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Counting Restricted Integer Partitions

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Abstract

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The first chapter examines $p_b(n)$, the number of partitions of n into powers of b, along with a family of identities which can be deduced by iterating a recurrence satisfied by $p_b(n)$ in a suitable way. These identities can then be used to calculate $p_b(n)$ for large values of n.

The second chapter restricts these types of partitions even further, limiting the multiplicity of each part. Its object of study is $p_{b,d}(n)$, that is, the number of partitions of n into powers of b repeating each power at most d times. The methods of the first chapter are applied, and the self-similarity of these sequences is discussed in detail.

The third chapter focuses on $p_{A,M}(n)$, the number of partitions of n with parts in A and multiplicities in M. A construction of Alon which produces infinite sets A and M so that $p_{A,M}(n) = 1$ is generalized so that A can be chosen to be a subset of powers of a given base.

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Chapter 1

Partitions into powers of a base

1.1 History

A *partition* of a nonnegative integer n is an expression consisting of a sum of positive integers whose value is n. A *b-ary partition* of n is a partition of n where each term in the sum is a powers of a base b. Denote the number of partitions of n as $p(n)$ and the number of b-ary partitions[1] as $p_b(n)$. For example, the partitions of 4 are $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$ therefore $p(4) = 5$ and $p_2(4) = 4$. The problem of calculating $p_b(n)$ dates to Euler [Eul53] who first studied $p_2(n)$ in his celebrated 1748 paper *De partitione numerorum*. In 1918, Tanturri [Tan18] examined the $p_2(n)$ problem, stating its recurrence and proving several identities. In that same year Hardy and Ramanujan [HR18] published their asymptotic formula for

[1] See Table 1.1 for values of $p_b(bn)$ for small values of b and n. The expression $p_b(bn)$ is chosen because by Theorem 1.3.1 the value of $p_b(n)$ is constant on runs of b.

n	$p_2(2n)$	$p_3(3n)$	$p_4(4n)$	$p_5(5n)$	$p_6(6n)$	$p_7(7n)$	$p_8(8n)$	$p_9(9n)$
1	2	2	2	2	2	2	2	2
2	4	3	3	3	3	3	3	3
3	6	5	4	4	4	4	4	4
4	10	7	6	5	5	5	5	5
5	14	9	8	7	6	6	6	6
6	20	12	10	9	8	7	7	7
7	26	15	12	11	10	9	8	8
8	36	18	15	13	12	11	10	9
9	46	23	18	15	14	13	12	11
10	60	28	21	18	16	15	14	13
11	74	33	24	21	18	17	16	15
12	94	40	28	24	21	19	18	17
13	114	47	32	27	24	21	20	19
14	140	54	36	30	27	24	22	21
15	166	63	40	34	30	27	24	23
16	202	72	46	38	33	30	27	25
17	238	81	52	42	36	33	30	27
18	284	93	58	46	40	36	33	30
19	330	105	64	50	44	39	36	33
20	390	117	72	55	48	42	39	36
21	450	132	80	60	52	46	42	39
22	524	147	88	65	56	50	45	42
23	598	162	96	70	60	54	48	45
24	692	180	106	75	65	58	52	48
25	786	198	116	82	70	62	56	51
26	900	216	126	89	75	66	60	54
27	1014	239	136	96	80	70	64	58
28	1154	262	148	103	85	75	68	62
29	1294	285	160	110	90	80	72	66
30	1460	313	172	119	96	85	76	70
31	1626	341	184	128	102	90	80	74
32	1828	369	199	137	108	95	85	78

Table 1.1: Values of $p_b(n)$ for $2 \leqslant b \leqslant 9$ and $1 \leqslant n \leqslant 32$.

the general partition function $p(n)$. To achieve this, they pioneered the circle method, noting that for the generating function for $p(n)$:

> Every point of the circle is an essential singularity of the function, and no part of the contour of integration can be deformed in such a manner as to make its contribution obviously negligible. Every element of the contour requires special study; and there is no obvious method of writing down a "dominant term."

In a 1940 paper Mahler [Mah40] established an oft-cited estimate which implies that $p_b(n)$ has intermediate growth, namely

$$\log p_b(n) \sim \frac{(\log n)^2}{2 \log b}.$$

Later, in 1948, de Bruijn [dB48] made use of residue calculations to refine Mahler's work on the asymptotics of $p_b(n)$. Subsequently in 1966, Knuth [Knu66] refined the asymptotic estimates of $p_2(n)$. Churchhouse [Chu69] in 1969 proved theorems regarding congruences of $p_2(n)$ by iterating the recurrence. He also posited a conjecture related to these congruences. Then Rødseth [Rød70] in 1970 proved Churchhouse's conjecture as well as congruences in the cases where b is a prime. Many later authors adapted Rødseth's method, about which he says:

> The method we use below in proving the above results goes back to Ramanujan, and has been exploited since then by many writers, notably Watson. We use the technique of Atkin and O'Brien.

Building on Rødseth's work, Andrews [And71] used generating function algebra to generalize Churchhouse's conjecture to all bases. This year also saw an independent proof of Churchhouse's conjecture by Gupta [Gup71]. Then, in 1972, Gupta [Gup72] proved Churchhouse's result in a simpler way by making use of Kemmer's identity. In 1975, Hirschhorn and Loxton [HL75] proved several congruences for $p_2(n)$ for n along certain arithemetic progressions. Dirdal [Dir75b, Dir75a] also proved congruences for $p_b(n)$ realizing these as limits of congruences of $p_{b,d}(n)$, the number of partitions of n into powers of b repeating each power at most d times. Gupta and Pleasants [GP79] used Kemmer's identity and matrix methods in 1979 to prove properties of $p_b(n)$ based on the base b expansion of n. Then, in 1990, Reznick [Rez90] proved properties of $p_{2,d}(n)$ while relating them to $p_2(n)$. His terrific bibliography in that paper is an excellent resource on the history of this subject. In a 2011 paper, Rødseth and Sellers [RS01] gave the problem a fresh look and prove congruences for $p_b(n)$ along the lines of Hirschhorn and Loxton.

1.2 Notation

Denote the set of integers by $\mathbb{Z}$ and the nonnegative integers by $\mathbb{N}$. Let $p_b(n)$ be the number of b-ary partitions of n, that is, the number of partitions of n whose parts are powers of b with no restriction on how often each power is used. Let $B_b(m, q)$ be the generating function of $p_b(b^m n)$, that is,

$$B_b(m, q) = \sum_{n \in \mathbb{Z}} p_b(b^m n) q^n.$$

Consider a sequence $(a_i)_{i \in I}$. Denote the length of the sequence as $|I|$ and note that a sequence can be interpreted as a $1 \times |I|$ matrix. Given a matrix M, denote its transposition as M^T.

The subsequent notations follow those of Graham, Knuth and Patashnik [GKP94]. If S is any statement then let $[\![S]\!]$ denote the Iverson bracket:

$$[\![S]\!] = \begin{cases} 1 \text{ if } S \text{ is true;} \\ 0 \text{ if } S \text{ is false.} \end{cases}$$

Denote the nth falling power of x as $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$. Let $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ be Stirling numbers of the first and second kind, respectively. In particular, define

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{Bmatrix} n \\ 0 \end{Bmatrix} = [\![n = 0]\!] \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix} = \begin{Bmatrix} n \\ k \end{Bmatrix} = [\![k = 0]\!]$$

and

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$$
$$\left[{n \atop k} \right] = (n-1) \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}.$$

Further, let $\left\langle {n \atop k} \right\rangle$ deonte the Eulerian numbers, that is,

$$\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n.$$

1.3 The recurrence

This section concerns itself with proving basic identities for $p_b(n)$.

Theorem 1.3.1. *The b-ary partition function satisfies the following recurrence:*

$$p_b(n) = 0 \text{ for } n < 0$$

$$p_b(n) = 1 \text{ for } 0 \leqslant n < b$$

$$p_b(bn+i) = p_b(bn) \text{ for } 0 \leqslant i < b \qquad \textbf{(RI)}$$

$$p_b(bn) = p_b(bn-1) + p_b(n) \qquad \textbf{(RII)}$$

Proof. Let $0 \leqslant i < b$. Consider a b-ary partition of $bn + i$. Such a partition must contain at least i copies of 1. Let f be a map which removes i ones from a b-ary partition of $bn + i$, and similarly let g be a map which adds i ones to a b-ary partition of bn. These operations are inverses since removing i ones from a b-ary partition of $bn+i$ and then adding i ones to the result produces the initial b-ary partition, that is fg is the identity map and therefore f is a bijection. Thus $p_b(bn) = p_b(bn+i)$ which proves **RI**.

To see **RII**, partition the set of b-ary partitions of bn into two sets: those involving ones and those not. For those involving ones, removing a single one will result in a b-ary partition of $bn-1$, and vice versa. Note that there are $p_b(bn-1)$ such partitions. For those not, each part is a positive power of b, from which b may be factored out, and the resulting sum will be a b-ary partition of n. Similarly for any b-ary partition of n, multiplying each part by b will result in a b-ary partition of bn. As before, this defines a bijection from b-ary partitions of bn without ones to the b-ary partitions of n. Therefore the number of b-ary partitions of bn without ones is $p_b(n)$. Consequently, $p_b(bn) = p_b(bn-1) + p_b(n)$. □

The following corollary is the primary way the recurrence for $p_b(n)$ will be used in what is to follow.

Corollary 1.3.2. *The b-ary partition counting function $p_b(n)$ satisfies the following identity:*

$$p_b(bn) = p_b(bn-b) + p_b(n). \qquad \textbf{(RIII)}$$

Proof. Combining **RI** and **RII** reveals

$$\begin{aligned} p_b(bn) &= p_b(bn-1) + p_b(n) \text{ by } \mathbf{RII} \\ &= p_b(b(n-1) + b - 1) + p_b(n) \text{ by } \mathbf{RI} \\ &= p_b(b(n-1)) + p_b(n) \end{aligned}$$

and hence the corollary. □

1.4 Generalizations of Tanturri and Churchhouse

The following lemma is a generalization of an identity which goes back to Tanturri.

Lemma 1.4.1. *The b-ary partition counting function $p_b(n)$ satisfies the following identity:*

$$p_b(bn) = \sum_{k=0}^{n} p_b(n-k)$$

Proof. By **RIII**, $p_b(n) = p_b(bn) - p_b(b(n-1))$, so

$$\sum_{k=0}^{n} p_b(b(n-k)) - p_b(b(n-1-k)) = \sum_{k=0}^{n} p_b(n-k)$$

where the left hand side is a telescoping sum, leaving

$$p_b(bn) - p_b(-b) = \sum_{k=0}^{n} p_b(n-k)$$

hence

$$p_b(bn) = \sum_{k=0}^{n} p_b(n-k)$$

as desired. □

Churchhouse extended this for $b = 2$ to calculate $p_2(2^m n)$. This may be futher extended to all b.

Theorem 1.4.2. *There exist positive integers $C_{b,m}(k)$ such that*

$$p_b(b^m n) = \sum_{k=0}^{n} C_{b,m}(k) p_b(n-k). \qquad (\mathrm{IH}(m))$$

Proof. The proof proceeds by induction on m, with the case $m = 1$ being provided by Lemma 1.4.1. The assertion IH($m+1$) can be shown by assuming IH(m), applying this to $p_b(b^m(bn))$, separating the first term, reindexing the remaining terms by setting $k = bj - i$ and using **RI**:

$$\begin{aligned}
p_b(b^{m+1}n) &= p_b(b^m(bn)) \\
&= \sum_{k=0}^{bn} C_{b,m}(k) p_b(bn-k) \\
&= C_{b,m}(0) p_b(bn) + \sum_{k=1}^{bn} C_{b,m}(k) p_b(bn-k) \\
&= C_{b,m}(0) p_b(bn) + \sum_{j=1}^{n} \sum_{i=0}^{b-1} C_{b,m}(bj-i) p_b(bn-bj+i) \\
&= C_{b,m}(0) p_b(bn) + \sum_{j=1}^{n} \sum_{i=0}^{b-1} C_{b,m}(bj-i) p_b(bn-bj).
\end{aligned}$$

Now applying Lemma 1.4.1 and reindexing by setting $h = n - j - \ell$ reveals

$$\begin{aligned} p_b(bn - bj) &= p_b(b(n-j)) \\ &= \sum_{\ell=0}^{n-j} p_b(n - j - \ell) \\ &= \sum_{h=0}^{n-j} p_b(h) \end{aligned}$$

therefore this yields

$$\begin{aligned} p_b(b^{m+1}n) &= C_{b,m}(0)p_b(bn) + \sum_{j=1}^{n}\sum_{i=0}^{b-1} C_{b,m}(bj - i)p_b(bn - bj) \\ &= C_{b,m}(0)p_b(bn) + \sum_{j=1}^{n}\sum_{i=0}^{b-1} C_{b,m}(bj - i)\sum_{h=0}^{n-j} p_b(h). \end{aligned}$$

This sum may be reordered by factoring out the sum indexed by h, extending the range of the sum indexed by h, making the subsitution $s = n - h$, interchanging the sums indexed by j and s, limiting the range of the sum indexed by j and recalling that $k = bj - i$, that is,

$$\begin{aligned}
\sum_{j=1}^{n}\sum_{i=0}^{b-1} C_{b,m}(bj-i)\sum_{h=0}^{n-j} p_b(h) &= \sum_{j=1}^{n}\sum_{h=0}^{n-j} p_b(h)\sum_{i=0}^{b-1} C_{b,m}(bj-i)\\
&= \sum_{j=1}^{n}\sum_{h=0}^{n} [\![h \leqslant n-j]\!] p_b(h)\sum_{i=0}^{b-1} C_{b,m}(bj-i)\\
&= \sum_{j=1}^{n}\sum_{s=0}^{n} [\![n-s \leqslant n-j]\!] p_b(n-s)\sum_{i=0}^{b-1} C_{b,m}(bj-i)\\
&= \sum_{j=1}^{n}\sum_{s=0}^{n} [\![j \leqslant s]\!] p_b(n-s)\sum_{i=0}^{b-1} C_{b,m}(bj-i)\\
&= \sum_{s=0}^{n} p_b(n-s)\sum_{j=1}^{n} [\![j \leqslant s]\!]\sum_{i=0}^{b-1} C_{b,m}(bj-i)\\
&= \sum_{s=0}^{n} p_b(n-s)\sum_{j=1}^{s}\sum_{i=0}^{b-1} C_{b,m}(bj-i)\\
&= \sum_{s=0}^{n} p_b(n-s)\sum_{k=1}^{sb} C_{b,m}(k).
\end{aligned}$$

Finally, the first term may be combined with this sum using Lemma 1.4.1:

$$\begin{aligned}
p_b(b^{m+1}n) &= C_{b,m}(0)p_b(bn) + \sum_{s=0}^{n} p_b(n-s)\sum_{k=1}^{sb} C_{b,m}(k)\\
&= C_{b,m}(0)\sum_{s=0}^{n} p_b(n-s) + \sum_{s=0}^{n} p_b(n-s)\sum_{k=1}^{sb} C_{b,m}(k)\\
&= \sum_{s=0}^{n} p_b(n-s)\sum_{k=0}^{sb} C_{b,m}(k)\\
&= \sum_{s=0}^{n}\sum_{k=0}^{bs} C_{b,m}(k)p_b(n-s)
\end{aligned}$$

Thus

$$p_b(b^{m+1}n) = \sum_{s=0}^{n} C_{b,m+1}(s)p_b(n-s)$$

where

$$C_{b,m+1}(s) = \sum_{k=0}^{bs} C_{b,m}(k).$$

proving IH$(m+1)$ and hence the theorem. □

The coefficients $C_{b,m}(k)$ are, in fact, more than simply coefficients, they are indeed polynomials of degree $m-1$.

Theorem 1.4.3. *The values $C_{b,m}(k)$ are polynomials of degree at most $m-1$ evaluated at k.*

Proof. Note that $C_{b,1} = 1$, a degree 0 polynomial in k. By the inductive hypothesis $C_{b,m}(k) = \sum_{i=0}^{m-1} \alpha_{m,i} k^i$, therefore[2]

$$\begin{aligned} C_{b,m+1}(k) &= \sum_{j=0}^{bk} C_{b,m}(j) \\ &= \sum_{j=0}^{bk} \sum_{i=0}^{m-1} \alpha_{m-1,i} j^i \\ &= \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{j=0}^{bk} j^i \\ &= \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{j=0}^{bk} \sum_{l=0}^{i} \left\{ {i \atop l} \right\} j^{\underline{l}} \end{aligned}$$

by an identity in [GKP94, p. 264] which gives powers as a sum of falling powers. Then by interchanging the order of summation and using the power

[2]Note that the Stirling numbers $\left[{n \atop k} \right]$ and $\left\{ {n \atop k} \right\}$ are defined on page 5.

rule for falling powers (*Ibid.*, p. 50 (2.50)):

$$\begin{aligned} C_{b,m+1}(k) &= \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{l=0}^{i} \left\{ {i \atop l} \right\} \sum_{j=0}^{bk} j^{\underline{l}} \\ &= \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{l=0}^{i} \left\{ {i \atop l} \right\} \frac{(bk)^{\underline{l+1}}}{l+1}. \end{aligned}$$

Now using an identity writing falling powers as a sum of powers (*Ibid.*, p. 264), noting that $l+1 \leqslant m$ and interchanging the order of summation reveals:

$$\begin{aligned} C_{b,m+1}(k) &= \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{l=0}^{i} \frac{1}{l+1} \left\{ {i \atop l} \right\} \sum_{j=0}^{l+1} \left[{l+1 \atop j} \right] (-1)^{l+1-j} (bk)^j \\ &= \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{l=0}^{i} \frac{1}{l+1} \left\{ {i \atop l} \right\} \sum_{j=0}^{l+1} k^j b^j \left[{l+1 \atop j} \right] (-1)^{l+1-j} \\ &= \sum_{j=0}^{m} k^j b^j \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{l=0}^{i} \frac{1}{l+1} \left\{ {i \atop l} \right\} \left[{l+1 \atop j} \right] (-1)^{l+1-j} \end{aligned}$$

Thus $C_{b,m+1}(k)$ is a polynomial in k of degree at most m with coefficients

$$\alpha_{m+1,j} = b^j \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{l=0}^{i} \frac{1}{l+1} \left\{ {i \atop l} \right\} \left[{l+1 \atop j} \right] (-1)^{l+1-j}$$

concluding the proof. □

Recall that the generating function for $p_b(b^m n)$ is $B_b(m,q) = \sum_{n \in \mathbb{Z}} p_b(b^m n) q^n$.

Lemma 1.4.4. *The generating function for $p_b(bn)$ satisfies the identity:*

$$(1-q)B_b(1,q) = B_b(0,q)$$

Proof. By **RIII**, $p_b(bn) = p_b(b(n-1)) + p_b(n)$ therefore $p_b(n) = p_b(bn) - p_b(b(n-1))$, so multiplying by q^n on both sides and summing over all integers n

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} p_b(n)q^n &= \sum_{n\in\mathbb{Z}} p_b(bn)q^n - \sum_{n\in\mathbb{Z}} p_b(b(n-1))q^n \\
&= \sum_{n\in\mathbb{Z}} p_b(bn)q^n - q\sum_{n\in\mathbb{Z}} p_b(bn)q^n \\
&= (1-q)\sum_{n\in\mathbb{Z}} p_b(bn)q^n \\
B_b(0,q) &= (1-q)B_b(1,q)
\end{aligned}$$

establishing the claim. □

1.5 A family of generating function identities

This section contains a proof of the main theorem which reveals a family of generating function identities. These identities correspond to a sequence of polynomials which have suggestive connections to Eulerian polynomials.

First, this lemma shows the recurrence may be iterated to express any value of $p_b(n)$ as the sum of multiples of $p_b(b^m)$ for suitable m.

Lemma 1.5.1. *For all $n, m \geqslant 1$ and $1 \leqslant k < b^m$*

$$p_b(b^m n + ub) = p_b(b^m n) + \sum_{k=1}^{u} p_b(b^{m-1}n + k).$$

Proof. Let $k = ub + v$ with $0 \leqslant v < b$. It may be assumed that $v = 0$ because if $v > 0$ then by **RI**

$$p_b(b^m n + k) = p_b(b^m n + ub + v) = p_b(b^m n + ub).$$

Therefore applying **RIII** once, twice and finally a total of u times iteratively to the leading term it may be seen that

$$\begin{aligned} p_b(b^m n + ub) &= p_b(b^m n + (u-1)b) + p_b(b^{m-1}n+u) \\ &= p_b(b^m n + (u-2)b) + p_b(b^{m-1}n+u-1) + p_b(b^{m-1}n+u) \\ &= p_b(b^m n + (u-2)b) + \sum_{j=0}^{1} p_b(b^{m-1}n+u-j) \\ p_b(b^m n + ub) &= p_b(b^m n) + \sum_{j=0}^{u-1} p_b(b^{m-1}n + u - j). \end{aligned}$$

Letting $\ell = u - j$ then reveals

$$p_b(b^m n + ub) = p_b(b^m n) + \sum_{\ell=1}^{u} p_b(b^{m-1}n + \ell)$$

concluding the proof. □

Lemma 1.5.2. *For all n, and $m \geqslant 2$*

$$\begin{aligned} p_b(b^m n) = p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) \\ + [\![m > 2]\!] b \sum_{u=1}^{b^{m-2}-1} p_b(b^{m-1}(n-1) + ub) \end{aligned}$$

Proof. First apply **RIII** to $p_b(b^m n)$ to obtain

$$\begin{aligned} p_b(b^m n) &= p_b(b^m n - b) + p_b(b^{m-1}n) \\ &= p_b(b^m(n-1) + b^m - b) + p_b(b^{m-1}n). \end{aligned}$$

Then Lemma 1.5.1 may be applied to the first term resulting in

$$p_b(b^m(n-1) + (b^{m-1}-1)b) = p_b(b^m(n-1)) + \sum_{k=1}^{b^{m-1}-1} p_b(b^{m-1}(n-1)+k)$$

which can then be subsituted into the previous expression. Then, note that the first $(b-1)$ terms in the sum are identical by **RI**. When $m = 2$ these are the only terms, but if $m > 2$ there are more terms in the sum which is indicated by the factor $[\![m > 2]\!]$ below.

$$\begin{aligned} p_b(b^m n) &= p_b(b^m(n-1) + b^m - b) + p_b(b^{m-1}n) \\ &= p_b(b^m(n-1)) + p_b(b^{m-1}n) + \sum_{k=1}^{b^{m-1}-1} p_b(b^{m-1}(n-1)+k) \\ &= p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}n) \\ &\qquad + [\![m > 2]\!] \sum_{k=b}^{b^{m-1}-1} p_b(b^{m-1}(n-1)+k) \end{aligned}$$

When $m > 2$ the summation stratifies by **RI**:

$$\begin{aligned} \sum_{k=b}^{b^{m-1}-1} p_b(b^{m-1}(n-1)+k) &= \sum_{u=1}^{b^{m-2}-1} \sum_{v=0}^{b-1} p_b(b^{m-1}(n-1)+ub+v) \\ &= b \sum_{u=1}^{b^{m-2}-1} p_b(b^{m-1}(n-1)+ub) \end{aligned}$$

Therefore in the general case the expression becomes

$$\begin{aligned} p_b(b^m n) = p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) \\ + [\![m > 2]\!] b \sum_{u=1}^{b^{m-2}-1} p_b(b^{m-1}(n-1) + ub) \end{aligned}$$

as claimed. □

Corollary 1.5.3. *The generating function for $p_b(b^2 n)$ satisfies the identity:*

$$(1-q)^2 B_b(2,q) = (1 + (b-1)q) B_b(0,q)$$

Proof. When $m = 2$, Lemma 1.5.2 becomes

$$p_b(b^2 n) = p_b(b^2(n-1)) + p_b(bn) + (b-1)p_b(b(n-1))$$

that is,

$$p_b(b^2 n) - p_b(b^2(n-1)) = p_b(bn) + (b-1)p_b(b(n-1)).$$

By passing to generating functions, the result is achieved.

$$\begin{aligned} \sum_{n\in\mathbb{Z}} p_b(b^2 n)q^n - \sum_{n\in\mathbb{Z}} p_b(b^2(n-1))q^n &= \sum_{n\in\mathbb{Z}} p_b(bn)q^n + \sum_{n\in\mathbb{Z}} (b-1)p_b(b(n-1))q^n \\ B_b(2,q) - qB_b(2,q) &= B_b(1,q) + (b-1)qB_b(1,q) \\ (1-q)B_b(2,q) &= (1 + (b-1)q)B_b(1,q) \\ (1-q)B_b(2,q) &= (1 + (b-1)q)(1-q)^{-1}B_b(0,q) \end{aligned}$$

Therefore

$$(1+(b-1)q)B_b(0,q) = (1-q)^2 B_b(2,q)$$

as stated. □

Lemma 1.5.4. *For all $n, m \geqslant 1$ and $1 \leqslant k < b^m$ there exist polynomials $g_{m,k,j}(x) = g_j(x)$ of degree j with integer coefficients for $0 \leqslant j \leqslant m-1$ such that*

$$p_b(b^m n + k) = p_b(b^m n) + \sum_{j=1}^{m-1} g_j(b) p_b(b^j n). \qquad (\mathrm{IH}(m))$$

Proof. The proof proceeds by induction on m. For $m = 1$ the induction hypothesis says $p_b(n+k) = p_b(n)$ for $1 \leqslant k < b$ which is true by **RI**. Assume that IH(m') is true for all $m' < m$. From Lemma 1.5.1

$$p_b(b^m n + ub) = p_b(b^m n) + \sum_{k=1}^{u} p_b(b^{m-1} n + k).$$

Then by the induction hypothesis at $m-1$

$$p_b(b^{m-1} + k) = p_b(b^{m-1} n) + \sum_{l=1}^{m-2} g_{m,k,l}(b) p_b(b^l n).$$

Therefore

$$\begin{aligned} p_b(b^m n + ub) &= p_b(b^m n) + \sum_{k=1}^{u} \left(p_b(b^{m-1} n) + \sum_{l=1}^{m-2} g_{m,k,l}(b) p_b(b^l n) \right) \\ &= p_b(b^m n) + u p_b(b^{m-1} n) + \sum_{k=1}^{u} \sum_{l=1}^{m-2} g_{m,k,l}(b) p_b(b^l n). \end{aligned}$$

Finally, switching the order of summation reveals

$$p_b(b^m n + ub) = p_b(b^m n) + up_b(b^{m-1}n) + \sum_{l=1}^{m-2}\left(\sum_{k=1}^{u} g_{m,k,l}(b)\right)p_b(b^l n).$$

Let $w = b^{m-1} - u$, $g_{m-1}(x) = x^{m-1} - w$ and $g_j(x) = \left(\sum_{k=0}^{u-1} g_{k,j}(x)\right)$ for $1 \leqslant j \leqslant m-2$. Then $u = b^{m-1} - w$ and $g_{m-1}(b) = b^{m-1} - w = u$, therefore

$$p_b(b^m n + ub + v) = p_b(b^m n) + \sum_{j=1}^{m-1} g_j(b)p_b(b^j n)$$

as stated. □

With this preparation, the main theorem may be proven. This allows the generating function $B_b(m,q)$ to be written in terms of $B_b(0,q)$.

Theorem 1.5.5. *For all m there exists a polynomial $f_m(x,q)$ of degree $m-1$ in q and degree $\binom{m}{2}$ in x such that*

$$f_m(b,q)B_b(0,q) = (1-q)^m B_b(m,q).$$

Proof. The proof proceeds by induction on m. The base case $m = 0$ is trivial, that is, $f_0(x,q) = 1$. Assume that the theorem holds for all $m' < m$. Applying **RIII** to $p_b(b^m n)$ yields the following:

$$\begin{aligned} p_b(b^m n) &= p_b(b^m n - b) + p_b(b^{m-1}n) \\ &= p_b(b^m(n-1) + b^m - b) + p_b(b^{m-1}n). \end{aligned}$$

By Lemma 1.5.4

$$p_b(b^m(n-1)+b^m-b) = p_b(b^m(n-1)) + \sum_{j=1}^{m-1} g_j(b)p_b(b^j(n-1))$$

therefore

$$p_b(b^m n) = p_b(b^m(n-1)) + \left(\sum_{j=1}^{m-1} g_j(b)p_b(b^j(n-1))\right) + p_b(b^{m-1}n)$$

Then multiplying by q^n on both sides and summing:

$$\begin{aligned} B_b(m,q) &= \sum_{n\in\mathbb{Z}} p_b(b^m n)q^n \\ &= \sum_{n\in\mathbb{Z}} p_b(b^m(n-1))q^n + \sum_{n\in\mathbb{Z}} p_b(b^{m-1}n)q^n \\ &\quad + \sum_{n\in\mathbb{Z}}\sum_{j=1}^{m-1} g_j(b)p_b(b^j(n-1))q^n \end{aligned}$$

and by reindexing in sums involving $n-1$ and combining the b^{m-1} terms

$$\begin{aligned} B_b(m,q) &= q\sum_{n\in\mathbb{Z}} p_b(b^m n)q^n + \sum_{n\in\mathbb{Z}} p_b(b^{m-1}n)q^n + q\sum_{n\in\mathbb{Z}}\sum_{j=1}^{m-1} g_j(b)p_b(b^j n)q^n \\ &= qB_b(m,q) + (1+g_{m-1}(b)q)\sum_{n\in\mathbb{Z}} p_b(b^{m-1}n)q^n \\ &\quad + q\sum_{j=1}^{m-2} g_j(b)\sum_{n\in\mathbb{Z}} p_b(b^j n)q^n \\ &= qB_b(m,q) + (1+g_{m-1}(b)q)B_b(m-1,q) + q\sum_{j=1}^{m-2} g_j(b)B_b(j,q) \end{aligned}$$

Therefore

$$(1-q)B_b(m,q) = (1+g_{m-1}(b)q)B_b(m-1,q) + q\sum_{j=1}^{m-2} g_j(b)B_b(j,q).$$

The induction hypothesis provides

$$(1-q)^j B_b(j,q) = B_b(0,q) f_j(q).$$

that is,

$$B_b(j,q) = \frac{(1-q)^{m-j-1} f_j(q)}{(1-q)^{m-1}} B_b(0,q).$$

hence substituting this into the previous sum and multiplying by $(1-q)^{m-1}$ reveals

$$\begin{aligned}(1-q)^m B_b(m,q) &= (1+g_{m-1}(b)q) f_{m-1}(q) B_b(0,q) \\ &\quad + q \sum_{j=1}^{m-2} g_j(b)(1-q)^{m-j-1} f_j(q) B_b(0,q) \\ &= \Bigg((1+g_{m-1}(b)q) f_{m-1}(q) \\ &\quad + q \sum_{j=1}^{m-2} g_j(b)(1-q)^{m-j-1} f_j(q) \Bigg) B_b(0,q).\end{aligned}$$

Consequently

$$f_m(x,q) = \left((1+g_{m-1}(x)q) f_{m-1}(x,q) + q \sum_{j=1}^{m-2} g_j(x)(1-q)^{m-j-1} f_j(x,q) \right)$$

which is a polynomial of degree $m-1$ in q and degree $\binom{m}{2}$ in x, therefore

$$f_m(b,q) B_b(0,q) = (1-q)^m B_b(m,q)$$

which proves the theorem. □

1.6 The polynomial data

The polynomials in Theorem 1.5.5 provide a bridge between large values of $p_b(n)$ and its generating function identities. Lacking further theorems, evaluating these large values quickly exceeds the computational power of pencil and paper, but computers are ideally suited to calculating these large values. Each $f_m(b,q)$ provides an identity which provides a new way to calculate values of the form $p_b(b^m n)$. Theorem 1.4.2 provides an alternate way of computing these numbers. The tools used for this work were primarily Python and Sage with doublechecking provided by Mathematica. The City University of New York High Performance Computing Center at the College of Staten Island helpfully provided hardware for long-running computations, but with the optimizations provided by Theorems 1.4.2 and 1.5.5 retail consumer hardware is capable of calculating $f_m(b,q)$ for high values of m. Tables 1.2 and 1.3 contain identities of the form

$$f_m(q)B_b(0,q) = (1-q)^m \sum_{n\in\mathbb{Z}} p_b(b^m n)q^n$$

for various specific m and b.

Since $f_m(b,q)$ is a polynomial of degree $\binom{m}{2}$ in b, then so is each coefficient in q. Therefore, for a given m, by calculating $f_m(b,q)$ for $\binom{m}{2}+1$ values of b it is possible to determine a polynomial in b for each coefficient of q.

b	$f_2(b,q)$
2	$1{+}q$
3	$1{+}2q$

b	$f_3(b,q)$
2	$1{+}6q + q^2$
3	$1{+}19q + 7q^2$
4	$1{+}42q + 21q^2$
5	$1{+}78q + 46q^2$

b	$f_4(b,q)$
2	$1{+}31q + 31q^2 + q^3$
3	$1{+}234q + 447q^2 + 47q^3$
4	$1{+}1081q + 2635q^2 + 379q^3$
5	$1{+}3702q + 10218q^2 + 1704q^3$
6	$1{+}10335q + 30735q^2 + 5585q^3$
7	$1{+}24896q + 77801q^2 + 14951q^3$
8	$1{+}53669q + 173747q^2 + 34727q^3$

Table 1.2: The polynomials $f_m(b,q)$ for $2 \leqslant m \leqslant 4$ and $2 \leqslant b \leqslant \binom{m}{2} + 2$

b	$f_5(b,q)$
2	$1{+}196q + 630q^2 + 196q^3 + q^4$
3	$1{+}5822q + 33504q^2 + 19040q^3 + 682q^4$
4	$1{+}79320q + 561714q^2 + 387600q^3 + 19941q^4$
5	$1{+}642451q + 5055891q^2 + 3835861q^3 + 231421q^4$
6	$1{+}3649340q + 30621390q^2 + 24573740q^3 + 1621705q^4$
7	$1{+}16077981q + 140871555q^2 + 117324441q^3 + 8201271q^4$
8	$1{+}58573732q + 529473294q^2 + 452753140q^3 + 32941657q^4$
9	$1{+}184174970q + 1704597594q^2 + 1486613030q^3 + 111398806q^4$
10	$1{+}515009556q + 4855552326q^2 + 4299866676q^3 + 329571441q^4$
11	$1{+}1308822280q + 12524820930q^2 + 11227696630q^3 + 876084760q^4$
12	$1{+}3072329216q + 29763241530q^2 + 26948358536q^3 + 2133434941q^4$

Table 1.3: The polynomials $f_5(b,q)$ for $2 \leqslant b \leqslant 12$

This data determines $f_m(b,q)$ for a given m and all b. An alternate way of calculating this polynomial is to continue to iterate the recurrence. This method is demonstrated for $m = 3$ in Theorem 1.10.1. For the case $m = 4$ this approach works, but the argument is significantly longer than the $m = 3$ case.

Table 1.4 shows $f_m(b,q)$ for $1 \leqslant m \leqslant 4$, written out brutally as polyno-

m	$f_m(b,q)$
1	1
2	$bq - q + 1$
3	$\frac{1}{2}b^3q^2 + \frac{1}{2}b^3q - \frac{1}{2}b^2q^2 + \frac{1}{2}b^2q - bq^2 + bq$ $+q^2 - 2q + 1$
4	$\frac{1}{6}b^6q^3 + \frac{2}{3}b^6q^2 - \frac{1}{4}b^5q^3 + \frac{1}{6}b^6q - \frac{1}{6}b^4q^3 + \frac{1}{4}b^5q$ $-\frac{1}{6}b^4q^2 - \frac{1}{4}b^3q^3 + \frac{1}{3}b^4q - \frac{1}{2}b^3q^2 + \frac{1}{2}b^2q^3 + \frac{3}{4}b^3q$ $-b^2q^2 + bq^3 + \frac{1}{2}b^2q - 2bq^2 - q^3 + bq$ $+3q^2 - 3q + 1$

Table 1.4: The polynomials $f_m(b,q)$ for $1 \leqslant m \leqslant 4$.

mials. This representation does not not, at first glance, appear particularly illuminating, but it may be the case something may be learned from it. Along these lines, tables of coefficients for the monomials in $f_m(b,q)$ for $5 \leqslant m \leqslant 8$ are presented in Tables 1.5, 1.6, 1.7 and 1.8. These tables may also be thought of as matrices M_m so that

$$f_m(b,q) = Q_m M_m B_m$$

where $Q_m = ((q^i)_{i=0}^{m-1})^T$ and $B_m = (b^i)_{i=0}^{\binom{m}{2}}$. Perhaps this representation will suggest a combinatorial intepretation of these coefficients.

The polynomials $f_m(b,q)$ may be seen from an alternate viewpoint as polynomials in b where each coefficient of b is a polynomial in q. This viewpoint (see Tables 1.9, 1.10, 1.11 and 1.12) proves its usefulness in revealing certain repeating structures. These polynomials have been cal-

	q^0	q^1	q^2	q^3	q^4
b^0	1	-4	6	-4	1
b^1	0	1	-3	3	-1
b^2	0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$
b^3	0	$\frac{3}{4}$	$-\frac{5}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
b^4	0	$\frac{11}{24}$	$-\frac{7}{8}$	$\frac{3}{8}$	$\frac{1}{24}$
b^5	0	$\frac{11}{24}$	$-\frac{5}{8}$	$-\frac{1}{8}$	$\frac{7}{24}$
b^6	0	$\frac{3}{8}$	$\frac{5}{24}$	$-\frac{13}{24}$	$-\frac{1}{24}$
b^7	0	$\frac{5}{24}$	$\frac{1}{8}$	$-\frac{3}{8}$	$\frac{1}{24}$
b^8	0	$\frac{1}{8}$	$\frac{5}{24}$	$-\frac{7}{24}$	$-\frac{1}{24}$
b^9	0	$\frac{1}{12}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{12}$
b^{10}	0	$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{1}{24}$

Table 1.5: Coefficients of $f_5(b,q)$

	q^0	q^1	q^2	q^3	q^4	q^5
b^0	1	-5	10	-10	5	-1
b^1	0	1	-4	6	-4	1
b^2	0	$\frac{1}{2}$	-2	3	-2	$\frac{1}{2}$
b^3	0	$\frac{3}{4}$	-2	$\frac{3}{2}$	0	$-\frac{1}{4}$
b^4	0	$\frac{11}{24}$	$-\frac{4}{3}$	$\frac{5}{4}$	$-\frac{1}{3}$	$-\frac{1}{24}$
b^5	0	$\frac{25}{48}$	$-\frac{4}{3}$	$\frac{7}{8}$	$\frac{1}{6}$	$-\frac{11}{48}$
b^6	0	$\frac{1}{2}$	$-\frac{13}{24}$	$-\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{24}$
b^7	0	$\frac{13}{36}$	$-\frac{35}{72}$	$-\frac{5}{24}$	$\frac{31}{72}$	$-\frac{7}{72}$
b^8	0	$\frac{7}{24}$	$-\frac{1}{8}$	$-\frac{5}{8}$	$\frac{11}{24}$	0
b^9	0	$\frac{11}{48}$	$\frac{1}{12}$	$-\frac{19}{24}$	$\frac{5}{12}$	$\frac{1}{16}$
b^{10}	0	$\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{6}$	0
b^{11}	0	$\frac{23}{240}$	$\frac{17}{60}$	$-\frac{41}{120}$	$-\frac{1}{20}$	$\frac{1}{80}$
b^{12}	0	$\frac{1}{16}$	$\frac{7}{24}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$\frac{1}{48}$
b^{13}	0	$\frac{5}{144}$	$\frac{17}{72}$	$-\frac{1}{12}$	$-\frac{13}{72}$	$-\frac{1}{144}$
b^{14}	0	$\frac{1}{48}$	$\frac{5}{24}$	0	$-\frac{5}{24}$	$-\frac{1}{48}$
b^{15}	0	$\frac{1}{120}$	$\frac{13}{60}$	$\frac{11}{20}$	$\frac{13}{60}$	$\frac{1}{120}$

Table 1.6: Coefficients of $f_6(b,q)$

	q^0	q^1	q^2	q^3	q^4	q^5	q^6
b^0	1	-6	15	-20	15	-6	1
b^1	0	1	-5	10	-10	5	-1
b^2	0	$\frac{1}{2}$	$-\frac{5}{2}$	5	-5	$\frac{5}{2}$	$-\frac{1}{2}$
b^3	0	$\frac{3}{4}$	$-\frac{11}{4}$	$\frac{7}{2}$	$-\frac{3}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$
b^4	0	$\frac{11}{24}$	$-\frac{43}{24}$	$\frac{31}{12}$	$-\frac{19}{12}$	$\frac{7}{24}$	$\frac{1}{24}$
b^5	0	$\frac{25}{48}$	$-\frac{89}{48}$	$\frac{53}{24}$	$-\frac{17}{24}$	$-\frac{19}{48}$	$\frac{11}{48}$
b^6	0	$\frac{17}{32}$	$-\frac{115}{96}$	$\frac{23}{48}$	$\frac{7}{16}$	$-\frac{17}{96}$	$-\frac{7}{96}$
b^7	0	$\frac{125}{288}$	$-\frac{331}{288}$	$\frac{109}{144}$	$\frac{41}{144}$	$-\frac{119}{288}$	$\frac{25}{288}$
b^8	0	$\frac{19}{48}$	$-\frac{13}{16}$	$\frac{1}{24}$	$\frac{19}{24}$	$-\frac{7}{16}$	$\frac{1}{48}$
b^9	0	$\frac{17}{48}$	$-\frac{71}{144}$	$-\frac{11}{18}$	$\frac{5}{4}$	$-\frac{67}{144}$	$-\frac{5}{144}$
b^{10}	0	$\frac{7}{24}$	$\frac{1}{16}$	$-\frac{23}{24}$	$\frac{7}{12}$	0	$\frac{1}{48}$
b^{11}	0	$\frac{313}{1440}$	$\frac{1}{32}$	$-\frac{127}{144}$	$\frac{113}{144}$	$-\frac{13}{96}$	$-\frac{23}{1440}$
b^{12}	0	$\frac{31}{180}$	$\frac{35}{144}$	$-\frac{71}{72}$	$\frac{19}{36}$	$\frac{5}{72}$	$-\frac{19}{720}$
b^{13}	0	$\frac{61}{480}$	$\frac{151}{480}$	$-\frac{69}{80}$	$\frac{21}{80}$	$\frac{27}{160}$	$-\frac{1}{96}$
b^{14}	0	$\frac{67}{720}$	$\frac{7}{18}$	$-\frac{53}{72}$	$-\frac{1}{18}$	$\frac{43}{144}$	$\frac{1}{90}$
b^{15}	0	$\frac{91}{1440}$	$\frac{43}{96}$	$-\frac{19}{144}$	$-\frac{49}{144}$	$-\frac{1}{32}$	$-\frac{11}{1440}$
b^{16}	0	$\frac{7}{180}$	$\frac{11}{36}$	$-\frac{11}{72}$	$-\frac{19}{72}$	$\frac{5}{72}$	$\frac{1}{360}$
b^{17}	0	$\frac{7}{288}$	$\frac{343}{1440}$	$-\frac{7}{240}$	$-\frac{181}{720}$	$\frac{23}{1440}$	$\frac{1}{480}$
b^{18}	0	$\frac{7}{480}$	$\frac{19}{96}$	$\frac{1}{16}$	$-\frac{13}{48}$	$-\frac{1}{96}$	$\frac{1}{160}$
b^{19}	0	$\frac{11}{1440}$	$\frac{13}{96}$	$\frac{19}{144}$	$-\frac{29}{144}$	$-\frac{7}{96}$	$-\frac{1}{1440}$
b^{20}	0	$\frac{1}{240}$	$\frac{5}{48}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{5}{48}$	$-\frac{1}{240}$
b^{21}	0	$\frac{1}{720}$	$\frac{19}{240}$	$\frac{151}{360}$	$\frac{151}{360}$	$\frac{19}{240}$	$\frac{1}{720}$

Table 1.7: Coefficients of $f_7(b,q)$

	q^0	q^1	q^2	q^3	q^4	q^5	q^6	q^7
b^0	1	-7	21	-35	35	-21	7	-1
b^1	0	1	-6	15	-20	15	-6	1
b^2	0	$\frac{1}{2}$	-3	$\frac{15}{2}$	-10	$\frac{15}{2}$	-3	$\frac{1}{2}$
b^3	0	$\frac{3}{4}$	$-\frac{7}{2}$	$\frac{25}{4}$	-5	$\frac{5}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$
b^4	0	$\frac{11}{24}$	$-\frac{9}{4}$	$\frac{35}{8}$	$-\frac{25}{6}$	$\frac{15}{8}$	$-\frac{1}{4}$	$-\frac{1}{24}$
b^5	0	$\frac{25}{48}$	$-\frac{19}{8}$	$\frac{65}{16}$	$-\frac{35}{12}$	$\frac{5}{16}$	$\frac{5}{8}$	$-\frac{11}{48}$
b^6	0	$\frac{17}{32}$	$-\frac{83}{48}$	$\frac{161}{96}$	$-\frac{1}{24}$	$-\frac{59}{96}$	$\frac{5}{48}$	$\frac{7}{96}$
b^7	0	$\frac{259}{576}$	$-\frac{161}{96}$	$\frac{137}{64}$	$-\frac{113}{144}$	$-\frac{89}{192}$	$\frac{13}{32}$	$-\frac{41}{576}$
b^8	0	$\frac{7}{16}$	$-\frac{137}{96}$	$\frac{127}{96}$	$\frac{11}{48}$	$-\frac{11}{12}$	$\frac{35}{96}$	$-\frac{1}{96}$
b^9	0	$\frac{27}{64}$	$-\frac{85}{72}$	$\frac{307}{576}$	$\frac{89}{72}$	$-\frac{823}{576}$	$\frac{7}{18}$	$\frac{17}{576}$
b^{10}	0	$\frac{329}{864}$	$-\frac{43}{72}$	$-\frac{131}{288}$	$\frac{127}{108}$	$-\frac{155}{288}$	$\frac{5}{72}$	$-\frac{31}{864}$
b^{11}	0	$\frac{911}{2880}$	$-\frac{389}{720}$	$-\frac{289}{576}$	$\frac{19}{12}$	$-\frac{623}{576}$	$\frac{161}{720}$	$\frac{1}{2880}$
b^{12}	0	$\frac{1189}{4320}$	$-\frac{323}{1440}$	$-\frac{13}{12}$	$\frac{787}{432}$	$-\frac{245}{288}$	$\frac{19}{480}$	$\frac{47}{2160}$
b^{13}	0	$\frac{329}{1440}$	$-\frac{1}{240}$	$-\frac{125}{96}$	$\frac{59}{36}$	$-\frac{47}{96}$	$-\frac{19}{240}$	$\frac{11}{1440}$
b^{14}	0	$\frac{325}{1728}$	$\frac{293}{1440}$	$-\frac{859}{576}$	$\frac{595}{432}$	$-\frac{23}{576}$	$-\frac{67}{288}$	$-\frac{43}{8640}$
b^{15}	0	$\frac{427}{2880}$	$\frac{281}{720}$	$-\frac{641}{576}$	$\frac{1}{2}$	$\frac{29}{576}$	$\frac{7}{720}$	$\frac{41}{2880}$
b^{16}	0	$\frac{61}{540}$	$\frac{521}{1440}$	$-\frac{1607}{1440}$	$\frac{1177}{2160}$	$\frac{179}{720}$	$-\frac{223}{1440}$	$\frac{11}{4320}$
b^{17}	0	$\frac{31}{360}$	$\frac{289}{720}$	$-\frac{127}{144}$	$\frac{1}{8}$	$\frac{13}{36}$	$-\frac{67}{720}$	$\frac{1}{720}$
b^{18}	0	$\frac{23}{360}$	$\frac{587}{1440}$	$-\frac{199}{288}$	$-\frac{25}{144}$	$\frac{67}{144}$	$-\frac{97}{1440}$	$-\frac{7}{1440}$
b^{19}	0	$\frac{131}{2880}$	$\frac{109}{288}$	$-\frac{247}{576}$	$-\frac{7}{16}$	$\frac{241}{576}$	$\frac{37}{1440}$	$-\frac{1}{576}$
b^{20}	0	$\frac{277}{8640}$	$\frac{11}{32}$	$-\frac{679}{2880}$	$-\frac{1283}{2160}$	$\frac{347}{960}$	$\frac{131}{1440}$	$\frac{13}{8640}$
b^{21}	0	$\frac{61}{2880}$	$\frac{5}{16}$	$\frac{13}{64}$	$-\frac{4}{9}$	$-\frac{5}{64}$	$-\frac{1}{80}$	$-\frac{1}{576}$
b^{22}	0	$\frac{269}{20160}$	$\frac{107}{504}$	$\frac{401}{4032}$	$-\frac{187}{504}$	$\frac{107}{4032}$	$\frac{13}{630}$	$-\frac{5}{4032}$
b^{23}	0	$\frac{1}{120}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{30}$	0
b^{24}	0	$\frac{43}{8640}$	$\frac{43}{360}$	$\frac{521}{2880}$	$-\frac{61}{270}$	$-\frac{269}{2880}$	$\frac{1}{72}$	$\frac{1}{8640}$
b^{25}	0	$\frac{1}{360}$	$\frac{4}{45}$	$\frac{29}{144}$	$-\frac{1}{6}$	$-\frac{5}{36}$	$\frac{1}{90}$	$\frac{1}{720}$
b^{26}	0	$\frac{1}{720}$	$\frac{1}{18}$	$\frac{13}{72}$	$-\frac{1}{18}$	$-\frac{23}{144}$	$-\frac{1}{45}$	0
b^{27}	0	$\frac{1}{1440}$	$\frac{7}{180}$	$\frac{49}{288}$	0	$-\frac{49}{288}$	$-\frac{7}{180}$	$-\frac{1}{1440}$
b^{28}	0	$\frac{1}{5040}$	$\frac{1}{42}$	$\frac{397}{1680}$	$\frac{151}{315}$	$\frac{397}{1680}$	$\frac{1}{42}$	$\frac{1}{5040}$

Table 1.8: Coefficients of $f_8(b,q)$

$$
\begin{array}{rrrlll}
 & f_4(b,q) & = & (1-q)^3 & & \\
+ & & q & (1-q)^2 & b & \\
+ & (2)^{-1} & q & (1-q)^2 & b^2 & \\
+ & (4)^{-1} & q & (1-q) & (3+q)\ b^3 & \\
+ & (6)^{-1} & q & (1-q) & (2+q)\ b^4 & \\
+ & (4)^{-1} & q & (1-q) & (1+q)\ b^5 & \\
+ & (6)^{-1} & q & \cdot & (1+4q+q^2)\ b^6 &
\end{array}
$$

Table 1.9: The polynomial $f_4(b,q)$ as a polynomial in b

$$
\begin{array}{rrrlll}
 & f_5(b,q) & = & (1-q)^4 & & \\
+ & & q & (1-q)^3 & b & \\
+ & (2)^{-1} & q & (1-q)^3 & b^2 & \\
+ & (4)^{-1} & q & (1-q)^2 & (3+q)\ b^3 & \\
+ & (24)^{-1} & q & (1-q)^2 & (11+q)\ b^4 & \\
+ & (24)^{-1} & q & (1-q)^2 & (11+7q)\ b^5 & \\
+ & (24)^{-1} & q & (1-q) & (9+14q+q^2)\ b^6 & \\
+ & (24)^{-1} & q & (1-q) & (5+8q-q^2)\ b^7 & \\
+ & (24)^{-1} & q & (1-q) & (3+8q+q^2)\ b^8 & \\
+ & (12)^{-1} & q & (1-q) & (1+4q+q^2)\ b^9 & \\
+ & (24)^{-1} & q & \cdot & (1+11q+11q^2+q^3)\ b^{10} &
\end{array}
$$

Table 1.10: The polynomial $f_5(b,q)$ as a polynomial in b

$$
\begin{array}{rrcll}
 & f_6(b,q) & = & (1-q)^5 & \\
+ & & q & (1-q)^4 & b \\
+ & (2)^{-1} & q & (1-q)^4 & b^2 \\
+ & (4)^{-1} & q & (1-q)^3 & (3+q)\ b^3 \\
+ & (24)^{-1} & q & (1-q)^3 & (11+q)\ b^4 \\
+ & (48)^{-1} & q & (1-q)^3 & (25+11q)\ b^5 \\
+ & (24)^{-1} & q & (1-q)^2 & (12+11q+q^2)\ b^6 \\
+ & (72)^{-1} & q & (1-q)^2 & (26+17q-7q^2)\ b^7 \\
+ & (24)^{-1} & q & (1-q)^2 & (7+11q)\ b^8 \\
+ & (48)^{-1} & q & (1-q)^2 & (11+26q+3q^2)\ b^9 \\
+ & (6)^{-1} & q & (1-q) & (1+4q+q^2)\ b^{10} \\
+ & (240)^{-1} & q & (1-q) & (23+91q+9q^2-3q^3)\ b^{11} \\
+ & (48)^{-1} & q & (1-q) & (3+17q+5q^2-q^3)\ b^{12} \\
+ & (144)^{-1} & q & (1-q) & (5+39q+27q^2+q^3)\ b^{13} \\
+ & (48)^{-1} & q & (1-q) & (1+11q+11q^2+q^3)\ b^{14} \\
+ & (120)^{-1} & q & \cdot & (1+26q+66q^2+26q^3+q^4)\ b^{15}
\end{array}
$$

Table 1.11: The polynomial $f_6(b,q)$ as a polynomial in b

culated for values of m up to 23, unfortunately these pages are unable to contain them. Or, with apologies to Fermat, "Hanc *paginis* exiguitas non caperet." Fortunately, this data is available for download at the following URL: `http://dakota.tensen.net/2015/rp/`

The form of these polynomials suggests a conjecture containing an unexpected appearance of Eulerian numbers:

Conjecture 1.6.1. *The polynomial $f_m(b,q)$ has the form*

$$f_m(b,q) = \sum_{i=0}^{\binom{m}{2}} (1-q)^{m-y(i)} g_{m,i}(q) b^i$$

where $y(n) = \left\lfloor \frac{\sqrt{8n+1}}{2} \right\rfloor$ and $g_{m,i}(q)$ are polynomials. Further, with $\left\langle {n \atop k} \right\rangle$ denoting the Eulerian numbers[3]*:*

$$g_{m,\binom{m}{2}}(q) = \frac{q}{(m-1)!} \sum_{i=0}^{m-2} \left\langle {m-1 \atop i} \right\rangle q^i$$

Another conjecture also suggests itself:

Conjecture 1.6.2. *Let $h_i(q)$ be defined by $qh_i(q) = g_{i+1,i}(q)$. Then*

$$\begin{aligned} f_m(b,q) = (1-q)^{m-1} & \\ &+ q \sum_{i=0}^{m-1} (1-q)^{m-y(i)} h_i(q) b^i \\ &+ \sum_{i=m}^{\binom{m}{2}-1} (1-q)^{m-y(i)} g_{m,i}(q) b^i \\ &+ \frac{qb^{\binom{m}{2}}}{(m-1)!} \sum_{i=0}^{m-2} \left\langle {m-1 \atop i} \right\rangle q^i. \end{aligned}$$

Assuming these conjectures indicates that the polynomials $f_b(m,q)$ are completely determined by the polynomials $(g_{m,i})_{i=m}^{\binom{m}{2}-1}$. The data so far obeys this conjecture, so these polynomials are given for $8 \leqslant m \leqslant 10$ (in Tables

[3]Note that the Eulerian numbers $\left\langle {n \atop k} \right\rangle$ are defined on page 6.

1.13, 1.14, 1.15, 1.16, 1.17, 1.18) from which, by using the form above, one may construct $f_m(b,q)$.

1.7 A new congruence

The conjectured form of $f_m(b,q)$ suggests several conjectures, including some regarding congruences of $p_b(n)$. For instance, $f_m(b,q) \equiv (1-q)^{m-1} \pmod{b}$ seems likely, therefore Theorem 1.5.5 suggests

$$(1-q)^m B_b(m,q) \equiv (1-q)^{m-1} B_b(0,q) \pmod{b}$$

and hence

$$(1-q) B_b(m,q) \equiv B_b(0,q) \pmod{b}$$

Fortunately, this can be proven independently of the conjectured form of $f_m(b,q)$ and this statement appears below as Theorem 1.7.2. Reducing Lemma 1.5.2 modulo b reveals the following corollary:

Corollary 1.7.1. *The partition counting function $p_b(n)$ satisfies the congruence:*

$$p_b(b^m n) \equiv p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) \pmod{b}$$

This corollary can then be used to prove the following theorem.

Theorem 1.7.2. *The partition counting function $p_b(n)$ satisfies the congruence:*

$$p_b(b^m n) - p_b(b^m(n-1)) \equiv p_b(n) \pmod{b}$$

Proof. Beginning with the statement Corollary 1.7.1,

$$p_b(b^m n) \equiv p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) \pmod{b},$$

and applying Corollary 1.7.1 to the middle term $p_b(b^{m-1}n)$ reveals:

$$\begin{aligned} p_b(b^m n) &\equiv p_b(b^m(n-1)) + p_b(b^{m-1}(n-1)) + p_b(b^{m-2}n) + (b-1)p_b(b^{m-2}(n-1)) \\ &\quad + (b-1)p_b(b^{m-1}(n-1)) \pmod{b} \\ &\equiv p_b(b^m(n-1)) + p_b(b^{m-2}n) + (b-1)p_b(b^{m-2}(n-1)) \pmod{b} \end{aligned}$$

Subsequently applying the Corollary to the middle term $m-3$ more times produces

$$p_b(b^m n) \equiv p_b(b^m(n-1)) + p_b(bn) + (b-1)p_b(b(n-1)) \pmod{b}$$

and finally applying **RIII** to $p_b(bn)$ yields

$$\begin{aligned} p_b(b^m n) &\equiv p_b(b^m(n-1)) + p_b(b(n-1)) + p_b(n) + (b-1)p_b(b(n-1)) \pmod{b} \\ &\equiv p_b(b^m(n-1)) + p_b(n) \pmod{b} \end{aligned}$$

that is,

$$p_b(b^m n) - p_b(b^m(n-1)) \equiv p_b(n) \pmod{b}$$

as stated. ∎

1.8 Sellers' question

In a Spring 2014 talk at the New York Number Theory Seminar, Sellers presented the following identites:

$$\sum_{n\in\mathbb{Z}} p_3(81n+42)q^n = \frac{27(8q^2+17q+2)}{(1-q)^4}B_3(0,q)$$
$$\sum_{n\in\mathbb{Z}} p_3(81n+78)q^n = \frac{27(2q^2+17q+8)}{(1-q)^4}B_3(0,q)$$

and asked, "Why do the polynomial factors in the numerator come in such natural pairs as 'reciprocal polynomials'?" Given that $8q^2+17q+2$ appears, why should its reciprocal polynomial, the polynomial with its coefficients reversed, that is, $2q^2+17q+8$, appear?

Why should one expect that these sorts of identities exist in the first place? Some combinatorial insight is desired, but failing that, Lemma 1.5.4 and Theorem 1.5.5 guarantee that *some* relationship exists, although they fall short of explaining why such reciprocal polynomials appear.

By Lemma 1.5.4, applying **RI** and **RIII** to an expression like $p_b(b^m n+k)$ will produce identies between its generating function and $B_b(0,q)$. Consider the results when Lemma 1.5.4 applied to Sellers' example:

$$p_3(81n+42) = p_3(81n) + 14p_3(27n) + 30p_3(9n) + 9p_3(3n)$$

$$p_3(81n+78) = p_3(81n) + 26p_3(27n) + 108p_3(9n) + 81p_3(3n)$$

Then upon passing to generating functions

$$\begin{aligned}\sum_{n\in\mathbb{Z}} p_3(81n+42)q^n &= B_3(4,q) + 14B_3(3,q) + 30B_3(2,q) + 9B_3(1,q)\\ &= \left(\frac{f_4(3,q)}{(1-q)^4} + 14\frac{f_3(3,q)}{(1-q)^3} + 30\frac{f_2(3,q)}{(1-q)^2} + 9\frac{f_1(3,q)}{1-q}\right) B_3(0,q)\\ &= \frac{27(8q^2+17q+2)}{(1-q)^4} B_3(0,q)\end{aligned}$$

and

$$\begin{aligned}\sum_{n\in\mathbb{Z}} p_3(81n+78)q^n &= B_3(4,q) + 26B_3(3,q) + 108B_3(2,q) + 81B_3(1,q)\\ &= \left(\frac{f_4(3,q)}{(1-q)^4} + 26\frac{f_3(3,q)}{(1-q)^3} + 108\frac{f_2(3,q)}{(1-q)^2} + 81\frac{f_1(3,q)}{1-q}\right) B_3(0,q)\\ &= \frac{27(2q^2+17q+8)}{(1-q)^4} B_3(0,q).\end{aligned}$$

A full understanding of identities like these seems to require a thorough understanding of the polynomials $f_m(b,q)$ as well as the polynomials $g_{m,k,j}(b)$ from Lemma 1.5.4.

1.9 Some computations

Doing this work without computing $p_b(n)$ for large values of n would be a waste, so here are the values for a few choice n and their prime factorization.

$$\begin{aligned}p_2(2^{10}) &= 2320518948\\ &= 2^2 \cdot 3 \cdot 11 \cdot 197 \cdot 89237\end{aligned}$$

See also [Chu69]

$$
\begin{aligned}
p_2(2^{30}) &= 1525223521662612652482573042270879062244863772153300\backslash \\
&\quad 7375091793655998185220930656974338568054217947023338 0 \\
&= 2^2 \cdot 5 \cdot 19 \cdot 31 \cdot 79 \cdot 1217 \cdot 46553987 \cdot 719224073 \\
&\quad \cdot 882439652751991 21 \cdot 1201364132790744647 \\
&\quad \cdot 3793933910711600253501418262383058570580931
\end{aligned}
$$

$$
\begin{aligned}
p_3(3^{27}) &= 350364423551707258416807382080740574025054741900008\backslash \\
&\quad 66860012688287861568320207570189878528238814549 7481\backslash \\
&\quad 0418192030384012393566952227798779 8995852 \\
&\quad 2^2 \cdot 8759110588792681460420184552018514350626368547500 2\backslash \\
&\quad 1671500317207196539208005189254746963205970363743 7\backslash \\
&\quad 0260454800759600309839173805694969497489 63
\end{aligned}
$$

1.10 Proving the case $m = 3$

This section gives an iterative construction of the polynomial $f_3(b, q)$. The methods used here can be used to prove the $m = 4$ case, but the argument becomes significantly longer. It is likely that this method can be used to construct $f_m(b, q)$ for any fixed m, but the length of the argument becomes unwieldy.

Theorem 1.10.1. *The generating function for $p_b(b^3n)$ satisfies the identity:*

$$f_3(b,q)B_b(q) = (1-q)^3 B_b(3,q)$$

where

$$f_3(b,q) = (1-q)^2 + q(1-q)b - \tfrac{1}{2}q(1-q)b^2 + \tfrac{1}{2}q(q+1)b^3$$

Proof. Begin as before, by iterating the recurrence via Lemma 1.5.2:

$$\begin{aligned} p_b(b^3n) = p_b(b^3(n-1)) + p_b(b^2n) + (b-1)p_b(b^2(n-1)) \\ + b\sum_{u=1}^{b-1} p_b(b^2(n-1)+ub) \end{aligned}$$

The sum in the final term can be simplified further via Lemma 1.5.1:

$$\begin{aligned} \sum_{u=1}^{b-1} p_b(b^2(n-1)+ub) &= \sum_{u=1}^{b-1} p_b(b^2(n-1)) + \sum_{k=1}^{u} p_b(b(n-1)+k) \\ &= (b-1)p_b(b^2(n-1)) + \sum_{u=1}^{b-1}\sum_{k=1}^{u} p_b(b(n-1)+k) \end{aligned}$$

Now $1 \leqslant k \leqslant b-1$ so by **RI**

$$\begin{aligned} \sum_{u=1}^{b-1}\sum_{k=1}^{u} p_b(b(n-1)+k) &= \sum_{u=1}^{b-1}\sum_{k=1}^{u} p_b(b(n-1)) \\ &= p_b(b(n-1)) \sum_{u=1}^{b-1}\sum_{k=1}^{u} 1 \\ &= p_b(b(n-1)) \sum_{u=1}^{b-1} u \\ &= \binom{b}{2} p_b(b(n-1)) \end{aligned}$$

Finally, the original expression becomes

$$\begin{aligned} p_b(b^3 n) &= p_b(b^3(n-1)) + p_b(b^2 n) + (b-1)p_b(b^2(n-1)) \\ &\quad + b(b-1)p_b(b^2(n-1)) + b\binom{b}{2}p_b(b(n-1)) \end{aligned}$$

Passing to generating functions by multiplying this identity by q^n and summing over all n yields

$$\begin{aligned} \sum_{n\in\mathbb{Z}} p_b(b^3 n)q^n &= \sum_{n\in\mathbb{Z}} p_b(b^3(n-1)) + \sum_{n\in\mathbb{Z}} p_b(b^2 n) + \sum_{n\in\mathbb{Z}}(b-1)p_b(b^2(n-1)) \\ &\quad + \sum_{n\in\mathbb{Z}} b(b-1)p_b(b^2(n-1)) + \sum_{n\in\mathbb{Z}} b\binom{b}{2}p_b(b(n-1)) \\ B_b(3,q) &= qB_b(3,q) + B_b(2,q) + (b-1)qB_b(2,q) \\ &\quad + b(b-1)qB_b(2,q) + b\binom{b}{2}qB_b(1,q) \end{aligned}$$

After moving terms of $B_b(3,q)$ to the right hand side the above equation becomes

$$\begin{aligned} (1-q)B_b(3,q) &= (1+(b-1)q+b(b-1)q)\,B_b(2,q) + b\binom{b}{2}qB_b(1,q) \\ &= (1+(b^2-1)q)B_b(2,q) + b\binom{b}{2}qB_b(1,q) \end{aligned}$$

Substituting in the results for $B_b(2,q)$ and $B_b(1,q)$ in Lemma 1.4.4, Corollary

1.5.3 and multiplying by $(1-q)^2$ yields

$$\begin{aligned}(1-q)^3 B_b(3,q) &= (1+(b^2-1)q)\left((1+(b-1)q)B_b(0,q)\right) + b\binom{b}{2}q(1-q)B_b(0,q)\\ &= \left((1+(b^2-1)q)\left((1+(b-1)q) + b\binom{b}{2}q(1-q)\right)\right)B_b(0,q)\\ &= \left(1+\tfrac{1}{2}(b-1)\left((b^2+2b+4)q+(b^2-2)q^2\right)\right)B_b(0,q)\end{aligned}$$

Therefore

$$(1-q)^3 B_b(3,q) = \left((1-q)^2 + q(1-q)b - \tfrac{1}{2}q(1-q)b^2 + \tfrac{1}{2}q(q+1)b^3\right)B_b(0,q)$$

as desired. $\square$

$$
\begin{array}{rrclll}
 & f_7(b,q) & = & (1-q)^6 & & \\
+ & & q & (1-q)^5 & b & \\
+ & (2)^{-1} & q & (1-q)^5 & b^2 & \\
+ & (4)^{-1} & q & (1-q)^4 & (3+q)\ b^3 & \\
+ & (24)^{-1} & q & (1-q)^4 & (11+q)\ b^4 & \\
+ & (48)^{-1} & q & (1-q)^4 & (25+11q)\ b^5 & \\
+ & (96)^{-1} & q & (1-q)^3 & (51+38q+7q^2)\ b^6 & \\
+ & (288)^{-1} & q & (1-q)^3 & (125+44q-25q^2)\ b^7 & \\
+ & (48)^{-1} & q & (1-q)^3 & (19+18q-q^2)\ b^8 & \\
+ & (144)^{-1} & q & (1-q)^3 & (51+82q+5q^2)\ b^9 & \\
+ & (48)^{-1} & q & (1-q)^2 & (14+31q+2q^2+q^3)\ b^{10} & \\
+ & (1440)^{-1} & q & (1-q)^2 & (313+671q-241q^2-23q^3)\ b^{11} & \\
+ & (720)^{-1} & q & (1-q)^2 & (124+423q+12q^2-19q^3)\ b^{12} & \\
+ & (480)^{-1} & q & (1-q)^2 & (61+273q+71q^2-5q^3)\ b^{13} & \\
+ & (720)^{-1} & q & (1-q)^2 & (67+414q+231q^2+8q^3)\ b^{14} & \\
+ & (1440)^{-1} & q & (1-q) & (91+736q+546q^2+56q^3+11q^4)\ b^{15} & \\
+ & (360)^{-1} & q & (1-q) & (14+124q+69q^2-26q^3-q^4)\ b^{16} & \\
+ & (1440)^{-1} & q & (1-q) & (35+378q+336q^2-26q^3-3q^4)\ b^{17} & \\
+ & (480)^{-1} & q & (1-q) & (7+102q+132q^2+2q^3-3q^4)\ b^{18} & \\
+ & (1440)^{-1} & q & (1-q) & (11+206q+396q^2+106q^3+q^4)\ b^{19} & \\
+ & (240)^{-1} & q & (1-q) & (1+26q+66q^2+26q^3+q^4)\ b^{20} & \\
+ & (720)^{-1} & q & \cdot & (1+57q+302q^2+302q^3+57q^4+q^5)\ b^{21} &
\end{array}
$$

Table 1.12: The polynomial $f_7(b,q)$ as a polynomial in b

m	i	$g_{m,i}(q)$	
8	8	$(96)^{-1}$	$(42 + 31q - q^2)$
8	9	$(576)^{-1}$	$(243 + 292q + 17q^2)$
8	10	$(864)^{-1}$	$(329 + 471q + 33q^2 + 31q^3)$
8	11	$(2880)^{-1}$	$(911 + 1177q - 647q^2 - q^3)$
8	12	$(4320)^{-1}$	$(1189 + 2598q - 453q^2 - 94q^3)$
8	13	$(1440)^{-1}$	$(329 + 981q + 81q^2 - 11q^3)$
8	14	$(8640)^{-1}$	$(1625 + 6633q + 2139q^2 + 43q^3)$
8	15	$(2880)^{-1}$	$(427 + 1978q + 324q^2 + 110q^3 + 41q^4)$
8	16	$(4320)^{-1}$	$(488 + 2539q - 231q^2 - 647q^3 + 11q^4)$
8	17	$(720)^{-1}$	$(62 + 413q + 129q^2 - 65q^3 + q^4)$
8	18	$(1440)^{-1}$	$(92 + 771q + 455q^2 - 111q^3 - 7q^4)$
8	19	$(2880)^{-1}$	$(131 + 1352q + 1338q^2 + 64q^3 - 5q^4)$
8	20	$(8640)^{-1}$	$(277 + 3524q + 4734q^2 + 812q^3 + 13q^4)$
8	21	$(2880)^{-1}$	$(61 + 961q + 1546q^2 + 266q^3 + 41q^4 + 5q^5)$
8	22	$(20160)^{-1}$	$(269 + 4549q + 6554q^2 - 926q^3 - 391q^4 + 25q^5)$
8	23	$(120)^{-1}$	$(1 + 21q + 41q^2 + q^3 - 4q^4)$
8	24	$(8640)^{-1}$	$(43 + 1075q + 2638q^2 + 686q^3 - 121q^4 - q^5)$
8	25	$(720)^{-1}$	$(2 + 66q + 211q^2 + 91q^3 - 9q^4 - q^5)$
8	26	$(720)^{-1}$	$(1 + 41q + 171q^2 + 131q^3 + 16q^4)$
8	27	$(1440)^{-1}$	$(1 + 57q + 302q^2 + 302q^3 + 57q^4 + q^5)$

Table 1.13: The polynomials $g_{8,i}$ for $8 \leqslant i \leqslant 27$

m	i	$g_{m,i}(q)$	
9	9	$(1152)^{-1}$	$(513 + 548q + 43q^2)$
9	10	$(3456)^{-1}$	$(1463 + 1587q + 285q^2 + 121q^3)$
9	11	$(17280)^{-1}$	$(6521 + 5607q - 3597q^2 + 109q^3)$
9	12	$(8640)^{-1}$	$(3018 + 4631q - 1066q^2 - 103q^3)$
9	13	$(1920)^{-1}$	$(597 + 1263q - 17q^2 - 3q^3)$
9	14	$(8640)^{-1}$	$(2369 + 6981q + 1551q^2 + 79q^3)$
9	15	$(8640)^{-1}$	$(2024 + 6103q - 513q^2 + 901q^3 + 125q^4)$
9	16	$(17280)^{-1}$	$(3379 + 11660q - 4458q^2 - 2020q^3 + 79q^4)$
9	17	$(17280)^{-1}$	$(2827 + 12542q - 732q^2 - 1750q^3 + 73q^4)$
9	18	$(17280)^{-1}$	$(2325 + 13126q + 2988q^2 - 1854q^3 - 25q^4)$
9	19	$(17280)^{-1}$	$(1867 + 12770q + 6684q^2 - 466q^3 + 25q^4)$
9	20	$(8640)^{-1}$	$(743 + 6185q + 4971q^2 + 319q^3 + 22q^4)$
9	21	$(5760)^{-1}$	$(383 + 3533q + 1974q^2 - 470q^3 + 331q^4 + 9q^5)$
9	22	$(60480)^{-1}$	$(3054 + 31679q + 15070q^2 - 18864q^3 - 788q^4 + 89q^5)$
9	23	$(120960)^{-1}$	$(4589 + 57611q + 54038q^2 - 21950q^3 - 3683q^4 + 115q^5)$
9	24	$(120960)^{-1}$	$(3371 + 49719q + 63010q^2 - 11342q^3 - 4077q^4 + 119q^5)$

Table 1.14: The polynomials $g_{9,i}$ for $9 \leqslant i \leqslant 24$

m	i	$g_{m,i}(q)$	
9	25	$(120960)^{-1}$	$(2417 + 42599q + 74486q^2 + 5818q^3 - 4295q^4 - 65q^5)$
9	26	$(120960)^{-1}$	$(1689 + 34745q + 75466q^2 + 20322q^3 - 1187q^4 + 5q^5)$
9	27	$(120960)^{-1}$	$(1157 + 28445q + 79022q^2 + 39334q^3 + 3181q^4 + 61q^5)$
9	28	$(120960)^{-1}$	$(761 + 21712q + 64153q^2 + 27480q^3 + 4667q^4 + 2168q^5 + 19q^6)$
9	29	$(120960)^{-1}$	$(481 + 15384q + 47073q^2 + 8608q^3 - 10677q^4 - 408q^5 + 19q^6)$
9	30	$(120960)^{-1}$	$(301 + 11232q + 40293q^2 + 15328q^3 - 6417q^4 - 288q^5 + 31q^6)$
9	31	$(10080)^{-1}$	$(15 + 680q + 3011q^2 + 1856q^3 - 419q^4 - 104q^5 + q^6)$
9	32	$(120960)^{-1}$	$(103 + 5528q + 28775q^2 + 25632q^3 + 1069q^4 - 632q^5 + 5q^6)$
9	33	$(20160)^{-1}$	$(9 + 632q + 4097q^2 + 4832q^3 + 667q^4 - 152q^5 - 5q^6)$
9	34	$(60480)^{-1}$	$(13 + 1112q + 8861q^2 + 14496q^3 + 5431q^4 + 328q^5 - q^6)$
9	35	$(10080)^{-1}$	$(1 + 120q + 1191q^2 + 2416q^3 + 1191q^4 + 120q^5 + q^6)$

Table 1.15: The polynomials $g_{9,i}$ for $25 \leqslant i \leqslant 35$

m	i	$g_{m,i}(q)$	
10	10	$(3456)^{-1}$	$(1508 + 1479q + 366q^2 + 103q^3)$
10	11	$(17280)^{-1}$	$(6971 + 4662q - 3057q^2 + 64q^3)$
10	12	$(17280)^{-1}$	$(6736 + 8167q - 1772q^2 - 171q^3)$
10	13	$(25920)^{-1}$	$(9437 + 15513q - 207q^2 + 97q^3)$
10	14	$(17280)^{-1}$	$(5828 + 13307q + 2582q^2 + 243q^3)$
10	15	$(103680)^{-1}$	$(31513 + 66086q - 10716q^2 + 15722q^3 + 1075q^4)$
10	16	$(34560)^{-1}$	$(9287 + 22450q - 12300q^2 - 2258q^3 + 101q^4)$
10	17	$(17280)^{-1}$	$(4103 + 12979q - 3099q^2 - 1091q^3 + 68q^4)$
10	18	$(34560)^{-1}$	$(7159 + 28984q + 210q^2 - 3208q^3 - 25q^4)$
10	19	$(51840)^{-1}$	$(9217 + 45515q + 12507q^2 - 1507q^3 + 148q^4)$
10	20	$(17280)^{-1}$	$(2619 + 15673q + 7953q^2 + 315q^3 + 80q^4)$
10	21	$(32400)^{-1}$	$(4111 + 25660q + 1585q^2 - 1870q^3 + 2915q^4 - q^5)$
10	22	$(120960)^{-1}$	$(12709 + 90224q - 1458q^2 - 43888q^3 + 2869q^4 + 24q^5)$
10	23	$(362880)^{-1}$	$(31337 + 267179q + 92282q^2 - 116054q^3 - 2579q^4 - 5q^5)$
10	24	$(120960)^{-1}$	$(8470 + 84599q + 53646q^2 - 29488q^3 - 1388q^4 + 81q^5)$

Table 1.16: The polynomials $g_{10,i}$ for $10 \leqslant i \leqslant 24$

m	i	$g_{m,i}(q)$	
10	25	$(241920)^{-1}$	$(13535+158685q+157158q^2-30134q^3-6837q^4-87q^5)$
10	26	$(120960)^{-1}$	$(5334+71929q+93434q^2+1956q^3-1360q^4+67q^5)$
10	27	$(362880)^{-1}$	$(12460+195325q+332566q^2+75308q^3+3926q^4+335q^5)$
10	28	$(241920)^{-1}$	$(6355+108566q+147413q^2-43284q^3+14869q^4+8062q^5-61q^6)$
10	29	$(362880)^{-1}$	$(7178+138609q+210603q^2-121954q^3-55980q^4+3057q^5-73q^6)$
10	30	$(120960)^{-1}$	$(1781+39601q+78646q^2-15078q^3-15247q^4+1013q^5+4q^6)$
10	31	$(725760)^{-1}$	$(7807+200214q+493533q^2+11980q^3-106623q^4-2178q^5+67q^6)$
10	32	$(80640)^{-1}$	$(623+18268q+54323q^2+15184q^3-7435q^4-332q^5+9q^6)$
10	33	$(362880)^{-1}$	$(1970+66843q+237309q^2+114242q^3-23184q^4-4029q^5-31q^6)$
10	34	$(60480)^{-1}$	$(226+8760q+36747q^2+27718q^3+2274q^4-126q^5+q^6)$

Table 1.17: The polynomials $g_{10,i}$ for $25 \leqslant i \leqslant 34$

m	i	$g_{m,i}(q)$	
10	35	$(725760)^{-1}$	$(1831 + 82308q + 403011q^2 + 395512q^3 + 80853q^4 + 4068q^5 + 97q^6)$
10	36	$(241920)^{-1}$	$(399 + 20449q + 106983q^2 + 95433q^3 + 10453q^4 + 6939q^5 + 1269q^6 - 5q^7)$
10	37	$(725760)^{-1}$	$(763 + 43703q + 236763q^2 + 167071q^3 - 72007q^4 - 15363q^5 + 1969q^6 - 19q^7)$
10	38	$(120960)^{-1}$	$(79 + 5281q + 33561q^2 + 33127q^3 - 6983q^4 - 4617q^5 + 31q^6 + q^7)$
10	39	$(362880)^{-1}$	$(143 + 11137q + 80253q^2 + 99011q^3 + q^4 - 9297q^5 + 179q^6 + 13q^7)$
10	40	$(241920)^{-1}$	$(55 + 5137q + 43875q^2 + 69397q^3 + 10597q^4 - 7533q^5 - 575q^6 + 7q^7)$
10	41	$(1209600)^{-1}$	$(151 + 16753q + 163431q^2 + 316465q^3 + 117805q^4 - 8181q^5 - 1643q^6 + 19q^7)$
10	42	$(80640)^{-1}$	$(5 + 723q + 8577q^2 + 20519q^3 + 10719q^4 + 9q^5 - 229q^6 - 3q^7)$
10	43	$(241920)^{-1}$	$(7 + 1217q + 17163q^2 + 51757q^3 + 41957q^4 + 8595q^5 + 265q^6 - q^7)$
10	44	$(80640)^{-1}$	$(1 + 247q + 4293q^2 + 15619q^3 + 15619q^4 + 4293q^5 + 247q^6 + q^7)$

Table 1.18: The polynomials $g_{10,i}$ for $35 \leqslant i \leqslant 44$

Chapter 2

Stern-like sequences

2.1 History

Limiting the multiplicities of powers appearing in b-ary partitions has also been studied quite extensively. Moreover, this problem can inform the discussion of the general problem. Recall that $p_{b,d}(n)$ is the number of partitions of n into powers of b repeating each power at most d times. The function $p_{2,2}(n)$ is sometimes referred to as the hyperbinary counting function, as well as the Stern diatomic sequence. In 2000, Calkin and Wilf [CW00] studied a tree of rationals, the Calkin-Wilf tree, whose entries are ratios of consecutive terms of the Stern sequence. This tree encodes information related to continued fractions and has many beautiful properties. In 1858, Stern [Ste58] first studied the recurrence for the Stern sequence. In 1860, Brocot [Bro62] studied a tree of rationals which has come to be known as the the

Stern-Brocot tree. Later, in 1877, Lucas [Luc78] studied the Stern-Brocot array, recording various properties which also relate to the Stern sequence. Subsequently in 1929, Lehmer [Leh29] studied the Stern sequence and observed many properites which connect it to the Calkin-Wilf tree, while also proving the connection between elements in the sequence and continued fractions. The work of Reznick in 1990 also contributed many relations in this regard. This included several theorems about $p_b(n)$ shown to be limits of congruences of $p_{b,d}(n)$, In 2000, Protasov [Pro00] studied the Stern sequence and sequences related to similar partition problems. He developed a method to explicitly calculate the growth rate of $p_{b,d}(n)$.

Definition 2.1.1 (Stern-like sequences)**.** Let $b \geqslant 2$ be an integer. Define $S_b(n)$ recursively with

$$S_b(n) = 0 \text{ for } n < 0$$

$$S_b(n) = 1 \text{ for } 0 \leqslant n < b$$

$$S_b(bn + r) = S_b(n) \text{ for } 0 < r < b \tag{\textbf{SI}}$$

$$S_b(bn) = S_b(n) + S_b(n - 1). \tag{\textbf{SII}}$$

The following theorem can be established by a similar argument to Theorem 1.3.1. The case $b = 2$ can be found in [Rez90, Theorem 5.2]. An alternate viewpoint can also be found in [Pro00].

Theorem 2.1.2. *For every integer n, the sequence $S_b(n)$ counts partitions of n into powers of b using each power at most b times, that is $S_b(n) = p_{b,b}(n)$.*

Proof. Let $n = ub + v \geqslant b$ where $0 \leqslant v < b$.

If $v > 0$ then a partition of n counted by $p_{b,b}(n)$ has exactly v ones. For such a partition consider the following process. Removing ones from this partition results in a partition counted by $p_{b,b}(ub)$ whose parts are all divisible by b. Cancelling a factor of b from each part yields a partition of u counted by $p_{b,b}(u)$. This procedure defines a map f from partitions of n counted by $p_{b,b}(n)$ to partitions of u counted by $p_{b,b}(u)$. For any partition of n counted by $p_{b,b}(n)$, applying f yields a partition of u, and multiplying each term in this partition by b and subsequently adding v ones results in the original partition of n. Therefore f is a bijection which implies $p_{b,b}(ub+v) = p_{b,b}(u)$. Thus

$$p_{b,b}(ub+v) = p_{b,b}(u) \text{ for } 0 < v < b.$$

If $v = 0$ then the set of partitions counted by $p_{b,b}(ub)$ can be partitioned into partitions with ones and those without. For those with ones, a bijection similar to f exists between partitions of ub counted by $p_{b,b}(ub)$ and partitions of $(u-1)$ counted by $p_{b,b}(u-1)$. For a partition of ub counted by $p_{b,b}(ub)$ without ones each term is divisible by b. Cancelling a factor of b from each term yields a partition of u counted by $p_{b,b}(u)$. Again, multiplying each term in this partition by b results in the original partition showing that $p_{b,b}(u)$ is the number of partitions of ub counted by $p_{b,b}(ub)$ with no ones. Therefore

$$p_{b,b}(bu) = p_{b,b}(u) + p_{b,b}(u-1).$$

Then $p_{b,b}(n)$ satisfies the same recurrence as $S_b(n)$ and since they are identical on the values of $n < b$, they are the same sequence by induction. □

2.2 The iterated recurrence

The form of this recurrence allows much faster calculation of $p_{b,b}$ for large values of n than the correspnding one for $p_b(n)$. This can be seen by taking the b-ary expansion of n and iteratively applying the recurrence to runs of zeroes and runs of nonzero digits. The following theorem address the nonzero case.

Theorem 2.2.1. *Let $n > 0$ such that $n = n'b^m + k$ where $k < b^m$ and $b \nmid k$. Then*

$$S_b(n) = S_b(n'b^m + k) = S_b(n'). \qquad \text{(IH}(m))$$

Proof. Proceed by induction on m. The base case $m = 0$ is true since $S_b(n) = S_b(n)$ for all n. Assume IH$(m-1)$ is true. Let $k' = \left\lfloor \frac{k}{b} \right\rfloor$. Note that $k' < b^{m-1}$ and $b \nmid k'$. Then by **SI** and IH$(m-1)$

$$S_b(n) = S_b(n'b^m + k) = S_b(n'b^{m-1} + k') = S_b(n')$$

as desired. □

Corollary 2.2.2. *Let $n > 0$. If the b-ary expansion of n contains no zeroes then $S_b(n) = 1$.*

Proof. If n contains no zeroes in its b-ary expansion then $n = 0 \cdot b^m + k$ for some m and $n = k$ where $b \nmid k$. Thus $S_b(n) = S_b(0) = 1$. □

This theorem addresses the case when the b-ary expansion of n terminates with a run of zeroes, that is, $n = b^m n'$.

Theorem 2.2.3. *Let n and m be integers such that $m > 0$. Then*

$$S_b(b^m n) = S_b(n) + mS_b(n-1) \qquad (\mathrm{IH}(m))$$

hence

$$(1 + mq) \sum_{n \in \mathbb{Z}} S_b(n) q^n = \sum_{n \in \mathbb{Z}} S_b(b^m n) q^n$$

Proof. Note that $b^m n - 1 = (n-1)b^m + b^m - 1$. Assume IH$(h)$ for $1 \leqslant h < m$.

Then

$$\begin{aligned} S_b(b^m n) &= S_b(b^{m-1}n) + S_b(b^{m-1}n - 1) \\ &= S_b(b^{m-1}n) + S_b(b^{m-1}(n-1) + b^{m-1} - 1) \\ &= S_b(b^{m-1}n) + S_b(n-1) \text{ by Theorem 2.2.1} \\ &= S_b(n) + (m-1)S_b(n-1) + S_b(n-1) \text{ by } \text{IH}(m-1) \\ S_b(b^m n) &= S_b(n) + mS_b(n-1) \end{aligned}$$

Therefore $S_b(b^m n) = S_b(n) + mS_b(n-1)$ as stated. □

2.3 Self-similarity in Stern-like sequences

The Stern-like sequences $S_b(n)$ share properties with the original Stern sequence, such as having a simple recurrence and corresponding to the same type of combinatorial problem, but the sequence of ratios $S_b(n+1)/S_b(n)$ is not a unique enumeration of the rationals for $b > 2$. Unfortunately this is not the only property lost, but it is unclear whether there is another generalization of the Stern sequence which retains more or all of these properties. Nevertheless new patterns and features arise in this context which are of interest.

These patterns are most easily seen by considering the b-ary expansion of integers. In any base, denote the integer whose b-ary expansion consists of

$n+1$ repetitions of one by $\text{ones}_b(n) = \frac{b^{n+1}-1}{b-1}$ and the integer corresponding to n repetitions of $b-1$ by $\text{bmos}_b(n) = b^n - 1$. Similarly in any base $b > 2$, the b-ary expansion of $\text{osat}(n) = \frac{b^n-1}{b-1} + 1$ is $n-1$ repetitions of one followed by a two. Note that in the b-ary expansion the identity

$$\text{osat}_b(n+1) = b\,\text{osat}_b(n) - b + 2 \qquad (\star)$$

indicates that n ones followed by a two may be obtained from $n-1$ ones followed by a two by shifting to the left and then adding $-b+2$. Algebraically this is expressed by

$$\begin{aligned}
b\,\text{osat}_b(n) &= b\left(\frac{b^n-1}{b-1} + 1\right) = b\left(1 + \sum_{i=0}^{n-1} b^i\right) \\
&= b + \sum_{i=1}^{n} b^i = b - 2 + \left(1 + \sum_{i=0}^{n} b^i\right) \\
&= b - 2 + \left(\frac{b^{n+1}-1}{b-1} + 1\right) \\
b\,\text{osat}_b(n) &= b - 2 + \text{osat}_b(n+1).
\end{aligned}$$

Let $\mathbb{Z}^*$ be the free monoid generated by the integers under concatenation.

Definition 2.3.1. (Layer and related values) Finally define the nth layer of the base b Stern-like sequence as the following product in $\mathbb{Z}^*$:

$$\text{Layer}_b(n) = \prod_{i=\text{osat}_b(n)}^{\text{ones}_b(n)} S_b(i).$$

For an example, consider $\text{Layer}_3(1)$ and $\text{Layer}_3(2)$:

$$\text{Layer}_3(1) = 1, 2, 1$$

$$\text{Layer}_3(2) = 1, 2, 1, 1, 3, 2, 2, 3, 1$$

where the comma represents concatenation in $\mathbb{Z}^*$. Note that $1, 2, 1$ and $1, 3, 2, 2, 3, 1$ are palindromes and $1, 2, 1$ appears as an initial segment of both. In the argument to prove this pattern persists in general, the following lemma will be used:

Lemma 2.3.2. *Let $b > 2$, then*

$$S_b(bn) = S_b(bn - 1) + S_b(bn + 1).$$

Proof. By **SII** $S_b(bn) = S_b(n - 1) + S_b(n)$ and by **SI**

$$S_b(bn + 1) = S_b(n)$$

$$S_b(bn - 1) = S_b(b(n - 1) + b - 1) = S_b(n - 1)$$

Hence

$$S_b(bn) = S_b(bn + 1) + S_b(bn - 1)$$

as stated. □

Lemma 2.3.3. *For $b > 2$ and some $X_m \in \mathbb{Z}^*$*

$$\text{Layer}_b(m) = \text{Layer}_b(m - 1)X_m.$$

Proof. The following equivalent result may be seen by induction. For all i such that $i \leqslant \mathrm{bmos}_b(m+1)$

$$S_b(\mathrm{osat}_b(m+1)+i) = S_b(\mathrm{osat}_b(m)+i). \qquad (\mathrm{IH}(m))$$

In the base case $m = 0$ and $S_b(1) = S_b(2) = \cdots = S_b(b-1) = 1$ is true by definition. Assume that $m > 0$, that $\mathrm{IH}(m)$ is true and $i \leqslant \mathrm{bmos}_b(m+1)$ and of the form $i = bj + b - 2$ for some integer j. To apply the induction hypothesis it must be shown that $j \leqslant \mathrm{bmos}_b(m)$. To do this consider:

$$\begin{aligned} i = bj + b - 2 &\leqslant \mathrm{bmos}_b(m+1) \\ bj + b - 1 &\leqslant \mathrm{bmos}_b(m+1) + 1 = b^{m+1} \\ j + \frac{b-1}{b} &\leqslant b^m \\ j - \frac{1}{b} &\leqslant \mathrm{bmos}_b(m) \end{aligned}$$

and since j is an integer this implies $j < \mathrm{bmos}_b(m)$. Now note $(\star)$ may be applied, followed by **SII** resulting in:

$$\begin{aligned} S_b(\mathrm{osat}_b(m+2)+i) &= S_b(\mathrm{osat}_b(m+2)+bj+b-2) \\ &= S_b(b(\mathrm{osat}_b(m+1)+j)) \\ &= S_b(\mathrm{osat}_b(m{+}1){+}j) + S_b(\mathrm{osat}_b(m{+}1){+}j{-}1) \end{aligned}$$

At this point the induction hypothesis may be applied, followed by **SII** and then $(\star)$:

$$\begin{aligned}
S_b(\mathrm{osat}_b(m+2)+i) &= S_b(\mathrm{osat}_b(m)+j) + S_b(\mathrm{osat}_b(m)+j-1)\\
&= S_b(b(\mathrm{osat}_b(m)+j))\\
&= S_b(b\,\mathrm{osat}_b(m)+bj)\\
&= S_b(\mathrm{osat}_b(m+1)+bj+b-2)\\
S_b(\mathrm{osat}_b(m+2)+i) &= S_b(\mathrm{osat}_b(m+1)+i).
\end{aligned}$$

This establishes $\mathrm{IH}(m+1)$ is true for all $i \equiv b-2 \pmod b$. In the general case, if $i \leqslant \mathrm{bmos}_b(m+1)$ then by division $i-b+2 = ub+v$ for some integers u and v such that $0 \leqslant v < b$. By Corollary 2.2.2, $S_b(\mathrm{osat}_b(m+1)) = 1$ for all m. Then by **SI** for $0 \leqslant j < b-2$ the value of $S_b(\mathrm{osat}_b(m+1)+i)$ is the same. Since $\mathrm{osat}_b(m) \equiv 2 \pmod b$ for all m, when $j = b-2$ then $b \mid \mathrm{osat}_b(m)+j$, therefore by Lemma 2.3.2

$$\begin{aligned}
S_b(\mathrm{osat}_b(m)+j) &= S_b(\mathrm{osat}_b(m+1)+j)\\
&= S_b(\mathrm{osat}_b(m+1)+j-1) + S_b(\mathrm{osat}_b(m+1)+j+1)
\end{aligned}$$

which determines the next $b-1$ values by **SI**. Repeating this process u times yields

$$S_b(\mathrm{osat}_b(m)+i) = S_b(\mathrm{osat}_b(m+1)+i)$$

which is $\mathrm{IH}(m)$. Therefore the first $\mathrm{bmos}_b(m-1)$ values of $\mathrm{Layer}_b(m)$ are the same as those from $\text{Layer}_b(m-1)$, as stated. □

Another aspect of this self-similarity is that for every m, an initial segment of $\text{Layer}_b(m)$ is a palindrome.

Lemma 2.3.4. *For all $m \geqslant 0$ let $0 \leqslant j \leqslant \text{osat}_b(m)$, then*

$$S_b(\text{bmos}_b(m) + j) = S_b(\text{ones}_b(m) - j).$$

Proof. Proceed by induction on m. For $m = 0$ the following values establish the base case: $\text{osat}_b(0) = 0, \text{ones}_b(0) = 1, \text{bmos}_b(0) = 0$ and $S_b(0) = S_b(1)$. Now consider the case when $m > 0$. Let h and h' be such that

$$\text{bmos}_b(m) \leqslant h \leqslant h' \leqslant \text{ones}_b(m),$$

and $h + h' = \text{bmos}_b(m) + \text{ones}_b(m)$. Therefore if $j = h - \text{bmos}_b(m)$ then

$$h + h' = \text{bmos}_b(m) + j + \text{ones}_b(m) - j.$$

hence this condition is equivalent to the premise of the assertion.

Assume h is not a multiple of b and define u, u', v and v' by $h = ub + v$, $h' = u'b + v'$ where $\{v, v'\} \subset \{0, 1, \ldots, b-1\}$. Therefore $v + v' = b$ since $h + h'$ is a multiple of b. Note that this implies $u + u' = \text{ones}_b(m-1) + \text{bmos}_b(m-1)$, hence by the induction hypothesis applies and states $S_b(u) = S_b(u')$. By **SI**, $S_b(h) = S_b(u)$ and $S_b(h') = S_b(u')$, so $S_b(h) = S_b(h')$.

On the other hand if h and h' are multiples of b then there is an integer t such that $h = \text{bmos}_b(m) + bt + 1$ and $h' = \text{ones}_b(m) - bt - 1$. Thus

$$S_b(h) = S_b(\text{bmos}_b(m-1) + t + 1) + S_b(\text{bmos}_b(m-1) + t)$$
$$S_b(h') = S_b(\text{ones}_b(m-1) - t) + S_b(\text{ones}_b(m-1) - t - 1)$$

and by the induction hypothesis

$$S_b(\text{ones}_b(n-1) - t) = S_b(\text{bmos}_b(n-1) + t)$$
$$S_b(\text{ones}_b(n-1) - t - 1) = S_b(\text{bmos}_b(n-1) + t + 1)$$

hence $S_b(h) = S_b(h')$. Consequently, for all $n \geqslant 0$ if $0 \leqslant j \leqslant \text{osat}_b(n)$ then $S_b(\text{bmos}_b(n) + j) = S_b(\text{ones}_b(n) - j)$. $\square$

Chapter 3

On a problem of Canfield and Wilf

3.1 History

The focus of the discussion now drops the restriction that the parts of a partition of n are powers of a given base. Instead, the function $p_{A,M}(n)$, the number of partitions of n with parts in A and multiplicities in M, is considered, that is,

$$p_{A,M}(n) = \left| \left\{ n = \sum_{i \in I} m_i a_i \;\middle|\; \{a_i\}_{i \in I} \subset A, \{m_i\}_{i \in I} \subset M, I \subset \mathbb{Z} \right\} \right|.$$

In 2012, Canfield and Wilf [CW12] proved that $p_{A,M}(n)$ grows faster than any polynomial when A is any infinite set of parts and $M = \mathbb{Z}$. They also asked whether there exists infinite sets A and M such that $p_{A,M}(n)$ grows like a polynomial. In a partial answer to this question That same year Ljujić and Nathanson [LN12] constructed such an M where $p_{A,M}(n)$ is 1 infinitely often. Soon after, Alon [Alo13] gave a construction of infinite sets A and M such that $p_{A,M}(n) = 1$ for all n. This construction is based on partitioning the integers by their binary expansion. The results in this chapter generalize this construction, in a way, to expansions of any base.

3.2 A construction

Given a set A, let $\mathrm{FS}(A)$ be all finite sums of elements of A, that is

$$\mathrm{FS}(A) = \left\{ \sum_{i \in I} a_i \;\middle|\; \{a_i\}_{i \in I} \subset A, I \subset \mathbb{Z} \right\}.$$

Let $\mathbb{N} = S_0 \oplus S_1$ where $S_i \subset \mathbb{N}$ for $i \in \{0,1\}$. That is, every $n \in \mathbb{N}$ has a unique representation as $n = s_0 + s_1$ where $s_0 \in S_0$ and $s_1 \in S_1$. Therefore there is a bijective function $S : \mathbb{N} \to S_0 \times S_1$ such that $S(n) = (s_0, s_1)$.

The following lemma gives an example construction of such a function.

Lemma 3.2.1 ([Alo13]). *Given $B \subset \mathbb{N}$ there exists B_0 and B_1 such that $\mathbb{N} = B_0 \oplus B_1$, that is, every nonnegative integer n has a unique representation $n = b_0 + b_1$ such that $b_0 \in B_0$ and $b_1 \in B_1$. Moreover there exists a bijective function $S_B : \mathbb{N} \to B_0 \times B_1$ such that $S_B(n) = (b_0, b_1)$.*

Proof. Let $C_0 = \{2^i\}_{i \in B}$ and $C_1 = \{2^i\}_{i \in \mathbb{N} \setminus B}$ and $B_j = \mathrm{FS}(C_j)$ for $j \in \{0,1\}$. Let $n \in \mathbb{N}$ and consider its binary expansion

$$n = \sum_{i \in \mathbb{N}} \epsilon_i 2^i$$

where $\epsilon_i \in \{0,1\}$ for all i and $\epsilon_i = 0$ for all but finitely many i. Note that

$$\sum_{i\in\mathbb{N}} \epsilon_i 2^i = \sum_{i\in C_0} \epsilon_i 2^i + \sum_{i\in C_1} \epsilon_i 2^i.$$

therefore every integer has a representation $b_0 + b_1$ where $b_0 \in B_0$ and $b_1 \in B_1$. This representation is unique because this rearrangement defines a one-to-one map, say S_B, where

$$S_B((\epsilon_i)_{i\in\mathbb{N}}) = ((\epsilon_i)_{i\in C_0}, (\epsilon_i)_{i\in C_1}).$$

To show that S_B is one-to-one assume

$$((\epsilon_i)_{i\in C_0}, (\epsilon_i)_{i\in C_1}) = ((\epsilon_i')_{i\in C_0}, (\epsilon_i')_{i\in C_1}).$$

Then this implies $\epsilon_i = \epsilon_i'$ for $i \in C_0$ and $\epsilon_i = \epsilon_i'$ for $i \in C_1$. Futher $\mathbb{N} = C_0 \cup C_1$ and $C_0 \cap C_1 = \varnothing$ therefore $\epsilon_i = \epsilon_i'$ for $i \in \mathbb{N}$, so $(\epsilon_i)_{i\in\mathbb{N}} = (\epsilon_i')_{i\in\mathbb{N}}$ hence $S_B((\epsilon_i)_{i\in\mathbb{N}}) = S_B((\epsilon_i')_{i\in\mathbb{N}})$.

Therefore every nonnegative integer n has a unique representation as $n = b_0 + b_1$ and S_B is a bijective function $S_B : \mathbb{N} \to B_0 \times B_1$ such that $S_B(n) = (b_0, b_1)$. □

With this gadget it is now possible to give a construction that takes a set B and produces A and M so that $p_{A,M}(n) = 1$.

Theorem 3.2.2. *Given a set $B \subset \mathbb{N}$, a function S_B as above and a nonnegative integer $b > 1$ there exists sets A and M such that $A \subset \{b^i\}_{i\in\mathbb{N}}$ and $p_{A,M}(n) = 1$ for all $n \in \mathbb{N}$.*

Moreover, if B and $\mathbb{N}\backslash B$ are infinite, then so are A and M.

Corollary 3.2.3. *Given a set $B \subset \mathbb{N}$ and a nonnegative integer $b > 1$ such that B is infinite and $\mathbb{N}\backslash B$ is infinite there exist infinite sets $A \subset \{b^i\}_{i\in\mathbb{N}}$ and M such that $p_{A,M}(n) = 1$ for all $n \in \mathbb{N}$.*

Proof of Theorem 3.2.2. As in the lemma let $B_0 = \mathrm{FS}(\{2^i\}_{i\in B})$ and further define $A = \{b^i\}_{i\in B_0}$,

$$M = \left\{ \sum_{i=0}^{l-1} [\![i \notin B_0]\!] \epsilon_i b^i \;\middle|\; 0 \leqslant \epsilon_i < b, l \in \mathbb{N} \right\}$$

and let S_B be the function given by the lemma when applied to B. By the definition of S_B every nonnegative integer n has a unique representation as $n = b_0 + b_1$ where $b_0 \in A$. Consider the b-adic expansion

$$n = \sum_{i\in\mathbb{N}} \epsilon_i b^i$$

where $0 \leqslant \epsilon_i < b$ for all i and $\epsilon_i = 0$ for all but finitely many i. Given i, define $(b_{0,i}, b_{1,i}) = S_B(i)$. Then since S_B is one-to-one, n has a unique representation as

$$n = \sum_{i\in\mathbb{N}} \epsilon_i b^{b_{0,i}+b_{1,i}}.$$

Now let $A_n = \{b^{b_{0,i}}\}_{i\in\mathbb{N}} = \left\{a_j \,\middle|\, 0 \leqslant j < |A_n|\right\}$, that is, the a_j are the distinct powers of b of the form $b_{0,i}$ appearing in terms of the b-adic expansion of n. Note that A_n is uniquely determined by the choice of B, b and n. Now define

$$m_j = \sum_{i\in\mathbb{N}} [\![b_{0,i} = j]\!]\epsilon_i b^{i-j}.$$

Note $m_j \in M$ since for each i the difference $i - j = i - b_{0,i} = b_{1,i}$ is not in B_0, so these conditions are equivalent, that is, $[\![i - j \notin B_0]\!] = [\![b_{0,i}{=}j]\!]$, hence

$$m_j = \sum_{i\in\mathbb{N}} [\![i - j \notin B_0]\!]\epsilon_i b^{i-j}.$$

Moreover this value is similarly uniquely defined by B, b, n and j. Therefore, $a_j \in A, m_j \in M$ and

$$n = \sum_{j\in A_n} m_j a_j.$$

Thus every nonnegative n has such a representation, that is, $p_{A,M}(n) \geqslant 1$. Further since $A_n \subset A$ is uniquely determined by B, b and n and $m_j \in M$ is uniquely defined by B, b, n and j this representation is unique, therefore $p_{A,M}(n) = 1$. □

The steps of this construction can be seen at work in the following example. Choose the nonnegative even integers as $B = \{2i\}_{i\in\mathbb{N}}$, and let $b = 10$ and $n = 1234$. Then

$$B_0 = \mathrm{FS}(\{2^i\}_{i\in B}) = \mathrm{FS}(\{2^{2i}\}_{i\in\mathbb{N}}) = \{0, 1, 4, 5, 64, 65, 68, 69, \ldots\}$$

and $B_1 = \mathrm{FS}(\{2^i\}_{i \notin B}) = \{0, 2, 8, 10, \ldots\}$. Thus $A = \{1, 10, 10^4, 10^5, \ldots\}$ and $M = \{1 \cdot 10^2, 2 \cdot 10^2, \ldots, 9 \cdot 10^2, 1 \cdot 10^3, 2 \cdot 10^3, \ldots 9 \cdot 10^3, \ldots\}$. Using S_B note that n is

$$1234 = 1 \cdot 10^{1+2} + 2 \cdot 10^{0+2} + 3 \cdot 10^{1+0} + 4 \cdot 10^{0+0}.$$

Hence $A_n = \{a_0, a_1\} = \{10^0, 10^1\}$, and the multiples are $m_0 = 2 \cdot 10^2 + 4 \cdot 10^0 = 204$ and $m_1 = 1 \cdot 10^3 + 3 \cdot 10 = 103$. Therefore this choice of B, b in this construction produces the following partition of n:

$$n = m_0 a_0 + m_1 a_1 = 204 \cdot 1 + 103 \cdot 10 = 1234$$

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