} a_n \frac{\partial^n}{\partial t^n} \right) S_H(k, t) = 0
\]  

(11.16)

which is the sum of modal functions in the wavenumber domain

\[ S_H(k, t) = \sum_{\ell=1}^{M} S_\ell(k, t) \]  

(11.17)

and \( S_F(k, t) \) is the source response (a particular solution) of Eq. (11.12). We can see from Eq. (10.12), that as Eq. (10.20) suggests, each wavenumber modal function satisfies

\[ i \frac{\partial}{\partial t} S_\ell(k, t) = \omega_\ell(k) S_\ell(k, t) \]  

(11.18)

and the part is

\[ S_\ell(k, t) = S_\ell(k, t_0) e^{-i\omega_\ell(k)(t-t_0)} \]  

(11.19)

The homogeneous solution, Eq. (11.17), is then

\[ S_H(k, t) = \sum_{\ell=1}^{M} S_\ell(k, 0) e^{-i\omega_\ell(k)t} \]  

(11.20)

This is a significant simplification compared to the position representation.

If we define the momentum propagator, \( K_\ell \), in analogy to Eq. (10.21), by

\[ S_\ell(k, t) = \int_{-\infty}^{\infty} K_\ell(k - k', t - t_0) S_\ell(k', t_0) dk' \]  

(11.21)
then

\[ K_\ell(k - k', t - t_0) = \delta(k - k')e^{-i\omega_\ell(k)(t-t_0)} \] (11.22)

### 11.1 Obtaining the Modes

As before, first consider the two-mode case

\[ S(k, t) = S_f(k, t) + S_1(k, t) + S_2(k, t) \] (11.23)

Given the initial conditions at time \( t = 0 \)

\[ S(k, 0) = S_f(k, 0) + S_1(k, 0) + S_2(k, 0) \] (11.24)

and

\[ \frac{\partial S(k, 0)}{\partial t} = \frac{\partial S_f(k, 0)}{\partial t} + \frac{1}{i}\omega_1(k)S_1(k, 0) + \frac{1}{i}\omega_2(k)S_2(k, 0) \] (11.25)

we define the analog of Eq. (10.32) in the wavenumber representation

\[ \varphi(k, t) = S(k, t) - S_f(k, t) \] (11.26)

and Eqs. (11.24) and (11.25) become

\[ \varphi(k, 0) = S_1(k, 0) + S_2(k, 0) \] (11.27)

\[ \frac{\partial \varphi(k, 0)}{\partial t} = \frac{1}{i}\omega_1(k)S_1(k, 0) + \frac{1}{i}\omega_2(k)S_2(k, 0) \] (11.28)
Solving gives

\[ S_2(k) = \frac{\omega_1(k) + \frac{1}{i} \frac{\partial}{\partial t} \varphi(k,0)}{\omega_1(k) - \omega_2(k)} \]  

(11.29)

\[ S_1(k) = \frac{\omega_2(k) + \frac{1}{i} \frac{\partial}{\partial t} \varphi(k,0)}{\omega_2(k) - \omega_1(k)} \]  

(11.30)

which match equations (10.38) and (10.39). This two mode case has been previously given by Whithem [18].

### 11.2 Multi-Mode (Wavenumber) Case

Now, generalizing to the multi-mode case, we define the analog of Eq. (10.43)

\[ \varphi(j)(k,t) = \frac{\partial^j}{\partial t^j} \varphi(k,t) = \sum_{\ell=1}^{M} \frac{\partial^j}{\partial t^j} S_\ell(k,t) = \sum_{\ell=1}^{M} \left[ \frac{1}{i} \omega_\ell (k) \right]^j S_\ell(k,t) \]  

(11.31)

Then in general we have

\[
\begin{pmatrix}
\varphi(k,0) \\
\varphi^{(1)}(k,0) \\
\vdots \\
\varphi^{(M-1)}(k,0)
\end{pmatrix}
= 
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
\frac{1}{i} \omega_1(k) & \frac{1}{i} \omega_2(k) & \cdots & \frac{1}{i} \omega_M(k) \\
\vdots & \vdots & \ddots & \vdots \\
\left[ \frac{1}{i} \omega_1(k) \right]^{M-1} & \left[ \frac{1}{i} \omega_2(k) \right]^{M-1} & \cdots & \left[ \frac{1}{i} \omega_M(k) \right]^{M-1}
\end{pmatrix}
\begin{pmatrix}
S_1(k,0) \\
S_2(k,0) \\
\vdots \\
S_M(k,0)
\end{pmatrix}
\]  

(11.32)

yielding

\[
\begin{pmatrix}
S_1(k,0) \\
S_2(k,0) \\
\vdots \\
S_M(k,0)
\end{pmatrix}
= 
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
\frac{1}{i} \omega_1(k) & \frac{1}{i} \omega_2(k) & \cdots & \frac{1}{i} \omega_M(k) \\
\vdots & \vdots & \ddots & \vdots \\
\left[ \frac{1}{i} \omega_1(k) \right]^{M-1} & \left[ \frac{1}{i} \omega_2(k) \right]^{M-1} & \cdots & \left[ \frac{1}{i} \omega_M(k) \right]^{M-1}
\end{pmatrix}
^{-1}
\begin{pmatrix}
\varphi(k,0) \\
\varphi^{(1)}(k,0) \\
\vdots \\
\varphi^{(M-1)}(k,0)
\end{pmatrix}
\]  

(11.33)
Chapter 12

The Source Term

In this chapter, we review the propagation effects of the source-response term and its formulation in terms of a Green’s function [8, 25]. We write the response of the source term as

\[ u_f(x, t) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dt' G(x, x'; t, t') f(x', t') \]  (12.1)

The Green’s function is the sum of two functions

\[ G(x-x', t-t') = G_R(x-x', t-t') + G_A(x-x', t-t') \]  (12.2)

where

\[ G_R(x, x'; t, t') = \eta(t-t')G(x, x'; t, t') \]  (12.3)

\[ G_A(x, x'; t, t') = \eta(t'-t)G(x, x'; t, t') \]  (12.4)

These are called the ‘retarded’ and ‘advanced’ Green’s functions, respectively. They are used if one wants to consider the effects of the sources due only to past or to future times.
For instance, for philosophical reasons, if one does not want to consider the effects on the wave at some time due to sources at a later time, then one would use the retarded Green’s function. We mention however, that these are not Green’s functions at all. We believe that the proper way of handling such situations is to truncate the source. A truncated source is a valid source, and using the full Green’s function will give a valid wave. We consider retarded and advanced Green’s functions an equivalent, yet misleading viewpoint.

Also, without going into too much detail, we note that in the case of wave equations that are first order in derivative, the Green’s function actually is truncated in time, a fact which probably contributes to this confusion.

The response due to the past and future may be written as

\[
    u_f(x, t) = u_f^R(x, t) + u_f^A(x, t)
\]  

(12.5)

where

\[
    u_f^R(x, t) = \int_{-\infty}^{t} dt' \int G(x - x', t - t') f(x', t') dx'
\]  

(12.6)

\[
    u_f^A(x, t) = \int_{t}^{\infty} dt' \int G(x - x', t - t') f(x', t') dx'
\]  

(12.7)

**Finite Starting Time.** Of particular interest is the case where the source is turned on at a particular time, say \( t = 0 \). Hence in Eq. (12.6) and (12.7) we substitute \( \eta(t) f(x, t) \), where \( \eta(t) \) is the step function, for \( f(x, t) \) to obtain

\[
    u_f^R(x, t) = \begin{cases} 
        \int_{0}^{t} dt' \int G(x - x', t - t') f(x', t') dx' & t > 0 \\
        0 & t < 0 
    \end{cases}
\]  

(12.8)
and

\[
    u^A_f(x, t) = \begin{cases} 
    \int_t^\infty dt' \int G(x - x', t - t') f(x', t') dx' & t > 0 \\
    \int_0^\infty dt' \int G(x - x', t - t') f(x', t') dx' & t < 0 
\end{cases} \tag{12.9}
\]

**First Order Equations.** If a differential equation is first order in time then it has only a single mode, and one can write the solution as

\[
    u(x, t) = \int K(x, x'; t, t_0) u(x', t_0) dx' + \int G(x, x'; t, t') f(x', t') dx' dt' \tag{12.10}
\]

In such a case the retarded Green’s function and propagator are related by

\[
    G_R(x, x'; t, t') = \eta(t - t') K(x, x'; t, t') \tag{12.11}
\]
Chapter 13

Examples

We now consider a number of examples to illustrate the mode formulation of solving wave equations. We will explicitly carry out the procedure for finding the modes and modal functions for different wave equations, including the one dimensional wave equation (for a string) and the beam equation, and we also describe how to find the effects due to sources. The advantage of the mode formulation is that the procedure is direct as the examples will show.

13.1 Example 1: The Free Space Wave Equation with Source Terms

Consider the case of the free space wave equation where the solution (the d’Alembert solution) is well known [18]. We derive it here using the above formulation.

The wave equation with sources is given by

\[
\left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u(x, t) = f(x, t) \quad (13.1)
\]
For the homogeneous equation

\[
\left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u_H(x, t) = 0 \quad (13.2)
\]

upon substituting \( e^{ikx-i\omega t} \) we find the dispersion relation

\[
\omega^2 = c^2 k^2 \quad (13.3)
\]

which has two modes

\[
\omega_1(k) = ck \quad \omega_2(k) = -ck \quad (13.4)
\]

For the first mode we have the modal function

\[
u_1(x, t) = \frac{1}{\sqrt{2\pi}} \int S_1(k, 0) e^{ikx-i\omega_1(k)t} dk = \frac{1}{\sqrt{2\pi}} \int S_1(k, 0) e^{ik(x-ct)} dk \quad (13.5)\]

Therefore

\[
u_1(x, t) = \nu_1(x - ct, 0) \quad (13.6)
\]

and similarly, the second mode corresponds to a modal function of the form

\[
u_2(x, t) = \nu_2(x + ct, 0) \quad (13.7)
\]

Using Eqs. (10.27), (13.6), and (13.7),

\[
u(x, t) = \nu_f(x, t) + \nu_1(x - ct, 0) + \nu_2(x + ct, 0) \quad (13.8)
\]

We now determine the initial modal functions in terms of the initial conditions for the wave and its first time derivative (assuming that the sources are known). Using Eq. (10.38)
we have

$$u_2(x, 0) = \frac{\omega_1(K) + \frac{1}{i} \frac{\partial}{\partial t} \psi(x, 0)}{\omega_1(K) - \omega_2(K)} \psi(x, 0) \quad (13.9)$$

$$= \frac{1}{2} \psi(x, 0) - \frac{i}{2cK} \frac{\partial}{\partial t} \psi(x, 0) \quad (13.10)$$

$$= \frac{1}{2} \psi(x, 0) + \frac{1}{2c} \int_0^x \frac{\partial \psi(x', 0)}{\partial t} dx' \quad (13.11)$$

where we have used the fact that for any function $f(x)$

$$\frac{1}{K} f(x) = \int_0^x f(x') dx' \quad (13.12)$$

where the lower limit of integration $x' = 0$ contributes a constant term which will be discussed in what follows. We have from (10.39) that the first modal function is

$$u_1(x, 0) = \frac{1}{2} \left[ \psi(x, 0) - \frac{1}{c} \int_0^x \frac{\partial \psi(x', 0)}{\partial t} dx' \right] \quad (13.13)$$

and substituting Eq. (13.11) and (13.13) into Eq. (13.8) (and using def. (10.32)) we have

$$\psi(x, t) = \frac{1}{2} \left[ \psi(x - ct, 0) - \frac{1}{c} \int_0^{x-ct} \frac{\partial \psi(x', 0)}{\partial t} dx' \right]$$

$$+ \frac{1}{2} \left[ \psi(x + ct, 0) + \frac{1}{c} \int_0^{x+ct} \frac{\partial \psi(x', 0)}{\partial t} dx' \right] \quad (13.14)$$

So the propagator which takes $\psi(x, 0) \rightarrow \psi(x, t)$ for this two-mode case is

$$\psi(x, t) = \frac{1}{2} \left[ \psi(x - ct, 0) + \psi(x + ct, 0) + \frac{1}{c} \int_{x-ct}^{x+ct} \frac{\partial \psi(x', 0)}{\partial t} dx' \right] \quad (13.15)$$
But we want the full wave, which is

\[ u(x, t) = u_f(x, t) + \frac{1}{2} \left[ \psi(x - ct, 0) + \psi(x + ct, 0) + \frac{1}{c} \int_{x-ct}^{x+ct} \frac{\partial \psi(x', 0)}{\partial t} dx' \right] \quad (13.16) \]

Notice that the constant terms which came from the integrations' lower limit \( x' = 0 \) in Eq. (13.11) and in Eq. (13.13) cancelled each other; so the lower limit \( x' = 0 \) in Eq. (13.12) did not matter. Explicitly, the wave is given by

\[ u(x, t) = \frac{1}{2} \begin{bmatrix} u(x - ct, 0) + u(x + ct, 0) - u_f(x - ct, 0) - u_f(x + ct, 0) \\ + \frac{1}{c} \int_{x-ct}^{x+ct} \left[ \frac{\partial u(x', 0)}{\partial t} - \frac{\partial u_f(x', 0)}{\partial t} \right] dx' \end{bmatrix} + u_f(x, t) \quad (13.17) \]

If we consider the standard textbook case where there is no source term, then \( u_f(x, t) = 0 \), and \( \psi(x, t) = u(x, t) \), so we have

\[ u(x, t) = \frac{1}{2} \left[ u(x - ct, 0) + u(x + ct, 0) + \frac{1}{c} \int_{x-ct}^{x+ct} \frac{\partial u(x', 0)}{\partial t} dx' \right] \quad (13.18) \]

which is the d’Alembert solution. We consider the more general case where the source \( f \) is non zero.

### 13.1.1 Wavenumber Representation

We now obtain the result in the wavenumber representation. From Eqs. (11.29) and (11.30) we have

\[ S_2(k, 0) = \frac{\varphi(k, 0)}{2} - i \frac{\partial \varphi(k, 0)}{2ck \partial t} \quad (13.19) \]
\[ S_1(k, 0) = \frac{\varphi(k, 0)}{2} + i \frac{\partial \varphi(k, 0)}{2ck \partial t} \quad (13.20) \]
and using Eq. (11.19) we obtain

\[ \varphi(k, t) = S_1(k, 0) e^{-i\omega_1(k)t} + S_2(k, 0) e^{-i\omega_2(k)t} \]  
\[ = S_1(k, 0) e^{-ickt} + S_2(k, 0) e^{ickt} \]  

(13.21) (13.22)

which using Eqs. (13.19) and (13.20) evaluates to

\[ \varphi(k, t) = \cos(ckt)\varphi(k, 0) + \frac{\sin(ckt)}{ck} \frac{\partial \varphi(k, 0)}{\partial t} = \left( \cos(ckt) + \frac{\sin(ckt)}{ck} \frac{\partial}{\partial t} \right) \varphi(k, 0) \]  

(13.23)

Since the two modes are similar, we define \( \Omega \) to be

\[ \Omega = ck = \omega_1(k) = -\omega_2(k) \]  

(13.24)

so \( \varphi(k, t) \) is determined from the initial conditions by

\[ \varphi(k, t) = \left( \cos(\Omega t) + \frac{\sin(\Omega t)}{\Omega} \frac{\partial}{\partial t} \right) \varphi(k, 0) \]  

(13.25)

This can also be written as

\[ \varphi(k, t) = \frac{\partial}{\partial t} \left( \frac{\sin(\Omega t)}{\Omega} \varphi(k, 0) \right) \]  

(13.26)

where we define

\[ \frac{\partial}{\partial t} \varphi(k, 0) = \varphi^{(1)}(k, 0) \]  

(13.27)

as an initial condition.
13.1.2 The Wave Equation with Source Terms in the Spatial Representation

The wave equation with source terms is given by Eq. (13.1)
\[ \mathcal{L} u(x, t) = \left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u(x, t) = f(x, t) \] (13.28)

To calculate the response of the wave to the source, we use the Green’s function for this particular situation.

The Green’s function for this problem is the solution of
\[ \mathcal{L} G(x - x', t - t') = \left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(x - x', t - t') = \delta(x - x')\delta(t - t') \] (13.29)

and the retarded response is then
\[ u^R_f(x, t) = \int_{-\infty}^{t} dt' \int dx' G(x - x', t - t') f(x', t') \] (13.30)

The Green’s function for this problem is given by [8]
\[ G(x - x', t - t') = \frac{c^2}{4\pi} \int \frac{e^{ik|x-x'|}}{i k} e^{ick(t-t')} dk \] (13.31)

In the retarded case, \( t > t' \), the Green’s function evaluates to
\[ G^R(x, x'; t, t') = \int c \frac{e^{ik|x-x'|}}{2ik} e^{-ick(t-t')} dk = c \int_{c-i\infty}^{c+i0} dc' \int \frac{\partial}{\partial c} \frac{e^{ik|x-x'|}}{2ik} e^{-ic'k(t-t')} dk \] (13.32)
\[ = -\frac{c}{2} \int_{c-i\infty}^{c+i0} dc' \int e^{ik|x-x'|-ic'k(t-t')} dk \] (13.33)
We cancel the denominator by performing a derivative and an integration with respect to $c'$. The limits of integration are chosen so that the action of the integral following the derivative will return the original integrand. These took into account that we are considering the retarded case ($t' < t$).

Then by definition of absolute value

$$G_R(x, x'; t, t') = -c \frac{t - t'}{2} \int_{c-i\infty}^{c+i0} dc' \int \left\{ \eta(x' - x)e^{ik[(x' - x) - c'(t-t')]} + \eta(x - x')e^{-ik[(x' - x) + c'(t-t')]} \right\} dk$$

(13.34)

Performing the integration over $k$

$$G^R(x, x'; t, t') = -2\pi c \frac{t - t'}{2} \int_{c-i\infty}^{c+i0} dc' \left\{ \eta(x' - x) \cdot \delta[(x' - x) - c'(t-t')] + \eta(x - x') \cdot \delta[(x' - x) + c'(t-t')] \right\}$$

(13.35)

$$= -\pi c \int_{c-i\infty}^{c+i0} dc' \left\{ \eta(x' - x) \cdot \delta \left[ \frac{x' - x}{t - t'} - c' \right] + \eta(x - x') \cdot \delta \left[ \frac{x' - x}{t - t'} + c' \right] \right\}$$

(13.36)

Since $x, x', t, t'$ are all real, the Dirac delta functions are non-zero only if $c'$ is real; then we deform the contour so that we get some integration along the Real line (fig. 13.1).

Performing the contour integration, the integral evaluates to
\[ G^R(x, x'; t, t') = -\pi c \int_C dc' \left\{ \eta(x' - x) \cdot \delta \left[ \frac{x' - x}{t - t'} - c' \right] + \eta(x - x') \cdot \delta \left[ \frac{x' - x}{t - t'} + c' \right] \right\} \]

\[ = -\pi c \int_{-\infty}^{c=t} dc' \left\{ \eta(x' - x) \cdot \delta \left[ \frac{x' - x}{t - t'} - c' \right] + \eta(x - x') \cdot \delta \left[ \frac{x' - x}{t - t'} + c' \right] \right\} \]

\[ = -\pi c \eta (x' - x) \eta \left( \frac{c(t - t') - x' + x}{t - t'} \right) - \pi c \eta (x - y) \eta \left( \frac{c(t - t') - x + x'}{t - t'} \right) \]
Using the Green’s function which we found above, the source-response is

\[
uf(x, t) = \int_{-\infty}^{t} dt' \int G(x, x'; t, t') f(x', t') dx'
\]

(13.40)

\[
= -\pi c \int_{-\infty}^{t} dt' \int \left\{ \eta [x' - x] \eta \left( c - \frac{x' - x}{t - t'} \right) + \eta [x - x'] \eta \left( c - \frac{x - x'}{t - t'} \right) \right\} f(x', t') dx'
\]

(13.41)

\[
= -\pi c \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} \eta \left( c - \frac{x - x'}{t - t'} \right) s(x', t') dx' + \pi c \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} \eta \left( c - \frac{x - x'}{t - t'} \right) f(x', t') dx'
\]

(13.42)

\[
= -\pi c \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} \eta \left( c - \frac{x - x'}{t - t'} \right) f(x', t') dx' + \pi c \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} \eta \left( c - \frac{x - x'}{t - t'} \right) f(x', t') dx'
\]

(13.43)

Now, considering each term separately, the first term, Eq. (13.43) is

\[
\int_{-\infty}^{t} dt' \int_{x}^{\infty} \eta \left( \frac{c(t - t') - (x' - x)}{t - t'} \right) f(x', t') dx'
\]

(13.45)

\[
= \int_{-\infty}^{t} dt' \int_{x}^{x + c(t - t')} \eta \left( \frac{x + c(t - t') - x'}{t - t'} \right) f(x', t') dx'
\]

(13.46)

and the second term, Eq. (13.44) is

\[
\int_{-\infty}^{t} dt' \int_{-\infty}^{x} \eta \left( \frac{c(t - t') - (x - x')}{t - t'} \right) f(x', t') dx'
\]

(13.47)

\[
= \int_{-\infty}^{t} dt' \int_{x - c(t - t')}^{x} \eta \left( \frac{x' - \{x - c(t - t')\}}{t - t'} \right) f(x', t') dx'
\]

(13.48)
Combining the two terms, Eqs. (13.46) and (13.48), we finally get

\[
uf(x, t) = -\pi c \int_{-\infty}^{t} dt' \int_{x}^{x+c(t-t')} f(x', t')dx' \\
- \pi c \int_{-\infty}^{t} dt' \int_{x-c(t-t')}^{x} f(x', t')dx' \\
= -\pi c \int_{-\infty}^{t} dt' \int_{x-c(t-t')}^{x+c(t-t')} f(x', t')dx' 
\]

(13.49)

(13.50)

Finally, the retarded Green’s function is

\[
G_R(x, x'; t, t' < t) = \int c \frac{e^{ik|x-x'|}}{2ik} e^{-i\mathcal{K}(t-t')} dk = -\pi c \eta(t-t')\eta(x + c(t-t') - x')\eta\left(x' - [x - c(t-t')]\right) 
\]

(13.51)

Notice that this is a function of the difference in position \(\Delta x = x - x'\) and the time difference \(\Delta t = t - t'\)

\[
G_R(x - x', t - t') = -\pi c \left[\eta(x' - x)\eta \left(c + \frac{x - x'}{t - t'}\right) + \eta(x - x')\eta \left(c - \frac{x - x'}{t - t'}\right)\right] 
\]

(13.52)

or

\[
G_R(\Delta x, \Delta t) = -\pi c \left[\eta(-\Delta x)\eta \left(c + \frac{\Delta x}{\Delta t}\right) + \eta(\Delta x)\eta \left(c - \frac{\Delta x}{\Delta t}\right)\right] 
\]

(13.53)

where as before, \(\eta(x)\) is the step function.

This expression is similar to the integral term in Eq. (13.17); only that in Eq. (13.50), we also have an additional integration over \(t'\). The integral term in Eq. (13.17) has to do with the initial condition for the wave, whereas Eq. (13.50) is dealing with the effects of the source.
13.1.3 The Wave Equation with Sources in the Wavenumber Representation

In the wavenumber representation, the wave equation operator becomes

\[ \mathcal{L} = -k^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \]  \hfill (13.54)

and the wave equation with sources is

\[ \mathcal{L} S(k, t) = F(k, t) \]  \hfill (13.55)

To calculate the response to this source, we use a Green’s function that satisfies

\[ \mathcal{L} G_k(k, k'; t, t') = \delta(k - k')\delta(t - t') \]  \hfill (13.56)

The Green’s function for this situation is

\[ G_k(k, k'; t, t') = -\frac{c^2}{2i ck} \delta(k - k')e^{ick|t-t'|} \]  \hfill (13.57)

which can be used to find the source term in a way that is analogous to that in Eq. (13.30)

\[ S_F(k, t) = \int G_k(k, k'; t, t')F(k', t')dt'dk' \]  \hfill (13.58)

However, in the wavenumber representation, we can also use an alternate version of the Green’s function Eq. (13.57)

\[ G_k'(k, t - t') = -\frac{c^2}{i2ck} e^{ick|t-t'|} \]  \hfill (13.59)
To calculate $S_F$ from $G'_k$

$$S_F(k,t) = \int G'_k(k,t-t') F(k,t') dt' \quad (13.60)$$

Note that the form of the solution is a single integral. Eq. (13.58) can be shown to be equivalent to Eq. (13.60) by performing the $k'$-integration.

The retarded (reduced) Green’s function is

$$G'^R_k(k,t-t') = -\frac{c^2}{2\imath k} e^{\imath k(t-t')} \quad (13.61)$$

and the retarded source response is

$$S_F(k,t) = \int_{-\infty}^{t} G'^R_k(k,t-t') F(k,t') dt' = -\int_{-\infty}^{t} \frac{c^2}{2\imath k} F(k,t') dt' \quad (13.62)$$

### 13.2 Example 2: The Beam Equation

The beam equation [18] with sources is given by

$$\mathcal{L} u(x,t) = \left( \frac{\partial^4}{\partial x^4} + \frac{4}{\alpha^2} \frac{\partial^2}{\partial t^2} \right) u(x,t) = f(x,t) \quad (13.63)$$

and has two modes

$$\omega_1(k) = \alpha k^2/2 \quad \omega_2(k) = -\alpha k^2/2 \quad (13.64)$$

Using Eqs. (10.20) and (13.64), the equation of motion for the modal function corresponding to the first mode is

$$i \frac{\partial u_1(x,t)}{\partial t} = \frac{\alpha}{2} \left( \frac{1}{i \frac{\partial}{\partial x}} \right)^2 u_1(x,t) \quad (13.65)$$
and in wavenumber space it is

\[ i \frac{\partial}{\partial t} S_1(k, t) = \alpha \frac{k^2}{2} S_1(k, t) \] (13.66)

This is the same as Schrödinger’s equation for a free particle with \( \alpha = 1/m \) and \( \hbar = 1 \). The position space propagator is

\[ K_1(x - x', t - t_0) = \frac{1}{\sqrt{2\pi \alpha (t - t_0)}} \exp \left[ -\frac{(x' - x)^2}{2i\alpha(t - t_0)} \right] \] (13.67)

and hence

\[ u_1(x, t) = \frac{1}{\sqrt{2\pi \alpha t}} \int u_1(x', 0) \exp \left[ -\frac{(x' - x)^2}{2i\alpha t} \right] dx' \] (13.68)

For the second mode, we note that its propagator can be obtained from the first by letting \( \alpha \to -\alpha \)

\[ K_2(x - x', t - t_0) = \frac{1}{\sqrt{-2\pi \alpha (t - t_0)}} \exp \left[ \frac{(x' - x)^2}{2i\alpha(t - t_0)} \right] \] (13.69)

To get the modal functions we use Eq. (10.38) and (10.39) to obtain

\[ u_2(x, 0) = \frac{\omega_1(K) - i \frac{\partial}{\partial \alpha}}{\omega_1(K) - \omega_2(K)} \psi(x, 0) \] (13.70)

\[ = \frac{1}{2} \psi(x, 0) - i \frac{1}{\alpha K^2} \frac{\partial}{\partial t} \psi(x, 0) \] (13.71)

\[ = \frac{1}{2} \psi(x, 0) - i \frac{1}{\alpha} \left( \frac{1}{i} \frac{\partial^2}{\partial x^2} \right) \frac{\partial}{\partial t} \psi(x, 0) \] (13.72)

and similarly,

\[ u_1(x, 0) = \frac{1}{2} \psi(x, 0) + i \frac{1}{\alpha} \left( \frac{1}{i} \frac{\partial^2}{\partial x^2} \right) \frac{\partial}{\partial t} \psi(x, 0) \] (13.73)
Therefore

\[
\psi(x, t) = \int K_1(x - x', t)u_1(x', 0)dx' + \int K_2(x - x', t)u_2(x', 0)dx' \tag{13.74}
\]

\[
= \int K_1(x - x', t) \left[ \frac{1}{2} \psi(x', 0) + \frac{i}{\alpha} \left( \frac{1}{\alpha^2} \frac{\partial^2}{\partial x'^2} \right) \frac{\partial}{\partial t} \psi(x', 0) \right] dx'
\]

\[
+ \int K_2(x - x', t) \left[ \frac{1}{2} \psi(x', 0) - \frac{i}{\alpha} \left( \frac{1}{\alpha^2} \frac{\partial^2}{\partial x'^2} \right) \frac{\partial}{\partial t} \psi(x', 0) \right] dx' \tag{13.75}
\]

\[
= \frac{1}{2} \int dx' \left[ K_1(x - x', t) + K_2(x - x', t) \right] \psi(x', 0)
\]

\[
- \frac{i}{\alpha} \int dx' \left[ K_1(x - x', t) - K_2(x - x', t) \right] \frac{1}{\left( \frac{\partial^2}{\partial x'^2} \right)} \frac{\partial}{\partial t} \psi(x', 0) \tag{13.76}
\]

### 13.2.1 Wavenumber Representation

We now discuss the modal functions of the beam equation in the wavenumber domain. In this case, Eq. (11.20) gives

\[
\varphi(k, t) = S_1(k, 0) e^{-i\alpha k^2 t/2} + S_2(k, 0) e^{+i\alpha k^2 t/2} \tag{13.77}
\]

Eqs. (11.29) and (11.30) give us the two initial modal functions

\[
S_2(k, 0) = \frac{\varphi(k, 0)}{2} - \frac{i}{\alpha k^2} \frac{\partial \varphi(k, 0)}{\partial t} \tag{13.78}
\]

\[
S_1(k, 0) = \frac{\varphi(k, 0)}{2} + \frac{i}{\alpha k^2} \frac{\partial \varphi(k, 0)}{\partial t} \tag{13.79}
\]

and hence, from Eqs. (13.77), (13.78), and (13.79)

\[ \varphi(k, t) = \cos(\alpha k^2 t/2) \varphi(k, 0) + \frac{\sin(\alpha k^2 t/2)}{\alpha k^2/2} \frac{\partial \varphi(k, 0)}{\partial t} \] (13.80)

\[ = \left( \cos(\alpha k^2 t/2) + \frac{\sin(\alpha k^2 t/2)}{\alpha k^2/2} \frac{\partial}{\partial t} \right) \varphi(k, 0) \] (13.81)

Since the two modes of the beam equation are similar, we define

\[ \Omega = \alpha k^2/2 = \omega_1(k) = -\omega_2(k) \] (13.82)

and \( \varphi(k, t) \) could be obtained from the initial conditions by

\[ \varphi(k, t) = \left( \cos(\Omega t) + \frac{\sin(\Omega t)}{\Omega} \frac{\partial}{\partial t} \right) \varphi(k, 0) \] (13.83)

which has the exact same form as Eq. (13.25). We can also write

\[ \varphi(k, t) = \frac{\partial}{\partial t} \left( \frac{\sin(\Omega t)}{\Omega} \varphi(k, 0) \right) \] (13.84)

where we define

\[ \frac{\partial}{\partial t} \varphi(k, 0) = \varphi^{(1)}(k, 0) \] (13.85)

to be an initial condition.

### 13.2.2 The Beam Equation: Source Term in the Spatial Domain

The beam equation with a source term is Eq. (13.63)

\[ \mathcal{L} u(x, t) = \left( \frac{\partial^4}{\partial x^4} + 4 \frac{\partial^2}{\alpha^2 \partial t^2} \right) u(x, t) = f(x, t) \] (13.86)
To calculate the response to the source, we need the Green’s function that satisfies

\[ \mathcal{L} G(x, x'; t, t') = \delta(x - x')\delta(t - t') \]  

(13.87)

The Green’s function for this situation is

\[ G(x, x'; t, t') = G(x - x', t - t') = \frac{c^2}{4\pi} \int \frac{e^{i\frac{k^2}{2\alpha^2}|t-t'|}}{i\alpha k^2} e^{i\alpha(x-x')}dk \]  

(13.88)

and the retarded response, \( u^R_f(x, t) \) is calculated by way of Eq. (13.30)

\[ u^R_f(x, t) = \int_{-\infty}^{t} dt' \int dx' G^R(x - x', t - t') f(x', t') \]  

(13.89)

### 13.2.3 The Beam Equation: Source Term in the Wavenumber Representation

In the wavenumber representation, the beam equation operator becomes

\[ \mathcal{L} = k^4 + \frac{4}{\alpha^2} \frac{\partial^2}{\partial t^2} \]  

(13.90)

and the beam equation with a source term becomes

\[ \mathcal{L} S(k, t) = \left( k^4 + \frac{4}{\alpha^2} \frac{\partial^2}{\partial t^2} \right) S(k, t) = F(k, t) \]  

(13.91)

We seek a Green’s function which satisfies

\[ \mathcal{L} G_k(k', t, t') = \delta(k - k')\delta(t - t') \]  

(13.92)
The solution is
\[ G_k(k, k'; t, t') = \frac{c^2}{2i\epsilon k^2} \delta(k - k')e^{ick^2|t - t'|} \] (13.93)
which can be used to calculate the response to our source
\[ S_F(k, t) = \int G_k(k, k'; t, t') F(k', t') dk' dt' \] (13.94)

However, we can also use the reduced/stripped version of our Green’s function
\[ G'_k(k, t - t') = \frac{c^2}{2i\epsilon k^2} e^{ick^2|t - t'|} \] (13.95)
and evaluate the response by way of
\[ S'_F(k, t) = \int G'_k(k, t - t') F(k, t') dt' \] (13.96)

Notice that the form of this solution has a single integral, compared with the double integral in Eq. (13.94). Eq. (13.94) can be shown to be equivalent to Eq. (13.96) by performing the \( k' \)-integration.

The retarded Green’s function is
\[ G'_R(k, t - t') = \frac{c^2}{2i\epsilon k^2} e^{ick^2(t - t')} \] (13.97)
and therefore, the retarded source response is
\[ S_R(k, t) = \int_{-\infty}^{t} G'_R(k, t - t') F(k, t') dt' = \int_{-\infty}^{t} \frac{c^2}{2i\epsilon k^2} e^{ick^2(t - t')} F(k, t') dt' \] (13.98)
### 13.3 Example 3: A Non-Separable Wave Equation

In this example we consider the case of the following wave equation which is given in reference [26] as an example where the method of separation of variables does not work

\[
\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial t^2 \partial x^2} + \frac{\partial^4}{\partial t^4}\right) u(x,t) = 0 \tag{13.99}
\]

We will consider the more general case

\[
\left(\frac{\partial^4}{\partial x^4} + b \frac{\partial^4}{\partial t^2 \partial x^2} + a \frac{\partial^4}{\partial t^4}\right) u(x,t) = 0 \tag{13.100}
\]

We now show that the solution can be achieved in a straightforward way by using modes. We note here that this equation is not of the form of Eq. (10.1), but of the more general structure allowed by Eq. (10.2). Substituting \(e^{ikx-i\omega t}\) in Eq. (13.100) results in the following dispersion relation

\[
a\omega^4 + b\omega^2 k^2 + k^4 = 0 \tag{13.101}
\]

which has four modes, assuming that \(b^2 - 4a \neq 0\)

\[
\omega = \pm k \sqrt{-\frac{b \pm \sqrt{b^2 - 4a}}{2a}} \tag{13.102}
\]

To find the modal components of the wave, one uses the same methods used in our previous examples. All of the modes in Eq. (13.102) are dispersionless because they are of the form \(\omega(k) = c_k k\). As we can see from this example, the method of modes is very effective because it allows equations such as Eq. (13.100) to be solved in a natural way, that is, we do not need a new methodology for each new equation.
13.4 Example 4: Two Different Types of Modes

Consider the following wave equation

\[
\left( \frac{\partial^2}{\partial t^2} + \alpha \frac{\partial^2}{\partial t \partial x} - i \beta \frac{\partial^3}{\partial t \partial x^2} - i \alpha \beta \frac{\partial^3}{\partial x^3} \right) u(x,t) = 0
\] (13.103)

which leads to the dispersion relation

\[
\omega^2 - (\alpha k + \beta k^2) \omega + \alpha \beta k^3 = 0
\] (13.104)

The dispersion relation has two modes

\[
\omega_1(k) = \alpha k
\] (13.105)

\[
\omega_2(k) = \beta k^2
\] (13.106)

The interest in this equation is that the modes are qualitatively different in that the function corresponding to the first mode, Eq. (13.105), propagates without dispersion while the function corresponding to the second mode, Eq. (13.106), propagates with dispersion. The explicit expressions for the modal functions may be obtained by using the results of sections 13.1 and section 13.2 for the dispersionless and dispersive mode respectively.
14.1 The Effect of Sources on Modal Functions

Consider the following situation. Suppose that we have a wave equation with corresponding modes evolving without a source function. Suppose a source function is activated, and at a later time, terminated. After the conclusion of the source, the wave is again the superposition of modes. We ask and answer: What is the relationship between the old and new modal functions? That is, does the source term alter the modal functions?

Consider a wave evolving under the effect of the wave equation operator $\mathcal{L}$

$$\mathcal{L}u(x,t) = f(x,t) \quad (14.1)$$

For example, $\mathcal{L}$ could be

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \quad (14.2)$$

Consider the situation in which a source is turned on at time $t = 0$, and subsequently turned
off at time \( t = T \). Then, before the source is turned on

\[
\mathcal{L} u(x, t) = 0 \quad \text{for } t < 0
\]  

(14.3)

and after the source is turned off,

\[
\mathcal{L} u(x, t) = 0 \quad \text{for } T < t
\]  

(14.4)

Thus, both before the source is turned on and after it is turned off, the wave must be homogeneous

\[
\mathcal{L} u(x, t) = 0 \quad \text{for } t < 0 \text{ or } T < t
\]  

(14.5)

Therefore, for \( t < 0 \) or for \( t > T \), the wave is the sum of modal functions, and we must be able to write

\[
\begin{align*}
\mathcal{L} u(x, t) &= \sum_{\ell=1}^{M} u_\ell(x, t) \quad \text{for } t < 0 \\
&= \sum_{\ell=1}^{M} v_\ell(x, t) \quad \text{for } t > T
\end{align*}
\]  

(14.6)

However, while the source is on, the wave satisfies

\[
\mathcal{L} u(x, t) = f(x, t) \quad \text{for } 0 < t < T
\]  

(14.7)

and so the wave is not the sum of modal functions only but is

\[
u(x, t) = \sum_{\ell=1}^{M} u_\ell(x, t) + u_f(x, t) \quad \text{for } 0 < t < T
\]  

(14.8)
For both Eqs. (14.6) and (14.8) to hold, it must be the case that after the source is turned off, the source term becomes a sum of modal functions

\[ u_f(x, t) = \sum_{\ell=1}^{M} \mu_{\ell}(x, t) \quad \text{for } t > T \]  

(14.9)

We call the new modal functions \( v_{\ell}(x, t) \), which are the superposition

\[ v_{\ell}(x, t) = u_{\ell}(x, t) + \mu_{\ell}(x, t) \quad \text{for } t > T \]  

(14.10)

to be the sum of the old modal functions, the \( u_{\ell} \)'s, and the modal functions produced by the source, the \( \mu_{\ell} \)'s.

For simplicity, we sketch the above argument for the two-mode case

\[
u(x, t) = \begin{cases} 
   u_1(x, t) + u_2(x, t) & \text{for } t < 0, \\
   u_1(x, t) + u_2(x, t) + u_f(x, t) & \text{for } 0 < t < T, \\
   v_1(x, t) + v_2(x, t) & \text{for } t > T.
\end{cases}
\]  

(14.11)

It is important to appreciate is that the \( u \)'s and \( v \)'s are not the same functions, although of course they are of the same functional form. That is, the effects of the source in general do not disappear, but turn into new modal functions. Considering that

\[ u_f^R(x, t) = \int_0^t dt' \int G_R(x, x'; t, t') f(x', t') dx' \]  

(14.12)

we see that it is the Green’s function that determines how the source should ‘populate’ each of the modes.
In particular, the $\mu_\ell$’s satisfy, after the source terminates

$$
\sum_\ell \mu_\ell(x,t) = \int_0^T dt' \int G^R(x,x';t,t') f(x',t') dx' \quad \text{for } t > T \tag{14.13}
$$

## 14.2 Example 1: Delta Function Source

In this section, we consider a source that is on for only the briefest length of time, and find the consequent modal functions $v_1(x,t)$ and $v_2(x,t)$. Consider a system described by the wave equation, Eq. (13.1)

$$
\mathcal{L}u(x,t) = \left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u(x,t) = f(x,t) \tag{14.14}
$$

In this example we take the source be

$$
f(x,t) = \frac{dF(x)}{dx} \delta(t - t_*) \tag{14.15}
$$

where $t_*$ is some time between $t = 0$ and $t = T$,

$$
0 < t_* < T \tag{14.16}
$$

We have expressed the source in terms of the general function $\frac{d}{dx} F(x)$ in anticipation of our result.

For times between $t = 0$ and $t = T$, the response is

$$
u^R_f(x,t) = \int_0^t dt' \int_{x-c(t-t')}^{x+c(t-t')} dx' \frac{dF(x')}{dx'} \delta(t' - t_*) \quad \text{for } 0 < t < T \tag{14.17}
$$
After time $t = T$, we have

$$u^R_f(x,t) = \int_0^t dt' \int_{x-c(t-t')}^{x+c(t-t')} dx' \frac{dF(x')}{dx'} \delta(t' - t_*) = F(x + c(t - t_*)) - F(x - c(t - t_*)) \quad (14.18)$$

which is the sum of two modal functions. Hence after the source term is turned off, the two modal functions are

$$v_1(x,t) = u_1(x - ct,0) - F(x - ct + ct_*) \quad (14.19)$$

$$v_2(x,t) = u_2(x + ct,0) + F(x + ct - ct_*) \quad (14.20)$$

Where $F(x - ct + ct_*)$ and $F(x + ct - ct_*)$ are $\mu_1(x,t)$ and $\mu_2(x,t)$ respectively. We note that indeed $v_1$ and $v_2$ are functions of $x - ct$ and $x + ct$ respectively.

### 14.3 Example 2: Step Function Source

In this section, we consider a source that is on only between $t = 0$ and $t = T$, and that at these times, the source is some function of position. We find the source response in general, and show that it degenerates into the superposition of modal functions after the source is off, as predicted in section 14.1.

$$f(x,t) = \frac{d^2F(x)}{dx^2} \eta(t)\eta(T - t) \quad (14.21)$$

We want to consider a system described by the wave equation, Eq. (13.1)

$$L u(x,t) = \left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u(x,t) = f(x,t) \quad (14.22)$$
The response to the source at \( t > T \), after it was turned off, is

\[
\begin{align*}
uf^R_f(x, t) &= \int_0^t dt' \int_{x-c(t-t')}^{x+c(t-t')} dx' \frac{d^2F(x')}{dx'^2} \eta(t') \eta(T - t') \\
&= \int_0^T dt' \int_{x-c(t-t')}^{x+c(t-t')} dx' \frac{d^2F(x')}{dx'^2} \\
&= \int_{t_1}^{t_2} dt' \left\{ \frac{d}{dx}F(x + c(t - t')) - \frac{d}{dx}F(x - c(t - t')) \right\}
\end{align*}
\] (14.23)

This evaluates to

\[
\begin{align*}
uf^R_f(x, t) &= \frac{1}{c} \left\{ F(x + ct) + F(x - ct) \right\} - \frac{1}{c} \left\{ F(x + c(t - T)) + F(x - c(t - T)) \right\}
\end{align*}
\] (14.24)

which is the sum of modes. Hence, after the source is turned off, we have

\[
\begin{align*}
v_1(x, t) &= u_1(x - ct, 0) + \frac{1}{c} F(x - ct) - \frac{1}{c} F(x - ct + cT) \\
v_2(x, t) &= u_2(x + ct, 0) + \frac{1}{c} F(x + ct) - \frac{1}{c} F(x + ct - cT)
\end{align*}
\] (14.25)

We again note that \( v_1 \) and \( v_2 \) are functions of \( x - ct \) and \( x + ct \) respectively.

Notice in Eq. (14.26) that the contribution to the modes only occurs at time \( t = 0 \) when the source is turned on, and at time \( t = T \) when it is turned off. Notice further that the contribution at time \( t = 0 \), when the source is turned on, comes in with a positive sign, while the contribution from \( t = T \), when the source is turned off, comes in with a negative sign.

### 14.4 Example 3: Source Activated at \( t = 0 \)

We now consider a source which is turned on at time \( t = 0 \) and otherwise is not a function of time. After its activation, the source persists forever. We show that in addition to the sum of modal functions, the source response has a term which serves to satisfy the inhomogeneous
equation, Eq. (14.30)

\[ f(x, t) = \frac{d^2 F(x)}{dx^2} \eta(t) \]  (14.29)

We are still considering a system described by the wave equation Eq. (13.1)

\[ \mathcal{L} u(x, t) = \left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u(x, t) = f(x, t) \]  (14.30)

Assume that we know the wave \( u(x, 0) \) and its time derivative at time \( t = 0 \). The source response after time \( t = 0 \) is

\[
    u^R_f(x, t) = \int_0^t dt' \int_{x - c(t - t')}^{x + c(t - t')} \frac{dx' d^2 F(x')}{dx'^2} 
    = \int_0^t dt' \left\{ \frac{d}{dx} F(x + c(t - t')) - \frac{d}{dx} F(x - c(t - t')) \right\} 
\]  (14.31)

which evaluates to

\[
    u^R_f(x, t) = \frac{1}{c} \left\{ F(x + ct) - 2F(x) + F(x - ct) \right\} 
\]  (14.32)

In the response to the source, Eq. (14.33), we have the two modal functions which the system can admit, both starting from time \( t = 0 \), which is the time at which the source was turned on. Note that the “\( F(x) \)” term is not functionally that of a mode. It appears because the source is always on.

### 14.5 Example 4: Persistent Source

We now consider a source which has always been on, and is a function of position only

\[
    f(x, t) = \frac{d^2 F(x)}{dx^2} 
\]  (14.34)
We are still considering a system described by the wave equation Eq. (13.1)

\[ \mathcal{L} u(x, t) = \left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u(x, t) = f(x, t) \]  

(14.35)

The source response is

\[ u^R_f(x, t) = \int_{-\infty}^{t} dt' \int_{x-c(t-t')}^{x+c(t-t')} dx' \frac{d^2 F(x')}{dx'^2} \]

(14.36)

\[ = \int_{-\infty}^{t} dt' \left\{ \frac{d}{dx} F(x + c(t - t')) - \frac{d}{dx} F(x - c(t - t')) \right\} \]

(14.37)

which evaluates to

\[ u^R_f(x, t) = \frac{1}{c} \left( F(-\infty) - 2F(x) + F(\infty) \right) \]

(14.38)

Thus, the source response is independent of time. We see in distinction to the previous example “Example 3: Source Activated at \( t = 0 \)”, that we do not get propagating modes. We can obtain this result from the previous result in Eq. (14.33) by substituting \( t = \pm \infty \), so in this sense, the “\( F(\pm \infty) \)” terms are modal functions which have propagated for an infinite time duration. Again, the “\( F(x) \)” term in Eq. (14.38) is not functionally that of a mode, and appears because the source is always on.
Chapter 15

Wigner Distribution Approximation using Modal Functions

In dispersive propagation, the wave changes in space and time as it propagates. Because of these changes, phase space methods, such as time-frequency analysis, have been widely applied to study dispersive wave propagation [27]-[37]. In this chapter, we consider linear dispersive wave propagation described by partial differential equations with constant coefficients. For simplicity of presentation, we initially consider the case of two modes, and then generalize to an arbitrary number of modes.

We give a procedure for obtaining a simple but insightful approximation of the quasi-distribution of the full wave (sum of modal functions), using a different approach than previously considered. This solution requires one to obtain (i) the initial modal functions from the given initial wave, and (ii) the cross-Wigner approximation between different modal functions. The approximation is obtained for general linear wave equations by transforming the equations to phase space, and then solving in the new domain. We show how to obtain the initial modal functions from the initial homogeneous wave in Chap. 10. We also make use of another result of the same chapter, which shows that each modal function of the wave
satisfies a Schrödinger type equation where the equivalent "Hamiltonian" operator is the dispersion relation corresponding to the mode and where the wavenumber is replaced by the wavenumber operator.

15.0.1 Wigner Distribution and Approximation

We consider the wave in phase space by taking the position-wavenumber Wigner distribution of the homogeneous wave, as a function of time, defined by [38, 39, 40]

\[
W(x, k, t) = \frac{1}{2\pi} \int \psi^*(x - \frac{1}{2}\theta, t) \psi(x + \frac{1}{2}\theta, t) e^{-i\theta k} d\theta \quad (15.1)
\]

\[
= \frac{1}{2\pi} \int \varphi^*(k + \frac{1}{2}\theta, t) \varphi(k - \frac{1}{2}\theta, t) e^{-i\theta x} d\theta \quad (15.2)
\]

The marginal distributions are given by

\[
\int W(x, k, t) dk = |\psi(x, t)|^2 \quad (15.3)
\]

\[
\int W(x, k, t) dx = |\varphi(k, t)|^2 \quad (15.4)
\]

The Wigner distribution for a single modal function \(W_\ell(x, k, t)\), is given by

\[
W_\ell(x, k, t) = \frac{1}{2\pi} \int u_\ell^*(x - \frac{1}{2}\theta, t) u_\ell(x + \frac{1}{2}\theta, t) e^{-i\theta k} d\theta \quad (15.5)
\]

\[
= \frac{1}{2\pi} \int S_\ell^*(k + \frac{1}{2}\theta, t) S_\ell(k - \frac{1}{2}\theta, t) e^{-i\theta x} d\theta \quad (15.6)
\]

It has been previously shown that \(W_\ell(x, k, t)\) evolves, approximately, in a simple manner in terms of the Wigner distribution of the initial modal Wigner function, namely [19, 20]

\[
W_\ell(x, k, t) \approx W_\ell(x - v_\ell(k)t, k, 0) \quad (15.7)
\]
where we have assumed real dispersion.

Note that the Wigner distribution of the total wave is not the sum of the single modal function Wigner distributions,

$$W(x, k, t) \neq \sum_{\ell} W_{\ell}(x, k, t)$$  \hspace{1cm} (15.8)

Additionally, initial conditions are usually given in terms of the initial wave, and not in terms of its initial modal functions. That is, we generally are not given all $u_{\ell}(x, 0)$ but rather $u(x, 0)$. We address both of these issues and extend the phase space formulation and approach to obtain an approximation of the total wave.

### 15.1 Phase-Space Formulation and Approximation

With the initial modes thus obtained we are in a position to extend the phase space approximation of a mode [20] to obtain an approximation for the wave. The approach we use here is different than that used in reference [20], in that we use the Weyl correspondence rule to transform the wave equation for each mode into Wigner phase space. Doing so yields phase space wave equations in terms of the auto and cross Wigner distributions of the modes. We then obtain approximation solutions to these phase space equations.

For simplicity, we consider the two-mode case. To transform the wave equation into phase space, we first substitute Eq. (10.15) into Eq. (15.1). The Wigner distribution of the wave is therefore given by

$$W(x, k, t) = W_{11}(x, k, t) + W_{22}(x, k, t) + W_{12}(x, k, t) + W_{21}(x, k, t)$$  \hspace{1cm} (15.9)
where
\[
W_{\ell m}(x, k, t) = \frac{1}{2\pi} \int u^*_\ell(x - \frac{1}{2}\theta, t) u_m(x + \frac{1}{2}\theta, t) e^{-i\theta k} d\theta
\] (15.10)

We note that
\[
W_{\ell m}(x, k, t) = W^*_{m\ell}(x, k, t)
\] (15.11)

Our aim now is to obtain the equations of motion for \(W_{\ell\ell}(x, k, t)\) and \(W_{\ell m}(x, k, t)\). We consider a more general case than Eq. (10.18), where each of the modal functions \(u_1(x, t)\) and \(u_2(x, t)\) satisfies a differential equation of the form [21]
\[
i \frac{\partial}{\partial t} u_\ell(x, t) = \mathcal{H}_\ell(x, \mathcal{K}) u_\ell(x, t)
\] (15.12)

Each mode has a different Hamiltonian because different modal functions are governed by different dispersion relations.

To obtain the corresponding phase space differential equation, the fundamental idea is to associate operators, \(\mathcal{H}(x, \mathcal{K})\), with a classical \(c\)-functions, \(H(x, k)\); this association is symbolized by
\[
\mathcal{H}(x, \mathcal{K}) \leftrightarrow H(x, k)
\] (15.13)

where \(\mathcal{K}\) is the wavenumber operator

\[
\mathcal{K} = \begin{cases} 
\frac{1}{i} \frac{\partial}{\partial x} & \text{in the position representation, and} \\
k & \text{in the wavenumber representation}
\end{cases}
\] (15.14)

For the Wigner distribution, the association is the Weyl correspondence, defined as [39, 41,
\[ H(x,k) = \int \hat{H}(\theta, \tau) e^{i\theta x + i\tau K} \, d\theta \, d\tau \]  \hspace{1cm} (15.15)
\[ = \int \hat{H}(\theta, \tau) e^{i\theta \tau/2} e^{i\theta x} e^{i\tau K} \, d\theta \, d\tau \]  \hspace{1cm} (15.16)

where \( \hat{H}(\theta, \tau) \) is the Fourier transform of \( H(x,k) \),

\[ \hat{H}(\theta, \tau) = \frac{1}{4\pi^2} \int H(x,k) e^{-i\theta x - i\tau k} \, dx \, dk \]  \hspace{1cm} (15.17)

The operation of \( H(x,k) \) on a function \( u(x,t) \) is then given by

\[ H(x,k)u(x,t) = \int \hat{H}(\theta, \tau) e^{i\theta \tau/2} e^{i\theta x} e^{i\tau K} u(x,t) \, d\theta \, d\tau \]  \hspace{1cm} (15.18)
\[ = \int \hat{H}(\theta, \tau) e^{i\theta \tau/2} e^{i\theta x} u(x + \tau, t) \, d\theta \, d\tau \]  \hspace{1cm} (15.19)

Also,

\[ H(x,k)u(x + \frac{1}{2}\tau', t) = \int \hat{H}(\theta, \tau) e^{i\theta \tau/2} e^{i\theta (x + \tau' / 2)} u(x + \tau + \frac{1}{2}\tau', t) \, d\theta \, d\tau \]  \hspace{1cm} (15.20)

Differentiating Eq. (15.10) with respect to time, and using Eq. (15.12), we have

\[ \frac{\partial}{\partial t} W_{12}(x,k,t) = \frac{1}{2\pi} \int u_1^*(x - \frac{1}{2}\tau, t) \frac{\partial}{\partial t} u_2(x + \frac{1}{2}\tau, t) e^{-i\tau k} \, d\tau 
+ \frac{1}{2\pi} \int u_2(x + \frac{1}{2}\tau, t) \frac{\partial}{\partial t} u_1^*(x - \frac{1}{2}\tau, t) e^{-i\tau k} \, d\tau \]  \hspace{1cm} (15.21)
\[ = \frac{1}{2\pi i} \left[ \int u_1^*(x - \frac{1}{2}\tau, t) H_2(x,k) u_2(x + \frac{1}{2}\tau, t) e^{-i\tau k} \, d\tau 
- \int u_2(x + \frac{1}{2}\tau, t) \{ H_1(x,k) u_1(x - \frac{1}{2}\tau, t) \}^* e^{-i\tau k} \, d\tau \right] \]  \hspace{1cm} (15.22)
Consider now the first term

\[
\frac{1}{2\pi i} \int u_1^*(x - \frac{1}{2}\tau', t) H_2(x, \mathcal{K}) u_2(x + \frac{1}{2}\tau', t) e^{-i\tau'k} d\tau
\]

\[
= \frac{1}{i} \frac{1}{2\pi} \int u_1^*(x - \frac{1}{2}\tau', t) \widehat{H}_2(\theta, \tau) e^{i\theta(x+\tau'/2)} e^{-i\tau'k} u_2 \left( x + \frac{1}{2}\tau' + \tau, t \right) d\theta d\tau d\tau' \quad (15.23)
\]

\[
= \frac{1}{i} \frac{1}{2\pi} \int u_1^*(x + \frac{1}{2}\tau - \frac{1}{2}\tau', t) \widehat{H}_2(\theta, \tau, t) e^{i\theta x + i\tau k} e^{-i\tau'(k-\theta/2)} u_2 \left( x + \frac{1}{2}\tau + \frac{1}{2}\tau', t \right) d\theta d\tau d\tau' \quad (15.24)
\]

Therefore

\[
\frac{1}{2\pi} \int u_1^*(x - \frac{1}{2}\tau', t) \frac{\partial}{\partial t} u_2(x + \frac{1}{2}\tau', t) e^{-i\tau'k} d\tau' = \frac{1}{i} \int W_{12}(x + \frac{1}{2}\tau, k - \frac{1}{2}\theta, t) \widehat{H}_2(\theta, \tau) e^{i\theta x + i\tau k} d\theta d\tau
\]

\[
(15.25)
\]

\[
= \frac{1}{i} \int \widehat{H}_2(\theta, \tau) e^{i\theta(x+\tau/2) + i\tau k} W_{12}(x, k, t) d\theta d\tau \quad (15.26)
\]

\[
= \frac{1}{i} \int e^{-\frac{\theta}{2} \frac{\partial}{\partial W} + \frac{\tau}{2} \frac{\partial}{\partial W}} e^{i\theta x + i\tau k} \widehat{H}_2(\theta, \tau) W_{12}(x, k, t) d\theta d\tau \quad (15.27)
\]

\[
= \frac{1}{i} \int e^{-\frac{\theta}{2} \frac{\partial}{\partial W} + \frac{\tau}{2} \frac{\partial}{\partial W} + \frac{\partial}{\partial H} \frac{\partial}{\partial W}} e^{i\theta x + i\tau k} \widehat{H}_2(\theta, \tau) W_{12}(x, k, t) d\theta d\tau \quad (15.28)
\]

where the derivatives \( \frac{\partial}{\partial W} \) and \( \frac{\partial}{\partial H} \) act on \( W_{12}(x, k, t) \) and \( e^{i\theta x + i\tau k} \widehat{H}_2(\theta, \tau) \) respectively and likewise for \( \frac{\partial}{\partial W} \) and \( \frac{\partial}{\partial H} \). Hence,

\[
\frac{1}{2\pi} \int u_1^*(x - \frac{1}{2}\tau', t) \frac{\partial}{\partial t} u_2(x + \frac{1}{2}\tau', t) e^{-i\tau'k} d\tau' = \frac{1}{i} e^{\frac{i}{2} \left[ \frac{\partial}{\partial W} \frac{\partial}{\partial W} - \frac{\partial}{\partial H} \frac{\partial}{\partial W} \right]} H_2(x, k) W_{12}(x, k)
\]

\[
(15.29)
\]

The second term in Eq. (15.21) can be obtained from the first term by complex conjugation and interchange of the indices 1 and 2. We therefore have

\[
\frac{1}{2\pi} \int u_2(x + \frac{1}{2}\tau', t) \frac{\partial}{\partial t} u_1^*(x - \frac{1}{2}\tau', t) e^{-i\tau'k} d\tau' = -\frac{1}{i} e^{-\frac{i}{2} \left[ \frac{\partial}{\partial W} \frac{\partial}{\partial W} - \frac{\partial}{\partial H} \frac{\partial}{\partial W} \right]} H_1^*(x, k) W_{21}(x, k)
\]

\[
(15.30)
\]
Hence,

$$\frac{\partial}{\partial t} W_{12}(x, k, t) = \frac{1}{i} e^{\frac{i}{2} \left[ \frac{\partial}{\partial k} \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \frac{\partial}{\partial k} \right]} H_2(x, k) W_{12}(x, k)$$

(15.31)

$$- \frac{1}{i} e^{-\frac{i}{2} \left[ \frac{\partial}{\partial k} \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \frac{\partial}{\partial k} \right]} H^*_1(x, k) W^*_{21}(x, k)$$

(15.32)

Keeping in mind that $W_{21}(x, k, t) = W^*_{12}(x, k, t)$, we have

$$\frac{\partial}{\partial t} W_{12}(x, k, t)$$

$$= \frac{1}{i} \left[ e^{\frac{i}{2} \left[ \frac{\partial}{\partial k} \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \frac{\partial}{\partial k} \right]} H_2(x, k) - e^{-\frac{i}{2} \left[ \frac{\partial}{\partial k} \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \frac{\partial}{\partial k} \right]} H^*_1(x, k) \right] W_{12}(x, k, t)$$

(15.33)

This equation expresses the evolution of the cross-Wigner distribution, where each of the wave functions is governed by a different Hamiltonian. It arises here because different modal functions are governed by different dispersion relations. [We note that in quantum mechanics this does not generally arise in that a system is governed by a single Hamiltonian; however, recently, E. Giese et al. [44] have considered a quantum mechanical situation where indeed two Hamiltonians have to be considered, and they derived an equation analogous to Eq. (15.33)].

Now, for our case

$$H_1(x, k) = \omega_1(k)$$

(15.34)

$$H_2(x, k) = \omega_2(k)$$

(15.35)
and hence

\[
\frac{\partial}{\partial t} W_{12}(x, k, t) = \frac{1}{i} \left[ e^{\frac{i}{2} \left[ -\text{Im} \omega \frac{\partial}{\partial k} \right] } \omega_2(k) - e^{-\frac{i}{2} \left[ -\text{Im} \omega \frac{\partial}{\partial k} \right] } \omega_1^*(k) \right] W_{12}(x, k, t)
\] (15.36)

\[
= \frac{1}{i} \left[ \omega_2 \left( k - i \frac{\partial}{\partial x} \right) - \omega_1^* \left( k + i \frac{\partial}{2 \partial x} \right) \right] W_{12}(x, k, t)
\] (15.37)

or

\[
\frac{\partial}{\partial t} W_{12}(x, k, t) = i \left[ \omega_1^* \left( k + i \frac{\partial}{2 \partial x} \right) - \omega_2 \left( k - i \frac{\partial}{2 \partial x} \right) \right] W_{12}(x, k, t)
\] (15.38)

It also readily follows that

\[
\frac{\partial}{\partial t} W_{11}(x, k, t) = i \left[ \omega_1^* \left( k + i \frac{\partial}{2 \partial x} \right) - \omega_1 \left( k - i \frac{\partial}{2 \partial x} \right) \right] W_{11}(x, k, t)
\] (15.39)

Similarly for \( W_{22}(x, k, t) \) and \( W_{21}(x, k, t) \). If we take real modes, then

\[
\frac{\partial}{\partial t} W_{12}(x, k, t) = i \left[ \omega_1^* \left( k + i \frac{\partial}{2 \partial x} \right) - \omega_2 \left( k - i \frac{\partial}{2 \partial x} \right) \right] W_{12}(x, k, t)
\] (15.40)

\[
\frac{\partial}{\partial t} W_{11}(x, k, t) = i \left[ \omega_1^* \left( k + i \frac{\partial}{2 \partial x} \right) - \omega_1 \left( k - i \frac{\partial}{2 \partial x} \right) \right] W_{11}(x, k, t)
\] (15.41)

These, then, are the phase space equations of motion to solve, corresponding to Eq. (10.18).

### 15.1.1 Approximation

To obtain an approximation, consider the expansions

\[
\omega \left( k + \frac{i}{2} \frac{\partial}{\partial x} \right) = \omega(k) + \frac{i}{2} \frac{\partial \omega(k)}{\partial k} \frac{\partial}{\partial x} + \cdots
\] (15.42)

\[
\omega \left( k - \frac{i}{2} \frac{\partial}{\partial x} \right) = \omega(k) - \frac{i}{2} \frac{\partial \omega(k)}{\partial k} \frac{\partial}{\partial x} + \cdots
\] (15.43)
Keeping terms up to first order and substituting into Eq. (15.40) yields

\[
\frac{\partial}{\partial t} W_{12}(x,k,t) \approx i(\omega_1 - \omega_2)W_{12}(x,k,t) - \frac{v_1(k) + v_2(k)}{2} \frac{\partial}{\partial x} W_{12}(x,k,t) \tag{15.44}
\]

The solution is

\[
W_{12}(x,k,t) \approx e^{i(\omega_1 - \omega_2)t} W_{12} \left( x - \frac{v_1(k) + v_2(k)}{2} t, k, 0 \right) \tag{15.45}
\]

If we make the indices equal, we obtain the single-mode approximation obtained previously by a different approach, [20]

\[
W_{nn}(x,k,t) \approx W_{nn} \left( x - v_n(k)t, k, 0 \right) \tag{15.46}
\]

Thus, the phase space Wigner approximation for a wave consisting of two modes is

\[
W(x,k,t) \approx W_{11} \left( x - v_1(k)t, k, 0 \right) + W_{22} \left( x - v_2(k)t, k, 0 \right) + 2 \Re \left\{ e^{i(\omega_1(k) - \omega_2(k))t} W_{12} \left( x - \frac{v_1(k) + v_2(k)}{2} t, k, 0 \right) \right\} \tag{15.47}
\]

15.1.2 The $M$ Mode case

For $M$ modes, the wave is given by Eq. (10.15) or (11.17), and the corresponding Wigner distribution is

\[
W(x,k,t) = \sum_{\ell=1}^{M} W_{\ell\ell}(x,k,t) + \sum_{\ell \neq m=1}^{M} W_{\ell m}(x,k,t) \tag{15.48}
\]

For real modes, the equation of motion is

\[
\frac{\partial}{\partial t} W_{\ell m}(x,k,t) = i \left[ \omega_{\ell} \left( k + \frac{i}{2} \frac{\partial}{\partial x} \right) - \omega_{m} \left( k - \frac{i}{2} \frac{\partial}{\partial x} \right) \right] W_{\ell m}(x,k,t) \tag{15.49}
\]
and the approximate Wigner distribution of the wave is

\[
W(x, k, t) \approx \sum_{\ell=1}^{M} W_{\ell\ell}(x - v_\ell(k)t, k, 0) + 2 \text{Re} \left\{ \sum_{\ell > m=1}^{M} e^{i(\omega_\ell(k) - \omega_m(k))t} W_{\ell m} \left( x - \frac{v_\ell(k) + v_m(k)}{2} t, k, 0 \right) \right\}
\]  

(15.50)

### 15.2 Example: Beam equation

We now apply the above formulation to the beam equation [18]

\[
\left( \frac{\partial^4}{\partial x^4} + \frac{4}{\gamma^2} \frac{\partial^2}{\partial t^2} \right) \psi(x, t) = 0
\]  

(15.51)

Substituting \( e^{ikx - i\omega t} \) into Eq. (15.51) gives the dispersion relation

\[
(ik)^4 + \frac{4}{\gamma^2} (-i\omega)^2 = 0
\]  

(15.52)

by which we have that there are two modes,

\[
\omega_1 = \frac{\gamma k^2}{2} \quad ; \quad \omega_2 = -\frac{\gamma k^2}{2}
\]  

(15.53)

with respective group velocities

\[
v_1 = \gamma k \quad ; \quad v_2 = -\gamma k
\]  

(15.54)
15.2.1 Exact Solution for the Wave

It is of interest to obtain the exact solution. We have, using Eqs. (11.30) and (11.29), that the initial modal functions are given by

\[ S_1(k, 0) = \frac{\gamma k^2}{2} + i \frac{\partial}{\partial t} \varphi(k, 0) \]  
\[ S_2(k, 0) = \frac{\gamma k^2}{2} - i \frac{\partial}{\partial t} \varphi(k, 0) \]  

Hence using Eqs. (10.12) and (11.17), the exact solution is

\[ \varphi(k, t) = S_1(k, 0) e^{-i \gamma t k^2 / 2} + S_2(k, 0) e^{i \gamma t k^2 / 2} \]  
\[ = \left\{ \frac{\gamma k^2}{2} + i \frac{\partial}{\partial t} \varphi(k, 0) \right\} e^{-i \gamma t k^2 / 2} + \left\{ \frac{\gamma k^2}{2} - i \frac{\partial}{\partial t} \varphi(k, 0) \right\} e^{i \gamma t k^2 / 2} \]  
\[ = \frac{1}{2} \varphi(k, 0) e^{-i \gamma t k^2 / 2} + \frac{1}{2} \varphi(k, 0) e^{i \gamma t k^2 / 2} + \frac{1}{\gamma k^2} \left\{ i \frac{\partial}{\partial t} \varphi(k, 0) \right\} e^{-i \gamma t k^2 / 2} + \frac{1}{\gamma k^2} \left\{ -i \frac{\partial}{\partial t} \varphi(k, 0) \right\} e^{i \gamma t k^2 / 2} \]  

which gives

\[ \varphi(k, t) = \left( \varphi(k, 0) \right) \cos \left( \frac{1}{2} \gamma k^2 t \right) + \left( \frac{\partial \varphi(k, 0)}{\partial t} \right) \sin \left( \frac{1}{2} \gamma k^2 t \right) \]  

To obtain \( \psi(x, t) \), one must then evaluate Eq. (11.9). Next, we consider the exact and approximate Wigner solutions, to illustrate the simplifications that can arise when one considers wave propagation in phase space.
15.2.2 Phase-Space Solution and Approximation

For the example considered above, the exact Wigner distribution is given by

\[ W(x, k, t) = W_{11}(x - \gamma kt, k, 0) + W_{22}(x + \gamma kt, k, 0) + \frac{2}{\pi} \text{Re} \left\{ e^{i\gamma k^2 t} \int W_{12}(x', k, 0) e^{-i\theta(x-x')} e^{i\gamma t\theta^2/4} d\theta dx' \right\} \]  

(15.61)

The \( \theta \) integration may be done and we have that

\[ W(x, k, t) = W_{11}(x - \gamma kt, k, 0) + W_{22}(x + \gamma kt, k, 0) + 2 \text{Re} \left\{ \frac{1}{\sqrt{-i\gamma t \pi}} e^{i\gamma k^2 t} \int W_{12}(x', k, 0) e^{(x-x')^2/4t} dx' \right\} \]  

(15.62)

By comparison, using eq. (15.47), the approximate Wigner distribution is given by

\[ W(x, k, t) \approx W_{11}(x - \gamma kt, k, 0) + W_{22}(x + \gamma kt, k, 0) + 2 \text{Re} \left\{ e^{i\gamma k^2 t} W_{12}(x, k, 0) \right\} \]  

(15.63)

For this example, the self terms \( W_{11} \) and \( W_{22} \) of the approximation are in fact exact.

How accurate is the approximation? Using the fact that

\[ \lim_{\gamma t \to 0} \frac{1}{\sqrt{-i\gamma t \pi}} e^{(x-x')^2/4t} = \delta(x - x') \]  

(15.64)

we see that in this case the approximation is good when \( \gamma t << 1 \). However, a more general condition for the accuracy of the approximation can be obtained, that depends on the spatial extent of the initial pulse in addition to the dispersion relation. Specifically, compare the Fourier transform of the approximation cross-term to the Fourier transform of the (convolution) integral in the exact expression. The Fourier transform of the approximation, Eq.
(15.63), is

\[
\frac{1}{\sqrt{2\pi}} \int e^{i\gamma k^2 t} W_{12}(x, k, 0) e^{-ix\kappa} = \frac{e^{i\gamma k^2 t}}{\sqrt{2\pi}} \hat{W}_{12}(\kappa, k, 0)
\]  

(15.65)

and the Fourier transform of the cross-term in the exact expression, Eq. (15.62), is

\[
\frac{1}{\sqrt{2\pi}} \int e^{i\gamma k^2 t} \frac{1}{\sqrt{-i\gamma t \pi}} W_{12}(x', k, 0) e^{(x-x')^2 / i\gamma t} dx' e^{-ix\kappa} dx = \frac{e^{i\gamma k^2 t}}{\sqrt{2\pi}} \hat{W}_{12}(\kappa, k, 0) D(\kappa, t)
\]  

(15.66)

where

\[
D(\kappa, t) = \frac{1}{\sqrt{-i\gamma t \pi}} \int e^{-i\frac{x^2}{2\gamma t}} e^{-ix\kappa} dx = e^{i\gamma t \kappa^2 / 4}
\]  

(15.67)

Thus, comparing Eqs. (15.65) and (15.66), we see that the approximation will be accurate when

\[
\hat{W}_{12}(\kappa, k, 0) D(\kappa, t) \approx \hat{W}_{12}(\kappa, k, 0)
\]  

(15.68)

This will occur when

\[
\hat{W}_{12}(\kappa, k, 0) \approx 0 \quad \text{for } |\kappa| > K
\]  

(15.69)

and

\[
D(\kappa, t) \approx 1 \quad \text{for } |\kappa| < K
\]  

(15.70)

Thus, for a given (nonlinear) dispersion relation, the approximation will generally be more accurate for broader initial pulses (smaller $K$). We illustrate this result next with a specific initial pulse.
15.2.3 Specific Case

Consider the case with initial conditions

\[ \psi(x, 0) = Ae^{-\alpha x^2/2 + ik_0 x} \]  
\[ \frac{\partial}{\partial t} \psi(x, 0) = 0 \]

In the wavenumber domain, the initial conditions are

\[ \varphi(k, 0) = \frac{A}{\sqrt{\alpha}} e^{-(k-k_0)^2/(2\alpha)} \]  
\[ \frac{\partial}{\partial t} \varphi(k, 0) = 0 \]

Note that the extent in \( x \) of the initial pulse is proportional to \( 1/\alpha \), and hence based on the considerations above, we expect the approximation to be more accurate for smaller \( \alpha \).

The exact cross-Wigner distribution at time zero is

\[ W_{12}(x, k, 0) = \frac{1}{4\sqrt{\pi\alpha}} |A|^2 e^{-\alpha x^2} e^{-(k-k_0)^2/\alpha} \]

and the evolved exact cross-Wigner at time \( t \) is

\[ W_{12}(x, k, t) = \frac{|A|^2}{4\sqrt{\pi\alpha}} \sqrt{\frac{1}{1 - i\gamma t\alpha}} \exp \left[ -\frac{\alpha x^2}{1 - i\gamma t\alpha} - \frac{(k - k_0)^2 - i\gamma \alpha k^2 t}{\alpha} \right] \]

Using our approximation Eq. (15.45) we have that

\[ W_{12}(x, k, t) \approx \frac{|A|^2}{4\sqrt{\pi\alpha}} \exp \left[ -\alpha x^2 - \frac{(k - k_0)^2 - i\gamma \alpha k^2 t}{\alpha} \right] \]

By considering the magnitude of the exact and approximate Wigner distributions, it follows
that the approximation is accurate for

\[ \alpha \gamma t << 1 \quad (15.78) \]

which is consistent with the considerations above; namely, for a given value of the dispersion parameter \( \gamma \), the approximation will be accurate for a longer time extent for a broader initial pulse \( u(x,0) \) (smaller \( \alpha \)) than for a narrower initial pulse (larger \( \alpha \)).
Chapter 16

Spectrogram Approximation using Modal Functions

In the previous chapter, we have shown that defining phase space by way of the Wigner distribution leads to considerable insight and also to an effective approximation scheme. In this chapter we review the phase-space quasi-distribution called the Spectrogram [1, 2], and use it to develop a simple approximation scheme.

16.1 Windowed Waves and the Spectrogram

Here we briefly review the Spectrogram. For a homogeneous wave $\psi(x, t)$ with corresponding spectral function $\varphi(k, t)$ defined by

$$\varphi(k, t) = \frac{1}{\sqrt{2\pi}} \int \psi(k, t) e^{-i k x} \, dx$$

(16.1)

the intensity in wavenumber $(k)$ space is $|\varphi(k, t)|^2$. This does not give an indication as to what is happening locally, that is, at or around some position $x$. In analogy with the short
time Fourier transform, one defines the windowed wave $\psi(x', t) h(x' - x)$ where $h(x' - x)$ is a function (typically called the “window”) chosen so that

$$
\psi(x', t) h(x' - x) \sim \begin{cases} 
\psi(x', t) & \text{for } x' \text{ near } x \\
0 & \text{for } x' \text{ far away from } x
\end{cases} \quad (16.2)
$$

The windowed wave is a function of two variables, the position we are focused on, $x$, and the variable position, $x'$. Since the windowed wave emphasizes the wave around the position $x$, its Fourier transform, a phase space wave function, will reflect the distribution of $k$ around that position,

$$
\mu(x, k, t) = \frac{1}{\sqrt{2\pi}} \int \psi(x', t) h(x' - x) e^{-ikx'} dx' \quad (16.3)
$$

The intensity at position $x$ is then defined by

$$
F(x, k, t) = |\mu(x, k, t)|^2 \quad (16.4)
$$

which can be considered as a joint position-wavenumber distribution.

One can alternatively formulate the problem in $k$ space where we window a wavenumber function, $\varphi(k, t)$, with a window function, $\hat{h}(k)$, and define the short-$k$ position wave by

$$
\eta(x, k, t) = \frac{1}{\sqrt{2\pi}} \int e^{ik'x} \varphi(k', t) \hat{h}(k - k') dk' \quad (16.5)
$$

If we take the wavenumber window $\hat{h}(k)$ to be the Fourier transform of the position space window function, $h(x)$,

$$
\hat{h}(k) = \frac{1}{\sqrt{2\pi}} \int dx e^{-ikx} h(x) \quad (16.6)
$$

then, for $\varphi(k, t)$ and $\psi(x, t)$ related via Eq. (11.1), the relationship between $\eta(x, k, t)$ and
\( \mu(x, k, t) \) is,
\[
\mu(x, k, t) = e^{-ikx} \eta(x, k, t)
\] (16.7)
by which it follows that
\[
F(x, k, t) = |\eta(x, k, t)|^2 = |\mu(x, k, t)|^2
\] (16.8)

### 16.2 Equation of Motion for Windowed Modal Functions

We now define the windowed modal function by
\[
\eta_\ell(x, k, t) = \frac{1}{\sqrt{2\pi}} \int e^{ik'x} S_\ell(k', t) \hat{h}(k - k') dk'
\] (16.9)
\[
= \frac{1}{\sqrt{2\pi}} \int e^{ik'x} S_\ell(k', 0) e^{-i\omega_\ell(k') t} \hat{h}(k - k') dk' 
\] (16.10)
\[
= \frac{1}{\sqrt{2\pi}} e^{ikx} \int e^{ik'x} S_\ell(k' + k, 0) e^{-i\omega_\ell(k') t} \hat{h}(-k') dk' 
\] (16.11)

Differentiating and using Eq. (16.11) we have
\[
i \frac{\partial}{\partial t} \eta_\ell(x, k, t) = \frac{1}{\sqrt{2\pi}} e^{ikx} \int \omega_\ell(k') e^{ik'x} S_\ell(k' + k, 0) e^{-i\omega_\ell(k') t} \hat{h}(-k') dk' 
\] (16.12)
\[
= e^{ikx} \omega_\ell \left( k + \frac{1}{i} \frac{\partial}{\partial x} \right) e^{-ikx} \eta_\ell(x, k, t)
\] (16.13)
Hence
\[
i \frac{\partial}{\partial t} e^{-ikx} \eta_\ell(x, k, t) = \omega_\ell \left( k + \frac{1}{i} \frac{\partial}{\partial x} \right) e^{-ikx} \eta_\ell(x, k, t)
\] (16.14)
or
\[
i \frac{\partial}{\partial t} e^{-ikx} \eta_\ell(x, k, t) = \omega_\ell (k + K) e^{-ikx} \eta_\ell(x, k, t)
\] (16.15)
This is the equation of motion for the windowed modal function. Using Eq. (16.7) we also have that

\[ i \frac{\partial}{\partial t} \mu_\ell(x, k, t) = \omega_\ell (k + \mathcal{K}) \mu_\ell(x, k, t) \]  

(16.16)

The solutions to these equation can be written symbolically as

\[ e^{-ikx} \eta_\ell(x, k, t) = e^{-i\omega_\ell(k+\mathcal{K})t} e^{-ikx} \eta_\ell(x, k, 0) \]  

(16.17)

and

\[ \mu_\ell(x, k, t) = e^{-i\omega_\ell(k+\mathcal{K})t} \mu_\ell(x, k, 0) \]  

(16.18)

where analogously to Eq. (16.9), we define the \( \mu_\ell \)'s as

\[ \mu_\ell(x, k, t) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx'} u_\ell(x', t) h(x' - x) \, dx' \]  

(16.19)

Since \( \psi \) and \( \varphi \) could be single mode waves \( u_\ell, S_\ell \), then it follows from Eq. (16.7) that

\[ \mu_\ell(x, k, t) = e^{-ikx} \eta_\ell(x, k, t) \]  

(16.20)

### 16.2.1 Obtaining Windowed Modes from the Windowed Wave

We now show that we can obtain the windowed modal functions directly from \( \eta(x, k, t) \). Since

\[ \omega \left( \frac{1}{i} \frac{\partial}{\partial x} \right) \eta(x, k, t) = \frac{1}{\sqrt{2\pi}} \int e^{ik\prime x} [\omega(k') \varphi(k', t)] \hat{h}(k - k') \, dk' \]  

(16.21)

and

\[ \frac{\partial}{\partial t} \eta(x, k, t) = \frac{1}{\sqrt{2\pi}} \int e^{ik\prime x} \left[ \frac{\partial}{\partial t} \varphi(k', t) \right] \hat{h}(k - k') \, dk' \]  

(16.22)
the windowed modal functions (in the two-mode case) are

\[ \eta_1(x, k, 0) = \frac{\omega_2 \left( \frac{1}{i} \frac{\partial}{\partial x} \right) - i \frac{\partial}{\partial t}}{\omega_2 \left( \frac{1}{i} \frac{\partial}{\partial x} \right) - \omega_1 \left( \frac{1}{i} \frac{\partial}{\partial x} \right)} \eta(x, k, 0) \] (16.23)

\[ \eta_2(x, k, 0) = \frac{\omega_1 \left( \frac{1}{i} \frac{\partial}{\partial x} \right) - i \frac{\partial}{\partial t}}{\omega_1 \left( \frac{1}{i} \frac{\partial}{\partial x} \right) - \omega_2 \left( \frac{1}{i} \frac{\partial}{\partial x} \right)} \eta(x, k, 0) \] (16.24)

Also, since

\[ \int e^{ikx'} h(x' - x) \frac{\partial}{\partial x'} \psi(x', 0) dx' = -\int \psi(x', 0) \frac{\partial}{\partial x'} \left( e^{ikx'} h(x' - x) \right) dx' \] (16.25)

\[ = -\int \psi(x', 0) \left( ik - \frac{\partial}{\partial x} \right) e^{ikx'} h(x' - x) dx' \] (16.26)

\[ = \left( \frac{\partial}{\partial x} - ik \right) \mu(x, k, t) \] (16.27)

then

\[ \int e^{ikx'} h(x' - x) \omega \left( \frac{1}{i} \frac{\partial}{\partial x'} \right) \psi(x', 0) dx' = \omega \left( \frac{1}{i} \frac{\partial}{\partial x} + k \right) \mu(x, k, t) = \omega(K + k) \mu(x, k, t) \] (16.28)

Therefore

\[ \mu_1(x, k, 0) = \frac{\omega_2 (K + k) - i \frac{\partial}{\partial t}}{\omega_2 (K + k) - \omega_1 (K + k)} \mu(x, k, 0) \] (16.29)

\[ \mu_2(x, k, 0) = \frac{\omega_1 (K + k) - i \frac{\partial}{\partial t}}{\omega_1 (K + k) - \omega_2 (K + k)} \mu(x, k, 0) \] (16.30)

which could be seen directly from Eq. (16.7) since

\[ \frac{1}{i} \frac{\partial}{\partial x} \eta(x, k, t) = \frac{1}{i} \frac{\partial}{\partial x} e^{-ikx} \mu(x, k, t) = -ke^{-ikx} \mu(x, k, t) + e^{-ikx} \left( \frac{1}{i} \frac{\partial}{\partial x} \right) \mu(x, k, t) \] (16.31)

and hence

\[ K \eta(x, k, t) = e^{-ikx} (K - k) \mu(x, k, t) \] (16.32)
16.2.2 Approximation for Windowed Modes

We now show we can obtain a simple approximation. Expand

\[ \omega_{\ell}(k + K) = \omega_{\ell}(k) + \omega'_{\ell}(k) K + \cdots \]  

(16.33)

\[ \approx \omega_{\ell}(k) + v_{\ell}(k) K \]  

(16.34)

Accordingly, Eq. (16.18) becomes

\[ \mu_{\ell}(x, k, t) = e^{-i\omega_{\ell}(k + K)t} \mu_{\ell}(x, k, 0) \approx e^{-i\omega_{\ell}(k)t} e^{-v_{\ell}(k)t \frac{\partial}{\partial x}} \mu_{\ell}(x, k, 0) \]  

(16.35)

However, \( e^{-v_{\ell}(k)t \frac{\partial}{\partial x}} \) is the translation operator, and hence we have that

\[ \mu_{\ell}(x, k, t) \approx e^{-i\omega_{\ell}(k)t} \mu_{\ell}(x - v_{\ell}(k)t, k, 0) \]  

(16.36)

Applying the same approximation to Eq. (16.17)

\[ e^{-ikx} \eta_{\ell}(x, k, t) \approx e^{-i(\omega_{\ell}(k) + \omega'_{\ell}(k)K)t} e^{-ikx} \eta_{\ell}(x, k, 0) \]  

(16.37)

\[ = e^{-i\omega_{\ell}(k)t} e^{-v_{\ell}(k)t \frac{\partial}{\partial x}} e^{-ikx} \eta_{\ell}(x, k, 0) \]  

(16.38)

\[ = e^{-i\omega_{\ell}(k)t} e^{-ik(x - v_{\ell}(k)t)t} \eta_{\ell}(x - v_{\ell}(k)t, k, 0) \]  

(16.39)

by which we have

\[ \eta_{\ell}(x, k, t) \approx e^{-i\omega_{\ell}(k)t} e^{ikv_{\ell}(k)t} \eta_{\ell}(x - v_{\ell}(k)t, k, 0) \]  

(16.40)

Notice that the approximations to \( \eta_{\ell}(x, k, t) \) and \( \mu_{\ell}(x, k, t) \) are not the same but the corresponding quasi-distributions are the same since

\[ |\mu_{\ell}(x, k, t)|^2 \approx |\mu_{\ell}(x - v_{\ell}(k)t, k, 0)|^2 \]  

[real dispersion relation]  

(16.41)
and

\[ |\eta_\ell(x, k, t)|^2 \approx |\eta_\ell(x - v_\ell(k) t, k, 0)|^2 \quad \text{[real dispersion relation]} \quad (16.42) \]

## 16.3 Approximation of the Full Wave

We are now in the position to obtain the approximation for the full wave, that is \( \psi(x, t) \) or \( \varphi(k, t) \). Starting with

\[ \psi(x, t) = u_1(x, t) + u_2(x, t) \quad (16.43) \]

and using the approximation for each mode (Eq. (16.40)) we have

\[ \eta(x, k, t) \approx e^{-i\omega_1(k)t} e^{ikv_1(k)t} \eta_1(x - v_1(k) t, k, 0) + e^{-i\omega_2(k)t} e^{ikv_2(k)t} \eta_2(x - v_2(k) t, k, 0) \quad (16.44) \]

and therefore

\[ F(x, k, t) = |\eta(x, k, t)|^2 \approx |\eta_1(x - v_1(k) t, k, 0)|^2 + |\eta_2(x - v_2(k) t, k, 0)|^2 \]

\[ + e^{i[\omega_2(k) - \omega_1(k)]t} e^{-ik(v_2(k) - v_1(k))t} \eta_2^*(x - v_2(k) t, k, 0) \eta_1(x - v_1(k) t, k, 0) + \text{c.c.} \]

\[ (16.45) \]

Hence the steps to obtain the approximation are as follows:

1. From the initial condition for the full wave calculate the initial modes as per Eq. (10.38) or its generalization, Eq. (10.45).

2. To find the windowed modal functions, multiply by the window, and take the Fourier transform of the product.

3. We could alternatively find the initial full windowed wave, and use Eq. (10.38) or its
generalization, Eq. (10.45), to obtain the windowed modal functions directly.

4. Then calculate the approximation for each of the modal functions as given by Eq. (16.36) or (16.40).

5. Finally, just write Eq. (16.44) and Eq. (16.45).

Note that for the calculation of the approximations (in steps 4 and 5) one merely performs simple substitutions. For example to calculate the approximate $\eta_1(x, k, t)$ one merely substitutes $x - v_1(k)t$ for $x$ in $\eta_1(x, k, 0)$ to obtain that $\eta_1(x, k, t) \approx \eta_1(x - v_1(k)t, k, 0)$.

### 16.4 Example

To demonstrate the approximation, consider the beam equation [18],

$$
\left( \frac{\partial^4}{\partial x^4} + \frac{4}{\gamma^2} \frac{\partial^2}{\partial t^2} \right) \psi(x, t) = 0
$$

(16.46)

which exhibits dispersion, as determined by substituting $e^{ikx - i\omega t}$ into Eq. (16.46), which gives

$$(ik)^4 + \frac{4}{\gamma^2} (-i\omega)^2 = 0$$

(16.47)

The solutions are the two modes

$$
\omega_1 = \frac{\gamma k^2}{2} ; \quad \omega_2 = -\frac{\gamma k^2}{2}
$$

(16.48)

and hence we have dispersive behavior. The corresponding group velocities are

$$
v_1 = \gamma k ; \quad v_2 = -\gamma k
$$

(16.49)
Consider the specific case with initial conditions

\[ \psi(x, 0) = Ae^{-\alpha x^2/2 + ik_0 x} \quad (16.50) \]
\[ \frac{\partial}{\partial t} \psi(x, 0) = 0 \quad (16.51) \]

The initial spectrum is given by

\[ \varphi(k, 0) = \frac{A}{\sqrt{\alpha}} e^{-(k-k_0)^2/(2\alpha)} \quad (16.52) \]
\[ \frac{\partial}{\partial t} \varphi(k, 0) = 0 \quad (16.53) \]

For the window function we take a Gaussian

\[ h(x) = Be^{-\beta x^2/2} \quad (16.54) \]
\[ \hat{h}(k) = \frac{1}{\sqrt{2\pi}} \int B e^{-\beta x^2/2} e^{-ikx} dx = \frac{B}{\sqrt{\beta}} e^{-k^2/2\beta} \quad (16.55) \]

Using Eqs. (10.38) and (10.39), the modal functions are

\[ u_1(x, 0) = \frac{1}{2} \psi(x, 0) \quad (16.56) \]
\[ u_2(x, 0) = \frac{1}{2} \psi(x, 0) \quad (16.57) \]

Then the windowed modal functions Eq. (16.19) are

\[ \mu_1(x, k, 0) = \mu_2(x, k, 0) = \frac{AB}{\sqrt{2\pi}} \int e^{-ikx'} e^{-\alpha x'^2/2 + ik_0 x'} e^{-\beta(x-x')^2/2} dx' \quad (16.58) \]
which works out to be

\[
\mu_1(x, k, 0) = \mu_2(x, k, 0) = \frac{AB}{\sqrt{\alpha + \beta}} e^{-\frac{\alpha^2}{4(\alpha + \beta)} x^2 - \frac{(k-k_0)^2}{4(\alpha + \beta)} - i\beta \frac{k-k_0}{\alpha+\beta} x} 
\]  

(16.59)

Applying the approximation given by Eq. (16.44) we obtain

\[
\mu(x, k, t) \approx e^{i\gamma k^2 t} \mu_1(x - \gamma k t, k, 0) + e^{-i\gamma k^2 t} \mu_2(x + \gamma k t, k, 0) 
\]  

(16.60)

and the quasi probability distribution is therefore given by

\[
F(x, k, t) = |\mu(x, k, t)|^2 \approx |\mu_1(x - \gamma k t, k, 0)|^2 + |\mu_2(x + \gamma k t, k, 0)|^2 
\]  

\[
+ e^{2i\gamma k^2 t} \mu_1(x - \gamma k t, k, 0) \mu_2^*(x + \gamma k t, k, 0) + e^{-2i\gamma k^2 t} \mu_1^*(x - \gamma k t, k, 0) \mu_2(x + \gamma k t, k, 0) 
\]  

(16.61)
Part III

Particle Motion in a Snell’s Law Medium
Chapter 17

Snell’s Law as a Conservation Law

We consider the motion of a “particle” in a medium of variable index of refraction whose motion is governed by Snell’s law. The trajectory of the particle traces a ray. Explicit expressions are derived for the velocity and acceleration components of the particle. These are derived directly from Snell’s law. It is shown that the propagation of a particle can be modeled in terms of Newtonian-like equations of motion and that momentum is conserved along the interface. It is shown that Snell’s law follows from this conservation law. We give a simple derivation of the Newtonian forces which dictate the motion and show that a position-dependent variable mass is necessary. Hamiltonian and Lagrangian formulations are also provided. The variable mass formulations of the Hamilton and Euler-Lagrange equations must be used with our Hamiltonian and Lagrangian respectively. Moreover the equations of motion we derive can be effectively applied to explain simply a number of phenomena and in particular we apply the equations to explain the SOFAR channel phenomenon. Properties of the motion are studied, and we finally present an example of particle trajectory for a specific case to illustrate the technique.

Consider a stratified medium where the index of refraction is a variable and depends on the depth. Suppose we are given the rule that the object travels according to Snell’s law.
Operationally, we evolve a particle by partitioning the medium into horizontal strips and applying Snell’s law, approximating each strip as having a constant index of refraction. The particle travels from strip to strip and then we take the limit as the width of the strips goes to zero.

Explanation of wave motion using rays is standard and what we are doing is deriving the dynamics of rays. Wave propagation in materials with variable index of refraction has become of importance in many fields including sound propagation in the ocean and metamaterials [45]. One can use these equations to not only calculate the ray path, which could alternatively be calculated by other means, such as the Bellhop algorithm [46] and Fermat’s principle, but also to obtain other information of interest. For example, one could obtain the response of a time-varying source as a function of delay time along the ray path.

If one assumes a constant mass for the particle, as Newton did, that leads the particle to have higher velocity in higher index of refraction. Newton was aware of this, but this has been experimentally rejected by Foucault in (1850) [47].

This chapter is largely adopted from the papers [48, 49].

17.1 Conservation Law, Velocities, and Accelerations from Snell’s Law

Consider a stratified medium where the speed, $c$, is only a function of depth,

$$c(x, z) = c(z) \quad (17.1)$$

and consider for a moment the two medium case in which Snell’s law is

$$\frac{\sin \theta_1}{c_1} = \frac{\sin \theta_2}{c_2} \quad (17.2)$$
and where $c_1$ and $c_2$ are the speeds in the two media and $\theta_1$ and $\theta_2$ are the usual angles measured with respect to the normal to the interface. In this case, the normal points parallel or antiparallel to the $z$ direction. Snell’s law implies that for any position, $x$, we have a conservation law

$$\frac{\sin \theta(x, z)}{c(z)} = \text{constant} = 2\eta$$

(17.3)

where we have taken the constant to be $2\eta$ for convenience. The velocities in the $x$ and $z$ directions, $v_x$ and $v_z$, are

$$v_x(x, z) = c(z) \sin \theta$$

(17.4)

$$v_z(x, z) = c(z) \cos \theta$$

(17.5)

In addition we have that

$$v_x^2(x, z) + v_z^2(x, z) = c^2(z)$$

(17.6)

Using Eq. (17.3) we obtain

$$\sin \theta = 2\eta c(z)$$

(17.7)

$$\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - 4\eta^2 c^2(z)}$$

(17.8)

Substituting these values into Eqs. (17.4) and (17.5) gives

$$v_x(x, z) = 2\eta c^2(z)$$

(17.9)

$$v_z(x, z) = \pm c(z) \sqrt{1 - 4\eta^2 c^2(z)}$$

(17.10)

This shows that both $v_x$ and $v_z$ are functions of $z$ only. Eq. (17.9) was previously obtained in a somewhat different form [50].
The constant of motion can be expressed in two equivalent ways, in terms of $v_x$ or $v_z$

$$\eta = \frac{1}{2} \frac{v_x(z)}{c^2(z)}$$  \hspace{1cm} (17.11)

$$= \frac{\sqrt{c^2(z) - v_z^2(z)}}{2c^2(z)}$$ \hspace{1cm} (17.12)

For later use we note that

$$v_z v_x = \pm 2\eta c^2 \sqrt{c^2 - 4\eta^2 c^4}$$ \hspace{1cm} (17.13)

To obtain the acceleration equations we can simply differentiate Eqs. (17.9) or (17.10). Alternatively we can differentiate Eq. (17.11) and (17.12) and solve for the accelerations. One obtains that

$$a_x = \frac{d}{dt}v_x(z) = \frac{2}{c} v_x v_z = 4\eta c^2 v_x$$ \hspace{1cm} (17.14)

and

$$a_z = \frac{d}{dt}v_z(z) = \frac{c'}{c} (v_z^2 - v_x^2) = \frac{c'}{c} (v_z^2 - 4\eta^2 c^4)$$ \hspace{1cm} (17.15)

### 17.1.1 Acceleration from the Constraint

Alternatively we now show that one can obtain the equations of motion from the Snell’s law constraint $\eta$, as given by Eq. (17.86). Differentiating Eq. (17.86) we have

$$\frac{d}{dt} \eta = 0 = \frac{1}{2} \frac{d}{dt} \frac{v_x}{c^2(z)} = \frac{d}{dt} \frac{v_x}{2c^2(z)} - \frac{v_x}{c^3(z)} c' \frac{d}{dt} z$$ \hspace{1cm} (17.16)

and hence

$$\frac{dv_x}{dt} = \frac{2c'}{c} v_x v_z$$ \hspace{1cm} (17.17)
which is the acceleration equation in the horizontal direction, Eq. (17.14). Rewriting \( \eta \) as

\[
\eta = \sqrt{\frac{c^2(z) - v_z^2}{2c^2(z)}}
\]  

(17.18)

differentiation gives

\[
\frac{d}{dt} \eta = 0 = \frac{cc'v_z - v_z \frac{dv_z}{dt}}{2c^2(z)v_x} - \frac{v_x v_z c'}{c^3(z)}
\]  

(17.19)

which leads to the acceleration equation in the vertical direction, Eq. (17.15)

\[
\frac{d^2 z}{dt^2} = \frac{c'}{c} (v_z^2 - v_x^2)
\]  

(17.20)

### 17.2 Mass and Forces

We now ask whether it is possible to have both a constant mass, and forces which arise from a potential function. Assume that the mass is constant. Then Newton’s laws states that

\[
m \frac{dv_x}{dt} = - \frac{\partial}{\partial x} \varphi(x, z)
\]  

(17.21)

\[
m \frac{dv_z}{dt} = - \frac{\partial}{\partial z} \varphi(x, z)
\]  

(17.22)

Differentiating the first equation with respect to \( \frac{\partial}{\partial z} \) and the second with respect to \( \frac{\partial}{\partial x} \) we obtain that

\[
m \frac{\partial}{\partial z} \frac{dv_x}{dt} = m \frac{\partial}{\partial x} \frac{dv_z}{dt}
\]  

(17.23)

But the right hand side is zero because \( \frac{dv_z}{dt} \) has no \( x \) dependence. Hence

\[
m \frac{\partial}{\partial z} \frac{dv_x}{dt} = 0
\]  

(17.24)
and therefore
\[ \frac{\partial}{\partial z} \frac{d}{dt} v_x = 0 \] (17.25)

Now, substitute Eq. (17.14) into Eq. (17.25) to obtain
\[ \frac{\partial}{\partial z} \frac{d}{dt} v_x(z) = \frac{\partial}{\partial z} \frac{2c'}{c} v_x v_z = \frac{\partial}{\partial z} 4 \eta c c' v_z = \pm \frac{\partial}{\partial z} 4 \eta c c' \sqrt{c^2 - 4 \eta^2 c^4} = 0 \] (17.26)

We have expressed the left hand side of Eq. (17.25) in terms of a constant of the motion and the speed in the medium. Since we can have arbitrary speeds \( c(z) \) for different media, and arbitrary \( \eta \)'s for each situation (determined by initial conditions), then \( \frac{\partial}{\partial z} 4 \eta c c' \sqrt{c^2 - 4 \eta^2 c^4} \) cannot be in general always zero, and we have a contradiction. We show in the next section that if the mass is taken to be variable, then we have no contradiction.

17.2.1 Variable mass

We now show that the difficulty indicated by Eq. (17.26) may be removed if we have a variable mass, and that one can obtain a dynamics given by Newton’s second law. For the variable mass case, Newton’s equations are written in terms of momentum [51].

\[ \frac{d}{dt} m(z) v_x = - \frac{\partial}{\partial x} \varphi(x, z) \] (17.27)
\[ \frac{d}{dt} m(z) v_z = - \frac{\partial}{\partial z} \varphi(x, z) \] (17.28)

Now, instead of Eq. (17.23) we have
\[ \frac{\partial}{\partial z} \frac{d}{dt} m(z) v_x = \frac{\partial}{\partial x} \frac{d}{dt} m(z) v_z \] (17.29)

and again we must have
\[ \frac{\partial}{\partial z} \frac{d}{dt} m(z) v_x = 0 \] (17.30)
since the right hand side of Eq. (17.29) still does not have any \( x \) dependence. Carrying out the differentiation in Eq. (17.30) we have

\[
\frac{\partial}{\partial z} \left( m'(z)v_zv_x + m \frac{d}{dt}v_x \right) = 0 \quad (17.31)
\]

and using Eq. (17.14) we obtain

\[
\frac{\partial}{\partial z} \left( m'(z)v_zv_x + 2m \frac{c'}{c}v_zv_x \right) = 0 \quad (17.32)
\]

Therefore

\[
v_zv_x \left( m'(z) + 2m \frac{c'}{c} \right) = \text{constant} \quad (17.33)
\]

and using Eq. (17.13) we have

\[
\pm 2\eta c^2 \sqrt{c^2 - 4c'^2} \left( m'(z) + 2m \frac{c'}{c} \right) = \text{constant} \quad (17.34)
\]

The only way this can be satisfied for arbitrary \( c(z) \) and \( \eta \) is if

\[
m'(z) + 2m \frac{c'}{c} = 0 \quad (17.35)
\]

which implies that (up to a constant factor)

\[
m(z) = \frac{1}{2c^2(z)} \quad (17.36)
\]
17.2.2 Forces

Having obtained the mass we now obtain the forces. For the force in the $x$ direction we have

$$F_x = \frac{d}{dt} m(z) v_x = \frac{d}{dt} \frac{1}{2c^2} v_x = \frac{d}{dt} \eta$$  \hspace{1cm} (17.37)

But $\eta$ is a constant of the motion and hence

$$F_x = 0$$  \hspace{1cm} (17.38)

This agrees with Eq. (17.30). For the $z$ direction

$$F_z = \frac{d}{dt} m(z) v_z = \frac{d}{dt} \frac{1}{2c^2} v_z = -\frac{c'}{c^3} v_z + \frac{1}{2c^2} \frac{d}{dt} v_z$$  \hspace{1cm} (17.39)

Using Eqs (17.15) and (17.6), gives

$$F_z = -\frac{c'}{2c}$$  \hspace{1cm} (17.40)

For the potential we want

$$F_x = -\frac{\partial}{\partial x} \varphi(x, z) = 0$$  \hspace{1cm} (17.41)

$$F_z = -\frac{\partial}{\partial z} \varphi(x, z) = -\frac{c'}{2c}$$  \hspace{1cm} (17.42)

and therefore

$$\varphi(x, z) = \frac{1}{2} \ln c(z)$$  \hspace{1cm} (17.43)
17.3 Properties of the motion

Velocity equations. From Eq. (17.9) we see that the sign of \( \frac{dx}{dt} \) is constant and given by the sign of the constant of the motion \( \eta \), which is obtained from the initial conditions,

\[
\eta = \frac{v_x(z_0)}{2c^2(z_0)} \quad (17.44)
\]

Hence the particle always moves to the right or to the left depending on the sign of \( \eta \). We point out that the equation for \( \frac{dx}{dt} \) in Eq. (17.9) has been previously given in reference [50].

Notice that for the \( \frac{dz}{dt} \) equation in Eq. (17.10) one would run into difficulty if the square root became imaginary. We now show that indeed it will never become imaginary.

For convenience we define

\[
\Delta(z) = 1 - 4\eta^2c^2(z) \quad (17.45)
\]

and define the point \( z_* \) where

\[
\Delta(z_*) = 0 \quad (17.46)
\]

At \( z_* \) we have that

\[
4\eta^2c^2(z_*) = 1 \quad (17.47)
\]

and from Eq. (17.10) we have that

\[
\frac{dz}{dt} \bigg|_{z=z_*} = 0 \quad (17.48)
\]

which implies that

\[
\frac{d}{dt} \left( \Delta \right) \bigg|_{z=z_*} = 0 \quad (17.49)
\]

Hence we have to look at the second derivative of \( z \) at \( z = z_* \). We obtain

\[
\frac{d^2}{dt^2} \Delta \bigg|_{z=z_*} = \frac{1}{2} \eta^2 c \frac{d^2}{dz^2} \bigg|_{z=z_*} > 0 \quad (17.50)
\]
which shows that if \( \Delta(z) = 0 \) at a point, then it must increase at that point and hence can never go negative.

**Equation for the path.** We obtain the equation for the path in the usual way, wherein one eliminates time to obtain a relationship between \( x \) and \( z \) [52, 53]. We have

\[
\frac{dz}{dx} = \frac{dz}{dt} \frac{dt}{dx}
\] (17.51)

and using Eqs. (17.9) and (17.10), we obtain

\[
\frac{dz}{dx} = \sqrt{\frac{1}{4\eta^2 c^2(z)} - 1}
\] (17.52)

### 17.4 Lagrangian formulation

Lagrange’s equation involves positions and velocities; however, the standard forms of Lagrange’s equations have to be modified when we have a variable mass. This has been done in many fields and in particular we use the formulation given in references [51, 54, 56, 57].

For a two dimensional case with position dependent mass, Lagrange’s equations are

\[
\frac{d}{dt} \frac{\partial L}{\partial v_x} - \frac{\partial L}{\partial x} = -\frac{1}{2} (v_x^2 + v_z^2) \frac{\partial m}{\partial x}
\] (17.53)

\[
\frac{d}{dt} \frac{\partial L}{\partial v_z} - \frac{\partial L}{\partial z} = -\frac{1}{2} (v_x^2 + v_z^2) \frac{\partial m}{\partial z}
\] (17.54)

The Lagrangian is the kinetic energy minus the potential energy,

\[
L = T - \varphi(x, z) = \frac{1}{2} m(v_x^2 + v_z^2) - \varphi(x, z)
\] (17.55)
where $\varphi(x, z)$ and $T$ are the potential and kinetic energy respectively. We point out that if the mass and potential $\varphi(x, z)$ are only functions of position we also have

$$\frac{d}{dt} \frac{\partial T}{\partial v_x} - \frac{\partial T}{\partial x} = -\frac{\partial \varphi(x, z)}{\partial x} - \frac{1}{2}(v_x^2 + v_z^2) \frac{\partial m}{\partial x}$$ (17.56)

$$\frac{d}{dt} \frac{\partial T}{\partial v_z} - \frac{\partial T}{\partial z} = -\frac{\partial \varphi(x, z)}{\partial z} - \frac{1}{2}(v_x^2 + v_z^2) \frac{\partial m}{\partial z}$$ (17.57)

which are the equations given in references [51, 54]. We apply these equations to our case where the mass is a function of position and where the potential is

$$\varphi(x, z) = \frac{1}{2} \ln c(z) = -\frac{1}{4} \ln(2m(z))$$ (17.58)

The Lagrangian is then

$$L = \frac{v_x^2 + v_z^2}{4c^2} - \frac{1}{2} \ln c(z)$$ (17.59)

We can see from Eq. (17.36) that the mass varies with position as

$$\frac{\partial m}{\partial x} = 0 ; \quad \frac{\partial m}{\partial z} = -\frac{c'}{c^3}$$ (17.60)

which shows that the right hand side of Eq. (17.56) is zero. Furthermore

$$F_z = -\frac{\partial \varphi}{\partial z} = \frac{1}{2}(v_x^2 + v_z^2) \frac{\partial m}{\partial z}$$ (17.61)

which shows that the right hand side of Eq. (17.57) is also zero. Therefore Lagrange's equations, Eqs. (17.56) and (17.57), become

$$\frac{\partial T}{\partial x} - \frac{d}{dt} \frac{\partial T}{\partial v_x} = 0 ; \quad \frac{\partial T}{\partial z} - \frac{d}{dt} \frac{\partial T}{\partial v_z} = 0$$ (17.62)
These equations now lead to the equations of motion as given by Eqs. (17.14), (17.15) and the constraint Eq. (17.3).

### 17.5 Hamiltonian Formulation

The Hamiltonian formulation involves positions and canonical momenta. As with the Lagrangian formulation, Hamilton’s equations have to be modified to take variable mass into account. For the general two dimensional case with position-dependent mass, Hamilton’s equations become

\[
\frac{dx}{dt} = \frac{\partial H}{\partial p_x} ; \quad \frac{dz}{dt} = \frac{\partial H}{\partial p_z} \tag{17.63}
\]

\[
\frac{dp_x}{dt} = -\frac{\partial H}{\partial x} - \frac{1}{2} \frac{p_x^2 + p_z^2}{m} \frac{\partial m}{\partial x} ; \quad \frac{dp_z}{dt} = -\frac{\partial H}{\partial z} - \frac{1}{2} \frac{p_x^2 + p_z^2}{m} \frac{\partial m}{\partial z} \tag{17.64}
\]

where \( p_x \) and \( p_z \) are the canonical momenta and are given by \( p_i = \frac{\partial}{\partial v_i} L \). For our case, the canonical momenta are

\[
p_x = \frac{\partial}{\partial v_x} L = mv_x ; \quad p_z = \frac{\partial}{\partial v_z} L = mv_z \tag{17.65}
\]

and hence the Hamiltonian is\(^1\)

\[
H = \frac{p_x^2 + p_z^2}{2m} + \varphi(x, z) = (p_x^2 + p_z^2)c^2 + \frac{1}{2} \ln c(z) \tag{17.66}
\]

with the same potential, Eq. (17.58), used for our Lagrangian. We now derive the equations of motion. From the first pair of Hamilton’s equations, Eqs. (17.63), we obtain

\[
\frac{dx}{dt} = \frac{\partial H}{\partial p_x} = 2p_x c^2 = v_x ; \quad \frac{dz}{dt} = \frac{\partial H}{\partial p_z} = 2p_z c^2 = v_z \tag{17.67}
\]

\(^1\)This Hamiltonian has also been considered in [55].
and from the second pair of the modified Hamilton’s equations, Eqs. (17.64), we obtain

\[
\frac{dp_x}{dt} = -\frac{\partial H}{\partial x} - \frac{1}{2} \left( \frac{p_x^2 + p_z^2}{m^2} \right) \frac{\partial m}{\partial x} = 0 \tag{17.68}
\]

\[
\frac{dp_z}{dt} = -\frac{\partial H}{\partial z} - \frac{1}{2} \left( \frac{p_x^2 + p_z^2}{m^2} \right) \frac{\partial m}{\partial z} = -2cc' (p_x^2 + p_z^2) - \frac{\partial}{\partial z} \varphi(x, z) - \frac{1}{2} \frac{(p_x^2 + p_z^2)}{m^2} \frac{\partial m}{\partial z} \tag{17.69}
\]

Using Eqs. (17.58) and (17.60), these lead straightforwardly to

\[
\frac{dp_x}{dt} = 0 \tag{17.70}
\]

\[
\frac{dp_z}{dt} = -2cc' (p_x^2 + p_z^2) \tag{17.71}
\]

which lead to the equations of motion as given by Eqs. (17.14), (17.15) and (17.3).

### 17.5.1 Energy Production

Using Eq. (17.66) and taking into account variable mass we obtain

\[
\frac{dH}{dt} = -\frac{T \, dm}{m \, dt} \tag{17.72}
\]

Eq. (17.72) is general. For our case we have

\[
\frac{dm}{dt} = \frac{d}{dt} \frac{1}{2c^2(z)} = -\frac{c'}{c^3} v_z \tag{17.73}
\]

and substituting into Eq. (17.72) we obtain

\[
\frac{dH}{dt} = \frac{c'}{2c} v_z = -F_z v_z \tag{17.74}
\]
where $F_z v_z$ can be interpreted as the power. This shows that energy is not conserved although it may be conserved over a time interval.

### 17.6 Snell’s Law from Momentum Conservation

We now show that Snell’s law follows from conservation of momentum in the horizontal direction. Consider the usual horizontal two-medium situation with respective velocities $c_1$ and $c_2$. As we have shown, momentum in the horizontal direction is conserved and hence for the two medium case we have $p_x^{(1)} = p_x^{(2)}$, where $p_x^{(1)}$ and $p_x^{(2)}$ are the horizontal components of the momenta in media 1 and 2 respectively. Using Eqs. (17.36), (17.65), and (17.70) we have

$$\frac{1}{2c_1^2} v_x^{(1)} = \frac{1}{2c_2^2} v_x^{(2)} \quad (17.75)$$

But because we are considering a horizontally-stratified medium where the speed is only a function of $z$ we have that

$$v_x(z) = c(z) \sin \theta(z) \quad ; \quad v_z(z) = c(z) \cos \theta(z) \quad (17.76)$$

where the angle $\theta$ is measured from the horizontal. Hence, from Eq. (17.75)

$$\frac{1}{2c_1^2} c_1 \sin \theta_1 = \frac{1}{2c_2^2} c_2 \sin \theta_2 \quad (17.77)$$

giving

$$\frac{\sin \theta_1}{c_1} = \frac{\sin \theta_2}{c_2} \quad (17.78)$$

which is Snell’s law for angles $\theta$ measured from the horizontal.
17.7 SOFAR channels

A SOFAR sound channel is a situation where sound is trapped within a limited range of depths. SOFAR stands for SOund Fixing And Ranging channel and comes about when there is a local minimum of the sound speed profile. In the ocean, the speed of sound depends mainly on temperature, pressure, and salinity. In certain places in the ocean there is a local minimum sound speed, around a depth of 700-800 meters; this creates a channel containing the minimum, through which sound travels significant distances. We now show how such a channel can be understood in a simple manner using our particle formulation.

Consider a sound speed which has a single minimum. For convenience we take \( z = 0 \) to be the location of the minimum and hence,

\[
\frac{d}{dz} c \equiv c' (0) = 0 \quad (17.79)
\]

Suppose that

\[
\frac{d}{dz} c > 0 \quad \text{for } z > 0 \quad (17.80)
\]

Consider the case where the particle is on the positive \( z \) side of the minimum, and that it is traveling in the direction of increasing \( z \). We do not exclude motion in the \( x \)-direction and in fact, motion in the \( x \) direction is necessary for the SOFAR channel phenomenon. We show that there is a turning point, \( z_T \), that is, a point where the velocity goes to zero and the acceleration points downward. At the turning point, \( z_T \), the velocity in the \( z \) direction is zero

\[
v_z (z_T) = 0 \quad (17.81)
\]

and therefore, Eqs. (17.6) and (17.9) imply that

\[
|v_x (z_T)| = c (z_T) = \frac{1}{2 |\eta|} \quad (17.82)
\]
Furthermore, using Eq. (17.15) the acceleration is then

\[ a_z(z_T) = -c(z_T)c'(z_T) \]  

(17.83)

Since the speed \( c \) is always positive, and \( c' \) is positive, the acceleration at \( z_T \) is negative, and the particle turns around.

Similar considerations apply to a particle traveling downward. We note that the turning point can be found from

\[ c(z_T) = \frac{1}{2\eta} \]  

(17.84)

As an example consider the case where

\[ c(z) = a + bz^2 \]  

(17.85)

with \( b \) positive (so we have a minimum), and where \( c' \) equals zero at \( z = 0 \). The turning point is obtained by solving

\[ c(z_T) = a + b z_T^2 = \frac{1}{2\eta} \]  

(17.86)

giving

\[ z_T = \pm \sqrt{\frac{1}{b} \left( \frac{1}{2\eta} - a \right)} \]  

(17.87)

The velocity in the \( z \) direction at these points is zero, and the acceleration is

\[ a_z(z_T) = \mp 2b|z_T|(a + b(z_T)^2) \]  

(17.88)

which points toward the minimum \( (z = 0) \) of the speed function \( c(z) \).

Notice that equation Eq. (17.86) implies that it important that \( \eta \neq 0 \) in order to have a SOFAR channel. This is a sensible result since \( \eta = 0 \) means that \( v_x = 0 \), which tells us
that the particle is moving in the $z$ direction. In the ordinary Snell’s law formulation, this means that the ray is parallel to the normal to the interfaces, in which case, the rays travel unimpeded (the rays are perfectly straight and continue forever without reversing course). In such a case, we do not have a SOFAR channel.

17.8 Method of Solution

As typical with mechanics problems, the formulation that is used depends on the circumstances of the particular problem. We have found that the use of the velocity equations, Eqs. (17.9) and (17.10), is the simplest from an analytic point of view, where $\eta$ is obtained from the initial conditions,

$$\eta = \frac{v_x(z_0)}{2c^2(z_0)} \quad (17.89)$$

The general approach is to first solve the $\frac{dz}{dt}$ equation in Eq. (17.10),

$$\int_{z_0}^{z(t)} \frac{dz}{c(z)\sqrt{1 - 4\eta^2c^2(z)}} = t - t_0 \quad (17.90)$$

This gives $z(t)$ which is then substituted into the $\frac{dx}{dt}$ equation

$$\frac{dx}{dt} = 2\eta c^2(z(t)) \quad (17.91)$$

to give

$$x(t) - x_0 = 2\eta \int_{t_0}^{t} c^2(z(t)) dt \quad (17.92)$$
17.9 Example

As an example consider the case where speed is a linear function of depth, \( c(z) = a + bz \). The velocity equations are then

\[
\frac{dx}{dt} = 2\eta(a + bz)^2
\]

(17.93)

\[
\frac{dz}{dt} = (a + bz)\sqrt{1 - 4\eta^2(a + bz)^2}
\]

(17.94)

These equations can be solved exactly and we give here only the final result,

\[
z(t) = \frac{1}{2b\eta} \frac{1}{\cosh(\beta - bt)} - \frac{a}{b}
\]

(17.95)

and

\[
x(t) = x_0 - \frac{1}{2b\eta} (\tanh(\beta - bt) - \tanh \beta)
\]

(17.96)

where \( \beta \) is a constant determined by the initial conditions

\[
\beta = \cosh^{-1} \left( \frac{1}{2\eta c(z_0)} \right)
\]

(17.97)

We note that it follows that

\[
c(t) = \frac{1}{2\eta} \frac{1}{\cosh(\beta - bt)}
\]

(17.98)

This is a useful relation in manipulating the above equations. For the energy and its rate of change one obtains, respectively

\[
H(t) = \frac{1}{4} - \frac{1}{2} \ln(2\eta) - \frac{1}{2} \ln [\cosh(\beta - bt)]
\]

(17.99)
and

\[ \frac{dH}{dt} = \frac{b}{2} \tanh(\beta - bt) \]  

(17.100)
Appendices
Appendix A

Green’s Function Solutions of Fourier Pairs

In this appendix, we show that if our Green’s function is of the form

\[ G(x, x'; t, t') = G(x, x'; t - t') \quad (A.1) \]

then the Green’s function equation for the Fourier transform of the wave has only a single integration, as opposed to the two which the original wave had

\[ \frac{1}{\sqrt{2\pi}} \hat{u}_f(x, \omega) = \int H(x, x'; \omega) \hat{f}(x', \omega) dx' \quad (A.2) \]

We start by defining the Fourier transform pairs

\[ u_f(x, t) = \frac{1}{\sqrt{2\pi}} \int \hat{u}_f(x, \omega)e^{i\omega t} d\omega \quad (A.3) \]

\[ f(x, t) = \frac{1}{\sqrt{2\pi}} \int \hat{f}(x, \omega)e^{i\omega t} d\omega \quad (A.4) \]
\[ G(x, x'; t - t') = \frac{1}{\sqrt{2\pi}} \int H(x, x'; \omega) e^{i\omega(t-t')} d\omega \quad (A.5) \]

Then, using the Green’s function solution for \( u_f \), along with the above definitions, we see that

\[
\begin{align*}
    u_f(x, t) &= \int G(x, x'; t - t') f(x', t') dx' dt' \\
    &= \int dx' dt' \frac{1}{\sqrt{2\pi}} \int d\omega_1 e^{i\omega_1(t-t')} H(x, x'; \omega_1) \frac{1}{\sqrt{2\pi}} \int d\omega_2 e^{i\omega_2 t} \hat{f}(x', \omega_2) \\
    &= \frac{1}{2\pi} \int dx' d\omega_1 d\omega_2 H(x, x'; \omega_1) \hat{f}(x', \omega_2) e^{i\omega_1 t} \int dt' e^{it'(\omega_2 - \omega_1)} \\
    &= \int dx' d\omega_1 d\omega_2 H(x, x'; \omega_1) \hat{f}(x', \omega_2) e^{i\omega_1 t} \delta(\omega_2 - \omega_1) \\
    &= \int d\omega e^{i\omega t} \int dx' H(x, x'; \omega) \hat{f}(x', \omega) \\
\end{align*}
\quad (A.6)\]

To reconcile Eq. (A.10) with the definition Eq. (A.3), we must have that the Fourier transform of the wave has its own Green’s function solution involving \( H \) and \( \hat{f} \) (defined in Eqs. (A.4) and (A.5))

\[
    \frac{1}{\sqrt{2\pi}} \hat{u}_f(x, \omega) = \int H(x, x'; \omega) \hat{f}(x', \omega) dx' \\
\quad (A.11)\]
Appendix B

Main Result for Single-Frequency Green’s Function

In this appendix we derive our main result for the case where one of the variables, in this case $\omega$, is not Wigner-transformed. This means that we are studying the intensity in the frequency coordinate. This case arises in Chap. 9.

We start with the result of App. A

$$\frac{1}{\sqrt{2\pi}} \hat{u}_f(x, \omega) = \int H(x, x'; \omega) \hat{f}(x', \omega)dx'$$  \hspace{1cm}  (B.1)

The Wigner distribution of $\hat{u}_f$ is

$$W_{\hat{u}_f}(x, k|\omega) = \int \hat{f}^*(x_1, \omega) \hat{f}(x_2, \omega) H^*(x - \theta/2, x_1; \omega)H(x + \theta/2, x_2; \omega)e^{-ik\theta}dx_1dx_2d\theta$$  \hspace{1cm}  (B.2)
Changing variables, we have that

\[ W_{\tilde{u},j}(x,k|\omega) = \int \hat{f}^*(x' - \psi/2, \omega) \hat{f}(x' + \psi/2, \omega) H^*(x - \theta/2, x' - \psi/2; \omega) \]
\[ \times H(x + \theta/2, x' + \psi/2; \omega) e^{-i\theta k} dx' d\psi d\theta \]  \hspace{1cm} (B.3)

Now, using the definition of \( W_{\tilde{f}} \), we have that

\[ \hat{f}^*(x' - \psi/2, \omega) \hat{f}(x' + \psi/2, \omega) = \int W_{\tilde{f}}(x', k'|\omega) e^{+i\psi k'} dk' \]  \hspace{1cm} (B.4)

Using Eq. (B.4) we have that Eq. (B.3) becomes

\[ W_{\tilde{u},j}(x,k|\omega) = (2\pi)^2 \int dx' dk' W_{\tilde{f}}(x', k'|\omega) \frac{1}{(2\pi)^2} \int H^*(x - \theta/2, x' - \psi/2; \omega) \]
\[ \times H(x + \theta/2, x' + \psi/2; \omega) e^{-i\theta k} e^{+i\psi k'} d\psi d\theta \]
\[ = (2\pi)^2 \int W_{\tilde{f}}(x', k'|\omega) W_H(x, k; x', k'|\omega) dx' dk' \]  \hspace{1cm} (B.5)

This is an intuitive result, since here \( \omega \) is a parameter, and the rest of the variables are treated in the usual way.

In the case where

\[ H(x, x'; \omega) = H(x - x'; \omega) \]  \hspace{1cm} (B.7)

we have that Eq. (B.6) becomes

\[ W_{\tilde{u},j}(x, k|\omega) = (2\pi)^2 \int W_{\tilde{f}}(x', k'|\omega) W_H(x - x', k|\omega) dx' \]  \hspace{1cm} (B.8)
Appendix C

Main Result for Green’s Functions
Requiring Less Integrations

In deriving our main result, Eq. (5.12), we assumed that the Green’s function equation had an integration for each of the wave’s coordinates

$$u(x, t) = \int G(x, x'; t, t') f(x', t') dx' dt'$$  \hspace{1cm} (C.1)

But what if, as we saw in the previous appendix, the Green’s function equation for a wave requires less integrations?

In this appendix, we show that the analog of our main result for the case where the Green’s function equation has one less integration is

$$W_{\tilde{u}_f}(x, k, t, \omega) = (2\pi)^2 \int W_{\tilde{f}}(x', k', t', \omega) W_H(x, k; x', -k', t - t', \omega) dx' dk' dt'$$  \hspace{1cm} (C.2)
Defining the Fourier pair

\[ u_f(x, t) = \frac{1}{\sqrt{2\pi}} \int \hat{u}_f(x, \omega) e^{i\omega t} d\omega \]  
(C.3)

and using the result from App. A

\[ \frac{1}{\sqrt{2\pi}} \hat{u}_f(x, \omega) = \int H(x, x'; \omega) \hat{f}(x', \omega) dx' \]  
(C.4)

then the Wigner distribution of \( \hat{u}_f \) is

\[
W_{\hat{u}_f}(x, k, t, \omega) = \frac{1}{(2\pi)^2} \int \hat{u}_f^*(x - \theta/2, \omega - \tau/2) \hat{u}_f(x + \theta/2, \omega + \tau/2) e^{-i\theta k} e^{i\tau t} d\theta d\tau 
\]
(C.5)

\[
= \frac{1}{(2\pi)^2} \int d\theta d\tau e^{-i\theta k} e^{i\tau t} \sqrt{2\pi} \int dx_1 H^*(x - \theta/2, x_1; \omega - \tau/2) \hat{f}^*(x_1, \omega - \tau/2) \\
\times \sqrt{2\pi} \int dx_2 H(x + \theta/2, x_2; \omega + \tau/2) \hat{f}(x_2, \omega + \tau/2) 
\]
(C.6)

\[
= \frac{1}{2\pi} \int d\theta d\tau e^{-i\theta k} e^{i\tau t} \int dx_1 dx_2 \hat{f}^*(x_1, \omega - \tau/2) \hat{f}(x_2, \omega + \tau/2) \\
\times H^*(x - \theta/2, x_1; \omega - \tau/2) H(x + \theta/2, x_2; \omega + \tau/2) 
\]
(C.7)

Changing variables

\[ x_1 = x' - \phi/2 \]  
(C.8)

\[ x_2 = x' + \phi/2 \]  
(C.9)
then

\[ W_{\hat{u}_f}(x, k, t, \omega) = \frac{1}{2\pi} \int d\theta d\tau e^{-\imath \theta k} e^{\imath \tau t} \int dx' d\phi \hat{f}^*(x' - \phi/2, \omega - \tau/2) \hat{f}(x' + \phi/2, \omega + \tau/2) \]

\[ \times H^*(x - \theta/2, x' - \phi/2; \omega - \tau/2) H(x + \theta/2, x' + \phi/2; \omega + \tau/2) \quad (C.10) \]

Now, the Wigner distribution of \( \hat{f} \) is

\[ W_{\hat{f}}(x', k', t', \omega) = \frac{1}{(2\pi)^2} \int d\phi d\tau e^{-\imath \phi k'} e^{\imath \tau t'} \hat{f}^*(x' - \phi/2, \omega - \tau/2) \hat{f}(x' + \phi/2, \omega + \tau/2) \quad (C.11) \]

and (double) inverse Fourier transforming, we see that the factor in Eq. (C.10) can be expressed as

\[ \hat{f}^*(x' - \phi/2, \omega - \tau/2) \hat{f}(x' + \phi/2, \omega + \tau/2) = \int W_{\hat{f}}(x', k', t', \omega) e^{\imath \phi k'} e^{-\imath \tau t'} dk' dt' \quad (C.12) \]

The Wigner distribution of \( \hat{u}_f \) is

\[ W_{\hat{u}_f}(x, k, t, \omega) = (2\pi)^2 \int W_{\hat{f}}(x', k', t', \omega) W_H(x, k; x', -k'; t - t', \omega) dx' dk' dt' \quad (C.13) \]
Appendix D

Obtaining the Full Wigner Distribution of the Green’s Function from its Double Fourier Transform in the Case of Variable Speed

In this appendix we give an alternate derivation of Eq. (6.5).

We obtain the Wigner of the Green’s function which we obtained in Chap. 6, but this time, we do so by using the Green’s function’s double Fourier-transform (reproduced here for convenience). Consider Eqs. (4.21), (4.28), and (4.40)

\[
G(\rho, \rho'; z, z', t - t') = \frac{1}{2\pi} \int h(\rho, \rho'; z, z'; \omega) e^{i\omega(t-t')} d\omega
\]  
\[\text{Eq. (D.1)}\]

\[
h(\rho, \rho'; z, z'; \omega) = \frac{1}{2\pi} \int g(\kappa; z, z'; \omega) e^{i\kappa \cdot (\rho - \rho')} d\kappa
\]  
\[\text{Eq. (D.2)}\]

\[
g(\kappa; z, z'; \omega) = \sum_n \frac{\phi_n(z', \omega)\phi_n(z, \omega)}{\lambda_n - \kappa^2}
\]  
\[\text{Eq. (D.3)}\]
The Wigner distribution of the above function is

\[
W_g(\rho, \kappa; z, k_z; z', k'_z; \omega, t) = \frac{1}{(2\pi)^5} \int d\theta_\kappa d\theta_z d\theta'_z e^{-i\theta_\kappa \cdot \rho} e^{-i\theta_z k_z} e^{-i\theta'_z k'_z} e^{-i\tau t} \times \sum_n \frac{\phi_n(z' - \theta'_z, \omega - \tau/2) \phi_n^*(z - \theta_z/2, \omega - \tau/2)}{\lambda_n^* - (\kappa + \theta_\kappa/2)^2} \times \sum_m \frac{\phi_m^*(z' + \theta'_z/2, \omega + \tau/2) \phi_m(z + \theta_z/2, \omega + \tau/2)}{\lambda_m - (\kappa - \theta_\kappa/2)^2} \tag{D.4}
\]

Notice that because \(\kappa\) is in the wavenumber domain, we performed its Wigner transform in the conjugate way. This becomes

\[
W_g(\rho, \kappa; z, k_z; z', k'_z; \omega, t) = \frac{1}{(2\pi)^5} \sum_n \sum_m \int d\psi_\kappa \frac{e^{-i\theta_\kappa \cdot \rho}}{\lambda_n^* - (\kappa + \theta_\kappa/2)^2 \lambda_m - (\kappa - \theta_\kappa/2)^2} \int d\tau e^{-i\tau t} \times \int \phi_n(z - \theta_z/2, \omega - \tau/2) \phi_m(z + \theta_z/2, \omega + \tau/2)e^{-i\theta_z k_z} d\theta_z \times \int \phi_n(z' - \theta'_z/2, \omega - \tau/2) \phi_m^*(z' + \theta'_z/2, \omega + \tau/2)e^{-i\theta'_z k'_z} d\theta'_z \tag{D.5}
\]

Changing variables

\[
\theta_\kappa = 2\psi_\kappa - 2\kappa \tag{D.6}
\]

the Wigner distribution of \(g\) becomes

\[
W_g(\rho, \kappa; z, k_z; z', k'_z; \omega, t) = \frac{4}{(2\pi)^5} \sum_n \sum_m \int d\psi_\kappa \frac{1}{\lambda_n^* - \psi_\kappa^2 \lambda_m - (2\kappa - \psi_\kappa)^2} \int d\tau e^{-i\tau t} \times \int \phi_n(z - \theta_z/2, \omega - \tau/2) \phi_m(z + \theta_z/2, \omega + \tau/2)e^{-i\theta_z k_z} d\theta_z \times \int \phi_n(z' - \theta'_z/2, \omega - \tau/2) \phi_m^*(z' + \theta'_z/2, \omega + \tau/2)e^{-i\theta'_z k'_z} d\theta'_z \tag{D.7}
\]
which is the expression we obtained from \( g \)'s double Fourier transform in Chap. 6, Eq. (6.5).

Remember again, that the integrations over \( \theta_z \) and \( \theta'_z \) are effectively not over the infinite space, but somewhat more complicated [10]. This is considered explicitly in App. 7.4.
Appendix E

Single-Frequency Wigner Distribution of Green’s Function

In this chapter we calculate directly the single-frequency Wigner distribution of the Green’s function for an arbitrary depth-dependent sound speed. That is, we do not Wigner-transform the frequency coordinate $\omega$ in Eq. (4.41).

We can only use the single-frequency (intensity) Wigner distribution if it is appropriate, that is, only if the autocorrelation function of the noise source is also only concerned with a single frequency\(^1\) (i.e. it is given in terms of the frequency-intensity of the noise). In App. B, we show how our main result (5.12) would change for such a case in order to find the Wigner spectrum of the source-response. In App. H, we provide the results for different degrees of stationarity of the noise source. Some approximations are applied, and general results are provided. We use the results of App. H to reproduce the well-known results of Kuperman and Ingenito in chapter 9.

We note that the single-frequency (intensity) Wigner distribution of $h$ (Eq. (4.41)) does not correspond to the Wigner distribution of the Green’s function $G$ (Eq. (4.21)) of the

\(^1\)See App. B.
three-dimensional scalar wave equation in the time domain. This can be seen immediately since it is time independent.

Recall that the Green’s function of the three-dimensional scalar wave equation with depth dependent speed $c(z)$ was derived in Chap. 4 in the frequency domain, and is given by Eq. (4.41) (reproduced below for convenience)

$$h(\rho, z, z' | \omega) = \frac{1}{2\pi} \int d\kappa e^{i\kappa \cdot \rho} \sum_n \frac{\phi_n^*(z', \omega) \phi_n(z, \omega)}{\lambda_n - \kappa^2}$$  \hspace{1cm} (E.1)

Our intermediate goal is to find the Wigner distribution of the single-frequency Green’s function\(^2\)

$$W_h(\rho, \kappa, z, k_z, z', k'_z | \omega) = \frac{1}{(2\pi)^4} \int H^*(\rho - \theta\rho/2, z - \theta z/2, z' - \theta' z'/2 | \omega)$$

$$\times H(\rho + \theta\rho/2, z + \theta z/2, z' + \theta' z'/2 | \omega) e^{-i\theta\rho \cdot \kappa} e^{-i\theta z k_z} e^{-i\theta' z' k'_z} d\theta \rho d\theta z d\theta z'$$  \hspace{1cm} (E.2)

Plugging-in Eq. (E.1) into Eq. (E.2), we have

$$W_h(\rho, \kappa, z, k_z, z', k'_z | \omega)$$

$$= \frac{4}{(2\pi)^2} \sum_n \sum_m W_{nm}^{\phi}(z, k_z | \omega) W_{nm}^{\phi}(z', k'_z | \omega) \int \frac{1}{\lambda_n - \theta^2} e^{i2\rho(\kappa - \theta\rho)} \frac{1}{\lambda_m - (2\kappa - \theta\rho)^2} d\theta \rho$$  \hspace{1cm} (E.3)

where $W_{nm}$ are

$$W_{nm}^{\phi}(z, k_z | \omega) = \frac{1}{2\pi} \int \phi_n^*(z - \theta z/2, \omega) \phi_m(z + \theta z/2, \omega) e^{-i\theta k_z} d\theta z$$  \hspace{1cm} (E.4)

Care should be taken when calculating the $W_{nm}^{\phi}$'s, since the depth of the waveguide is finite, and therefore the limits of integration are effectively not $-\infty$ to $\infty$.

\(^2\)This could alternatively be calculated directly from $g$ of Eq. (4.28). See App. F.
As mentioned in Chap. 6, Eq. (E.3) could be obtained by integration over all time. This means that we are looking at the intensity with respect frequency of the full Wigner distribution of the Green’s function, Eq. (6.5).
Appendix F

Calculation of the Wigner Distribution of the Green’s Function for Arbitrary Speed in the Fourier Domain

In this appendix, we give an alternate derivation of Eq. (E.3).

Using Eq. (4.41) for the single-frequency Green’s function \( h \), we can deduce its double Fourier-transform, \( g \). Recalling Eq. (4.41)

\[
\begin{aligned}
  h(\rho - \rho'; z, z'|\omega) &= \frac{1}{2\pi} \int d\kappa e^{i\kappa (\rho - \rho')} \sum_n \frac{\phi_n^*(z', \omega) \phi_n(z, \omega)}{\lambda_n - \kappa^2} \\
  \end{aligned}
\]  

(F.1)

Defining the Green’s function’s double Fourier-transform

\[
\begin{aligned}
  h(\rho, \rho', z, z'|\omega) &= \frac{1}{2\pi} \int g'(\kappa, \rho', z, z'|\omega)e^{i\kappa \cdot \rho} d\kappa \\
  \end{aligned}
\]  

(F.2)
Then \( g \) is
\[
g(\kappa, \rho', z, z'|\omega) = e^{-i\kappa \cdot \rho'} \sum_n \frac{\phi_n(z', \omega)\phi_n(z, \omega)}{\lambda_n - \kappa^2}
\]  
\text{(F.3)}

Notice that \( g \) is a function of the spatial coordinates \( z \) and \( z' \), and of the wavenumber coordinates \( \kappa \). We know that the Wigner distributions of \( G \) and \( h \) are equal, and also, those of \( h \) and \( g \) are equal
\[
W_G = W_h = W_g
\]  
\text{(F.4)}

We calculate the Wigner distribution of \( g \) to be
\[
W_g(\rho, \kappa, z, k_z, z', k'_z|\omega)
= \frac{1}{(2\pi)^2} \int g^*(\kappa - \theta, z - \theta z/2, z' - \theta' z'/2|\omega) g(\kappa + \theta, z + \theta z/2, z' + \theta' z'/2|\omega) d\theta d\theta' dz dz'
\times e^{i\theta \cdot \rho} e^{-i\theta z k_z} e^{-i\theta' z' k'_z}
\]  
\text{(F.5)}

Notice that as the prescription called for, the complex exponentials have a ‘+i’ for \( g \)’s wavenumber coordinates, and have a ‘-i’ for \( g \)’s spatial coordinates.

Solving, we get that
\[
W_g(\rho, \kappa, z, k_z, z', k'_z|\omega)
= \frac{1}{(2\pi)^2} \sum_n \sum_m \int \frac{e^{i\theta \cdot \rho}}{\lambda_n^* - (\kappa - \theta/2)^2} \frac{1}{\lambda_m - (\kappa + \theta/2)^2} W_{nm}(z, k_z|\omega) W_{nm}(z', k'_z|\omega)
\]  
\text{(F.6)}

where the \( W_{nm}^\phi \)’s are as in Eq. (E.4)
\[
W_{nm}^\phi(z, k_z|\omega) = \frac{1}{2\pi} \int \phi_n^*(z - \theta z/2, \omega)\phi_m(z + \theta z/2, \omega) d\theta
\]  
\text{(F.7)}

Recall that the integration in \( W_{nm}^\phi \) is not trivially over an infinite space, since the waveg-
uide is finite in depth. This is discussed in App. G.

Changing variables

\[ \theta = -2\phi + 2\kappa \]  

we obtain

\[ W_g(\rho, \kappa, z, k_z, z', k'_z | \omega) = \frac{1}{(2\pi)^2} \sum_n \sum_m W_{nm}^\phi (z, k_z | \omega) W_{nm}^\phi (z', k'_z | \omega) \int \frac{1}{\lambda^* - \phi^2} \frac{e^{i(\kappa - \phi) \cdot \rho}}{\lambda - (2\kappa - \phi)^2} d\phi \]  

which is Eq. (E.3).
Appendix G

Calculation of Wigner Distribution of Green’s Function: Constant Speed

From Eq. (7.4), \( h(r, r'; \omega) \) is

\[
h(r, r'; \omega) = \frac{1}{2\pi^2 L} \int d\eta_x d\eta_y \sum_{\eta_z} e^{i\eta_x (x-x')} e^{i\eta_y (y-y')} \frac{w^2}{c^2 - \eta_x^2 - \eta_y^2 - \eta_z^2} \sin(\eta_z z') \sin(\eta_z z) \quad (G.1)
\]

In Chap. 7 we found that the Wigner of the constant speed Green’s function could be calculated in pieces. In this appendix we provide the details of the calculation.

The \( x, x' \)-Piece.

We begin with the transformation of the \( x \) and \( x' \)-relevant terms in \( h \).

\[
W_{h}^{(x, x')} = \frac{1}{(2\pi)^2} \int d\theta_x d\theta'_x e^{-i\eta_x [(x-\theta_x)/2] - (x'-\theta'_x)/2]} e^{-i\theta_x k_x} e^{-i\theta'_x k'_x} e^{i\eta'_x [(x+\theta_x)/2] - (x'+\theta'_x/2)} \quad (G.2)
\]

\[
e^{i2(k_x - \eta_x)(x-x')} \delta \left( k_x - \frac{\eta_x + \eta'_x}{2} \right) \delta \left( k'_x + \frac{\eta_x + \eta'_x}{2} \right) \quad (G.3)
\]

\[
e^{i2(k_x - \eta_x)(x-x')} \delta \left( k_x - \frac{\eta_x + \eta'_x}{2} \right) \delta (k'_x + k_x) \quad (G.4)
\]
Since the $y$ and $y'$-dependence of $h$ is very similar to the $x$ and $x'$-dependence, we can just write

$$W_h^{(y,y')} = e^{i2(k_y - n_y)(y - y')} \delta \left( k_y - \frac{\eta_y + \eta'_y}{2} \right) \delta \left( k'_y + \frac{\eta_y + \eta'_y}{2} \right)$$  \hspace{1cm} (G.5)$$

$$W_h^{(y,y')} = e^{i2(k_y - n_y)(y - y')} \delta \left( k_y - \frac{\eta_y + \eta'_y}{2} \right) \delta \left( k'_y + k_y \right)$$  \hspace{1cm} (G.6)

The $\Delta x$-Piece.

Noting that the $x$ and $x'$ dependence of our Green's function is really a function of the difference $\Delta x = (x - x')$, we rederive the $x$-piece of $W_h$

$$W_h^{(\Delta x)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta_x e^{-i\eta_x(\Delta x - \theta_x/2)} e^{-i\theta_x k_x} e^{+i\eta'_x(\Delta x + \theta_x/2)}$$  \hspace{1cm} (G.7)$$

$$= e^{i\Delta x(\eta'_x - \eta_x)} \int_{-\infty}^{\infty} d\theta_x \frac{e^{i\theta_x(\frac{\eta'_x + \eta_x}{2} - k_x)}}{2\pi}$$  \hspace{1cm} (G.8)$$

$$= e^{i\Delta x(\eta'_x - \eta_x)} 2\delta(\eta'_x + \eta_x - 2k_x)$$  \hspace{1cm} (G.9)$$

$$= 2e^{i2\Delta x(k_x - \eta_x)} \delta(\eta'_x + \eta_x - 2k_x)$$  \hspace{1cm} (G.10)

Since the $\Delta y$-dependence of $h$ is very similar to the $\Delta x$-dependence, we can just write

$$W_h^{(\Delta y)} = 2e^{i2\Delta y(k_y - n_y)} \delta(\eta'_y + \eta_y - 2k_y)$$  \hspace{1cm} (G.11)

The relationship between $W_h^{(x,x')}$ and $W_h^{(\Delta x)}$ is consistent with Eq. (L.6) in App. L.

The $z,z'$-Piece.

Let’s look at the $z$-relevant pieces of $W_h$

$$W_h^{(z)} = \frac{1}{(2\pi)^2} \int \sin[\eta_z(z - \theta_z/2)] \sin[\eta'_z(z + \theta_z/2)] e^{-i\theta_z k_z} d\theta_z$$  \hspace{1cm} (G.12)$$
Since we are interested in waveguides, which have finite extent in the $z$-direction, the limits of integration in the above integral are non-trivial.

In references [10, 11] we are given the solution for the case of pressure-release surface and bottom. For general functions $\phi$,

$$W_{n,m}(z, k_z) = \frac{1}{2\pi} \int_{-2\min(L-z,z)}^{2\min(L-z,z)} \phi_n^*(z - \theta/2) \phi_m(z + \theta/2)e^{i\theta k_z} d\theta \quad \text{for } 0 \leq z \leq L \quad (G.13)$$

Recall that in the constant speed case,

$$\phi(z) = \sin(\eta_z z) \quad (G.14)$$

and

$$\eta_z = \frac{n\pi}{L} = \eta_n \quad (G.15)$$

$$\eta'_z = \frac{m\pi}{L} = \eta_m \quad (G.16)$$

Hence, Eq. (G.13) becomes

$$W_{nm}(z, k_z) = \frac{1}{\pi L} \left[ \left( e^{i(n-m)\pi z/L} \sin[(2k_z + (n + m)\pi/L)X] \right) \frac{2k_z + (n + m)\pi/L}{2k_z - (n + m)\pi/L} \right]$$

$$- \left( e^{i(n+m)\pi z/L} \sin[(2k_z - (n - m)\pi/L)X] \right) \frac{2k_z - (n - m)\pi/L}{2k_z + (n - m)\pi/L} \right] \quad (G.17)$$

where $X = 2\min(L - z, z)$. 
The \( m = n \) terms are

\[
W_{nm}(z, k_z)
= \frac{1}{2\pi L}\left[\left(\frac{\sin[(k_z + n\pi/L)2X]}{k_z + n\pi/L} + \frac{\sin[(k_z - n\pi/L)2X]}{k_z - n\pi/L}\right) - 2\cos[2n\pi z/L]\frac{\sin[2k_z X]}{k_z}\right]
\]

(G.18)

Noticing that \( X = 2\min(L - z, z) = L - 2|z - L/2| \), we have

\[
W_{nm}(z, k_z)
= -\frac{1}{\pi L}\cos[(n - m)\pi]\left[\left(e^{i(n-m)\pi z/L}\frac{\sin[(2k_z + (n + m)\pi/L)2|z - L/2| - 2Lk_z]}{2k_z + (n + m)\pi/L} + e^{-i(n-m)\pi z/L}\frac{\sin[(2k_z - (n + m)\pi/L)2|z - L/2| - 2Lk_z]}{2k_z - (n + m)\pi/L}\right)\right.
- \left.\left(e^{i(n+m)\pi z/L}\frac{\sin[(2k_z + (n - m)\pi/L)2|z - L/2| - 2Lk_z]}{2k_z + (n - m)\pi/L} + e^{-i(n+m)\pi z/L}\frac{\sin[(2k_z - (n - m)\pi/L)2|z - L/2| - 2Lk_z]}{2k_z - (n - m)\pi/L}\right)\right]\]

(G.19)

and for the \( n = m \) case

\[
W_{nn}(z, k_z)
= -\frac{1}{2\pi L}\left[\left(\frac{\sin[(k_z + n\pi/L)4|z - L/2| - 2Lk_z]}{k_z + n\pi/L} + \frac{\sin[(k_z - n\pi/L)4|z - L/2| - 2Lk_z]}{k_z - n\pi/L}\right)\right.
- \left.2\cos[2n\pi z/L]\frac{\sin[4|z - L/2|k_z - 2Lk_z]}{k_z}\right]
\]

(G.20)

Similarly, for the \( z' \) terms, simply replace every \( z \) in the above equations by \( z' \).

The \( \omega \)-Piece.
\[ W_h^{(\omega)} = \frac{1}{2\pi} \int d\theta \frac{e^{-i\theta \tau}}{(\omega - \theta/2)^2 - \eta_x^2 - \eta_y^2 - \eta_z^2} \frac{1}{(\omega + \theta/2)^2 - (\eta'_x)^2 - (\eta'_y)^2 - (\eta'_z)^2} \]  
\[ = \frac{1}{2\pi} \int d\theta \frac{e^{-i\theta \tau}}{(\omega - \theta/2)^2 - \eta_x^2} \frac{1}{(\omega + \theta/2)^2 - \eta_y^2} \]  
\[ = \frac{1}{2\pi} \Delta t \frac{e^{\frac{1}{2}2\omega |\Delta t|}}{\eta'} \left\{ e^{-i2\eta \Delta t} \frac{(2\omega + c\eta)^2 - (c\eta')^2}{(2\omega - c\eta)^2 - (c\eta')^2} - e^{i2c\eta \Delta t} \right\} \]  
\[ = \frac{1}{2\pi} \frac{\Delta t}{|\Delta t|} \frac{e^{\frac{1}{2}2\omega |\Delta t|}}{\eta'} \left\{ e^{-i2\eta \Delta t} \frac{(2\omega + c\eta)^2 - (c\eta')^2}{(2\omega - c\eta)^2 - (c\eta')^2} - e^{i2c\eta \Delta t} \right\} \]  

where \( \eta^2 = \eta_x^2 + \eta_y^2 + \eta_z^2 \), and we will see that \( \eta' = \sqrt{(\eta_x - 2k_x)^2 + (\eta_y - 2k_y)^2 + (\eta_z)^2} \). This comes from the Dirac delta functions in \( W_h^{(x)}, W_h^{(|\Delta x|)}, W_h^{(y)}, \) and \( W_h^{(\Delta y)} \).
Appendix H

The Single-Frequency Wigner Spectrum of the Source Response with Variable Speed

Many authors, including Kuperman and Ingenito [9], use the single-frequency Green’s function and the corresponding single-frequency Green’s function equation, Eqs. (4.41) and (A.11) respectively, to calculate the source-response in the frequency domain. In App. B, we show the form of our main result for the single-frequency case. In this appendix we use our main result (in the single-frequency form), Eq. (B.8), to calculate the single-frequency (intensity) Wigner spectrum of a source-response from the Wigner spectrum of a noise source when we have a single frequency (intensity). Here we use the Wigner distribution of the single-frequency Green’s function for an arbitrary depth-dependent speed, Eq. (4.41), which was derived in Chap. 4 and is given in terms of a complete set of functions $\phi$. Then, we obtain the autocorrelation function of the source-response by the inverse Fourier transform of its Wigner spectrum.

Since we are using the single-frequency Green’s function, and since it depends on $\Delta \rho$,
our main result takes the form

$$
\mathcal{W}_{uf}(\rho, \kappa, z, k_z | \omega) = (2\pi)^4 \int \mathcal{W}_G(\rho', \kappa, z, k_z, k_z' | \omega) \mathcal{W}_f(\rho - \rho', \kappa, z', -k_z | \omega) d\rho' dz' dk_z' \quad (H.1)
$$

$$
= (2\pi)^4 \int \mathcal{W}_G(\rho - \rho', \kappa, z, k_z, k_z', k_z' | \omega) \mathcal{W}_f(\rho', \kappa, z', -k_z | \omega) d\rho' dz' dk_z' \quad (H.2)
$$

where the Wigner spectrum of the noise source is

$$
\mathcal{W}_f(\rho, \kappa, z, k_z | \omega) = \frac{1}{(2\pi)^3} \int R(\rho - \tau/2, \rho + \tau/2, z - \tau_z/2, z + \tau_z/2 | \omega) e^{-i\tau \cdot \kappa} e^{-i\tau \cdot k_z} d\tau d\tau_z \quad (H.3)
$$

and the Wigner distribution of the single-frequency Green’s function is

$$
\mathcal{W}_G(\Delta \rho, \kappa, z, k_z, k_z' | \omega) = \frac{1}{(2\pi)^4} \int G^*(\Delta \rho - \theta/2, z - \theta_z/2, z' - \theta_z'/2 | \omega) e^{-i\theta \cdot \kappa} e^{-i\theta \cdot k_z} e^{-i\theta \cdot k_z'} d\theta d\theta_z d\theta_z' \quad (H.4)
$$

Expressing the single-frequency Green’s function in terms of a general orthonormal basis, its
Wigner distribution becomes

\[
W_G(\Delta \rho, \kappa, z, z', \omega) = \frac{1}{(2\pi)^4} \int d\eta d\eta' \sum_n \sum_m \int e^{-i\theta \cdot \kappa} e^{-i\theta_z k_z} e^{-i\theta'_z k'_z} d\theta d\theta_z d\theta'_z d\eta d\eta' \\
\times \frac{e^{-i\eta (\Delta \rho - \theta/2)} \phi_n^*(z' - \theta'/2, \omega) \phi_n^*(z - \theta/2, \omega)}{\lambda_n^2 - \eta^2} \\
\times \frac{e^{i\eta' (\Delta \rho + \theta/2)} \phi_m(z' + \theta'/2, \omega) \phi_m(z + \theta/2, \omega)}{\lambda_m - (\eta')^2} \\
= 4 \sum_n \sum_m W_{nm}^\phi(z, k_z | \omega) W_{nm}^\phi(z', k_z' | \omega) \int \frac{1}{\lambda_n^2 - \eta^2} \frac{e^{i\Delta \rho (\kappa - \eta)}}{\lambda_m - (2\kappa - \eta)^2} d\eta \\
\tag{H.5}
\]

\[
W_{nm}^\phi(z, k_z | \omega) = \frac{1}{2\pi} \int \phi_n^*(z - \theta/2, \omega) \phi_m(z + \theta/2, \omega) d\theta_z \\
\tag{H.7}
\]

Note that the integration is not performed over an infinite region since the waveguide restricts the wave to the region of size \( L \).

### H.1 Wigner Spectrum of the Source Response

Now that we have expressions for \( W_G \) and \( W_f \), we use our main result in the appropriate form, Eq. (H.1), we find an expression for the Wigner spectrum of the source-response in terms of the autocorrelation function of the noise source. We proceed to specify to gradually more restrictive cases of the noise source autocorrelation function.

Our ultimate goal is to find the Wigner spectrum of the response, \( \overline{W}_{u_f} \). Using Eq. (H.1)

\[
\overline{W}_{u_f}(\rho, \kappa, z, k_z | \omega) = (2\pi)^4 \int W_G(\rho', \kappa, z, z', k_z' | \omega) \overline{W}_f(\rho - \rho', \kappa, z', -k_z' | \omega) d\rho' dz' dk_z' \\
\tag{H.8}
\]

Using our results for \( \overline{W}_f \) and \( W_G \), Eqs. (H.3) and (H.6) respectively, into Eq. (H.8), the
Wigner spectrum of the source-response is

\[
\bar{W}_{u_f}(\rho, \kappa, z, k_z|\omega) = \frac{4}{(2\pi)^2} \int d\rho' d\rho' \int d\kappa' d\kappa' e^{-i\tau' \kappa} e^{-i\tau' k_z} \\
\times R(\rho - \tau/2, \rho' + \tau/2, z - \tau_z/2, z' + \tau_z/2|\omega) \\
\times \int d\eta \sum n \sum m \frac{1}{\lambda_n^2 - \eta^2 \lambda_m - (2\kappa - \eta)^2} W_{nm}^\phi(z, k_z|\omega) W_{nm}^\phi(z', k'_z|\omega)
\]

(H.9)

Now, if the autocorrelation function \( R \) is only a function of the differences in position in the \( xy \) plane, that is, it is stationary within an \( \rho \) coordinates\(^1\)

\[
R(\rho - \tau/2, \rho + \tau/2, z - \tau_z/2, z + \tau_z/2|\omega) = R(\tau, z - \tau_z/2, z + \tau_z/2|\omega)
\]

(H.10)

Then the Wigner spectrum of the response becomes

\[
\bar{W}_{u_f}(\rho, \kappa, z, k_z|\omega) = \sum n \sum m \frac{1}{\lambda_n^2 - \kappa^2 \lambda_m - \kappa^2} W_{nm}^\phi(z, k_z|\omega) \int R(\tau, z' - \tau_z/2, z' + \tau_z/2|\omega) \\
\times e^{-i\tau' \kappa} e^{-i\tau' k_z} d\tau d\tau_z z d\tau d\tau_z=z' d\tau d\tau_z
\]

(H.11)

Notice that eventhough the autocorrelation of the noise, \( R \), depends on the difference in positions (\( \tau \), over which we integrate), \( W_{u_f} \) does not depend on the position \( \rho \).

\(^1\)where it is understood that in the above equation, the \( R \) on the left hand side is not the same \( R \) as on the right hand side (they have a different number of arguments).
Using the definition of the $W_{n\alpha}^\phi$, Eq. (E.4), we can perform the $k_z'$ integration

\[
W_{u_f}(\rho, \kappa, z, k_z|\omega) = \sum_n \sum_m \frac{1}{\lambda_n^* - \kappa^2} \frac{1}{\lambda_m - \kappa^2} W_{n\alpha}^\phi(z, k_z|\omega) \int dz' d\tau d\tau_z e^{-i\tau \kappa}
\]

\[
\times R(\tau, z' - \tau_z/2, z' + \tau_z/2|\omega)
\]

\[
\times \frac{1}{2\pi} \int d\theta' \phi_n^*(z' - \theta_z'/2, \omega) \phi_m(z' + \theta_z'/2, \omega) \int dk_z e^{+i(\tau_z - \theta_z')k_z}
\]

\[
= \sum_n \sum_m \frac{1}{\lambda_n^* - \kappa^2} \frac{1}{\lambda_m - \kappa^2} W_{n\alpha}^\phi(z, k_z|\omega) \int d\tau e^{-i\tau \kappa}
\]

\[
\times \int dz' \int d\tau_z \phi_n^*(z' - \tau_z/2, \omega) \phi_m(z' + \tau_z/2, \omega) R(\tau, z' - \tau_z/2, z' + \tau_z/2|\omega)
\]

\[
(H.12)
\]

\[
(H.13)
\]

\[
(H.14)
\]

**H.2 Autocorrelation Function of the Source Response**

Recall from Chap. 1 that the autocorrelation of the response is

\[
\langle \hat{u}_f^*(\rho_1, z_1, \omega) \hat{u}_f(\rho_2, z_2, \omega) \rangle = \int W_{u_f}(\rho, \kappa, z, k_z, \omega) e^{+i\theta \cdot \kappa} e^{+i\theta_z k_z} d\kappa dk_z
\]

\[
(H.15)
\]

where

\[
\rho_1 = \rho - \theta/2 \quad \rho_2 = \rho + \theta/2
\]

\[
(H.16)
\]

\[
z_1 = z - \theta_z/2 \quad z_2 = z + \theta_z/2
\]

\[
(H.17)
\]
In our case, the autocorrelation function of the response is

\[
\left\langle \hat{u}_f^*(\rho - \theta/2, z - \theta_z/2, \omega) \hat{u}_f(\rho + \theta/2, z + \theta_z/2, \omega) \right\rangle
= \int W_u(\rho, \kappa, z, k_z|\omega) e^{i \theta \cdot \kappa} e^{i \theta_z k_z} d\kappa dk_z
\]  
(H.18)

\[
= \int d\tau \int_{0}^{2\pi} d\varphi \int k dk \cos(\varphi) \sum_n \sum_m \frac{1}{\lambda_n^* - \kappa^2} \frac{1}{\lambda_m - \kappa^2} \int W_{nm}(z, k_z|\omega) e^{i \theta \cdot \kappa} e^{i \theta_z k_z} d\kappa
\times \int dz' \int d\tau' \phi_n^*(z' - \tau_z/2|\omega) \phi_m(z' + \tau_z/2|\omega) R(\tau, z' - \tau_z/2, z' + \tau_z/2|\omega)
\]  
(H.19)

\[
= 2\pi \sum_n \sum_m \phi_n^*(z - \theta_z/2, \omega) \phi_m(z + \theta_z/2, \omega) \int d\tau \int \frac{J_0(|\theta - \tau|\kappa)}{(\lambda_n^* - \kappa^2)(\lambda_m - \kappa^2)} k dk \kappa
\times \int dz' \int d\tau' \phi_n^*(z' - \tau_z/2, \omega) \phi_m(z' + \tau_z/2, \omega) R(\tau, z' - \tau_z/2, z' + \tau_z/2|\omega)
\]  
(H.20)

Consider the integral

\[
\int_{0}^{\infty} \frac{J_0(|\theta - \tau|\kappa)}{(\lambda_n^* - \kappa^2)(\lambda_m - \kappa^2)} k dk \kappa
\]  
(H.21)

which appears in Eq. (H.20). We note here that the answer is

\[
\int_{0}^{\infty} \frac{J_0(|\theta - \tau|\kappa)}{(\lambda_n^* - \kappa^2)(\lambda_m - \kappa^2)} k dk \kappa = \frac{\pi}{2} \frac{\lambda_n^* - \lambda_m}{\lambda_n^* - \lambda_m} \left( \frac{H_0^{(1)}(|\theta - \tau| \sqrt{\lambda_m})}{2\sqrt{\lambda_m}} - \frac{H_0^{(1)}(-|\theta - \tau| \sqrt{\lambda_m})}{2\sqrt{\lambda_m}} \right)
\]  
(H.22)

Further, the integral is approximately

\[
\int_{0}^{\infty} \frac{J_0(|\theta - \tau|\kappa)}{(\lambda_n^* - \kappa^2)(\lambda_m - \kappa^2)} k dk \kappa \simeq -\frac{\pi}{4} \frac{\beta_n(\alpha_n)^2}{\beta_m(\alpha_m)^2} J_0(|\theta - \tau| \cdot \alpha_n)
\]  
(H.23)

\footnote{Assuming that \(\sqrt{\lambda_m} = \alpha_m + i \beta_m\) where \(\alpha, \beta > 0\).}

\footnote{If \(\beta \ll \alpha\), where \(\sqrt{\lambda_m} = \alpha_m + i \beta_m\).}
Using the above approximation, the autocorrelation function of the source-response is

\[
\langle \hat{u}_f(\rho - \theta/2, z - \theta z/2, \omega) \hat{u}_f(\rho + \theta/2, z + \theta z/2, \omega) \rangle \\
\simeq -2\pi \sum_n \phi_n^*(z - \theta z/2, \omega) \phi_n(z + \theta z/2, \omega) \int d\tau \frac{\pi/4}{\beta_n(\alpha_n)^2} J_0(|\theta - \tau| \cdot \alpha_n) \\
\times \int dz' \int d\tau' \phi_n^*(z' - \tau z/2, \omega) \phi_n(z' + \tau z/2, \omega) R(\tau, z' - \tau z/2, z' + \tau z/2|\omega) \quad (H.24)
\]
Appendix I

The Hankel and a Delta Function at the Coordinate Origin

Some authors claim without proof that the zeroth order Hankel function of the first kind, $H_0^{(1)}(kr)$, satisfies

$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + k^2\right) H_0^{(1)}(kr) \sim i\frac{\delta(r)}{r}$ \hspace{1cm} (I.1)

We have never seen a direct proof of Eq. (I.1) so in this appendix we prove directly that

$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + k^2\right) H_0^{(1)}(kr) = i\frac{\pi + 4\delta(r)}{2\pi} \frac{1}{r}$ \hspace{1cm} (I.2)

This equation is well known and usually just stated [7]. We suspect that it is assumed from indirect considerations.

We begin by using a definition of the Hankel function given (for $0 \leq r$ and $0 < k$) in [7]

$H_0^{(1)}(kr) = \frac{1}{i\pi} \int_{-\infty}^{\infty} dy \frac{e^{ik\sqrt{r^2+y^2}}}{\sqrt{r^2+y^2}} = \frac{2}{i\pi} \int_{0}^{\infty} dy \frac{e^{ik\sqrt{r^2+y^2}}}{\sqrt{r^2+y^2}} = \frac{2}{i\pi} \int_{0}^{\infty} dy \frac{e^{ikR}}{R}$ \hspace{1cm} (I.3)
where

\[ R = \sqrt{r^2 + y^2} \]  
(I.4)

\[ RdR = ydy \]  
(I.5)

For convenience we define the operator

\[ \mathcal{L}_k = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + k^2 = \mathcal{L}_0 + k^2 \]  
(I.6)

where

\[ \mathcal{L}_0 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \]  
(I.7)

Note that the integrand in Eq. (I.3) is a function of \( R \) (as defined in Eq. (I.4)). For any function of \( R \), \( f(R) \), we have that

\[ \frac{\partial}{\partial r} f(R) = \frac{r}{R} \frac{\partial}{\partial R} \]  
(I.8)

Using Eq. (I.8), we transform the operator of Eq. (I.7) into \( R \)-space. If we are operating on a function of \( R \) only, then the operator in Eq. (I.7) becomes

\[ \mathcal{L}_0 f(R) = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} f(R) = 2 \frac{1}{R} \frac{\partial}{\partial R} f(R) + r \frac{\partial}{\partial r} \frac{1}{R} \frac{\partial}{\partial R} f(R) \]  
(I.9)

But \( \frac{1}{R} \frac{\partial}{\partial R} f(R) \) is also a function of just \( R \), so Eq. (I.9) becomes

\[ \mathcal{L}_0 f(R) = 2 \frac{1}{R} \frac{\partial}{\partial R} f(R) + r^2 \frac{1}{R} \frac{\partial}{\partial R} \frac{1}{R} \frac{\partial}{\partial R} f(R) \]  
(I.10)

Now we can evaluate \( \mathcal{L}_k H_0^{(1)}(kr) \). We begin by evaluating \( \mathcal{L}_0 H_0^{(1)}(kr) \) and show that
when \( r \neq 0 \),
\[
\mathcal{L}H_0^{(1)} = -k^2 H_0^{(1)} \quad \text{for } r \neq 0
\]  
(I.11)

but that when \( r = 0 \),
\[
\mathcal{L}H_0^{(1)} = \frac{\delta(r)}{r} - k^2 H_0^{(1)} \quad \text{for } r = 0
\]  
(I.12)

### I.1 \( \mathcal{L}_k H_0^{(1)}(kr) \) In General

Operating with \( \mathcal{L}_0 \) on Eq. (I.3)
\[
\mathcal{L}_0 H_0^{(1)}(kr) = \frac{2}{i\pi} \int_0^\infty \frac{dy}{y} \frac{\partial}{\partial r} \frac{e^{ikR}}{R} \int_0^\infty \frac{dy}{y} \frac{e^{ikR}}{R} 
\]  
(I.13)

\[
= \frac{2}{i\pi} \int_0^\infty dy \left[ 2 \frac{1}{R} \frac{\partial}{\partial R} + r^2 \frac{1}{R} \frac{\partial}{\partial R} \frac{1}{R} \right] \frac{e^{ikR}}{R} 
\]  
(I.14)

Now changing our variable of integration to \( R \), we get
\[
\mathcal{L}_0 H_0^{(1)}(kr) = \frac{2}{i\pi} \int_0^\infty dy \frac{1}{y} \frac{\partial}{\partial R} \left[ 2 \frac{1}{R} \frac{\partial}{\partial R} + r^2 \frac{1}{R} \frac{\partial}{\partial R} \frac{1}{R} \right] \frac{e^{ikR}}{R} \int_0^\infty \frac{dy}{y} \frac{e^{ikR}}{R} 
\]  
(I.15)

\[
= \frac{2}{i\pi} \int_0^\infty dR \left[ \left( 3 \frac{r^2}{yR^4} - 2 \frac{1}{yR^2} \right) e^{ikR} + \left( 2 \frac{1}{yR} - 3 \frac{r^2}{yR^3} \right) \frac{\partial}{\partial R} e^{ikR} + \left( \frac{r^2}{yR^2} \right) \frac{\partial^2}{\partial R^2} e^{ikR} \right] 
\]  
(I.16)

\[
= J_0 + J_1 + J_2 \]  
(I.17)

Notice that the term in the first pair of parentheses is
\[
\left( 3 \frac{r^2}{yR^4} - 2 \frac{1}{yR^2} \right) = \frac{\partial}{\partial R} \frac{y}{R^3} 
\]  
(I.18)
Integrating $J_0$ by parts, we get

$$i \frac{\pi}{2} J_0 = \int_r^\infty dR \, e^{ikR} \left( 3 \frac{r^2}{yR^4} - 2 \frac{1}{yR^2} \right) = \int_r^\infty dR \, e^{ikR} \frac{\partial y}{\partial R} \frac{y}{R^3}$$  \hspace{1cm} (I.19)

$$= e^{ikR} \frac{y}{R^3} \bigg|_{R \to \infty} - \int_r^\infty dR \, \frac{y}{R^3} \frac{\partial}{\partial R} e^{ikR}$$  \hspace{1cm} (I.20)

The sum of the first two terms in the integral Eq. (I.16) is then

$$i \frac{\pi}{2} (J_0 + J_1) = e^{ikR} \frac{y}{R^3} \bigg|_{R \to \infty} + \int_r^\infty dR \, \left( 2 \frac{1}{yR} - 3 \frac{r^2}{yR^3} - \frac{y}{R^3} \right) \frac{\partial}{\partial R} e^{ikR}$$  \hspace{1cm} (I.21)

Noticing that

$$\left( 2 \frac{1}{yR} - 3 \frac{r^2}{yR^3} - \frac{y}{R^3} \right) = -\frac{\partial}{\partial R} \frac{y}{R^2}$$  \hspace{1cm} (I.22)

We integrate by parts Eq. (I.21)

$$\int_r^\infty dR \, \left( 2 \frac{1}{yR} - 3 \frac{r^2}{yR^3} - \frac{y}{R^3} \right) \frac{\partial}{\partial R} e^{ikR} = -\int_r^\infty dR \, \frac{\partial y}{\partial R} \frac{y}{R^2} \frac{\partial}{\partial R} e^{ikR}$$  \hspace{1cm} (I.23)

$$= -\frac{y}{R^2} \frac{\partial}{\partial R} e^{ikR} \bigg|_{R \to \infty} + \int_r^\infty dR \, \frac{y}{R^2} \frac{\partial^2}{\partial R^2} e^{ikR}$$  \hspace{1cm} (I.24)

The full expression is then

$$i \frac{\pi}{2} L_0 H_0^{(1)}(kr) = e^{ikR} \frac{y}{R^3} \bigg|_{R \to \infty} - \frac{y}{R^2} \frac{\partial}{\partial R} e^{ikR} \bigg|_{R \to \infty} + \int_r^\infty dR \, \left( \frac{r^2}{yR^2} + \frac{y}{R^2} \right) \frac{\partial^2}{\partial R^2} e^{ikR}$$  \hspace{1cm} (I.25)
Now, we notice that

\[
\left( \frac{r^2}{yR^2} + \frac{y}{R^2} \right) = \frac{1}{y}
\]  

(I.26)

and since

\[
\frac{dR}{y} = \frac{dy}{R}
\]  

(I.27)

then we have that the integral in Eq. (I.25) is

\[
\int_R^\infty dR \left( \frac{r^2}{yR^2} + \frac{y}{R^2} \right) \frac{\partial^2}{\partial R^2} e^{ikR} = \int_R^\infty dR \frac{1}{y} \frac{\partial^2}{\partial R^2} e^{ikR} = -k^2 \int_0^\infty dy \frac{e^{ikR}}{R}
\]  

(I.28)

Finally we have that

\[
\mathcal{L}_0 H_0^{(1)}(kr) = -\frac{2}{i\pi} \sqrt{R^2 - r^2} \left( \frac{1}{R^3} - \frac{ik}{R^2} \right) e^{ikR} \bigg|_{R=r}^{R\to\infty}
\]  

(I.29)

and

\[
\mathcal{L}_k H_0^{(1)}(kr) = \frac{2}{i\pi} \sqrt{R^2 - r^2} \left( \frac{1}{R^3} - \frac{ik}{R^2} \right) e^{ikR} \bigg|_{R=r}^{R\to\infty}
\]  

(I.30)

The \( R \to \infty \) limit in Eq. (I.30) is zero, so the equation becomes

\[
\mathcal{L}_k H_0^{(1)}(kr) = \frac{2}{i\pi} \sqrt{R^2 - r^2} \left( \frac{1}{R^3} - \frac{ik}{R^2} \right) e^{ikR} \bigg|_{R=r}
\]  

(I.31)

I.2 If \( r \neq 0 \)

If \( r \neq 0 \), then we can evaluate Eq. (I.31)

\[
\mathcal{L}_k H_0^{(1)}(kr) = \frac{2}{i\pi} \sqrt{R^2 - r^2} \left( \frac{1}{R^3} - \frac{ik}{R^2} \right) e^{ikR} \bigg|_{R=r} = 0 \text{ for } r \neq 0
\]  

(I.32)
Therefore, in the case where \( r \neq 0 \), then the Hankel function of the first kind \( H_0^{(1)}(kr) \) satisfies the differential equation

\[
\mathcal{L}_k H_0^{(1)}(kr) = 0 \quad \text{for} \ r \neq 0 \quad (I.33)
\]

This agrees with Eq. (I.2).

**I.3 If \( r = 0 \)**

Now we expect that

\[
\mathcal{L}_k H_0^{(1)}(kr) \sim i \frac{\delta(r)}{r} \quad (I.34)
\]

Which is non-zero only at \( r = 0 \).

To check Eq. (I.34), consider an arbitrary continuous function \( f(r) \), and integrate

\[
I = \int_0^\infty r \, dr \, f(r) \mathcal{L}_k H_0^{(1)}(kr) \quad (I.35)
\]

We expect to get

\[
I \sim i \int_0^\infty r \, dr \, f(r) \frac{\delta(r)}{r} = if(0) \quad (I.36)
\]

Calculating, we get

\[
I = i \frac{2}{\pi} \int_0^\infty r \, dr \, f(r) \left[ \sqrt{R^2 - r^2} \left( \frac{1}{R^3} - \frac{ik}{R^2} \right) e^{ikR} \right]_{R=r} \quad (I.37)
\]

We know that the integrand is zero if \( r > 0 \), so let us truncate the integral at some positive
A, then

\[ I = \int_0^\infty r dr \ f(r) \tilde{B}_k H_0^{(1)}(kr) \]  
\[ = i \frac{2}{\pi} \int_0^\infty r dr \ f(r) \left[ \sqrt{R^2 - r^2} \left( \frac{1}{R^3} - \frac{ik}{R^2} \right) e^{ikr} \right] \bigg|_{R=r} \]  
\[ = i \frac{2}{\pi} \lim_{\epsilon \to 0} \int_\epsilon^A (r - \epsilon) dr \ f(r - \epsilon) \left[ \sqrt{R^2 - r^2} \left( \frac{1}{R^3} - \frac{ik}{R^2} \right) e^{ikr} \right] \bigg|_{R=r} \]

Now, since the integral starts at \( \epsilon > 0 \), then \( r \) is never zero and we can evaluate \( R = r \)

\[ I = i \frac{2}{\pi} \lim_{\epsilon \to 0} \int_\epsilon^A (s - \epsilon) dr \ f(r - \epsilon) \left( \frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} \sqrt{r^2 - (r - \epsilon)^2} \]
\[ = i \frac{2}{\pi} \lim_{\epsilon \to 0} \int_\epsilon^A (r - \epsilon) dr \ f(r - \epsilon) \left( \frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} \sqrt{2r\epsilon - \epsilon^2} \]  

Let

\[ r = \epsilon x \]  
\[ dr = \epsilon dx \]

then

\[ I = i \frac{2}{\pi} \lim_{\epsilon \to 0} \int_1^{A/\epsilon} (\epsilon x - \epsilon)dx \ f(\epsilon x - \epsilon) \left( \frac{1}{(\epsilon x)^3} - \frac{ik}{(\epsilon x)^2} \right) e^{ik\epsilon x} \sqrt{2\epsilon^2 x - \epsilon^2} \]
\[ = i \frac{2}{\pi} \lim_{\epsilon \to 0} \int_1^\infty (x - 1)dx \ f[\epsilon(x - 1)] \left( \frac{1}{x^3} - \frac{ik\epsilon}{x^2} \right) e^{ik\epsilon x} \sqrt{2x - 1} \]

Taking the limit, we have

\[ I = i \frac{2}{\pi} f(0) \int_1^\infty dx \ \frac{x - 1}{x^3} \sqrt{2x - 1} \]
The integral in Eq. (I.47) can be evaluated using

\[ \int \, dx \, \frac{x - 1}{x^3} \sqrt{2x - 1} = \frac{1 - 3x}{2x^2} \sqrt{2x - 1} + \arctan(\sqrt{2x - 1}) \]  

(I.48)

Finally, we have that

\[ I = \int_{0}^{\infty} dr \, f(r) r L_k H_0^{(1)}(kr) = i \frac{4 + \pi}{2\pi} f(0) \]  

(I.49)

which means that

\[ L_k H_0^{(1)}(kr) = i \frac{4 + \pi}{2\pi} \frac{\delta(r)}{r} \]  

(I.50)

which is Eq. (I.2).
Appendix J

Pressure Release Waveguide

Finding the normal modes of a waveguide may prove difficult if the wave equation has a variable speed

$$\mathcal{L} = \left( \nabla^2 - \frac{1}{c^2(z)} \frac{\partial^2}{\partial t^2} \right)$$  \hspace{1cm} (J.1)

However, in waveguides with pressure-release surface and bottom, we were able to find some general properties of the normal modes for the case with monotonic speeds, that is, for speeds that are uniformly increasing or decreasing functions of the depth. We believe that these results are new.

We separate variables, and examine the depth-dependent part of the wave. This part has the property that every inflection point is also a root of the function, except that there may be only one (or zero) inflection point that is not a zero of the depth function.

Consider a wave that obeys the three dimensional scalar wave equation, and suppose that the speed $c(z)$ depends on depth $z$

$$\left( \nabla^2 - \frac{1}{c^2(z)} \frac{\partial^2}{\partial t^2} \right) u(x, y, z, t) = \left( \nabla_{\rho}^2 + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2(z)} \frac{\partial^2}{\partial t^2} \right) u(\rho, z, t) = 0$$  \hspace{1cm} (J.2)
We suppose that the wave $u$ is the superposition of functions of the form

$$\psi(\rho, z, t) = R(\rho)Z(z)T(t) \quad (J.3)$$

As a consequence, $T(t)$ is necessarily

$$T(t) = e^{i\omega t} \quad (J.4)$$

and further, that

$$\nabla_\rho^2 R(\rho) = \lambda \quad (J.5)$$

and therefore

$$\left( \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2(z)} \right) Z(z) = -\lambda Z(z) \quad (J.6)$$

Now, consider a waveguide that has pressure-release boundaries, that is

$$u(z = 0) = u(z = L) = 0 \quad (J.7)$$

which implies that

$$Z(z = 0) = Z(z = L) = 0 \quad (J.8)$$

This implies that $Z(z)$ must have at least one extreme value $z^i$ at which

$$\frac{\partial}{\partial z} Z(z^i) = 0 \quad (J.9)$$

Consider the inflection points $z_j$, such that

$$\frac{\partial^2}{\partial z^2} Z(z_j) = 0 \quad (J.10)$$
At an inflection point $z_j$ (if there are any), we have

$$\frac{\omega^2}{c^2(z_j)} Z(z_j) = -\lambda Z(z_j) \quad \text{(J.11)}$$

so at $z = z_j$ we have

$$\frac{\omega^2}{c^2(z_j)} = -\lambda \quad \text{unless} \quad Z(z_j) = 0 \quad \text{(J.12)}$$

Assume that we have $N + 1$ inflection points ($j = 0, \cdots, N$).

### J.1 Uniformly Increasing or Decreasing Sound Speed

Consider a situation where the speed $c(z)$ is a uniformly increasing (or decreasing) function of depth. In such a case, $\frac{\omega^2}{c^2(z_j)} = -\lambda$ can only hold for a single $z_j$ or for no $z_j$ because then $\omega^2/c^2(z)$, corresponding to $c(z)$, is uniformly decreasing (or increasing). Therefore

$$Z(z_j) = 0 \quad \text{for all-but-a-single } z_j, \text{ or for all } z_j \quad \text{(J.13)}$$

In the case where there is no $z_j$ for which $\frac{\omega^2}{c^2(z_j)} = -\lambda$, then $Z(z_j) = 0$ for all $z_j$. In this case $Z(z)$ is zero at every inflection point. We know that when the speed $c(z)$ is a constant, $Z(z)$ is a sine function. In such a case, all inflection points are also roots of $Z(z)$.

If for exactly one inflection point, $\omega^2/c^2(z_j) = -\lambda$. Then we have that $Z(z_j) = 0$ for all-but-a-single $j$, call this inflection point $z_j = z^*$.

$$Z(z^*) \neq 0 \quad \text{(J.14)}$$
In this case, if we order the \( z_j \)'s by increasing magnitude of \( z \), then:

if at \( z_\ast \), \( Z(z_\ast) < 0 \), then

\[
\frac{\partial^2}{\partial z^2} Z(z) \begin{cases} \text{negative} & z_{j-1} < z < z_j = z_\ast \\ \text{positive} & z_\ast = z_j < z < z_{j+1} \end{cases}
\]  

(J.15)

and

\[
Z(z) < 0 \text{ for } z_{j-1} < z < z_{j+1}
\]  

(J.16)

Alternatively, if at \( z_\ast \), \( Z(z_\ast) > 0 \), then

\[
\frac{\partial^2}{\partial z^2} Z(z) \begin{cases} \text{positive} & z_{j-1} < z < z_j = z_\ast \\ \text{negative} & z_\ast = z_j < z < z_{j+1} \end{cases}
\]  

(J.17)

and

\[
Z(z) > 0 \text{ for } z_{j-1} < z < z_{j+1}
\]  

(J.18)

In both cases above, we know that \( z_{j-1} \) and \( z_{j+1} \) exist because of the pressure-release boundary conditions. We also know that

\[
\frac{\partial^2}{\partial z^2} Z(z = 0) = 0 = \frac{\partial^2}{\partial z^2} Z(z = L)
\]  

(J.19)

Further, to satisfy the boundary conditions, we know that \( z_\ast \) (if it exists) must be either \( z_1 \) or \( z_{N-1} \) (the second root, or the one before last). This is shown in the diagram fig. J.1 The proof is pictorial: one can see that otherwise, the boundary conditions cannot be satisfied.
Figure J.1: Example of allowed normal mode with a non-zero inflection point. Dots are inserted to emphasize inflection points.
Appendix K

Main Result in Three Dimensions:
Relation between $W_f$, $W_G$ and $W_{uf}$

In this appendix we derive the three-dimensional case of Eq. (5.12), in the text,

\[
W_u(r, k, t, \omega) = \int W_f(x, k', s, \omega')W_G(r, k, r', -k', t, \omega, t', -\omega')dr' dt' dk' d\omega'
\] (K.1)

where

\[
W_{uf}(r, k, t, \omega) = \frac{1}{(2\pi)^4} \int u_f^*(r - \theta/2, t - \tau/2)u_f(r + \theta/2, t + \tau/2)e^{-i\theta \cdot k - i\tau \omega}d\theta d\tau
\] (K.2)

and

\[
W_G(r, k; r', k'; t, \omega; t', \omega')
= \frac{1}{(2\pi)^8} \int G^*(r - \theta/2, r' - \phi/2, t - \tau/2, t' - \psi/2)
\times G(r + \theta/2, r' + \phi/2, t + \tau/2, t' + \psi/2)e^{i\phi \cdot k' + i\psi \omega'}e^{-i\theta \cdot k - i\tau \omega}d\phi d\psi d\theta d\tau
\] (K.3)
\[ W_f(r', k', t', \omega') = \frac{1}{(2\pi)^4} \int f^*(r' - \phi/2, t' - \psi/2) f(r' + \phi/2, t' + \psi/2) e^{-i\phi \cdot k' - i\psi \omega'} d\phi d\psi \]  

(K.4)

Starting with the Green’s function solution for the source response

\[ u_f(r, t) = \int G(r, r', t, t') f(r', t') dr'dt' \]  

(K.5)

We have

\[
W_{u_f}(r, k, t, \omega) = \frac{1}{(2\pi)^4} \int d\theta d\tau e^{-i\theta \cdot k - i\tau \omega} \int G^*(r - \theta/2, r_1, t - \tau/2, t_1) f^*(r_1, t_1) dr_1 dt_1 \\
\times \int G(r + \theta/2, r_2, t + \tau/2, t_2) f(r_2, t_2) dr_2 dt_2
\]  

(K.6)

We make a change of variables

\[ r_1 \rightarrow r' - \phi/2 \quad t_1 \rightarrow t' - \psi/2 \]  

(K.7)

\[ r_2 \rightarrow r' + \phi/2 \quad t_2 \rightarrow t' + \psi/2 \]  

(K.8)

The Jacobian for this transformation is \( J = 1 \). So we have

\[
W_{u_f}(r, k, t, \omega) = \int dr' dt' \int f^*(r' - \phi, t' - \psi) f(r' + \phi, t' + \psi) d\phi d\psi \\
\times \frac{1}{(2\pi)^4} \int G^*(r - \theta/2, r' - \phi, t - \tau/2, t' - \psi/2) \\
\times G(r + \theta/2, r' + \phi, t + \tau/2, t' + \psi/2) e^{-i\theta \cdot k - i\tau \omega} d\theta d\tau
\]  

(K.9)

Using the definition of the Wigner distribution of the source, Eq. (K.4), and inverse-Fourier-
transforming, we get
\[ f^*(r' - \phi/2, t' - \psi/2)f(r + \phi/2, t + \psi/2) = \int W_f(r', k', t', \omega') e^{i\Phi k' + i\psi \omega'} dk' d\omega' \quad (K.10) \]

So Eq. (K.9) becomes
\[
W_{uf}(r, k, t, \omega) = (2\pi)^4 \int dr'dt' \int dk'd\omega' W_f(r', k', t', \omega') \frac{1}{(2\pi)^8} \int d\phi \int d\psi e^{i\Phi k' + i\psi \omega'} \\
\times \int d\theta d\tau e^{-i\theta \cdot k - i\tau \omega'} G^*(r - \theta/2, r' - \phi/2, t - \tau/2, t' - \psi/2) \\
\times G(r + \theta/2, r' + \phi/2, t + \tau/2, t' + \psi/2) \quad (K.11)
\]

Using the definition of the Wigner distribution, we obtain the three-dimensional analog of our main result (Eq. (5.14)) is
\[
W_{uf}(r, k, t, \omega) = (2\pi)^4 \int W_f(r', k', t', \omega') W_G(r, k, r', -k', t, \omega, t', -\omega') dr'dt'dk'd\omega' \quad (K.12)
\]

**K.1 The Case of** \( G(r, r'; t, t') = G(r, r'; t - t') \)

We want to consider the case where \( G(r, r'; t, t') = G(r, r'; t - t') \). Using Eq. (K.10) the condition that \( G(r, r'; t, t') = G(r, r'; t - t') \), Eq. (K.9) becomes
\[
W_{uf}(r, k, t, \omega) = \int dr'dt'dk'd\omega' W_f(r', k', t', \omega') \frac{1}{(2\pi)^4} \int G^*(r - \theta/2, r' - \phi/2, t - t' - (\tau - \psi)/2) \\
\times G(r + \theta/2, r' + \phi/2, t - t' + (\tau - \psi)/2) e^{-i\theta \cdot k - i\tau \omega'} e^{i\Phi k' + i\psi \omega'} d\theta d\tau d\phi d\psi \quad (K.13)
\]
which leads to

$$W_{u_i}(r,k,t,\omega) = (2\pi)^4 \int W_f(r',k',t',\omega) W_G(r,k;r',-k';t-t',\omega)dr'dt'dk'$$

(K.14)
Appendix L

Wigner Distribution of

\[ f(x, y, z) = g(x, y - z) \]

In general, one can derive how variables combine under a Wigner transformation. In this appendix we show the relationship between the Wigner distribution of a function \( f(x, y, z) \) and a function \( g(x, y - z) \), where \( f(x, y, z) = g(x, y - z) \).

The variables in the Wigner distribution of the function \( f \), \( W_f \), are the coordinates \( x \), \( y \), and \( z \), and their respective wavenumbers \( k_x \), \( k_y \), and \( k_z \); however, this might not be the best way to represent the function, since it must be equivalent to another Wigner distribution, \( W_g \), which has less variables. That is, there is a property of \( f(x, y, z) \), namely, that it is a function of the difference \( y - z \), which imbues its Wigner distribution with the property that if you look at all positions with a constant difference \( y - z = \Delta y \), then the wavenumbers \( k_y \) and \( k_z \) are negatives of each other \( k_z = -k_y \). In this sense \( W_f \) is redundant.

If one is working with \( f(x, y, z) \) in the Cartesian coordinates \( x \), \( y \), \( z \), and wants to consider the Wigner distribution of \( g(x, \Delta y) \), \( W_g(x, k_z; \Delta y, k_y) \), then one must understand that \( \Delta y \) is really the difference \( y - z \), and that its conjugate variable is the same as \( k_y \). One must also understand that \( k_z = -k_y \), which may be the easiest fact to overlook.
\[ W_f(x, k_x; y, k_y; z, k_z) \]
\[ = \frac{1}{(2\pi)^3} \int f^*(x - \phi/2, y - \theta/2, z - \tau/2) f(x + \phi/2, y + \theta/2, z + \tau/2) \]
\[ \times e^{-i\phi k_x} e^{-i\tau k_z} d\phi d\theta d\tau \]  \hspace{1cm} (L.1)

\[ = \frac{1}{(2\pi)^3} \int g^*(x - \phi/2, (y - z) - (\theta - \tau)/2) g(x + \phi/2, (y - z) + (\theta - \tau)/2) \]
\[ \times e^{-i\phi k_x} \exp \left[ -i(\theta + \tau) \frac{k_y + k_z}{2} \right] \exp \left[ -i(\theta - \tau) \frac{k_y - k_z}{2} \right] d\phi d\theta d\tau \]  \hspace{1cm} (L.2)

Changing variables

\[ \theta - \tau = \alpha \quad \beta + \frac{\alpha}{2} = \theta \]  \hspace{1cm} (L.3)
\[ \frac{\theta + \tau}{2} = \beta \quad \beta - \frac{\alpha}{2} = \tau \]  \hspace{1cm} (L.4)

we have

\[ W_f(x, k_x; y, k_y; z, k_z) \]
\[ = \frac{1}{(2\pi)^2} \int g^*(x - \phi/2, (y - z) - \alpha/2) g(x + \phi/2, (y - z) + \alpha/2) \]
\[ \times e^{-i\phi k_x} e^{-i\alpha(k_y - k_z)/2} d\phi d\alpha \left( \int e^{-i\beta(k_y + k_z)} \frac{d\beta}{2\pi} \right) \]  \hspace{1cm} (L.5)
\[ = \delta(k_y + k_z) W_g(x, k_x; (y - z), k_y) \]  \hspace{1cm} (L.6)
Appendix M

Direct Derivation of Constant Speed Green’s Function

In Chap. 4 we presented the constant speed Green’s function by specializing the more general result for arbitrary speeds. In this appendix we present a direct construction of the Green’s function for the waveguides discussed in Chap. 3.

Consider waves that obey the three dimensional scalar wave equation with sources.

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u(r, t) = f(r, t) \tag{M.1}
\]

The response to the source, \( u_f(x, t) \), can be expressed in terms of a Green’s function

\[
u_f(r, t) = \int G(r, r'; t, t') f(r', t') dr' dt' \tag{M.2}\]

where the Green’s function \( G(r, r'; t, t') \) satisfies

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(r, r', t, t') = \delta^{(3)}(r - r') \delta(t - t') \tag{M.3}\]
Consider the Fourier-transform of the Green’s function’s time coordinate

\[ G(r, r'; t, t') = \frac{1}{\sqrt{2\pi}} \int H(r, r'; \omega, t') e^{+i\omega t'} d\omega \]  

(M.4)

Operating on Eq. (M.4)

\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(r, r'; t, t') = \frac{1}{\sqrt{2\pi}} \int \left( \nabla^2 + \frac{\omega^2}{c^2} \right) H(r, r'; \omega, t') e^{+i\omega t'} d\omega \]  

(M.5)

Ensuring that Eq. (M.3) is satisfied, we can obtain the time Dirac delta function by letting

\[ H(r, r'; \omega, t') = \frac{e^{-i\omega t'}}{\sqrt{2\pi}} h(r, r'; \omega) \]  

(M.6)

and the spatial Dirac delta function by forcing

\[ \left( \nabla^2 + \frac{\omega^2}{c^2} \right) h(r, r'; \omega) = \delta^{(3)}(r - r') \]  

(M.7)

Notice that \( G(r, r'; t, t') \) is a function of \( (t - t') \) since

\[ G(r, r'; t, t') = \frac{1}{\sqrt{2\pi}} \int e^{i\omega(t-t')} h(r, r'; \omega) d\omega = G(r, r'; t - t') \]  

(M.8)

In order to find \( h \), we begin by considering a complete set of functions compatible with our waveguide (with the boundary conditions). Such a set could be obtained by separation of variables, and it is called a complete set because every wave which satisfies the homogeneous wave equation is a linear combination of this set’s elements. We choose the following set, whose elements’ general form is

\[ \psi_\eta(r) = \psi_{\eta_x \eta_y \eta_z}(x, y, z) = X_{\eta_x}(x) Y_{\eta_y}(y) Z_{\eta_z}(z) = e^{i\eta_x x} e^{i\eta_y y} \sin(\eta_z z) \]  

(M.9)
where each function $\psi(r)$ satisfies the boundary conditions of the waveguide as long as

$$\eta_z = \frac{m}{L} \pi \quad \text{(pressure-release surface and bottom)} \quad (M.10)$$

$$\eta_z = \frac{2m - 1}{2L} \pi \quad \text{(pressure-release surface and hard bottom)} \quad (M.11)$$

Since we know that Green’s functions create valid waves, we guess that $h$ can also be expanded in terms of the $\psi$’s. Thus, we separate the variables in $h(r, r'; \omega)$, and express the $r$ dependence in terms of the $\psi$’s. We do so because after the integration over all other variables, the $r$-dependent parts of the Green’s functions will give the $r$-dependence of the wave.

$$h(r, r'; \omega) = \int_{-\infty}^{\infty} d\eta_x \int_{-\infty}^{\infty} d\eta_y \sum_{\eta_z} A(\eta, r'; \omega) \psi_{\eta}(r)$$

$$= \int_{-\infty}^{\infty} d\eta_x \int_{-\infty}^{\infty} d\eta_y \sum_{\eta_z} A(\eta, r'; \omega)e^{i\eta_x x}e^{i\eta_y y} \sin(\eta_z z)$$

We can think of this as the $\psi$-transform of $h(r, r'; \omega)$.

We now find the $\eta$, $r'$, and $\omega$-dependent coefficient $A$ for which $h$ will satisfy the differential equation Eq. (M.7)

$$\delta^{(3)}(r - r') = \left(\nabla^2 + \frac{w^2}{c^2}\right) h(r, r'; \omega)$$

$$= \left(\nabla^2 + \frac{w^2}{c^2}\right) \int d\eta_x d\eta_y \sum_{\eta_z} A(\eta, r'; \omega)e^{i\eta_x x}e^{i\eta_y y} \sin(\eta_z z)$$

$$= \int d\eta_x d\eta_y \sum_{\eta_z} \left(\frac{w^2}{c^2} - \eta_x^2 - \eta_y^2 - \eta_z^2\right) A(\eta, r'; \omega)e^{i\eta_x x}e^{i\eta_y y} \sin(\eta_z z)$$
multiplying both sides by $e^{-i\eta_x'x}e^{-i\eta_y'y}\sin(\eta_z'z)$ and integrating over all $\mathbf{r}$-space, we find that

$$A(\eta, \mathbf{r}; \omega) = \frac{1}{2\pi^2 L} \frac{e^{-i\eta_x'x}e^{-i\eta_y'y}\sin(\eta_z'z)}{w^2 - \eta_x'^2 - \eta_y'^2 - \eta_z'^2}$$  \hspace{1cm} (M.17)

So $h(\mathbf{r}, \mathbf{r}'; \omega)$ is

$$h(\mathbf{r}, \mathbf{r}'; \omega) = \frac{1}{2\pi^2 L} \int d\eta_x d\eta_y \sum_{\eta_z} \frac{e^{i\eta_x(x-x')}e^{i\eta_y(y-y')}}{w^2 - \eta_x^2 - \eta_y^2 - \eta_z^2} \sin(\eta_z'z) \sin(\eta_zz)$$  \hspace{1cm} (M.18)

$$= h(x-x', y-y', z, z'; \omega)$$  \hspace{1cm} (M.19)

Therefore, using Eq. (M.8), the Green’s function is

$$G(\mathbf{r}, \mathbf{r}'; t-t') = \frac{1}{(2\pi)^{5/2}} \sum_{\eta_z} \frac{\sin(\eta_z'z') \sin(\eta_zz)}{L/2} \int \frac{e^{i\eta_x(x-x')}e^{i\eta_y(y-y')}e^{i\omega(t-t')}}{w^2 - \eta_x^2 - \eta_y^2 - \eta_z^2} d\eta_x d\eta_y d\omega$$  \hspace{1cm} (M.20)
Bibliography


Publications of J. S. Ben-Benjamin


