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Carlos Arnoldo Morales
Université de Bourgogne

Maria José Pacífico
Universidade Federal do Rio de Janeiro

Enrique Ramiro Pujals
CUNY Graduate Center

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On C^1 robust singular transitive sets for three-dimensional flows

Carlos Arnoldo MORALES [a], **Maria José PACÍFICO** [b], **Enrique Ramiro PUJALS** [b]

[a] Laboratoire de Topologie, université de Bourgogne, B.P. 400, 21011 Dijon cedex, France
E-mail: cmorales@u-bourgogne.fr

[b] Dep. de Métodos Matemáticos IM-UFRJ, Caixa Postal 68530, Cep 21945-970, Rio de Janeiro, RJ, Brasil
E-mail: pacifico@impa.br, enrique@impa.br

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Abstract. The main goal of this paper is to study robust invariant transitive sets containing singularities for C^1 flows on three-dimensional compact boundaryless manifolds: they are partially hyperbolic with volume expanding central direction. Moreover, they are either attractors or repellers. Robust here means that this property cannot be destroyed by small C^1-perturbations of the flow.

Sur les ensembles transitifs singuliers C^1 robustes pour des flots en dimension trois

Résumé. *Le but de ce travail est d'étudier des ensembles invariants robustes ayant des singularités pour des flots C^1 sur des variétés tridimensionelles : ce sont des ensembles hyperboliques singuliers. « Robuste » veut dire ici que cette propriété ne peut être détruite par des perturbations C^1 du flot.*

Version française abrégée

La motivation de ce travail est l'équation de Lorenz (*voir* [6]). Des expérimentations numériques réalisées par Lorenz suggèrent l'existence d'un attracteur étrange (sensitif) robuste.

Avant d'annoncer nos résultats, quelques définitions et des notations sont nécessaires. On notera M une variété fermée de dimension trois, $\mathcal{X}^r(M)$ l'ensemble des champs de vecteurs de classe C^r, et X_t le flot induit par X. Étant donné un ensemble compact invariant Λ de X, on notera par $\mathcal{P}_X(\Lambda)$ l'ensemble des orbites périodiques de X dans Λ et $\mathrm{Sing}_X(\Lambda)$ l'ensemble des singularités de X dans Λ.

Un ensemble compact et invariant Λ de $X \in \mathcal{X}^1(M)$ est *transitif* s'il y a une orbite périodique de X dans Λ tel que Λ est le ensemble ω-limite de cette orbite. Un ensemble singulier transitif

Note présentée par Étienne GHYS.

de $X \in \mathcal{X}^1(M)$ est un ensemble compact et transitif invariant $\Lambda \subset M$ tel que $\bar{\mathcal{P}}_X(\Lambda) = \Lambda$, $\mathrm{Sing}_X(\Lambda) \neq \emptyset$ possédant un voisinage $U \supset \Lambda$ (bloc isolant) tel que $\Lambda = \bigcap_{t \in \mathbb{R}} X_t(U)$ et Λ est l'adhérence de l'ensemble récurrent.

Un ensemble compact et invariant B de $X \in \mathcal{X}^1(M)$ avec $\mathrm{Sing}(B) \neq \emptyset$ est *hyperbolique singulier* s'il y a une décomposition dominée invariante continue du fibré tangent $\mathrm{T}_B M = E^{\mathrm{s}} \oplus E^{\mathrm{cu}}$. De plus, E^{s} est contracté, E^{cu} est de volume expansif et les singularités de X dans Λ sont hyperboliques.

Un ensemble transitif A de $X \in \mathcal{X}^1(M)$ est C^1 robuste s'il existe un bloc isolant U de A tel que $\Lambda(Y) = \bigcap_{t \in \mathbb{R}} Y_t(U)$ est un ensemble transitif de tout Y proche de X.

On dit que $\sigma \in \mathrm{Sing}_X(\Lambda)$ est de *type Lorenz* si les valeurs propres de $DX(\sigma)$ sont réelles et satisfont $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$.

Voici nos résultats :

THÉORÈME A. – *Si Λ est un ensemble singulier transitif C^1 robuste de $X \in \mathcal{X}^1(M)$, alors, pour X ou $-X$, Λ est un attracteur qui est un ensemble hyperbolique singulier dont toutes les singularités sont de type Lorenz.*

Les démonstrations de ces théorèmes paraîtront ultérieurement. Certaines des idées principales des démonstrations sont présentées ici.

1. Introduction

The motivation for this work is the so-called Lorenz attractor (*see* [6]), given by the solutions of the polynomial vector field on $\mathbb{R}^3$:

$$\begin{cases} \dot{x} = -\alpha x - \alpha y \\ \dot{y} = -\beta x - y - xz \\ \dot{z} = -\gamma z + xy \end{cases}$$

where α, β, γ are real parameters. Numerical experiments performed by Lorenz for $\alpha = 10$, $\beta = 28$, and $\gamma = 8/3$, suggested the existence of a robust strange (sensitive) attractor, toward which tend all positive trajectories starting in a neighborhood of the origin $(0, 0, 0)$, which is a singularity for the system and part of the attractor. Amazingly, three and a half decades after this remarkable work, we still do not know whether the solutions of the above equation really have such a strange attractor for any values α, β, γ. However, this was proved to be true for the flow in [3] nowadays called the *geometric Lorenz attractor*: it exhibits an attracting transitive set with a singularity in a robust way. Recall that an invariant set by the flow is *transitive* if it is the ω-limit set of one of its orbits.

In recent papers we have been exhibiting new classes of robust transitive sets with singularities (*see* [11]). Also, in [10] a mechanism is presented through which the suspension of certain expanding attractors of the two-disk can be deformed into a robust strange attractor with singularities that is neither equivalent to a geometric Lorenz attractor nor to any other known example.

In the light of these results and intending to unify them, a natural question arises: is there a general structure for robust transitive sets? In [2] it is proved that when $\Lambda = M$, M a three-dimensional closed manifold, then Λ does not contain singularities and the flow is Anosov. So, we assume that Λ is properly contained in M. In this case, Λ can have singularities or not. If it has, it is well known that such sets do not have a hyperbolic structure: the presence of a singularity in the closure of periodic orbits is an obstruction for this. Recall that a compact, invariant set $\Lambda \subset M$ is *hyperbolic* if the tangent bundle to M over Λ admits a splitting, $E^{\mathrm{u}} \oplus E^X \oplus E^{\mathrm{s}}$, where E^X is tangent to the flow

X_t and, for some Riemannian metric on M, E^u and E^s are, respectively, exponentially expanded and contracted by the derivative DX_t. The bundle E^u (resp. E^s) is called *unstable* (resp. *stable*) bundle.

This paper answers this question in a positive way for compact, boundaryless three-manifolds. Indeed, Λ is partially hyperbolic with volume expanding central direction. That is, $T\Lambda$ can be written as $E^s \oplus E^{cu}$, E^s being a uniformly contracting one-dimensional bundle that dominates E^{cu}, and E^{cu} is volume expanding. This definition was independently presented in a conference attended by the first author of this article (*see* [14]). Moreover, if Λ has singularities, then each is hyperbolic and Λ is either an attractor or a repellor for the flow. Recall that E^s is uniformly contracting if there exist a real positive T and $c < 1$ so that $||DX_T/E^s|| < c$; E^s dominates E^{cu} if there exist $c < 1$ and a real number T so that $||DX_T/E^s|| \cdot ||DX_{-T}/E^{cu}|| < c$, and E^{cu} is volume expanding if there exist $c > 1, T > 0$ so that $|\det(DX_T/E^{cu})| > c$. Here $\det(A)$ means determinant of A.

Let $\mathcal{G}^1(M)$ be the set of C^1 flows on M such that each of its elements has a neighborhood $\mathcal{U}$ formed by flows with all periodic orbits and singularities being hyperbolic. Note that $\mathcal{G}^1(M)$ is the flow version of $\mathcal{F}^1(M)$ (*see* [8]). It is known that $\mathcal{F}^1(M)$ coincides with the set of hyperbolic diffeomorphisms of M (*see* [4]). The example in [3] says that this is not the case in the framework of flows: the geometric Lorenz attractor is in $\mathcal{G}^1(M)$. Thus, to understand this set, it is necessary first to describe the C^1 robust transitive sets with singularities. So, our result provides a step in the direction of the characterization of $\mathcal{G}^1(M)$ for three-dimensional manifolds. In higher dimensions the study of $\mathcal{G}^1(M)$ is wide open. An important result here is the one in [1] showing the existence of Lorenz-like attractors with a (unique) singularity with any number of expanding directions.

We also point out that this paper is part of a more ambitious program idealized by Palis (*see* [12]), a high point being the following conjecture: for a dense set of flows in the interior of the complement of the hyperbolic ones, every element has a finite number of attractors, either robust or at least persistent (positive Lebesgue measure in parametrized form) whose union of the basins of attraction has total probability.

Let us state our results more precisely. Throughout, M denotes a closed 3-manifold and $\mathcal{X}^r(M)$ is the set of C^r-vector fields in M, $r \geq 1$. Given a compact invariant set Λ of $X \in \mathcal{X}^r(M)$, $\mathcal{P}_X(\Lambda)$ and $\mathrm{Sing}_X(\Lambda)$ are the sets of periodic orbits and of singularities of X in Λ, respectively. If $p \in M$ and $B \subset M$, $\mathcal{O}_X(p)$ and $\overline{B}$ denote the orbit of p and the closure of B, respectively. An invariant set Λ, of X is *isolated* if there is an open set $U \supset \Lambda$ (*isolating block*) such that $\Lambda = \bigcap_{t\in\mathbb{R}} X_t(U)$, and *singular* if $\mathrm{Sing}_X(\Lambda) \neq \emptyset$.

The geometric Lorenz attractor (either expanding or contracting) and a singular horseshoe (either expanding or contracting) are examples of singular transitive sets (*see* [3], [7], and [13]).

DEFINITION 1. – A compact invariant singular set Λ of $X \in \mathcal{X}^r(M)$ is *singular hyperbolic* if it is partially hyperbolic with volume expanding central direction and any $\sigma \in \mathrm{Sing}_X(\Lambda)$ is hyperbolic.

DEFINITION 2. – An isolated transitive set Λ of $X \in \mathcal{X}^r(M)$ is C^r *robust transitive* if there is an isolating block U of Λ so that $\Lambda(Y) = \bigcap_{t\in\mathbb{R}} Y_t(U)$ is a transitive set for all Y C^r nearby X.

The most well-known example of a robust singular transitive set is the expanding geometric Lorenz attractor mentioned above. The contracting geometric Lorenz attractor and the singular horseshoe (either expanding or contracting) are not C^1 robust singular transitive sets. We denote $\mathrm{Sing}(X) = \mathrm{Sing}_X(M)$.

DEFINITION 3. – We say that $\sigma \in \mathrm{Sing}(X)$ is *Lorenz-like* if the eigenvalues of $DX(\sigma)$ are real and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$.

Our result is the following:

THEOREM A. – *Let Λ be a C^1 robust singular transitive set of $X \in \mathcal{X}^1(M)$. Then, either for X or $-X$, Λ is a singular hyperbolic set which is an attractor and any of its singularities is Lorenz-like.*

2. Proof of Theorem A

Let Λ be a C^1 robust singular transitive set of $X \in \mathcal{X}^r(M)$. Given an isolating block U of Λ, denote by $\mathcal{U}_{\Lambda,U}$ the open set in $\mathcal{X}^1(M)$ such that $\Lambda(Y,U) = \bigcap_{t\in\mathbf{R}} Y_t(U)$ is transitive for $Y \in \mathcal{U}_{\Lambda,U}$. We denote $\Lambda_Y = \Lambda(Y,U)$, $\mathcal{P}_Y = \mathcal{P}_Y(\Lambda_Y)$, and $\mathrm{Sing}(\Lambda_Y) = \mathrm{Sing}_Y(\Lambda_Y)$ when no confusion is possible. An attracting (resp. repelling) hyperbolic critical element of a vector field X is called *sink* (resp. *source*) of X. The lemma below is immediate.

Lemma 1. – *Let $Y \in \mathcal{U}_{\Lambda,U}$. Then Y has neither sinks nor sources in U and any $p \in \mathcal{P}_Y$ is hyperbolic.*

We shall use the following known result (*see* [5]).

Lemma 2 (C^1-connecting lemma). – *Let $X \in \mathcal{X}^1(M)$ and $p, q \in \mathrm{Sing}(X)$ be fixed. Assume that for any two neighborhoods U, V of p and q, respectively, there is a point $x \in U$ and $t \geq 0$ so that $X_t(x) \in V$. Then, there is Y C^1-near X so that if p_Y and q_Y are the continuations of p and q, respectively, then $W^u(p_Y) \cap W^s(q_Y) \neq \emptyset$.*

Now we divide the proof in three steps.

Step 1: *The singularities of Λ are Lorenz-like either for X or $-X$.*

The eigenvalues $\{\lambda_2, \lambda_3, \lambda_1\}$ of any $\sigma \in \mathrm{Sing}(\Lambda_Y)$ are real by Lemmas 1 and 2. If we write $\lambda_2 \leq \lambda_3 \leq \lambda_1$, then $-\lambda_2, \lambda_1 > 0$ by Lemma 1. If $\sigma \in \mathrm{Sing}(\Lambda_Y)$ is hyperbolic, $\Lambda_Y \cap W^{ss}(\sigma) = \{\sigma\}$ (resp. $\Lambda_Y \cap W^{uu}(\sigma) = \{\sigma\}$) if $\dim W^s(\sigma) = 2$ (resp. $\dim W^u(\sigma) = 2$). So the hyperbolic singularities in Λ_Y have the same number of negative eigenvalues (index), $\forall Y \in \mathcal{U}_{\Lambda,U}$. If any $\sigma_0 \in \Lambda_Y$ is not hyperbolic, then σ_0 is a saddle-node (after a C^1 perturbation of Y) and, unfolding it, we obtain two singularities with different index, a contradiction. So all the singularities in $\mathrm{Sing}(\Lambda_Y)$ are hyperbolic with the same indexes. Then, we can write $\lambda_2 < \lambda_3 < 0 < \lambda_1$ (changing X by $-X$ if necessary) and $-\lambda_3 < \lambda_1$ holds by Lemma 2. Step 1 is completed.

Step 2: *Λ is singular hyperbolic either for X or $-X$.*

Assume that $\dim W^s(\sigma) = 2$ for any $\sigma \in \mathrm{Sing}(\Lambda_Y)$. We shall prove that Λ_Y is singular hyperbolic for Y C^1 close to X. We give a sketch of the proof assuming that $\overline{\mathcal{P}}_Y = \Lambda_Y$. The general case follows using the methods in [2]. We divide this step in three parts.

2.1. *The splitting*. – To start, since any $p \in \mathcal{P}_Y$ is hyperbolic, we have that $T\mathcal{P}_Y = \widetilde{E}^s \oplus E^X \oplus \widetilde{E}^u$, where $\widetilde{E}^s$ is the stable bundle, $\widetilde{E}^u$ is the unstable bundle, and E^X is tangent to the flow direction. We set $\widetilde{E}^{cu} = E^X \oplus \widetilde{E}^u$ and define over $\mathcal{P}_Y$ the following splitting:

$$E^{cu} = E^X \oplus \widetilde{E}^u, \qquad E^s = \widetilde{E}^s$$

The idea is to extend continuously this splitting to $\overline{\mathcal{P}}_Y$ and to prove that this extension satisfies Definition 1. Observe that the presence of a singularity in $\overline{\mathcal{P}}_Y$ could be an obstruction for this extension. Indeed, near a singularity, the angle between $\widetilde{E}^u$ and the direction of the flow or between $\widetilde{E}^s$ and the direction of the flow could go to zero.

Let us introduce some definitions and results already proved elsewhere that we shall use.

Definition 4. – A point $x \in M \setminus \mathrm{Sing}(Y)$ is *δ-strongly closable* if there are $Z \in \mathcal{U}_{\Lambda,U}$, $z \in M$, and $T > 0, L > 0$ such that $Z_T(z) = z$, $Y = Z$ on $M \setminus B_\delta(Y_{[-L,0]}(x))$ and $\mathrm{dist}(Z_t(z), Y_t(x)) < \delta$ for all $0 \leq t \leq T$.

Here $B_\delta(Y_{[-L,0]}(x))$ is the δ-neigborhood along the segment of orbit $\{Y_t(x), -L \leq t \leq 0\}$. Denote by $\Sigma(Y)$ the set of points of M δ-strongly closable.

Theorem 1 (Ergodic closing lemma for flows, *see* [9] and [15]). – *Let μ be any Y-invariant Borel probability measure. Then $\mu(\mathrm{Sing}(Y) \cup \Sigma(Y)) = 1$.*

Now we prove that $E^s \oplus E^{cu}$ extends to $\overline{\mathcal{P}}_Y$. For this, by a result in [9], it is enough to prove that the splitting over $\mathcal{P}_Y$ defined above is a dominated splitting. So, we are left to prove:

THEOREM 2. – *For every $p \in \mathcal{P}_Y$ there are $\lambda < 1$ and $T > 0$ so that*

$$||DY_T/E^s|| \cdot ||DY_{-T}/E^{cu}_{Y_T}|| < \lambda$$

Given two subspaces A and B we denote by $\angle(A, B)$ the angle between A and B.
Let $\delta > 0$ and define

$$C_\delta(\mathrm{Sing}(\Lambda_Y)) = \bigcup_{\sigma \in \mathrm{Sing}(\Lambda_Y)} \mathrm{B}_\delta(\sigma)$$

where $\mathrm{B}_\delta(\sigma)$ is the ball of radius δ and center at σ.

Theorem 2 is a consequence from the fact that for $p \in \mathcal{P}_Y$, $p \notin C_\delta(\mathrm{Sing}(\Lambda_Y))$, the angle between E^s_p and E^{cu}_p is uniformly bounded away from zero. To verify this we have to prove that $\angle(E^s_p, E^{cu}_p)$ is bounded away from zero for periodic orbits whose orbit is far away from singularities and for periodic orbits converging to a singularity. Far away from singularities this is a consequence from the fact that $\angle(\widetilde{E}^s_p, \widetilde{E}^u_p)$ is bounded away from zero, the ergodic closing lemma and [15]. For orbits converging to a singularity σ, it is a consequence from the fact that $E^s_p \to E^{ss}_\sigma$ and $E^{cu}_p \to E^{cu}_\sigma$. To prove these facts we make use of three lemmas that are now briefly commented.

The first one is a fundamental tool and follows from the fact that Λ_Y is C^1 robust. The second says that if the angle between a vector v and E^{ss}_σ is bounded away from zero, then, through the passage near σ, $DX_t(v)$ lands on the direction of the flow. The third one says that, outside a neighborhood of the singularities, the angle between the stable and unstable bundles along the periodic orbits is bounded away from zero. We will only announce precisely the first one.

LEMMA 3. – *Given $\delta > 0$, there exist $\alpha = \alpha(\delta) > 0$ and $T_\delta > 0$ so that if $p \in \mathcal{P}_Y$, $\angle(E^s_p, E^{cu}_p) < \alpha$ and $t_p = \min\{s; Y_r(p) \notin C_\delta(\mathrm{Sing}(\Lambda_Y)), |r| \leq s\}$, then $t_p \leq T_\delta$.*

The proof of this lemma is a consequence from Theorem 1 and the fact that we have hyperbolicity far away from the singularities. Thus the extension of $E^s \oplus E^{cu}$ to $\overline{\mathcal{P}}_Y = \Lambda_Y$ exists and we still denote it by $E^s \oplus E^{cu}$. Clearly, $E^s \oplus E^{cu}$ is dominated and so we are left to prove that E^s is contracting and E^{cu} is volume expanding.

2.2. *E^s is contracting.* – Assume that there is a sequence $p_n \in \mathcal{P}_Y$ so that $E^s_{p_n}$ is not contracted. Let p be so that $p_n \longrightarrow p$ as $n \longrightarrow \infty$. We have that the continuity of the splitting obtained above implies that E^s is not much contracting along $\mathcal{O}_Y(p)$. By the ergodic closing lemma for flows, p can be strongly approximated by periodic orbits. Again by the continuity of the splitting, we have that E^s does not contract much along these periodic orbits. By C^1-perturbations we obtain a vector field $Y \in \mathcal{U}_{\Lambda,U}$ with a repelling periodic orbit, contradicting Lemma 1.

2.3. *E^{cu} is volume expanding.* – If E^{cu} is not volume expanding, by the ergodic closing lemma for flows, we obtain a periodic orbit $p \in \mathcal{P}_Y$ so that E^{cu} is not sufficiently expanding along the orbit of p. As $E^{cu}_p = [Y(p)] \oplus E^u_p$ and $[Y(p)]$ is the eigenspace associated to one, we obtain that E^u does not expand enough along the orbit of p. By a small C^1-perturbation we obtain a vector field $Y \in \mathcal{U}_{\Lambda,U}$ presenting an attracting periodic orbit near p, contradicting Lemma 1. Step 2 is completed.

Step 3: *Λ is an attractor either for X or $-X$.*

Reversing the flow direction if necessary, we can assume that Λ_Y is partially hyperbolic with volume expanding central direction and all its singularities are Lorenz-like for Y close to X. Let us prove that Λ_Y is an attractor for Y close to X (and so $\Lambda = \Lambda_X$ does for X). If it were not, there is a sequence of periodic orbits $x_n \to x \in \Lambda_Y$ as $n \to \infty$ and $t_n \geq 0$ so that $Y_{t_n}(x_n) \notin U$. There are two possibilities for the orbit of x, $\mathcal{O}(x)$: either $\sigma \in \overline{\mathcal{O}(x)}$ for some $\sigma \in \mathrm{Sing}(Y)$ or there exists $\delta > 0$ so that $\overline{\mathcal{O}(x)} \cap C_\delta(\mathrm{Sing}(Y)) = \emptyset$.

In the first case, taking a small cross section Σ through $W^s(\sigma)$ and denoting by $\ell = W^s(\sigma) \cap \Sigma$, we get strips B_n crossing Σ with $\mathrm{dist}(B_n, \ell) \to 0$ as $n \to \infty$, so that $Y_{t_n}(B_n) \cap U = \emptyset$. We perform a small perturbation of Y obtaining X with $W^u(\sigma_X) \cap W^s(\sigma_X) \neq \emptyset$ and so that there exists at least one strip $B_n(X)$ sufficiently near ℓ with the property that $Y_{t_n}(B_n(X)) \cap U = \emptyset$. We take another small perturbation $\bar{X}$ of Y in such a way that $W^u(\sigma_{\bar{X}}) \in B_n(\bar{X})$, implying that the ω-limit set of $W^u(\sigma_{\bar{X}})$ is in the complement of U, in a robust way. Thus, $\sigma_X \notin \overline{\mathcal{P}}_{\bar{X}}$, a contradiction.

In the second case, by the ergodic closing lemma, we can assume that x is strongly closable. As above, there are strips B_n and t_n so that $Y_{t_n}(B_n) \cap \overline{U} = \emptyset$ and $\mathrm{dist}(B_n, W^s(x)) \to 0$ as $n \to \infty$. Let $p \in \Lambda_Y$ be a periodic orbit strongly closed to $\mathcal{O}(x)$. Then $\mathcal{O}(p) \cap C_\delta(\sigma) = \emptyset$ for any $\sigma \in \mathrm{Sing}(Y)$ and so $W^u_\varepsilon(p)$ is well defined for some small ε. We can take a small perturbation $\bar{Y}$ in such away that there exists n with $\bar{Y}_{t_n}(B_n) \cap \overline{U} = \emptyset$, and so that $W^u_\varepsilon(p) \cap B_n \neq \emptyset$. This implies that there are strips S_n accumulating on $W^s_\varepsilon(p)$ and $t_n \geq 0$ so that $\bar{Y}_{t_n}(S_n) \cap \overline{U} = \emptyset$. Now, the transitivity of $\Lambda_{\bar{Y}}$ and the connecting lemma imply, as before, that we can perturb $\bar{Y}$ so that the ω-limit set of the unstable manifold of the corresponding singularity is outside $\overline{U}$ in a robust way, a contradiction. Step 3 is completed.

The complete proofs of these results will appear elsewhere.

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