Massey Products in String Topology

Aron Fischer

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Massey Products in String Topology

by

Aron Fischer

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

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Abstract

Massey Products in String Topology

by

Aron Fischer

Advisor: Martin Bendersky

Given a manifold $M$, Massey triple products exist in ordinary homology $h_\ast(M)$ for triples of classes $a, b, c$ when the pairwise intersection products $ab$ and $bc$ vanish. This means that the triple product $abc$ vanishes for two different reasons and this is captured by the Massey product. In this thesis we make a similar construction in string topology where we use the Chas-Sullivan product instead of the intersection product; we call it the Chas-Sullivan-Massey product. Our constructions use the Geometric Homology Theory description of String Topology and this allows us to obtain a very geometric picture of both the ordinary Massey Product and the Chas-Sullivan-Massey product. We then turn our attention to the Loop Homology Spectral Sequence and prove that it can be used to calculate Chas-Sullivan-Massey products. We find examples of non-zero products for even dimensional spheres and for complex projective spaces. Finally, we indicate how the definition can be extended to free sphere spaces, to generalised homology theories and to higher Massey-type products. We prove that the Loop Homology Spectral sequence can be used to calculate these products even in the most general case.
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## Contents

List of Figures .......................................................... vi

Introduction ............................................................. 1

1 Overview ....................................................................... 1

I Geometric Constructions .............................................. 5

2 Geometric Homology Theory ....................................... 5

2.1 Motivation: The Steenrod Problem .......................... 5

2.2 Geometric Homology Groups ................................... 6

2.3 The Intersection Product ......................................... 11

3 The Massey Product in Geometric Homology ............. 22

3.1 Prerequisites and Definition .................................. 22

4 The Geometry of the Massey Product ......................... 25

4.1 When is a product zero? ....................................... 26

4.2 Construction of the threefold intersections ............. 27

4.3 The triple intersections as boundary ..................... 30

4.4 A closed representative for the Massey Product ........ 33
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>The Chas-Sullivan-Massey Product in the LHSS</td>
<td>80</td>
</tr>
<tr>
<td>9.1</td>
<td>Convergence in the LHSS</td>
<td>80</td>
</tr>
<tr>
<td>10</td>
<td>Examples of non-zero Chas-Sullivan-Massey products</td>
<td>88</td>
</tr>
<tr>
<td>10.1</td>
<td>Example 1: The even Sphere</td>
<td>89</td>
</tr>
<tr>
<td>10.2</td>
<td>Example 2: The even Sphere - over $\mathbb{Q}$</td>
<td>91</td>
</tr>
<tr>
<td>10.3</td>
<td>Example 3: Complex Projective Space</td>
<td>93</td>
</tr>
<tr>
<td>III</td>
<td>Generalisations &amp; Outlook</td>
<td>97</td>
</tr>
<tr>
<td>11</td>
<td>Higher Chas-Sullivan-Massey Products</td>
<td>97</td>
</tr>
<tr>
<td>11.1</td>
<td>Defining higher Chas-Sullivan-Massey products</td>
<td>97</td>
</tr>
<tr>
<td>11.2</td>
<td>Convergence of higher products in the LHSS</td>
<td>99</td>
</tr>
<tr>
<td>12</td>
<td>Generalised Homology Theories</td>
<td>102</td>
</tr>
<tr>
<td>13</td>
<td>Beyond the Free Loop Space</td>
<td>104</td>
</tr>
<tr>
<td>Ω</td>
<td>Appendices</td>
<td>107</td>
</tr>
<tr>
<td>A</td>
<td>Signs and Orientations in the Intersection Product</td>
<td>107</td>
</tr>
<tr>
<td>A.1</td>
<td>Orienting the transverse intersection of submanifolds</td>
<td>107</td>
</tr>
<tr>
<td>A.2</td>
<td>Orienting the transverse intersection of singular manifolds</td>
<td>109</td>
</tr>
<tr>
<td>B</td>
<td>Associativity of the Intersection Product</td>
<td>111</td>
</tr>
<tr>
<td>Bibliography and References</td>
<td></td>
<td>113</td>
</tr>
</tbody>
</table>
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a_1$ represents a class in $E^1$</td>
<td>61</td>
</tr>
<tr>
<td>2</td>
<td>$a_2$ represents the class in $E^2$</td>
<td>63</td>
</tr>
<tr>
<td>3</td>
<td>$a_3$ surviving to $E^3$</td>
<td>65</td>
</tr>
<tr>
<td>4</td>
<td>The product $\alpha \bullet \beta$ jumps filtration.</td>
<td>69</td>
</tr>
<tr>
<td>5</td>
<td>The cycle $&lt;y_p&gt;$ over $\Omega S^n$ situated at a point $p \in S^n$</td>
<td>74</td>
</tr>
<tr>
<td>6</td>
<td>Vector field on $S^2$ with one dipole (RokerHRO/CC BY-SA 3.0)</td>
<td>78</td>
</tr>
<tr>
<td>9</td>
<td>$e'$ gets killed by a $d_{r+s+u}$ contradicting (*)</td>
<td>87</td>
</tr>
</tbody>
</table>
Introduction

1 Overview

The homology of a smooth, closed, oriented manifold $M$ has a product, called the intersection product, which is given geometrically by transverse intersection of cycles. This product has an associated homological Massey product, the most famous use of which is that it can detect the linking of the Borromean Rings. The intersection product is also one of the key ingredients of the Chas-Sullivan product in the homology of the free loop space of $M$. In this paper we combine the Massey Product and the Chas-Sullivan product into a Chas-Sullivan-Massey product.

The Massey Product is ordinarily defined in cohomology, however, if the space in question is has a dual intersection structure (for example a manifold, or a Poincarè Duality space), there is a description in homology as

$$\langle a, b, c \rangle = (-1)^{i+j}aX + (-1)^{i+1}Yc$$

where $|a| = i$, $|b| = j$ and $|c| = k$; $X$ is a $j + k - d + 1$ chain with boundary $\partial X = bc$ and $Y$ is an $i+j-d+1$ chain with $\partial Y = ab$. This is a class in $h_{i+j+k-2d+1}(M)$.

We do not use chains however but instead use the more geometric description called Geometric Homology Theory (developed by Jakob in [9],[10]) which we review in Section 2.
Homology classes are given by triples $[P,a,f]$ where $P$ is a smooth, oriented manifold, $a$ is a cohomology class on $P$ and $f$ is a map $P\to M$; subject to the equivalence relations of bordism and bundle modification. In the case that $P$ has boundary, the triple still represents a relative homology class (with the boundary mapping to the relative subspace).

We describe the intersection product in this language (section 2.3) and then proceed to a definition of the Massey product (section 3). The definition of the Massey product in geometric homology looks identical at first but requires careful interpretation in relative homology.

**Definition.** The *Massey Product in Geometric Homology Theory* is given by

$$\langle [P_1], [P_2], [P_3] \rangle = (-1)^{i+j} [P_1] [X] + (-1)^{j+1} [Y] [P_3].$$

where $\partial [X] = [P_2] [P_3]$ and $\partial [Y] = [P_1] [P_2]$.

There is an indeterminacy which comes about from the non-uniqueness of $X$ and $Y$, resulting in

$$\langle [P_1], [P_2], [P_3] \rangle \in h_{i+j+k-2d+1}(M) \bigotimes [P_1] * h_{j+k-d+1}(M) + h_{i+j-d+1}(M) * [P_3].$$

We continue in Section 4 to analyse the geometry behind this construction before we turn our attention in Section 5 to String Topology – the study of the free loop space of $M$.

When working over the free loop space $LM$ we can look at classes of pairs of loops in $LM\times LM$ and we can transversely intersect with the codimension-$d$ submanifold of composable loops $LM\times_M LM$. We use this to define a basepoint intersection product “$\ast$” (section 5.1)

**Preview.** The basepoint intersection product is a map:

$$\ast : h_i(LM) \otimes h_j(LM) \to h_{i+j-d}(LM\times_M LM).$$

2
Then we have a Massey-type product derived from this basepoint intersection.

**Definition.** The *Basepoint Intersection Massey Product* is defined by

\[ \langle [P_1], [P_2], [P_3] \rangle = (-1)^{i+j} [P_1] \ast [X] + (-1)^{j+1} [Y] \ast [P_3] \]

\[ \in h_{i+j-k-2d+1}(LM \times M LM \times M LM) \big/ \left[ P_1 \right] \ast h_*(LM \times M LM) + h_*(LM \times M LM) \ast \left[ P_3 \right] \]

where \( \partial [X] = [P_2] \ast [P_3] \) and \( \partial [Y] = [P_1] \ast [P_2] \).

Combining the basepoint intersection product with loop concatenation leads to the Chas-Sullivan product (section 5.2).

**Preview.** The *Chas-Sullivan product* is a map:

\[ \bullet : h_i(LM) \otimes h_j(LM) \to h_{i+j-d}(LM) \]

Then we have a Massey-type product derived from the Chas-Sullivan product. This is the subject of Section 6.

**Definition.** The *Chas-Sullivan-Massey product*:

\[ \langle [P_1], [P_2], [P_3] \rangle = (-1)^{i+j} [P_1] \bullet [X] + (-1)^{j+1} [Y] \bullet [P_3] \]

\[ \in h_{i+j+k-2d+1}(M) \big/ \left[ P_1 \right] \bullet h_*(LM) + h_*(LM) \bullet \left[ P_3 \right] \]

where \( \partial [X] = [P_2] \bullet [P_3] \) and \( \partial [Y] = [P_1] \bullet [P_2] \).

We proceed in Section 7 to analyse the geometry involved in the Chas-Sullivan-Massey product and conclude Chapter I by constructing closed manifold representatives.

In Chapter II we begin (section 8) by introducing the Loop Homology Spectral Sequence
and proceed to describe it geometrically. We present sample calculations along with their geometric interpretations. In Section 9 we turn to the question of whether the spectral sequence respects our newly defined Chas-Sullivan-Massey Product leading to the following

**Theorem.** Let $M$ be a closed, oriented manifold of dimension $d$. Let $\alpha \in \mathbb{H}_i(LM), \beta \in \mathbb{H}_j(LM)$ and $\gamma \in \mathbb{H}_k(LM)$ with $\alpha \bullet \beta = 0$ and $\beta \bullet \gamma = 0$. Suppose we have classes $a \in E^{1}_{p_{a},q_{a}}, b \in E^{1}_{p_{b},q_{b}}$ and $c \in E^{1}_{p_{c},q_{c}}$ converging to $\bar{\alpha}, \bar{\beta}$ and $\bar{\gamma}$ respectively and suppose that $\langle a, b, c \rangle$ is defined on the $r + 1$ page of the Loop Homology Spectral Sequence. Then, if condition (*) is satisfied, any representative of $\langle a, b, c \rangle$ converges to a representative of $\langle \bar{\alpha}, \bar{\beta}, \bar{\gamma} \rangle$ in the associated graded.

(*) Let $(-1)^{i+j} a \bullet X + (-1)^{j+1} Y \bullet z$ be any element in $\langle a, b, c \rangle$. Each of the constituent pieces $[a], [b], [c], [X], [Y]$ has a bigrading $(m, n)$ (from the $E^1$ page). For each pair $(m, n)$ we require

$$E^{r+u+1}_{m+u,n-u} \subset E^{r+u+1,\infty}$$

for all $u \geq 0$, where the superscript $\infty$ in $E^{r+u+1,\infty}$ indicates all classes surviving to $\infty$.

We conclude this Chapter (section 10) by calculating examples of non-zero Chas-Sullivan-Massey products in the loop homology of even dimensional spheres and of complex projective space.

Finally, in Chapter III we indicate how our construction generalises to free sphere spaces $Map(S^n, M)$ as well as to generalised homology theories. We define higher Chas-Sullivan-Massey products in this context via families of geometric cycles called defining systems and we prove in Theorem 11.4 that the Loop Homology Spectral Sequence can be used to calculate these higher products even in the most general context.
I Geometric Constructions

2 Geometric Homology Theory

2.1 Motivation: The Steenrod Problem

In singular homology, cycles are represented by formal sums of singular simplicies whose total boundary is zero. An \( n \)-cycle \( \alpha \) of the space \( X \) might be given as \( \alpha = \sum_{i=0}^{k} \sigma^i : \Delta^n \to X \) with \( \partial \alpha = \sum \partial \sigma^i = 0 \). This means that within the target space \( X \), all the boundary pieces of the simplicies \( \sigma^i(\Delta^n) \) match up and cancel out. We can thus imagine glueing all the boundaries together before we map to \( X \); and there may be many different ways we can glue. Roughly, the Steenrod problem asks if we can glue the simplices together to form a (closed, oriented) manifold. Schematically:

\[
\begin{array}{ccc}
\bigcup \Delta^n & \xrightarrow{\alpha} & X \\
\alpha & \exists & \alpha^? \\
M & \exists & \alpha^? \\
\end{array}
\]

The question can be precisely formulated as:

Q: Is every homology class \([\alpha] \in h_n(X)\) the image, under a map \( f : M \to X \), of the fundamental class of an \( n \)-manifold \([M] \in h_n(M)\) \( f_* \) \( [\alpha] \in h_n(X) \)?

A: The answer is known to be No. For \( n \geq 6 \) there are classes not realised by manifolds.
Getting to Yes

One approach to rectifying the situation is to enlarge our class of test spaces to include, besides manifolds, certain manifolds with singularities. For instance we could enlarge the class to include all oriented pseudomanifolds\(^1\). Then the answer is Yes, every homology class can be represented by an oriented pseudomanifold. Moreover, any homology can be realised as a bordism of oriented pseudomanifolds whence we have a natural isomorphism \(\Omega_p^{\text{pseudo}}(X) \cong h_n(X)\) from the singular pseudomanifold bordism group of \(X\) to the singular homology group of \(X\).

Another approach is to ask not whether every class can be represented by the fundamental class of a manifold, but to study the ways in which classes can be carried by manifolds. Thus we can ask about manifolds \(N\), maps \(f : N \to X\), and classes \([\beta] \in h_n(N)\) such that \(f_*([\beta]) = [\alpha]\). This is the approach we shall discuss further below.

2.2 Geometric Homology Groups

Geometric Homology theory as described by Jacob [9], [10] as well as by Chataur [1], is an approach to the Steenrod problem allowing us represent ordinary homology using a bordism type description while still using only manifolds instead of pseudomanifolds. We will sketch the ideas below, but refer the reader to Jacob and Chataur for a more detailed treatment.

Geometric Homology theory starts from the realisation that for every homology class \([\alpha] \in h_n(X)\) we can find a \(d\)-manifold \(P\), a class \([\beta] \in h_n(P)\) and a map \(f : N \to X\) such that \(f_*([\beta]) = [\alpha]\). In this case we say that \([\alpha]\) is carried by \(P\). Progressing from here we have two issues to confront. The first is that we must have a more nuanced version of the bordism

\(^1\)See also [14].
relation, one in which the bordism carries a homology class which ‘restricts’ properly at the boundary. The second is that we may have manifolds $P$ and $P'$ of different dimensions $d$ and $d'$ carrying the same $n$-class $[\alpha]$. This implies that a pure bordism approach is insufficient because $P$ and $P'$ can never be bordant and leads us to the concept of Bundle Modification. We deal with these issues in turn.

**Geometric Cycles**

It turns out that it is much simpler to formulate the correct notion of bordism by switching our view from the homology classes in $h_n(P)$ to their Poincarè duals in $H^{d-n}(P)$. Therefore we study triples $(P,a,f)$ where $P$ is a connected, oriented, smooth $d$-manifold, $a \in H^i(P)$, and $f : P \to X$.

**Definition 2.1.** The group of **Geometric Cycles** over a space $X$ is the free abelian group generated by such triples $\langle P,a,f \rangle$, modulo the relation

$$\langle P,\lambda a + \mu b,f \rangle = \lambda \langle P,a,f \rangle + \mu \langle P,b,f \rangle.$$  

This group is alternately referred to as the group of “geometric chains” or “geometric cycles”; We will regularly refer to a triple $\langle P,a,f \rangle$ as a *cycle* even if $P$ is a manifold with boundary. This is justified because in this cases the triple $\langle P,a,f \rangle$ will still represent a relative homology class.
The Bordism Relation

**Idea:** Two geometric cycles are bordant if there is a bordism between the manifolds together with a cohomology class on the bordism that restricts to the corresponding classes on the boundaries.

On the group of geometric cycles, we define the following relation

**Definition 2.2.** Two geometric cycles \( \langle P_1, a_1, f_1 \rangle \) and \( \langle P_2, a_2, f_2 \rangle \) are bordant if \( \dim(P_1) = \dim(P_2) = d \), \( \deg(a_1) = \deg(a_2) = i \) and if there exists a \( d+1 \) manifold \( B \) with \( \partial W \cong P_1 \cup (-P_2) \), a map \( F : B \to X \) such that \( F_{|P_1} = f_1 \) and \( F_{|P_2} = f_2 \), and a class \( b \in H^i(W) \) which restricts to \( a_1 \) and \( a_2 \) on the boundary components:

\[
\langle P_1, i_1^*(b), f_1 \rangle = \langle P_1, a_1, f_1 \rangle; \quad \langle P_2, i_2^*(b), f_2 \rangle = \langle P_2, a_2, f_2 \rangle
\]

where \( i_1 \) and \( i_2 \) are the inclusions of \( P_1 \) and \( P_2 \) into \( \partial W \). (Here \( -P_2 \) refers to \( P_2 \) with opposite orientation).

**Bundle Modification**

A geometric cycle \( \langle P, a, f \rangle \) with \( \dim(P) = d \) and \( \deg(a) = i \) will represent homology class \( x \in h_{d-i}(X) \). Note here that it is a class of degree \( d - i \); it is the image under \( f_* \) of the Poincaré dual \( \bar{a} \) of \( a \), that is \( f_*(\bar{a}) \) (see equation (2.1) below). Now suppose we are given a manifold \( P' \) of dimension \( d' = d + k \) for some \( k > 0 \), a map \( g : P' \to P \) and a class \( a' \in H^{i+k}(P') \) such that \( \bar{a}' \) maps to \( a \) under \( g_* \):

\[
\begin{array}{ccc}
P' & \xrightarrow{g} & P \xrightarrow{f} X \\
\downarrow & & \downarrow \\
h_{d+k-i-k}(P') & \xrightarrow{(gf)_*} & h_{d-k}(X)
\end{array}
\]

\[
\begin{array}{ccc}
\bar{a}' & \xrightarrow{g_*} & \bar{a} \\
\downarrow & & \downarrow \\
f_* & \xrightarrow{(gf)_*} & f_*(x)
\end{array}
\]
Thus $\langle P,a,f \rangle$ and $\langle P',a',fg \rangle$ represent the same homology class $x$ and should therefore be identified. However, because $P$ and $P'$ are not of the same dimension, there can be no bordism exhibiting this identification.

Further note that this issue is not rare but is indeed generic: every single class is carried by manifolds of varying dimensions. For example, consider a geometric cycle $\langle P,a,f \rangle$ and consider an embedding of $P$ into $\mathbb{R}^N$ for some large $N$ and then regard the sphere bundle of the normal bundle of this embedding $\langle \mathbb{S}(P) \overset{\pi}{\to} P \rangle$. If $N$ is large enough, $\pi$ has a section $\sigma: P \to \mathbb{S}(P)$. It follows that $\langle \mathbb{S}(P),\sigma^!(a),f\pi \rangle$ satisfies the conditions laid out for $\langle P',a',fg \rangle$ above. Ultimately we want to identify $\langle \mathbb{S}(P),\sigma^!(a),f\pi \rangle \sim \langle P,a,f \rangle$.

Jakob shows in [9], [10] that we only need to add equivalences arising from this construction.

**Definition 2.3.** For any sphere bundle $S \overset{\pi}{\to} P$ with section $s: P \to S$, the geometric cycles $\langle P,a,f \rangle$ and $\langle S,s^!(a),f\pi \rangle$ are said to differ by a **Bundle Modification**.

**Lemma 2.4.** (Jakob) The equivalence relation of Bundle Modification is generated by

$$\langle P,a,f \rangle \sim \langle S(P),\sigma^!(a),f\pi \rangle$$

where $S(P) \overset{\pi}{\to} P$ is the sphere-normal-bundle of any embedding of $P$ into $\mathbb{R}^N$ and $\sigma$ is a section of $\pi$.

**Definition of Geometric Homology Groups**

From the group of geometric cycles $\langle P,a,f \rangle$ over $X$ we divide out the equivalence relation generated by bordism and bundle modification to get homology:

---

2. Here $\sigma^!$ is the Gysin morphism in cohomology associated to the section $\sigma$. In our present context we can think of $\sigma^!$ as Poincaré dual to $\sigma$, i.e. $\sigma^!(a) = \sigma_*(\bar{a})$.

3. Jakob calls this relation “Vector Bundle Modification” but with our focus on sphere bundles, we shall instead simply refer to “Bundle Modification”.
Definition 2.5. The group of Geometric Homology classes of $X$ — written as $h'_*(X)$ — are defined as equivalence classes of geometric cycles, using both the bordism and bundle modification relations

$$h'_{d-i}(X) \ni [P,a,f] \equiv \langle P,a,f \rangle \sim_{\text{Bord}} \sim_{\text{Bund}}.$$ 

By verifying the Eilenberg Steenrod axioms, Jakob shows that these groups $h'_*(X)$ are isomorphic to the ordinary singular homology groups of $h_*(X)$ and the isomorphism is given by

$$h'_*(X) \xrightarrow{\sim} h_*(X)$$

$$[P,a,f] \mapsto f_*(a \cap [P]).$$

Throughout this paper we will no longer distinguish between homology and geometric homology except where explicitly stated otherwise.

Note. For later reference we note that the connecting homomorphism $\partial$ of the long exact sequence

$$\ldots \to h_i(X) \to h_i(X,A) \to h_{i-1}(A) \to h_{i-1}(X) \to \ldots$$

is given geometrically by restriction to boundary. Given a class $[P,a,f] \in h_i(X,A)$, we have $f: (P,\partial P) \to (X,A)$, and

$$\partial [P,a,f] = (-1)^{|a|} [\partial P, a_{|\partial P}, f_{|\partial P}].$$

The sign is needed to make restriction to boundary behave well with respect to bundle modification and be well defined on equivalence classes. This of course means that we will also have to distinguish between $\partial \langle P \rangle$ and $\langle \partial P \rangle$. 
For more details on this construction and also a description of many standard homology constructions imported into the geometric homology setting, we refer the reader to [9], [10] and [1]. Subsequently we will only highlight the constructions that we need in our quest for the Chas-Sullivan-Massey product.

2.3 The Intersection Product

In this section we define the Intersection Product on the homology of an oriented manifold in the language of Geometric Homology. In ordinary singular homology theory, the intersection product is obtained by intersecting singular cycles transversely and in geometric homology, unsurprisingly, the intersection product is described by transversely intersecting geometric cycles.

A lot of this section will focus on calculating orientations and signs. Sign calculations are notoriously dry and often ommited. We focus so much on signs here, because as far as we know, the correct sign is not written down anywhere and indeed published signs (for example in [1]) are incorrect. First time readers may wish to skip directly to section 2.3.3 and definition 2.10.

We are looking at cycles over an oriented, closed manifold $M$ of dimension $d$. Since our geometric cycles are carried by smooth manifolds, we have a good theory of transversality for geometric cycles; both over smooth manifolds and (as we shall make use of later) Hilbert Manifolds. This aspect of geometric homology is highlighted in [1]. Given cycles $\langle P_1, a_1, f_1 \rangle$ and $\langle P_2, a_2, f_2 \rangle$ if $f_1$ is not transverse to $f_2$ we can modify the maps by a homotopy to make them transverse to each other, and since any homotopy defines a bordism the resulting cycles are equivalent. We may thus assume that $f_1$ is transverse to $f_2$. This will give us the
transverse intersection \( f_1 \pitchfork f_2 \) which is a mainfold of dimension \( \dim(P_1) + \dim(P_2) - d \) and we will denote it by \( P_{12} \). Indeed this is the notation convention we will use throughout.

**Convention:** when we take the (transverse) intersection of some \( P_i \) and \( P_j \), we will write it as \( P_{ij} \). When we restrict the class \( a_i \times a_j \) to \( P_{ij} \) we write it as \( a_{ij} \) and so on. Thus for example we write

\[
\langle P_{12}, a_{12}, f_{12} \rangle = \langle f_1 \pitchfork f_2, (a_1 x a_2)|_{f_1 \times f_2}, (f_1 \times f_2)|_{f_1 \times f_2} \rangle.
\]

When there is no cause for confusion we may also use the shorthand \( P_{12} \) to refer to this triple.

Indeed, this is what the Geometric Homology description of the ordinary intersection product in homology looks like up to sign.

\[
* : h_i(M) \otimes h_j(M) \quad \xrightarrow{[P_1,a_1,f_1]} \quad h_{i+j-d}(M) \quad \xrightarrow{\pm[P_{12},a_{12},f_{12}]} \quad [P_2,a_2,f_2] \quad (2.3)
\]

The rest of this section will be devoted to determining the correct orientation of \( P_{12} \) and the correct sign in the above formula.

### 2.3.1 Orienting the Transverse Intersection

The intersection of \( P_1 \) and \( P_2 \) is usually defined by either of the following pull-back squares:

\[
\begin{array}{ccc}
P_{12} & \longrightarrow & P_1 \\
\downarrow f_{12} & & \downarrow f_1 \\
P_2 & \longrightarrow & M \\
\end{array}
\quad
\begin{array}{ccc}
P_{12} & \longrightarrow & P_1 \times P_2 \\
\downarrow f_{12} & & \downarrow f_1 \times f_2 \\
M & \longrightarrow & M \times M \\
\end{array}
\]

Note: Here we are using \( \Delta \) for the diagonal map \( M \to M \times M \). We will also use the same symbol \( \Delta \) to denote the subspace \((x,x) \in M \times M | x \in M \subset M \times M \). It will be clear from context which is meant.
The pull-back squares give us

\[ P_{12} \cong \{(x, y) \in P_1 \times P_2 | f_1(x) = f_2(y) \in M\} \text{ or} \]

\[ \cong \{(x, y, m) \in P_1 \times P_2 \times M | f_1(x) = f_2(y) = m \in M\} \]

and

\[ f_{12} : P_{12} \to M; \ (x, y) \mapsto m, \ (x, y, m) \mapsto m. \]

In the case where \( f_2 \) is an embedding, we could also view \( P_{12} \) as a submanifold of \( P_1 \) via

\[ P_{12} \cong \{x \in P_1 | f_1(x) \in \text{Im}(f_2)\} \]

and, since \( \Delta \) is an embedding

\[ P_{12} \cong \{x \in M | (x, x) \in \text{Im}(f_1 \times f_2)\}. \]

Completing the list, if both \( f_1 \) and \( f_2 \) are embeddings, we could further view \( P_{12} \) as being a submanifold of \( P_2 \) or of \( \Delta \) or of \( M \times M \).

Unfortunately, these contexts give different induced orientations for \( P_{12} \)! Indeed, although it is customary for example to say that the transverse intersection of two submanifolds is the same as the intersection of their product with the diagonal, this is only true if one disregards orientations. In general the induced orientation depends on context.

We proceed by comparing two simple cases explicitly and relegate the general case to Appendix A. We assume that we are in the situation in which \( f_1 \) and \( f_2 \) are embeddings of submanifolds of \( M \) and we compare the induced orientations of the intersection of \( P_1 \) and \( P_2 \) as a submanifold of \( M \) and the intersection of \( P_1 \times P_2 \) with \( \Delta \) as a submanifold of \( M \times M \):
Lemma 2.6. Let $M$ be a smooth oriented manifold of dimension $d$ and let $P_1$ and $P_2$ be oriented submanifolds that are transverse in $M$. Let $P_{12}$ denote the transverse intersection oriented as a submanifold of $M$ and let $P_{12}^\Delta$ denote the (diffeomorphic-to-$P_{12}$) transverse intersection of $P_1 \times P_2$ with $\Delta$ in $M \times M$. Then the induced orientations on $P_{12}$ and $P_{12}^\Delta$ differ by $(-1)^{d(d-\text{dim}(P_2))}$.

Proof. We have an orientation on the tangent bundle $\tau_M$ (since $M$ is an oriented manifold). Similarly, an orientation on $P_1$ and $P_2$ gives an orientation on $\tau_{P_1}$ and $\tau_{P_2}$. These in turn induce orientations on the normal bundles $\nu_{P_1}$ and $\nu_{P_2}$ of $P_1$ and $P_2$ in $M$ by requiring the following to be oriented isomorphisms:

\begin{align}
\tau_M &\cong \tau_{P_1} \oplus \nu_{P_1} \quad (2.4) \\
\tau_M &\cong \tau_{P_2} \oplus \nu_{P_2}. \quad (2.5)
\end{align}

Then the orientation of $P_{12} \subset M$ is defined (see Dold [4]) by the oriented isomorphism

\begin{align}
\tau_M &\cong \tau_{P_{12}} \oplus \nu_{P_{12}} \oplus \nu_{P_1}. \quad (2.6)
\end{align}

If on the other hand, instead of intersecting $P_1$ with $P_2$ to get $P_{12}$ in $M$, we intersect $P_1 \times P_2$ with the diagonal $\Delta$ to get $P_{12}^\Delta$ in $M \times M$, we would require

\begin{align}
\tau_{M \times M} &\cong \tau_{P_{12}}^\Delta \oplus \nu_{\Delta} \oplus \nu_{P_1 \times P_2} \quad (2.7)
\end{align}

to be an oriented isomorphism; defining orientations on the normal bundles of $P_1 \times P_2$ and $\Delta$ in $M \times M$ by

\begin{align}
\tau_{M \times M} &\cong \tau_{P_1 \times P_2} \oplus \nu_{P_1 \times P_2} \quad (2.8) \\
\tau_{M \times M} &\cong \tau_{\Delta} \oplus \nu_{\Delta} \quad (2.9)
\end{align}
in analogy to (2.4) and (2.5).

On the intersection with $\Delta$ we can write $\tau_{M\times M}$ as $\tau_{M} \oplus \tau_{M}$ which is also isomorphic to $\tau_{\Delta} \oplus \nu_{\Delta}$.

By switching factors we introduce a sign:

$$\nu_{\Delta} \oplus \nu_{P_{1}\times P_{2}} = (-1)^{d(P_{1}+P_{2})} \nu_{P_{1}\times P_{2}} \oplus \nu_{\Delta}$$

and therefore equation (2.7) becomes

$$\tau_{M} \oplus \tau_{M} \cong \tau_{P_{1}\times P_{2}} \oplus \nu_{P_{1}\times P_{2}}$$

$$\tau_{\Delta} \oplus \nu_{\Delta} \cong (-1)^{d(P_{1}+P_{2})} \tau_{P_{1}\times P_{2}} \oplus \nu_{P_{1}\times P_{2}} \oplus \nu_{\Delta}$$

so

$$\tau_{M} \cong \tau_{\Delta} \cong (-1)^{d(P_{1}+P_{2})} \tau_{P_{1}\times P_{2}} \oplus \nu_{P_{1}\times P_{2}} \oplus \nu_{\Delta}$$

(2.10)

We complete the calculation by showing that

$$\nu_{P_{1}\times P_{2}} \cong (-1)^{P_{2}(d-P_{1})+(d-P_{1})(d-P_{2})} \nu_{P_{2}} \oplus \nu_{P_{1}} :$$

The orientations of the normal bundles are defined by (2.4), (2.5) and (2.7), and thus

$$\tau_{P_{1}} \oplus \nu_{P_{1}} \oplus \tau_{P_{2}} \oplus \nu_{P_{2}} \cong \tau_{M} \oplus \tau_{M}$$

$$\tau_{P_{1}} \oplus \tau_{P_{2}} \oplus \nu_{P_{1}} \oplus \nu_{P_{2}} \cong (-1)^{P_{2}(d-P_{1})} \tau_{M} \oplus \tau_{M}$$

$$\tau_{P_{1}} \oplus \tau_{P_{2}} \oplus \nu_{P_{2}} \oplus \nu_{P_{1}} \cong (-1)^{P_{2}(d-P_{1})} (-1)^{d-P_{1})(d-P_{2})} \tau_{M} \oplus \tau_{M}$$

and since on the intersection

$$\tau_{P_{1}\times P_{2}} \cong \tau_{P_{1}} \oplus \tau_{P_{2}}$$
we have shown that

\[ \nu_{P_1 \times P_2} \cong (-1)^{P_2(d-P_1) + (d-P_1)(d-P_2)} \nu_{P_2} \oplus \nu_{P_1}. \]

Putting all these signs together with equation (2.6) and (2.10) leaves us with the result:

\[ P_{12} \cong (-1)^{d(d-P_2)} P_{12}^\Delta \]

(2.11)
as a isomorphism of oriented manifolds.

That a sign of this form appears here geometrically should come as no surprise to readers of Dold [4] where the same sign is deduced algebraically.

**Recall:** (c.f. Dold [4])

The intersection product in ordinary homology is defined as Poincaré dual to the cup product. Let \( \xi \) and \( \eta \) represent homology classes of degrees \( p \) and \( q \) over a \( d \)-manifold \( M \). Let \( \overline{\xi} \) and \( \overline{\eta} \) represent their Poincaré dual cohomology classes i.e. \( \overline{\xi} \cap [M] = \xi \) and \( \overline{\eta} \cap [M] = \eta \). Then we define

\[ \xi \cdot \eta := (\overline{\xi} \cup \overline{\eta}) \cap [M] \]

(2.12)
and it is a Lemma that this satisfies the formula

\[ \xi \cdot \eta = (-1)^{d(d-q)} \Delta_! (\xi \times \eta). \]

(2.13)
The sign in (2.13) has its origins in the interaction of the cross product with Poincaré duality. Observe that

\[ \overline{\xi \times \eta} = (-1)^{d(d-q)} \overline{\xi} \times \overline{\eta} \]
because
\[ \bar{\xi} \times \bar{\eta} \cap [M \times M] = \bar{\xi} \times \bar{\eta} \cap [M] \times [M] = (-1)^{d(M)} (\bar{\xi} \cap [M]) \times (\bar{\eta} \cap [M]) = (-1)^{d(M)} \xi \times \eta. \]

The definition of the intersection product as Poincaré dual to the cup product is related to the intersection \( P_{12} \) as the definition using \( \Delta_1 \) is to the intersection \( P_{12}^\Delta \).

For indeed by the definition of \( \Delta_1 \) as the dual of \( \Delta^* \) we see that
\[ (-1)^{d(Q)} \Delta_1(\xi \times \eta) = (-1)^{d(Q)} \Delta^*(\bar{\xi} \times \bar{\eta}) \cap [M] = \Delta^*(\bar{\xi} \times \bar{\eta}) \cap [M] = (\bar{\xi} \cup \bar{\eta}) \cap [M] = \xi \cdot \eta. \]

If we are to employ similar ideas\(^4\) in our quest to get the correct sign in equation (2.3) and define the intersection product using \( \Delta_1 \) and \( \times \) we have to proceed carefully. Firstly, the degree of a class \([P_2,a_2,f_2]\) is not the same as the dimension of \( P_2 \) and secondly, as we shall see next, the cross product of \([P_1,a_1,f_1]\) and \([P_2,a_2,f_2]\) is not simply given by \([P_1 \times P_2,a_1 \times a_2,f_1 \times f_2]\). We start by investigating the latter.

2.3.2 The Cross Product

Given two spaces \( X \) and \( Y \), let \( \tau \) be the switch map \( \tau : X \times Y \to Y \times X \). First we make the following

Definition 2.7. The Cross Product in Geometric Homology is defined by

\(^4\)Indeed, when we define the Chas-Sullivan product in Section 6, it becomes apparent that is the more useful approach.
\[
\times : \quad h_i(X) \otimes h_j(Y) \quad \rightarrow \quad h_{i+j}(X \times Y)
\]

\[
[P_1,a_1,f_1] \otimes [P_2,a_2,f_2] \quad \rightarrow \quad [P_1,a_1,f_1] \times [P_2,a_2,f_2]
\]

where, (following [1]) we have

\[
[P_1,a_1,f_1] \times [P_2,a_2,f_2] = (-1)^{P_1 a_2} [P_1 \times P_2, a_1 \times a_2, f_1 \times a_2]. \tag{2.15}
\]

In the equation above and in the rest of this paper we are using to following

**Notation Convention:**

Whenever a manifold or a cohomology class appears as an exponent to -1, then we mean its dimension and its degree respectively; thus \((-1)^{P_1 a_2}\) means \((-1)^{\dim(P_1) \deg(a_2)}\).

The reason for this particular sign in equation (2.15) is that the geometric homology class \([P_1,a_1,f_1]\) corresponds to the homology class \((f_1), (a_1 \cap [P_1])\) and

\[
(a_1 \times a_2) \cap ([P_1 \times P_2]) = (-1)^{P_1 a_2} (a_1 \cap [P_1]) \times (a_2 \cap [P_2]).
\]

**Lemma 2.8.** ([1]) With the sign defined as in (2.15) above,

\[
\tau_* : h_*(X \times Y) \rightarrow h_*(Y \times X)
\]

satisfies the relation

\[
\tau_*(\alpha \times \beta) = (-1)^{||\alpha|| \beta} (\beta \times \alpha).
\]

**Proof.** Observe that, when switching factors \(X\) and \(Y\) we switch from \(P_1 \times P_2\) to \(P_2 \times P_1\) and so we pick up a sign of \((-1)^{P_1 P_2}\) due to the orientation. Also, we pick up a sign of \((-1)^{a_1 a_2}\) when we switch from \(a_1 \times a_2\) to \(a_2 \times a_1\). Thus all together we have
The key is that $(−1)^{i_1}P_{1}a_2(−1)^{i_2}P_{1}b_1(−1)^{a_1}a_2$ is equal to $(−1)^{i+j}P_{1}a_1$. Indeed

$$(-1)^{i+j}P_{2}a_1 = (-1)^{(P_1-a_1)(P_2-a_2)+P_2a_1}$$

$$= (-1)^{P_2-a_1-a_2}$$

$$= (-1)^{P_2}(−1)^{a_1}a_2.$$

\[\square\]

### 2.3.3 The Intersection Product

Continuing with the approach sketched on page 17, now that we have a description of the cross product, the only missing ingredient is $\Delta$.

**Definition 2.9.** The umkehr map associated to the diagonal map $\Delta : M \to M \times M$ is given by

$$\Delta! : h_\ast(M \times M) \to h_{-d}(M) \quad [P,a,f] \mapsto [P_{\Delta},\Delta,a|_{P_{\Delta}},f|_{P_{\Delta}}]$$

which, in our situation, this translates to

$$[P_{12}P_1a_1a_2, f_1f_2] \xrightarrow{\Delta!} [P_{12}^\Delta, a_{12}, f_{12}]. \quad (2.16)$$

Combining equations (2.15) and (2.16) with Dold’s definition of the intersection product (2.12), (2.13) we finally arrive at
Definition 2.10. The Intersection Product in Geometric Homology is defined by

\[
[P_1, a_1, f_1] \ast [P_2, a_2, f_2] = (-1)^{d-j} \Delta_1 ([P_1, a_1, f_1] \times [P_2, a_2, f_2])
\] (2.17)

and is thus given by

\[
*: \quad h_i(M) \otimes h_j(M) \rightarrow h_{i+j-d}(M) \quad \rightarrow (-1)^{P_1 a_2 + da_2} [P_{12}, a_{12}, f_{12}]
\] (2.18)

When using the intersection product, we often suppress the ∗ from notation when it is clear by context. Thus here \([P_1] [P_2]\) would refer to \([P_1] \ast [P_2]\) which is equal to \((-1)^{P_1 a_2 + da_2} [P_{12}]\).

**Notation:**

The sign in formula (2.18) appears so often, the we introduce a new notation for it. We write

\[
\zeta_s = (-1)^{P_1 a_2 + da_2}
\] (2.19)

This sign does indeed correspond to (2.17) and thus to Dold’s sign in (2.13). The sign \((-1)^{P_1 a_2}\) comes from the cross product (2.15), while \((-1)^{d(d-j)}\) from (2.12) is equal to \((-1)^{d(d-P_2 - a_2)}\) in our context. This in turn is equal to \((-1)^{d(d-P_2)}\) \((-1)^{da_2}\) and the \((-1)^{d(d-P_2)}\) part of that sign gets absorbed in the orientation of \(P_{12}\) vs. the orientation of \(P_{12}^\Delta\) as per Lemma 2.6 and equation (2.11).

**Note 2.11.** This definition of intersection product can easily be extended to relative homology

\[
*: h_i(M, A) \otimes h_j(M, B) \rightarrow h_{i+j-d}(M, A \cup B).
\]

Indeed we may copy the formula (2.18) verbatim; the only added subtlety is that relative cycles are represented by manifolds with boundary and so we must take care to smooth corners when
taking products. (Relative cycles are discussed in [9], [10]; the idea is that the boundary has to land in the subspace $A$ and this must be respected by all bordisms and homotopies).

**Lemma 2.12.** The intersection product as defined in (2.18) is graded commutative on $\mathbb{H}_*(M) = h_{*+d}(M)$.

**Proof.** The key to this calculation is keeping track of the signs we incur as we switch from $[P_{12}, a_{12}, f_{12}]$ to $[P_{21}, a_{21}, f_{21}]$. Just as we incurred a sign of $(-1)^{P_1P_2}$ above when switching from $P_1 \times P_2$ to $P_2 \times P_1$, we incur a sign of $(-1)^{(d-P_1)(d-P_2)}$ when switching from $P_{12}$ to $P_{21}$.

There is a sign of $(-1)^{a_1a_2}$ appearing from switching the cohomology classes.

We start with the diagram:

$$
\begin{array}{ccc}
h_i(M) \otimes h_j(M) & \rightarrow & h_{i+j-d}(M) \\
\downarrow & & \downarrow \text{id} \\
h_j(M) \otimes h_i(M) & \rightarrow & h_{i+j-d}(M)
\end{array}
$$

and rewrite it as

$$
\begin{array}{ccc}
\mathbb{H}_{i'}(M) \otimes \mathbb{H}_{j'}(M) & \rightarrow & \mathbb{H}_{i'+j'}(M) \\
\downarrow & & \downarrow \text{id} \\
\mathbb{H}_{j'}(M) \otimes \mathbb{H}_{i'}(M) & \rightarrow & \mathbb{H}_{i'+j'}(M),
\end{array}
$$

which leads us to check the commutativity of

$$
[P_{1, a_1, f_1}] \otimes [P_{2, a_2, f_2}] \rightarrow (-1)^{P_1a_2}(-1)^{da_2}[P_{12}, a_{12}, f_{12}]
$$

by discussion above

and

$$
(-1)^{i'+j'}[P_{2, a_2, f_2}] \otimes [P_{1, a_1, f_1}] \rightarrow (-1)^{i'+j'}(-1)^{P_2a_1}(-1)^{da_1}[P_{21}, a_{21}, f_{21}].
$$
This is verified by a simple (mod 2) calculation

\[ i' + j' + P_2 a_1 + da_1 = (i - d)(j - d) + P_2 a_1 + da_1 \]

\[ = (P_1 - a_1 - d)(P_2 - a_2 - d) + P_2 a_1 + da_1 \]

\[ \equiv P_1 P_2 + P_1 a_2 + P_1 d + a_1 a_2 + P_2 d + da_2 + d^2 \]

\[ \equiv P_1 a_2 + da_2 + (d - P_1)(d - P_2) + a_1 a_2. \]

\[ \square \]

3 The Massey Product in Geometric Homology

We proceed using Geometric Homology to describe the Massey product in the homology of a manifold. We are given \( P_1,a_1,f_1 \), \( P_2,a_2,f_2 \) and \( P_3,a_3,f_3 \), cycles of degree \( i,j \) and \( k \) over the manifold \( M^d \). Since we are working with the intersection product, we shall assume that we have chosen representatives in general position, i.e. we assume that \( f_1 \) is transverse to \( f_2 \), that \( f_2 \) is transverse to \( f_3 \) and that \( f_1 \) (resp \( f_3 \)) is transverse to the intersection of the other two: \( f_{23} \) (resp. \( f_{12} \)). We begin discussing further prerequisites before we proceed to the definition. In the following section we then analyse the underlying geometry.

3.1 Prerequisites and Definition

When do we have a Massey product?

Let us recall briefly the idea behind the Massey product in singular homology. Given three singular cycles \( \alpha, \beta \) and \( \gamma \) over \( M \) such that both \( [\alpha][\beta] = 0 \) and \( [\beta][\gamma] = 0 \), then the triple intersection product \( [\alpha][\beta][\gamma] \) is zero for two different reasons; it is equal to \( 0 * [\gamma] \) and also \( [\alpha] * 0 \). The Massey product measures the difference between these two.
Recall: The Massey Product (for a manifold)

Given homology classes \([\alpha],[\beta]\) and \([\gamma]\) with \([\alpha][\beta] = 0\) and \([\beta][\gamma] = 0\); suppose that we have chosen singular chains \(X\) and \(Y\) such that \(\partial Y = \alpha \beta\) and \(\partial X = \beta \gamma\). Then the transverse intersection \((-1)^{|\beta|} \alpha \beta \gamma\) is the boundary of \((-1)^{|\alpha|+|\beta|} \alpha X\) and also the negative boundary of \((-1)^{|\beta|+1} Y \gamma\). So \((-1)^{|\alpha|+|\beta|} \alpha X\) and \((-1)^{|\beta|+1} Y \gamma\) taken together form a cycle, and this cycle represents the Massey Product:

\[
< [\alpha],[\beta],[\gamma] > \equiv \{((-1)^{|\beta|+1} Y \gamma + (-1)^{|\alpha|+|\beta|} \alpha X) | \partial Y = \alpha \beta, \partial X = \beta \gamma\} \quad (3.1)
\]

The choices of \(X\) and \(Y\) are what leads to the indeterminacy of the Massey product, whence it is a well defined element of the quotient

\[
< [\alpha],[\beta],[\gamma] > \in h_*(M) /[\alpha]h_*(M) + h_*(M)[\gamma].
\]

The prerequisites for a Massey Product are of course identical to any other description of homology. Thus we need three classes \([P_1,a_1,f_1],[P_2,a_2,f_2]\) and \([P_3,a_3,f_3]\) over a closed, oriented \(d\)-manifold \(M\) and we need the following products to vanish

\[
[P_1,a_1,f_1] \ast [P_2,a_2,f_2] = 0
\]
\[
[P_2,a_2,f_2] \ast [P_3,a_3,f_3] = 0.
\]

Defining a Massey Product in Geometric Homology

The definition of the Massey product in geometric homology is going to be similar to (3.1), but there are subleties to consider. For one we do not have chains and thus there is no immediate translation of \(X\) and \(Y\) from (3.1) into geometric cycles; we would have to use cycles-with-boundary representing relative homology classes and this in turn leads to further questions. Additionally, there is a difference (of signs) between \(\partial [X]\) and \(\langle \partial X \rangle\) and thus
between $\partial [X] = [\partial (\langle X \rangle)]$ and $[\partial X]$ that we do not have when using singular chains.

Our plan is to give the correct definition first and then subsequently analyse its geometric properties in detail in section 4.

**Definition 3.1.** Given cycles $[P_1], [P_2], [P_3]$ of dimensions $i, j$ and $k$ over a $d$-manifold $M$ with $[P_1][P_2] = 0$ and $[P_2][P_3] = 0$; then the Massey Product of the triple is defined by the formula:

$$
([P_1], [P_3], [P_3]) = \{( -1 )^{i+j} [P_1] [X] + ( -1 )^{j+1} [Y] [P_3] \} \quad (3.2)
$$

where $\partial [X] = [P_2][P_3]$ and $\partial [Y] = [P_1][P_2]$.

**Lemma 3.2.** The Massey Product is well defined as an element of the quotient:

$$
([P_1], [P_2], [P_3]) \in h_{i+j+k-2d+1}(M) \cap h_{j+k-d+1}(M) + h_{i+j-d+1}(M) \ast [P_3]. \quad (3.3)
$$

**Proof.** In order to make sense of the right hand side of equation (3.2) we have to interpret it in relative homology: since $X$ has boundary, the map $f_X : X \to M$ can be viewed as a map of pairs $(X, \partial X) \to (M, f_X(\partial X))$, and since $\partial [X] = [P_2][P_3]$, we can choose $f_X(\partial X) = f_{23}(P_{23})$. Therefore we can view

$$
[X, a_X, f_X] \in h_{j+k-d+1}(M, f_{23}(P_{23}))
$$

and similarly

$$
[Y, a_Y, f_Y] \in h_{i+j-d+1}(M, f_{12}(P_{12})).
$$

When we form the products in equation (3.2) we have for example\(^5\) $[P_1][X] \in h_{i+j+k-2d+1}(M, f_{23}(P_{23}))$, but by inspection of the geometry we see that it comes from a class in $h_{*}(M, f_{123}(P_{123}))$ \(^6\).

---

\(^5\) See Note 2.11

\(^6\) The product $[P_1][X]$ is represented up to sign by the transverse intersection of $P_1$ and $X$, the boundary of which maps to $f_{123}(P_{123})$. This will be fleshed out in section 4.3
The same conclusion holds for $[Y][P_3]$, whence the right hand side of equation (3.2) makes sense as a sum in the group $h_{i+j+k-2d+1}(M, f_{123}(P_{123}))$.

Next we look at the boundary map (connecting homomorphism) from the long exact sequence

$$\partial: h_{i+j+k-2d+1}(M, f_{123}(P_{123})) \rightarrow h_{i+j+k-2d}(f_{123}(P_{123})),$$

and we note that

$$((-1)^{i+j} [P_1][X] + (-1)^{j+1} [Y][P_3]) \mapsto 0 \quad (3.4)$$

and therefore, by exactness, it represents a unique class in $h_{i+j+k-2d+1}(M)$.

Finally, the indeterminacy of the Massey product comes about from the non-uniqueness of $X$ and $Y$. Had we chosen $X'$ and $Y'$ instead of $X$ and $Y$, then the differences would give us: $[X - X'] \in h_{j+k-d+1}(M)$ and $[Y - Y'] \in h_{i+j-d+1}(M)$. That is why we only have a well defined element in the quotient.

### 4 The Geometry of the Massey Product

In the previous section we managed to define the Massey product, but this does not tell us what it looks like or how we might represent it with a closed geometric homology class. Our definition is generic in that it hardly uses any features of geometric homology theory at all.

In this section we look at the underlying geometry of the Massey product and we construct such a closed representative.

One of the main difficulties in constructing a closed representative is in keeping track of signs and orientations. This is mostly due to the fact that the degree of a class $[P,a,f]$ is not the same as the dimension of $P$. The second difficulty has been alluded to before in section 2.3; namely the context-dependency of induced orientations.
The contents of this section will not be needed in the remainder of this thesis and may be skipped on first reading.

4.1 When is a product zero?

In order to talk of the Massey product of the triple \([P_1], [P_2], [P_3]\), we need the products \([P_1][P_2]\) and \([P_2][P_3]\) to be zero and this means that the intersection cycle represents the zero class:

\[
\begin{align*}
[P_1, a_1, f_1] \ast [P_2, a_2, f_2] &= \zeta_1 P_{12}, a_{12}, f_{12} = 0 \quad (4.1) \\
[P_2, a_2, f_2] \ast [P_3, a_3, f_3] &= \zeta_3 P_{23}, a_{23}, f_{23} = 0.
\end{align*}
\]

Let us assume that we can find \(\langle X, a_X, f_X \rangle\) and \(\langle Y, a_Y, f_Y \rangle\) such that \(\partial \langle X \rangle = \langle P_2 \rangle \langle P_3 \rangle = \zeta_3 \langle P_{23} \rangle\) and \(\partial \langle Y \rangle = \langle P_1 \rangle \langle P_2 \rangle = \zeta_1 \langle P_{12} \rangle\). Then we can imagine defining a single geometric cycle representing the Massey product by somehow glueing together the intersection cycles of \([P_1][X]\) and \([Y][P_3]\) modelled on equation (3.2).

However, unlike with singular chains, when using geometric cycles, even though \(P_{12}\) and \(P_{23}\) represent the zero class, we cannot assume that they are boundaries themselves. In other words, although we can always find \(X\) with \(\partial [X] = [P_2][P_3]\), we cannot assume that \(\partial \langle X \rangle = \langle P_2 \rangle \langle P_3 \rangle\) — or stated geometrically: there may not be an \(X\) with \(\partial X \cong P_{23}\).

Recall: (see section 2.2, Bundle Modification on page 8) In geometric homology theory, if a class \([P,a,f]\) is zero, that does not imply that \(\langle P,a,f \rangle\) is itself a boundary. What we do know is that there is some sphere bundle \(S(P)\) on \(P\), and that the resulting geometric cycle \(\langle S(P),\sigma^1(a),f_\pi \rangle\) is a boundary. More specifically, we know that we can find such a sphere bundle which is a sphere-normal bundle for some embedding of \(P\) into \(\mathbb{R}^n\).

The upshot is that while \(P\) may not be a boundary, we have some \(\langle B,b,F \rangle\) whose
boundary is \( \langle S(P), \sigma'(a), f \pi \rangle \):

\[
\partial \langle B, b, F \rangle = (-1)^b \langle \partial B, b|_{\partial B}, F|_{\partial B} \rangle = \langle S(P), \sigma'(a), f \pi \rangle.
\]

Given the situation in equation (4.1), there exist sphere-normal bundles \( S(P_{ij}) \) on \( P_{ij} \) and there exist \( \langle X, a_X, f_X \rangle \) and \( \langle Y, a_Y, f_Y \rangle \)

\[
\begin{align*}
X & \xleftarrow{\partial} \partial X \cong S(P_{23}) \\
& \downarrow f_X \quad \downarrow \pi_{23} \quad \downarrow \sigma_{23} \\
& \quad \quad \downarrow f_{23} \\
& \quad \quad \quad \quad \quad \quad M \\
Y & \xrightarrow{\partial} \partial Y \cong S(P_{12}) \\
& \downarrow f_Y \quad \quad \downarrow \pi_{12} \quad \quad \downarrow \sigma_{12} \\
& \quad \quad \downarrow f_{12} \\
& \quad \quad \quad \quad \quad \quad M
\end{align*}
\]

such that

\[
\begin{align*}
\partial \langle X, a_X, f_X \rangle = (-1)^{a_X} \langle \partial X, a_X|_{\partial X}, f_X|_{\partial X} \rangle \\
& = \zeta_3^2 \langle S(P_{23}), \sigma_{23}^1 (a_{23}), f_{23} \circ \pi_{23} \rangle 
\end{align*}
\]

and

\[
\begin{align*}
\partial \langle Y, a_Y, f_Y \rangle = (-1)^{a_Y} \langle \partial Y, a_Y|_{\partial Y}, f_Y|_{\partial Y} \rangle \\
& = \zeta_2^1 \langle S(P_{12}), \sigma_{12}^1 (a_{12}), f_{12} \circ \pi_{12} \rangle
\end{align*}
\]

In other words, \( \zeta_3^2 S(P_{23}) \) is oriented as \( (-1)^{a_2+a_3+r} \partial X \) and \( \zeta_2^1 S(P_{12}) \) as \( (-1)^{a_1+a_2+s} \partial Y \) where \( s \) and \( r \) are the dimensions of the sphere bundles. It is these bundle equivalences that exhibit the equality

\[
[\partial \langle Y \rangle] = [P_1] [P_2] \quad \text{and} \quad [\partial \langle X \rangle] = [P_2] [P_3].
\]

4.2 Construction of the threefold intersections

We wish to investigate the geometric cycles involved in the triple products when twofold products are represented using sphere bundles.

We have cycles \([P_1], [P_2] \text{ and } [P_3]\) of degrees \( i, j \) and \( k \) over a \( d \)-manifold \( M \). We assume
that pairwise products are zero (equation (4.1)) and we assume that our cycles are chosen such that \( f_1 \) is transverse to \( f_{23} \) and that \( f_3 \) is transverse to \( f_{12} \). Since transversality is a condition happening in \( M \); given the maps \( \pi_{12} \) and \( \pi_{23} \) in

\[
\mathbb{S}(P_{23}) \xrightarrow{\pi_{23}} P_{23} \xrightarrow{f_{23}} M
\]

\[
\mathbb{S}(P_{12}) \xrightarrow{\pi_{12}} P_{12} \xrightarrow{f_{12}} M
\]

we know that \( f_1 \) is also transverse to the composition \( f_{23} \circ \pi_{23} \) and that \( f_3 \) is also transverse to \( f_{12} \circ \pi_{12} \) (taking sphere bundle replacements does not “break” transversality since the \( \pi_i \) are submersions). We look at the resulting intersections.

We will denote by \( P_{12,3} \) the intersection \( \mathbb{S}(P_{12}) \upharpoonright P_3 \) of \( \mathbb{S}(P_{12}) \xrightarrow{f_{12} \circ \pi_{12}} M \) with \( P_3 \xrightarrow{f_3} M \). It is defined by either of the following pull-back diagrams:

\[
\begin{array}{ccc}
P_{12,3} := \mathbb{S}(P_{12}) \upharpoonright P_3 & \xrightarrow{\pi_{12}} & P_{12} \xrightarrow{f_{12}} M \\
\mathbb{S}(P_{12}) & \xrightarrow{\pi_{12}} & P_{12} \xrightarrow{f_{12}} M \\
\end{array}
\]

Yielding the manifold

\[
P_{12,3} \cong \left\{ ([x,y],z) \in \mathbb{S}(P_{12}) \times P_3 \left| \begin{array}{c}
\text{[x,y] is a point in the fibre above (x,y) \in P_{12},}
\text{and } f_1(x) = f_2(y) = f_3(z) = m \in M
\end{array} \right. \right\}
\]

and the map

\[
f_{12,3} : P_{12,3} \to M; \quad ([x,y],z) \mapsto m
\]

in the cycle \( \langle P_{12,3}, a_{12,3}, f_{12,3} \rangle \).

We proceed analogously, intersecting \( \mathbb{S}(P_{12}) \) with \( P_3 \) to obtain \( \langle P_{12,3}, a_{12,3}, f_{12,3} \rangle \), and we also define

\[
P_{123} := \{ (x,y,z) \in P_1 \times P_2 \times P_3 \mid f_1(x) = f_2(y) = f_3(z) \in M \}
\]
Notice that (at least up to sign), $P_{12,3}, P_{1,23}$ and $P_{123}$ all represent the same homology class over $M$; indeed all three maps factor via $P_{123} \xrightarrow{f_{123}} M$.

4.2.1 Orienting the threefold intersections

Similarly to the calculation following Lemma 2.6 and calculation (2.14) on page 17 we record the different signs and orientations for the triple intersections. We relegate the detailed calculations to appendix B.

It is not clear a priori that the induced orientation for a triple intersection is independent of the order. Ignoring the sphere bundles for the moment; how does the standard orientation on $P_{1-23} = P_{1} \upharpoonright (P_{2} \upharpoonright P_{3})$ compare to $P_{12-3} = (P_{1} \upharpoonright P_{2}) \upharpoonright P_{3}$? It turns out (c.f. appendix B) that the standard orientation is associative ($\zeta_{1}^{12} \zeta_{2}^{12} = \zeta_{12}^{1} \zeta_{3}^{2}$) but the orientation induced via the diagonal is not. Let us write $P_{12-3}^{\Delta}$ for the orientated intersection of $P_{12}^{\Delta} \times P_{3}$ with $\Delta$ and write $P_{1-23}^{\Delta}$ for the oriented intersection of $P_{1} \times P_{23}^{\Delta}$ with $\Delta$. The we have the oriented isomorphism (see calculations (B.1) and (B.2) on page 111)

\[
(-1)^{d(P_{3})}(-1)^{d(P_{2})} P_{12-3}^{\Delta} \cong (-1)^{d(P_{2}+P_{3}-d)}(-1)^{d(P_{2})} P_{1-23}^{\Delta}
\]

while using the standard orientation we have

\[
P_{12-3} = P_{1-23}
\]

and we shall henceforth write this as simply $P_{123}$.

In regards to the sphere bundles, the orientation on $S(P_{12})$ is such that $[S(P_{12})] = [P_{12}]$ and
similarly $[S(P_{23})] = [P_{23}]$. This implies that

$$[P_{12,3}] = (-1)^{(P_{12} + P_{3})} [S(P_{12})] [P_{3}]$$

$$= (-1)^{(P_{12} + P_{3})} [P_{12}] [P_{3}]$$

$$= (-1)^{s_{23}} [P_{123}]$$

$$[P_{1,23}] = (-1)^{P_{1} + (a_{23} + d_{a_{2}} + d_{a_{3}} + (a_{23} + d_{a_{3}} + (a_{2} + a_{3} + r))} [P_{1}] [S(P_{23})]$$

$$= (-1)^{P_{1} + (a_{2} + a_{3} + r) + d(a_{2} + a_{3} + r)} [P_{1}] [P_{23}]$$

$$= (-1)^{P_{1} + r + dr} [P_{123}]$$

In summary\(^7\):

$$(-1)^{s_{23}} [P_{12,3}] = [P_{123}] = [P_{1,23}] = (-1)^{P_{1} + r + dr}.$$

Now that we know that $(-1)^{s_{23}} P_{12,3}, P_{123}$ and $(-1)^{P_{1} + r + dr} P_{1,23}$ all represent the same class; we can conclude that there are (appropriately oriented) sphere-normal bundles on $P_{1,23}, P_{123}$ and $P_{12,3}$ as well as bordisms between them exhibiting this equality. We shall return to this later on in section 4.4.

### 4.3 The triple intersections as boundary

Let us return now to $X$ and $Y$ from section 3.1 and 4.1. We chose orientations for $X$ and $Y$ to satisfy (4.2) and (4.3), we obtained equation (4.4):$$[\partial\langle Y, a_{Y}, f_{Y} \rangle] = [P_{1}, a_{1}, f_{1}][P_{2}, a_{2}, f_{2}]$$

and

$$[\partial\langle X, a_{X}, f_{X} \rangle] = [P_{2}, a_{2}, f_{2}][P_{3}, a_{3}, f_{3}].$$

\(^7\)By choosing different sphere bundle replacements it would be possible to make $r$ and $s$ even.
Continuing, we may assume that we chose \( f_X \) transverse to \( f_3 \) and \( f_Y \) transverse to \( f_1 \). We form their intersections \( P_{1X} \) and \( P_{3Y} \) defined via diagrams

\[
P_{1X} := P_1 \downarrow X \longrightarrow P_1 \times X \quad \text{with boundary:} \quad P_1 \to \partial P_1 \cong P_1 \times S(P_{23})
\]

\[
P_{1X} : P_1 \downarrow X \longrightarrow P_1 \times X \quad \text{with boundary:} \quad P_1 \to \partial P_1 \cong P_1 \times S(P_{23})
\]

\[
P_{1Y} : \quad \text{with boundary:} \quad P_{12} \to \partial P_{12} \cong S(P_{12} \times P_3)
\]

\[
P_{1Y} : \quad \text{with boundary:} \quad P_{12} \to \partial P_{12} \cong S(P_{12} \times P_3)
\]

Once we include all the correct signs and orientations and deal with boundaries, we get

\[
\zeta_X^1 \left[ \hat{\partial} \langle P_{1X} ; a_{1X} , f_{1X} \rangle \right] = (-1)^{a_{1X} + P_{1X} + a_X} \zeta_X^1 \left[ P_{123} , a_{123} , f_{123} \right]
\]

\[
\zeta_X^1 \left[ \hat{\partial} \langle P_{1X} ; a_{1X} , f_{1X} \rangle \right] = (-1)^{P_{1X} + a_X} \left[ P_1 , a_1 , f_1 \right] \left[ P_2 , a_2 , f_2 \right] \left[ P_3 , a_3 , f_3 \right]
\]

and

\[
\zeta_Y^3 \left[ \hat{\partial} \langle P_{3Y} ; a_{3Y} , f_{3Y} \rangle \right] = (-1)^{Y_{a_{3Y} + a_{3Y}} + P_{a_{2Y} + a_{3Y}} + a_3} \left[ P_{123} , a_{123} , f_{123} \right]
\]

\[
\zeta_Y^3 \left[ \hat{\partial} \langle P_{3Y} ; a_{3Y} , f_{3Y} \rangle \right] = \left[ P_1 , a_1 , f_1 \right] \left[ P_2 , a_2 , f_2 \right] \left[ P_3 , a_3 , f_3 \right].
\]

So here we begin to see the problem when working in this context. There are too many ways to write the correct signs for a product or boundary. For reference, here is a complete list...
for $X$:

$$\partial [X] = [\partial \langle X \rangle] = (-1)^{a_X} [\partial X]$$

$$= [P_2] [P_3]$$

$$= \zeta_3^2 [P_{23}]$$

$$\partial [P_1] [X] = (-1)^i [P_1] [\partial \langle X \rangle] = (-1)^i (-1)^{a_X} [P_1] [\partial X]$$

$$= (-1)^i [P_1] [P_2] [P_3]$$

$$= (-1)^i \zeta_3^2 [P_1] [P_{23}]$$

$$= (-1)^i \zeta_3^2 \zeta_{23}^1 [P_{123}]$$

$$= (-1)^i P_2 a_3 + da_3 + P_1 (a_2 + a_3) + d(a_2 + a_3) [P_{123}]$$

$$= (-1)^i P_2 a_3 + da_3 + P_1 (a_2 + a_3) + d(a_2 + a_3) + P_1 r + dr [P_{1,23}]$$

$$= (-1)^{a_1 \chi} \zeta_X^1 [\partial P_{1X}]$$

$$= \zeta_X^1 [\partial \langle P_{1X} \rangle]$$
and for $Y$

$$
\partial [Y] = [\partial \langle Y \rangle] = (-1)^{a_Y} \partial [Y] \\
= [P_1] [P_2] \\
= \zeta^1_2 [P_{12}] \\
\partial [Y] [P_3] = [P_1] [P_2] [P_3] \\
= \zeta^1_2 [P_{12}] [P_3] \\
= \zeta^1_2 \zeta_{13}^{12} [P_{123}] \\
= (-1)^{P_1 a_2 + da_2 + P_{12} a_3 + da_3} [P_{123}] \\
= (-1)^{P_1 a_2 + da_2 + P_{12} a_3 + sa_3} [P_{12,3}] \\
= (-1)^{a_{Y3}} \zeta^Y_3 [\partial P_{Y3}] \\
= \zeta^Y_3 \partial [P_{Y3}].
$$

### 4.4 A closed representative for the Massey Product

At this point we can say that a Massey product element

$$
(-1)^{j+1} [Y] [P_3] + (-1)^{i+j} [P_i] [X]
$$

is equivalently given by

$$
\epsilon_{Y3} [P_{Y3}] + \epsilon_{1X} [P_{1X}],
$$

with signs $\epsilon_{Y3} := (-1)^{j+1} \zeta^Y_{3}$ and $\epsilon_{1X} := (-1)^{i+j} \zeta^1_{X}$. Recall: $\zeta^1_{X} = (-1)^{P_1 a_X + da_X}$ and $\zeta^Y_3 (-1)^{Y a_3 + da_3}$; $\dim Y = \dim P_1 + \dim P_2 - d + 1 + s$ and $\deg(a_X) = \deg(a_2) + \deg(a_3) + r$. 

33
In order to construct a closed representative, we would like to “glue” $P_{1X}$ to $P_{Y3}$ along their common boundary$^8$; however they don’t actually have a common boundary; the boundary of the former is $P_{1,23}$ and the boundary of the latter is $P_{12,3}$. Alternatively we could hope to glue $P_{1X}$ to $P_{Y3}$ together using a bordism from $P_{1,23}$ to $P_{12,3}$; however such a bordism may not exist (this is noted at the end of section 4.2.1).

However, there do exist sphere bundles $S(P_{1X})$ and $S(P_{Y3})$ with the following properties:

- $\partial S(P_{1X})$ is a sphere bundle on $P_{1,23}$ – let’s call it $S(P_{1,23})$
- $\partial S(P_{Y3})$ is a sphere bundle on $P_{12,3}$ – let’s call it $S(P_{12,3})$
- there exists a bordism from $S(P_{1,23})$ to $S(P_{12,3})$ – let’s call it $\Sigma$
- the image of $f_{\Sigma}$ lies entirely in the subspace $f_{123}(P_{123})$ (possible because of (3.4))
- Glueing these together$^9$ gives us our closed representative:

\[
[S(P_{1X})] + [S(P_{Y3})] = \left[ S(P_{1X}) \bigcup_{S(P_{1,23})} \Sigma \bigcup_{S(P_{12,3})} S(P_{Y3}) \right].
\]

Here is an overview of how all the pieces fit together (all inclusions are boundary inclusions):

$^8$Gluing is well behaved in this context – it is a bordism invariant; c.f [20] Theorem 4.18
$^9$With appropriate orientations/signs.
5 String Topology and Geometric Homology

The geometric idea underlying the Chas Sullivan Product is a combination of the intersection product and the loop concatenation product. We can use the intersection product from section 2; the only difference being, that we are intersecting in an infinite dimensional (Hilbert manifold) context. More precisely, our cycles map finite dimensional manifolds $P_1$, $P_2$ to the Hilbert manifold $LM$, but we are only intersecting in the base manifold $M$; the intersection is thus realised here by intersecting the finite dimensional product cycle $P_1 \times P_2$ in the Hilbert manifold $LM \times LM$ with the infinite dimensional submanifold of composable loops $LM \times_M LM$. The details are mostly worked out in Chataur [1]. Below we will complete the definition by including the correct signs.
5.1 Transverse Intersection at the basepoint

We begin with cycles $\langle P_1, a_1, f_1 \rangle$ and $\langle P_2, a_2, f_2 \rangle$ over the Hilbert Manifold $LM$. We do not ask that they are transverse to each other because this would simply mean that their images in $LM$ are disjoint and their intersections thus empty. We want transversality in $M$ of the basepoint cycles $P_i \xrightarrow{f_i} LM \xrightarrow{ev_0} M$. We therefore consider our intersections using only the basepoints in $M$, but end up with a cycle over $LM \times_M LM$. In particular we re-define $P_{12}$ and $f_{12}$ via the diagram

\[
\begin{array}{ccc}
P_{12} & \longrightarrow & P_1 \times P_2 \\
\downarrow^{ev_0 f_{12}} & & \downarrow^{ev_0 f_1 \times ev_0 f_2} \\
M & \longrightarrow \Delta & M \times M \\
\end{array}
\]

which factors as

\[
\begin{array}{ccc}
P_{12} & \longrightarrow & P_1 \times P_2 \\
\downarrow^{f_{12}} & & \downarrow^{f_1 \times f_2} \\
LM \times_M LM & \longrightarrow & LM \times LM \\
\downarrow^{ev_0} & & \downarrow^{ev_0 \times ev_0} \\
M & \longrightarrow \Delta & M \times M \\
\end{array}
\]

Formulated in this way we see clearly that there are two things going on. The bottom square picks out the submanifold of composable loops\textsuperscript{10} $LM \times_M LM$ of the space of pairs of loops $LM \times LM$, while the top square tells us that we are intersecting the singular cycle $P_1 \times P_2$ over $LM \times LM$ with the codimension-$d$ submanifold $LM \times_M LM$. The alternative diagram obscures this view:

\[
\begin{array}{ccc}
P_{12} & \longrightarrow & P_2 \\
\downarrow^{ev_0 f_2} & & \downarrow^{ev_0 f_1} \\
P_1 & \longrightarrow & M \\
\end{array}
\]

which factors as

\[
\begin{array}{ccc}
P_{12} & \longrightarrow & P_1 \times_M LM \\
\downarrow^{f_{12}} & & \downarrow^{f_1} \\
P_1 \times_M LM & \longrightarrow & LM \times_M LM \\
\downarrow^{ev_0} & & \downarrow^{ev_0} \\
P_1 & \longrightarrow & LM \\
\end{array}
\]

For this reason we will only be presenting intersections as pull-backs along the diagonal. But in any case, we can say: $P_{12}$ consists of pairs $(x, y)$ with $x \in P_1$ and $y \in P_2$ such that they

\textsuperscript{10}Loops are composable if and only if they have the same basepoint. The space of pairs of loops with the same basepoint is the space of composable loops.
map to composable loops, i.e. \( f_1(x)(0) = f_2(y)(0) \).

Analogously, \( P_{23} \subset P_2 \times P_3 \) consists of \((y, z)\) such that \( f_2(y)(0) = f_3(z)(0) \).

We shall also re-use previous notation conventions; writing \( a_{12} \) for \((a_1 \times a_2)|_{P_12} \) for example.

In light of this, when working over the free loop space we redefine the intersection product “\(*\)” as

**Definition 5.1.** *the basepoint intersection product:*

\[
*: \begin{array}{c}
h_i(LM) \otimes h_j(LM) \\
[P_1,a_1,f_1] \otimes [P_2,a_2,f_2]
\end{array} \longrightarrow h_{i+j-a}(LM \times M \times LM) \\
\zeta^A_{[P_12,a_{12},f_{12}]}.
\]

(5.1)

### 5.2 The Chas-Sullivan product

We recall the definitions from [1] of the Chas-Sullivan product described using geometric homology and complete it using the signs worked out in section 2.

**Recall: Loop Concatenation**

Given any space \( X \) and a point \( b \in X \), we have the *based loopspace* \( \Omega_b(X) \) of \( X \) consisting of all loops \( \alpha: I \rightarrow X \) with \( \alpha(0) = \alpha(1) = b \). On the space of based loops we have the *loop concatenation product:*

\[
\gamma : \Omega_b(X) \otimes \Omega_b(X) \rightarrow \Omega_b(X)
\]

where

\[
\gamma(\alpha, \beta)(t) = \begin{cases} 
\alpha(2t) & \text{if } t \leq 1/2 \\
\beta(2t - 1) & \text{if } t \geq 1/2.
\end{cases}
\]

This product induces the familiar *Pontryagin Product* on homology:

\[
\gamma_* : h_*(\Omega_b(X)) \otimes h_*(\Omega_b(X)) \rightarrow (h_*(\Omega_b(X))).
\]
Since the product relies only on loops sharing a basepoint, it extends to a map
\[
\gamma : LM \times M LM \to LM
\]
and
\[
\gamma_* : h_*(LM \times M LM) \to h_*(LM).
\]
It is by combining the basepoint intersection with loop concatenation that we arrive at the following

**Definition 5.2.** The *Chas-Sullivan product* of two classes \([P_1] \) and \([P_2] \) is given by concatenating the loops of their basepoint-intersection \(P_{12}: \)

\[
\begin{align*}
  h_*(LM) \otimes h_*(LM) & \stackrel{\ast}{\longrightarrow} h_{*-d}(LM \times M LM) \stackrel{\gamma_*}{\longrightarrow} h_{*-d}(LM) \\
  [P_1, a_1, f_1] \otimes [P_2, a_2, f_2] & \mapsto \pm [P_{12}, a_{12}, f_{12}] \quad \mapsto \pm [P_{12}, a_{12}, \gamma \circ f_{12}].
\end{align*}
\]

So we have our basepoint-intersection

\[
[P_1, a_1, f_1] \ast [P_2, a_2, f_2] = (-1)^{P_1 a_2 + da_2} [P_{12}, a_{12}, f_{12}] \in h_{i+j-d}(LM \times M LM)
\]

and the complete Chas Sullivan product

\[
[P_1, a_1, f_1] \bullet [P_2, a_2, f_2] = (-1)^{P_1 a_2 + da_2} [P_{12}, a_{12}, \gamma f_{12}] \in h_{i+j-d}(LM).
\]

Our goal in the following is to use this product in the Loop Homology of \(M\) and combine it with the Massey product ideas of section 3 and 4.

### 6 The Chas-Sullivan-Massey Product

In this section we define and construct the Chas-Sullivan-Massey product using the language of geometric homology theory. We have described the intersection products and homology
Massey products in this language in sections 2.3 and 3.1 respectively and we described the Chas-Sullivan product in section 5.2. Our goal below is to combine these ideas. The section is divided into the “simple case” and the “general case”. Roughly speaking, the simple case uses the intersection “∗” as the primary product whereas the general case uses the full Chas-Sullivan product “●”.

6.1 The simple case - (basepoint intersections are zero)

Prerequisites and Definition:

In analogy to section 3, we assume we have cycles \( \langle P_1,a_1,f_1 \rangle, \langle P_2,a_2,f_2 \rangle \) and \( \langle P_3,a_3,f_3 \rangle \) satisfying transversality along their basepoints (c.f. section 5.1). In constructing the Chas-Sullivan-Massey product, we can copy a lot the discussion from Section 3 verbatim. The prime difference is that a-priori our twofold-intersection cycles \( \langle P_{12} \rangle \) and \( \langle P_{23} \rangle \) are cycles over \( LM \times_M LM \) and only by postcomposing the maps with the composition \( \gamma \) do we get cycles over \( LM \). For example we will have

\[
P_{12} \xrightarrow{f_{12}} LM \times_M LM \xrightarrow{\gamma} LM.
\]

i.e.

\[
\llbracket P_{12}, a_{12}, f_{12} \rrbracket \in h_*(LM \times_M LM)
\]

and

\[
\llbracket P_{12}, a_{12}, \gamma \circ f_{12} \rrbracket \in h_*(LM)
\]

The simple case is to assume

\[
\llbracket P_1, a_1, f_1 \rrbracket * \llbracket P_2, a_2, f_2 \rrbracket = \zeta_1^1 \llbracket P_{12}, a_{12}, f_{12} \rrbracket = 0
\]

\[
\llbracket P_2, a_2, f_2 \rrbracket * \llbracket P_3, a_3, f_3 \rrbracket = \zeta_3^2 \llbracket P_{23}, a_{23}, f_{23} \rrbracket = 0
\]

(6.1)
Crucially, the equality \([P_{12}] = 0\) and \([P_{23}] = 0\) takes place in \(h_\ast(LM \times_M LM)\).

**Defining the Chas-Sullivan-Massey Product using the Basepoint Intersection Product**

The definition of the Massey product using basepoint-intersection is going to be very similar to (3.1); the main difference is that the target space is different. Thus we have

**Definition 6.1.** Let \(M\) be a closed oriented manifold of dimension \(d\) and let \([P_1],[P_2]\) and \([P_3]\) be geometric homology classes over the free loop space \(LM\) of dimensions \(i, j\) and \(k\). If \([P_1] \ast [P_2] = 0\) and \([P_2] \ast [P_3] = 0\), then we define the **Basepoint-Intersection Massey Product** by the formula

\[
\langle [P_1], [P_2], [P_3] \rangle := \{(-1)^{i+j} [P_1] \ast [X] + (-1)^{j+1} [Y] \ast [P_3]\}
\]  

(6.2)

where \(\partial [X] = [P_2] \ast [P_3]\) and \(\partial [Y] = [P_1] \ast [P_2]\); and we define the **Chas-Sullivan-Massey Product** (in this simplified case) by the formula:

\[
\langle [P_1], [P_2], [P_3] \rangle := \gamma_{3\ast} \{(-1)^{i+j} [P_1] \ast [X] + (-1)^{j+1} [Y] \ast [P_3]\}
\]  

(6.3)

where \(\gamma_{3\ast}\) is the map induced by composing triples of loops.

**Lemma 6.2.** The Basepoint-Intersection Massey Product of Definition 6.1 is well defined as an element of the quotient:

\[
h_{i+jk-2d+1}(LM \times_M LM) / \langle [P_1] \ast h_\ast(LM \times_M LM) + h_\ast(LM \times_M LM) \ast [P_3]\n\]

and the Chas-Sullivan-Massey product is a well defined element of the quotient:

\[
h_{i+jk-2d+1}(LM) / \langle [P_1] \bullet h_\ast(LM) + h_\ast(LM) \bullet [P_3]\n\]
Proof. The indeterminacy, i.e. the fact that the products are only well defined in a quotient is the result of the choices involved for \([X]\) and \([Y]\), just as in Definition 3.1. The big difference between equations (6.2) and (6.3) and equation (3.2) is in how we are to interpret the right hand side.

Since we are dealing with geometric cycles with boundary, the above sums live in relative homology. We have

\[
[X,a_X,f_X] \in h_{j+k-d+1}(LM \times_M LM, f_{23}(P_{23}))
\]

\[
[Y,a_Y,f_Y] \in h_{i+j-d+1}(LM \times_M LM, f_{12}(P_{12})).
\]

When we form the products in equations (6.2) and (6.3), we are using a relative version of the basepoint intersection product (5.1); thus for example

\[
[P_1] \ast [X] \in h_{i+j+k-2d+1}(LM \times_M LM \times_M LM, LM \times_M f_{23}(P_{23}))
\]

but by inspection of the geometry we see that it comes from a class in \(h_{*}(LM \times_M LM \times_M LM, f_{123}(P_{123}))\). The same conclusion holds for \([Y] \ast [P_3]\), whence the right hand side of equation (6.2) makes sense as a sum in the group \(h_{i+j+k+2d-1}(LM \times_M LM \times_M LM, f_{123}(P_{123}))\).

In the long exact sequence in homology of the pair \((LM \times_M LM \times_M LM, f_{123}(P_{123}))\), the boundary map

\[
\partial : h_{i+j+k-2d+1}(LM \times_M LM \times_M LM, f_{123}(P_{123})) \to h_{i+j+k-2d}(f_{123}(P_{123}))
\]

takes this sum to zero:

\[
((-1)^{i+j}[P_1] \ast [X] + (-1)^{j+1}[Y] \ast [P_3]) \xrightarrow{\partial} 0
\]

and therefore, by exactness, it represents a unique class in \(h_{i+j+k-2d+1}(LM \times_M LM \times_M LM)\). 

\[\square\]
6.2 The general case - (Chas-Sullivan products are zero)

Prerequisites and Definition:

In the general case we do not assume that \([P_1] \ast [P_2] = 0\) and \([P_2] \ast [P_3] = 0\), but only
\([P_1] \bullet [P_2] = 0\) and \([P_2] \bullet [P_3] = 0\).

The difference to before is that we assume \([P_{12}, a_{12}, \gamma_{f_{12}}] = 0 \in h_*(LM)\) but we allow the
possibility that \([P_{12}, a_{12}, f_{12}] \neq 0 \in h_*(LM \times M \times M)\).

Jumping ahead a little, we may say: there exists \(\langle Y, a_Y, \phi_Y \rangle\) such that \([\partial \langle Y, a_Y, \phi_Y \rangle] =
\zeta_1 [P_{12}, a_{12}, \gamma \circ f_{12}]\) but we can no longer assume that the map \(\phi_Y\) factors over \(LM \times M \times M\) (see
also section 7.1).

Example:

To illustrate this point let us look an an example of a cycle that is non-zero in the ho-
mology of composable loops, but is zero after composing.

Imagine a torus \(T^2\) with a Morse function (height). At the first critical level we have as
cross-section a figure 8. We think of this figure 8 as representing either two composable
loops or one loop which touches (but does not cross) itself. In the first case, i.e. when
this figure 8 is viewed as a 0-cycle in \(LT^2 \times \gamma \times LT^2\), it is clearly non-zero; however in the
second case, as a 0-cycle in \(LT^2\), it is null-homologous as it is homotopic to a trivial loop.

Defining the Chas-Sullivan-Massey product

**Definition 6.3.** Given cycles \([P_1], [P_2]\) and \([P_3]\) of dimensions \(i, j\) and \(k\) over the free
loop space \(LM\) of a d-manifold \(M\); if \([P_1] \bullet [P_2] = 0\) and \([P_2] \bullet [P_3] = 0\) then we define the
**Chas-Sullivan-Massey Product** by the formula

\[
< [P_1], [P_2], [P_3] >= \{(-1)^{i+j} [P_1] \bullet [X] + (-1)^{j+1} [Y] \bullet [P_3]\}
\] (6.4)
where \( \partial [X] = [P_2] \cdot [P_3] \) and \( \partial [Y] = [P_1] \cdot [P_3] \).

**Lemma 6.4.** The Chas-Sullivan-Massey of Definition 6.3 is well defined as an element of the quotient:

\[
h_{i+j-2d+1}(LM) / [P_1] \cdot h_*(LM) + h_*(LM) \cdot [P_3].
\]

Equation (6.4) differs from (6.3) by its use of the Chas-Sullivan product “\( \bullet \)” instead of the basepoint intersection product “\( \ast \)”. Consequently, the right hand side of (6.4) needs to be analysed differently. We postpone the proof until the end of section 7.7.

### 7 The Geometry of the Chas-Sullivan-Massey Product

In this section we investigate the geometry behind definitions 6.1 and 6.3. Our aim is to construct closed manifold representatives for the products and to complete the analysis of equation (6.4). Just as section 4, the contents of this section are not necessary for the remainder of this paper and may be skipped at first reading.

#### 7.1 When are products zero?

In the simple case we had \( [P_1] \ast [X] = 0 \) and \( [Y] \ast [P_3] = 0 \). Then there are sphere bundles \( S(P_{12}) \) on \( P_{12} \) and \( S(P_{23}) \) on \( P_{23} \) such that there exist \( \langle X, a_X, f_X \rangle \) and \( \langle Y, a_Y, f_Y \rangle \):

![Diagram](image)
such that
\[
\partial\langle X, a_X, f_X \rangle = (-1)^{a_X} \partial X |_{\partial X, f_X |_{\partial X}} \\
= \zeta_3^2 \mathbb{S}(P_{23}), \sigma^1_{23}(a_{23}), f_{23} \circ \pi_{23}
\]
and
\[
\partial\langle Y, a_Y, f_Y \rangle = (-1)^{a_Y} \partial Y |_{\partial Y, f_Y |_{\partial Y}} \\
= \zeta_2^1 \mathbb{S}(P_{12}), \sigma^1_{12}(a_{12}), f_{12} \circ \pi_{12}
\]
In other words, \( \zeta_3^2 \mathbb{S}(P_{23}) \) is oriented as \((-1)^{a_2+a_3+r} \partial X \) and \( \zeta_2^1 \mathbb{S}(P_{12}) \) as \((-1)^{a_1+a_2+s} \partial Y \) where \( s \) and \( r \) are the dimensions of the sphere bundles. It is these bundle equivalences that exhibit the equality
\[
[\partial \langle Y \rangle] = [P_1] \ast [P_2] \quad \text{and} \quad [\partial \langle X \rangle] = [P_2] \ast [P_3].
\] (7.3)

In the general case on the other hand we assume \([P_{12}, a_{12}, \gamma f_{12}] = 0 \in h_*(LM)\) but we allow the possibility that \([P_{12}, a_{12}, f_{12}] \neq 0 \in h_*(LM \times_y LM)\). Thus there exists \(\langle Y, a_Y, \phi_Y \rangle\) such that \([\partial \langle Y, a_Y, \phi_Y \rangle] = \zeta_2^1 [P_{12}, a_{12}, \gamma \circ f_{12}]\) but we can no longer assume that the map \(\phi_Y\) factors over \(LM \times_y LM\):

<table>
<thead>
<tr>
<th>(\partial Y \cong \mathbb{S}(P_{23}))</th>
<th>(P_{23})</th>
<th>(LM \times_M LM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Y)</td>
<td>(Y)</td>
<td>(Y)</td>
</tr>
<tr>
<td>(\partial Y)</td>
<td>(\partial Y)</td>
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</tr>
<tr>
<td>(\partial Y)</td>
<td>(\partial Y)</td>
<td>(\partial Y)</td>
</tr>
<tr>
<td>(\phi_Y)</td>
<td>(\phi_Y)</td>
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</tr>
<tr>
<td>(\phi_Y)</td>
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<tr>
<td>(\phi_Y)</td>
<td>(\phi_Y)</td>
<td>(\phi_Y)</td>
</tr>
</tbody>
</table>

The same is true for \([P_{23}]\) and \(X\):

<table>
<thead>
<tr>
<th>(\partial X \cong \mathbb{S}(P_{12}))</th>
<th>(P_{12})</th>
<th>(LM \times_M LM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X)</td>
<td>(X)</td>
<td>(X)</td>
</tr>
<tr>
<td>(\partial X)</td>
<td>(\partial X)</td>
<td>(\partial X)</td>
</tr>
<tr>
<td>(\partial X)</td>
<td>(\partial X)</td>
<td>(\partial X)</td>
</tr>
<tr>
<td>(\phi_X)</td>
<td>(\phi_X)</td>
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<td>(\phi_X)</td>
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<td>(\phi_X)</td>
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<td>(\phi_X)</td>
</tr>
</tbody>
</table>

with \([\partial \langle X, a_X, \phi_X \rangle] = \zeta_3^2 [P_{23}, a_{23}, \gamma \circ f_{23}]\). In other words we have
\[
[\partial \langle Y \rangle] = [P_1] \ast [P_2] \quad \text{and} \quad [\partial \langle X \rangle] = [P_2] \ast [P_3].
\] (7.4)
7.2 Construction of the threefold intersections - simple case

In the simple case, the loop-concatenations are completely independent from our intersection constructions; we must, first and foremost, construct geometric models for (6.2). We construct “triple intersections” in analogy to section 4.2, taking the product of one our cycles $[P_i]$ with the sphere-normal bundle on the product of the other two. In our current context however, each new intersection picks up a further copy of $LM$, and as such we expect to land in $LM \times_M LM \times_M LM$:

\[
\begin{align*}
P_{12,3} = S(P_{12}) \times P_3 & \longrightarrow S(P_{12}) \times P_3 \\
\downarrow & \downarrow \\
P_{123} = P_{12} \times P_3 & \longrightarrow P_1 \times P_2 \times P_3 \\
\downarrow & \downarrow & \downarrow \\
LM \times_M LM \times_M LM & \longrightarrow LM \times LM \times LM \\
\downarrow & \downarrow & \downarrow \\
M & \Delta & M \times M \quad \Delta \times id & \longrightarrow M \times M \times M.
\end{align*}
\]

whence $P_{12,3}$ consists of tuples $([x, y], z, m)$ where $[x, y]$ is some point in the fibre above $(x, y) \in P_{12}$ and $f_1(x)(0) = f_2(y)(0) = f_3(z)(0) = m$ and the map $f_{12,3}$ takes this tuple to $(f_1(x), f_2(y), f_3(z))$ as indicated. Similarly:

\[
\begin{align*}
P_{1,23} = P_1 \times S(P_{23}) & \longrightarrow P_1 \times S(P_{23}) \\
\downarrow & \downarrow & \downarrow \\
P_{123} = P_1 \times P_23 & \longrightarrow P_1 \times P_23 \longrightarrow P_1 \times P_2 \times P_3 \\
\downarrow & \downarrow & \downarrow \\
LM \times_M LM \times_M LM & \longrightarrow LM \times (LM \times_M LM) \longrightarrow LM \times LM \times LM \\
\downarrow & \downarrow & \downarrow \\
M & \Delta & M \times M \quad id \times \Delta & \longrightarrow M \times M \times M.
\end{align*}
\]
7.3 The triple products as boundary - simple case

Proceeding as before, we note that both of the triple intersections represent the same homology class \([P_{123}]\).

Continuing with \(X\) and \(Y\), let us recall that

\[
[\partial \langle X, a_X, f_X \rangle] = \zeta_3^2 [P_{23}, a_{23}, f_{23}] \in h_*(LM \times_M LM)
\]

and:

\[
[\partial \langle Y, a_Y, f_Y \rangle] = \zeta_2 [P_{12}, a_{12}, f_{12}] \in h_*(LM \times_M LM).
\]

We define \(P_{1X}\) and \(P_{Y3}\) as

\[
P_{1X} := P_1 \times X \quad \text{which has boundary } P_{1,23}
\]

\[
P_{Y3} := Y \times P_3 \quad \text{which has boundary } P_{12,3}
\]

defined via diagrams

\[
\begin{array}{ccc}
P_{1X} & \longrightarrow & P_1 \times X \\
\downarrow & & \downarrow \\
LM \times_M (LM \times_M LM) & \longrightarrow & LM \times (LM \times_M LM) \\
\downarrow & & \downarrow \\
M & \rightarrow & M \times M \\
\end{array}
\]

with boundary:

\[
\begin{array}{ccc}
P_{1,23} & \longrightarrow & P_1 \times S(P_{23}) \\
\downarrow & & \downarrow \\
LM \times_M (LM \times_M LM) & \longrightarrow & LM \times (LM \times_M LM) \\
\downarrow & & \downarrow \\
M & \rightarrow & M \times M \\
\end{array}
\]

and

\[
\begin{array}{ccc}
P_{Y3} & \longrightarrow & Y \times P_3 \\
\downarrow & & \downarrow \\
(LM \times_M LM) \times_M LM & \longrightarrow & (LM \times_M LM) \times_M LM \\
\downarrow & & \downarrow \\
M & \rightarrow & M \times M \\
\end{array}
\]

with boundary:

\[
\begin{array}{ccc}
P_{12,3} & \longrightarrow & S(P_{12}) \times P_3 \\
\downarrow & & \downarrow \\
(LM \times_M LM) \times_M LM & \longrightarrow & (LM \times_M LM) \times_M LM \\
\downarrow & & \downarrow \\
M & \rightarrow & M \times M \\
\end{array}
\]

46
Summary: Signs and orientations

\[
\begin{align*}
\bar{\partial}(X,a_X,f_X) &= (-1)^a \partial X, a_X|_{\partial X}, f_X|_{\partial X} = \zeta_3^2 [P_{23}, a_{23}, f_{23}] = [P_2, a_2, f_2] [P_3, a_3, f_3] \\
\bar{\partial}(Y,a_Y,f_Y) &= (-1)^a \partial Y, a_Y|_{\partial Y}, f_Y|_{\partial Y} = \zeta_2^1 [P_{12}, a_{12}, f_{12}] = [P_1, a_1, f_1] [P_2, a_2, f_2] \\
\langle P_1 \rangle \ast \langle X \rangle &= \zeta_X^1 \langle P_{1X} \rangle \\
\langle P_Y \rangle \ast \langle P_3 \rangle &= \zeta_3^Y \langle P_{3Y} \rangle 
\end{align*}
\]

and

\[
\begin{align*}
\bar{\partial}[P_1][X] &= (-1)^i [P_1] \partial [X] = (-1)^i [P_1][P_2][P_3] = (-1)^i \zeta_3^2 [P_1][P_{23}] \\
&= (-1)^i \zeta_3^2 \zeta_{23}^1 [P_{123}] = (-1)^a [\partial P_{1X}] \\
\bar{\partial}[Y][P_3] &= [P_1][P_2][P_3] = \zeta_2^1 [P_{12}][P_3] = \zeta_3^2 \zeta_2^1 [P_{123}]
\end{align*}
\]

7.4 A closed representative - simple case

In the simple case, in which the Chas-Sullivan-Massey product factors over \( h_* (LM \times_M LM \times_M LM) \), we need only find a representative for what we called in (6.2) the Basepoint-Intersection-Massey product. We start with the same formula as in section 4.4:

\[
\langle [P_1], [P_2], [P_3] \rangle = \epsilon_{1X} [P_{1X}] + \epsilon_{Y3} [P_{Y3}],
\]

and we make the following observation

Remark 7.1. The Chas-Sullivan-Massey Product of classes \([P_1],[P_2]\) and \([P_3]\) is given by

\[
\langle P_1, P_2, P_3 \rangle = \gamma_{3*} (\epsilon_{1X} [P_{1X}] + \epsilon_{Y3} [P_{Y3}]).
\]

The signs \( \epsilon_{1X} \) and \( \epsilon_{Y3} \) are \((-1)^i+1(-1)^{a_X+d_X}\) and \((-1)^i+1(-1)^{a_3+d_{33}}\) respectively (c.f. section 4.4).
After taking sphere bundles on $P_{1X}$ and $P_{Y3}$, there will be a bordism between their respective boundaries and as such we can glue them together to form a closed manifold representative.

Here is an overview of how all the pieces fit together (all inclusions are boundary inclusions):

7.5 The triple intersections - general case

Geometrically, when we form the triple products for the Chas-Sullivan-Massey product, the ingredients are all the same. We have manifolds $P_1, P_2$ and $P_3$, we have their (basepoint)-intersections $P_{12}$ and $P_{23}$ and sphere bundles $S(P_{12})$ and $S(P_{23})$ on them which are the boundaries of $Y$ and $X$ respectively; and even the resulting triple product manifolds $P_{1,23}$ and $P_{12,3}$ are the same as in the previous case. However in this context we want to have different maps to the target space. In particular we need to compose the loops on the image of $P_{12}$ and $P_{23}$ in order for the classes to coincide with the boundaries of $[X]$ and $[Y]$. We will indicate the fact that we are composing loops by adding brackets, writing $f_{(12),3}$ for
\((\gamma \times \text{id}) \circ f_{12,3}\) and writing \(P_{(12),3}\) or \(P_{12,3}\) depending on which map we are considering. The full details are summarised in the commutative diagram below.

So again \(P_{12,3}\) consists of tuples \(([x, y], z, m)\) where \([x, y]\) is some point in the fibre above \((x, y) \in P_{12}\) and \(f_1(x)(0) = f_2(y)(0) = f_3(z)(0) = m\) and

\[
f_{12,3}([x, y], z, m) = (f_1(x), f_2(y), f_3(z))
\]

\[
f_{(12),3}([x, y], z, m) = (\gamma(f_1(x), f_2(y)), f_3(z)).
\]

### 7.6 Triple Products as boundary - general case

In analogy to equation (7.5) we may be tempted to set

\[
\left[ P_{1X}, a_{1X}, \varphi_{1X} \right] = \zeta_1^X \left[ P_{1X}, a_1, f_1 \right] \ast \left[ X, a_X, \phi_X \right]
\]

\[
\left[ P_{Y^3}, a_{Y^3}, \varphi_{Y^3} \right] = \zeta_3^Y \left[ Y, a_Y, \phi_Y \right] \ast \left[ P_{3Y}, a_3, f_3 \right],
\]

only this time we’d have \([\hat{\partial} \langle P_{1X} \rangle] = \pm \left[ P_{1,(23)} \right]\) and \([\hat{\partial} \langle P_{Y^3} \rangle] = \pm \left[ P_{(12),3} \right]\).

It is not immediately obvious that these boundaries two should be the same. Indeed, when viewed in \(h_*(LM \times_M LM)\) they may not be. Then again, in this diagram:
the square on the right commutes up to homotopy and so we can conclude that

\[
\left[ P_{1,23}, a_{1,23}, \gamma \circ f_{1,(23)} \right] = \left[ P_{12,3}, a_{12,3}, \gamma \circ f_{(12),3} \right]
\]

or, to put it another way

\[
\gamma_* \left[ P_{(12),3} \right] = \gamma_* \left[ P_{1,(23)} \right] \in h_*(LM).
\]

Therefore the correct analogy to (7.5) should be

\[
\left[ P_{1X, a_{1X}, \gamma \varphi_{1X}} \right] = \zeta_X \left[ P_{1, a_1, f_1} \right] \cdot \left[ X, a_X, \phi_X \right] \\
\left[ P_{Y3, a_{Y3}, \gamma \varphi_{Y3}} \right] = \zeta_3 \left[ Y, a_Y, \phi_Y \right] \cdot \left[ P_{3, a_3, f_3} \right],
\]

using the same manifolds but different maps so that here up to sign

\[
\left[ \partial \langle P_{1X} \rangle \right] = \gamma_* \left[ P_{1,23} \right] = \gamma_* \left[ P_{(12),3} \right] = \left[ \partial \langle P_{Y3} \rangle \right].
\]

The relevant defining diagrams are

\[
\begin{array}{c}
\begin{array}{ccc}
P_{1X} & \to & P_1 \times X \\
\uparrow_{\gamma \varphi_{1X}} & & \uparrow_{f_1 \times \varphi_X} \\
LM \times_{a_X} LM & \to & LM \times LM \\
\uparrow_{\gamma} & & \uparrow_{\gamma} \\
LM & \to & LM \\
\uparrow_{ev_0 \times ev_0} & & \downarrow \\
M & \to & M \times M \\
\end{array}
\end{array}
\]

where:

\[
\begin{array}{ccc}
P_{1X} & \to & P_1 \times X \\
\uparrow_{\gamma \varphi_{1X}} & & \uparrow_{f_1 \times \varphi_X} \\
LM & \to & LM \\
\uparrow_{\gamma} & & \downarrow \\
M & \to & M \times M \\
\end{array}
\]

and

\[
\begin{array}{ccc}
P_{1,23} & \to & LM \times M \\
\uparrow_{\gamma \varphi_{1,23}} & & \uparrow_{\gamma} \\
LM \times M & \to & LM \\
\uparrow_{\gamma} & & \downarrow \\
LM & \to & LM \\
\end{array}
\]

and

50
7.7 A closed representative for the Chas-Sullivan-Massey Product

In this section we construct a closed representative for the Chas-Sullivan-Massey product and we complete the proof of Lemma 6.4 that we postponed at the end of section 6.2.

Proof. (of Lemma 6.4)

A Massey product element

\[ (-1)^{j+1} [Y] \bullet [P_3] + (-1)^{i+j} [P_1] \bullet [X] \]

is equivalently given by

\[ \epsilon_3 [P_Y] + \epsilon_1 [P_1], \quad (7.7) \]

where the signs are just as in section 4.4. In order to correctly interpret this sum let us carefully examine what happens to the boundaries of manifold \( P_1 \) and \( P_Y \).

\[ \partial P_Y / P_{1,23} \]

and

\[ \partial P_1 / P_{1,23} \]
To make sense of the sum in equation (7.7) we must carefully note our context. A priori the (relative) homology groups involved are

\[ h_*(LM \times_M LM, f_{1(23)}(P_{123})) \xrightarrow{\gamma_*} h_* (LM, f_{1(23)}(P_{123})) \]

\[ [P_{1X}, a_{1X}, \varphi_1X] \rightarrow [P_{1X}, a_{1X}, \gamma \varphi_1X] \]

and

\[ h_*(LM \times_M LM, f_{(12)3}(P_{123})) \xrightarrow{\gamma_*} h_* (LM, f_{(12)3}(P_{123})) \]

\[ [P_{Y3}, a_{Y3}, \varphi_3Y] \rightarrow [P_{Y3}, a_{Y3}, \gamma \varphi_3Y] \]

and \( f_{1(23)}(P_{123}) \) is a different subspace of \( LM \times_M LM \) than \( f_{(12)3}(P_{123}) \) and also \( f_{1(23)}(P_{123}) \) is a different subspace of \( LM \) than \( f_{(12)3}(P_{123}) \).

There is no way we can take the sum in the former pair but we can find a way to make sense of the latter pair. Indeed, let \( \Gamma \) be the subspace of \( LM \) consisting of all possible parametrisations of the concatenated loop \( f_1(x)f_2(y)f_3(z) \). Then we regard both \([P_{Y3}, a_{Y3}, f_{(12)3}]\) and \([P_{1X}, a_{1X}, f_{1(23)}]\) as classes in \( h_*(LM, \Gamma) \) and we can take their sum there. The boundary operator from the long exact sequence then takes \( \epsilon_{1X} [P_{1X}] + \epsilon_{Y3} [P_{Y3}] \) to zero in \( h_*(\Gamma) \) and so we have a well defined element in \( h_*(LM) \) and thus a Chas-Sullivan-Massey Product

\[ \langle P_1, P_2, P_3 \rangle \in h_{i+j-k-2d+1}(LM) \cdot [P_1] \cdot h_*(LM) + h_*(LM) \cdot [P_3]. \]

In order to construct a closed representative we wish to “glue” together \( P_{1X} \) and \( P_{Y3} \). As noted previously, this may not be possible, however, there are sphere bundles on \( P_{1X} \) and on \( P_{Y3} \) whose boundaries are cobordant via a cobordism \( \Sigma \). We may therefore glue these together to get a closed representative like so:

52
II Spectral Sequences & Calculations

8 The Loop Homology Spectral Sequence

The Loop Homology Spectral Sequence (LHSS) is a shifted homological Serre spectral sequence for the free loop fibration:

\[ \Omega_b M \rightarrow LM \]

The LHSS has a grading shift of \((-d, 0)\) relative to the Serre spectral sequence and is a multiplicative spectral sequence in which the product converges to the Chas-Sullivan product. The spectral sequence is described in Cohen et al. [3] and by Nora Seeliger in [21],[22]. We return to Seeliger’s calculations in section 8.4. Then in section 8.5 we use a picture by Nancy Hingston [8] to reinterpret Seeliger’s calculation geometrically. We will return to this geometric picture again when we analyse Chas-Sullivan-Massey products in the LHSS in section 9.

8.1 The Serre & LHSS using Geometric Homology

In this section we introduce the Serre and LHSS spectral sequences from a geometric homology point of view and we review how the Chas-Sullivan product fits into this discussion.
Recall: The Serre Spectral Sequence

We start with a Fibration $F \to E \to B$ and filter $B$ by its skeleta. Let $E^{(p)}$ be the preimage of the $p$-skeleton of $B$. Using this filtration of $E$ we obtain a spectral sequence with

$$E^1_{p,q} = h_{p+q} \left( E^{(p)}, E^{(p-1)} \right)$$

and

$$d^1 : E^1_{p,q} \to E^1_{p-1,q}$$

which converges to $h_{p+q}(E)$.

Following the proof in Dyer’s book [5] we get $E^1_{p,q} \cong \text{Cell}_p(B, h_q(F))$; that is the cellular chains on $B$ with values in the cohomology of the fiber (as a local coefficient system). This leads to the more common description of the spectral sequence:

$$E^2_{p,q} \cong H_p(B, h_q(F)).$$

How does the Serre Spectral Sequence look in Geometric Homology? Well $h_{p+q}(E^{(p)}, E^{(p-1)})$ contains classes $[P,a,f]$ where $P$ is a $d$-dimensional $h$-oriented Manifold with boundary, $a \in h^k(P)$ (where $d-k = p+q$) and $f : (P, \partial P) \to (E^{(p)}, E^{(p-1)})$. The isomorphism identifying $E^1_{p,q}$ with $\text{Cell}_p(B, h_q(F))$ uses both excision and the homotopy axiom. Let us recall excision in Geometric Homology.

Recall: Excision.

Let $[P,a,f] \in h_r(X, A)$ i.e. $f$ maps $P$ to $X$ taking $\partial P$ to $A$ and let $U \subset A$ and $\bar{U} \subset \hat{A}$.

We define

$$\Psi : h_r(X, A) \to h_r(X-U, A-U).$$
We note that the \( B \coloneqq P - f^{-1}(\hat{A}) \) and \( C \coloneqq f^{-1}(\bar{U}) \) are closed, disjoint subsets of \( P \) and so we can find a function \( \tau: P \to \mathbb{R} \) with \( \sup \{ \tau(b); b \in B \} < \inf \{ \tau(c); c \in C \} \). We have an \( \epsilon \) strictly between the supremum and the infimum which is also a regular value of \( f \) and \( f|_{\partial P} \). Let \( S \coloneqq \tau^{-1}(\infty, \epsilon] \). At this point we may need to use a straightening-the-angle argument, but essentially we now have a submanifold \( S \) of \( P \) that we can use to define \( \Psi \).

\[
\Psi: [P,a,f] \mapsto [S,a|_S,f|_S].
\]

Getting back to our fibration, filtration and spectral sequence, we were looking at \([P,a,f]\) with \( f: (P, \partial P) \to (E(\nu), E(\nu-1)) \). To end up with the familiar \( E_2 \)-term we must first fatten \( E(\nu-1) \) which we are allowed to do by the homotopy axiom. Then using excision we remove \( E(\nu-1) \) and we are left with a disjoint union - over all cells \( \alpha \) of \( B \) - of classes \([P_\alpha,a_\alpha,f_\alpha]\). Using arguments adapted from Dyer’s book [5], in each cell \( \alpha \) we restrict to (intersect transversely with) the fibre \( E_\alpha \) over a single point \( x_\alpha \), to get manifolds \( P_\alpha \) of dimension \( d - \nu \).

In sumary, we start with classes \([P,a,f] \in h_{p,*}(E(\nu), E(\nu-1))\) and end up with \( \Sigma_\alpha[P_\alpha,a_\alpha,f_\alpha] \) which is isomorphic\(^1\) to \( Cell_p(B,h_q(F)) \).

In his diploma thesis, Lennart Meier ([16], [17]) describes how to translate the multiplicative structure of the LHSS into this setting. Specifically he shows

**Theorem A. Intersecting on the Base**

Let \( \xi = (F \to E \xrightarrow{\pi} B) \) be a fibration over a finite dimensional manifold \( B \) and let \( A \subset B \) be a closed codimension \( r \) submanifold with \( h_* \)-oriented normal bundle. Then there is a morphism \( s_A \) of convergent spectral sequences from \( \text{Serre}(\xi) \) to \( \text{Serre}(\xi|_A) \) of degree \((-r,0)\) on the \( E^1 \)-

---

\(^1\)Each \([P_\alpha,a_\alpha,f_\alpha]\) is a class in \( h_q(E_\alpha) \) - and \( E_\alpha \) is the local fibre at \( x_\alpha \). The sum, being parametrised by the \( p \)-cells \( \alpha \) of \( B \) implies that any element of \( +_\alpha[P_\alpha,a_\alpha,f_\alpha] \) gives a function from \( p \) cells of \( B \) into the local system \( h_q(F) \). Finally, boundary maps on \([P_\alpha,a_\alpha,f_\alpha]\) agree with the cellular boundary map.
term and this morphism induces the standard Gysin\textsuperscript{2} map on the $E^2$ term: $h_p(B; h_q(F)) \to h_{p-r}(A; h_q(F))$.

The proof of this theorem deals with the transversality questions of intersecting $[P, a, f] \in h_{p+q}(E^{(p)}, E^{(p-1)})$ with $A$, or rather with $\pi^{-1}(A) = E|_A$. By transversely intersecting we get a manifold $Q = f^{-1}\pi^{-1}(A)$ and the map in question is just a restriction to this manifold.

$$[P, a, f] \mapsto [Q, a|_Q, f|_Q] \in h_{(p-r)+q}(E|_A^{(p-r)}, E|_A^{(p-r-1)})$$

Using this theorem, we describe the intersection product by taking $M \times M$ as our base manifold $B$ and the diagonal $\Delta(M)$ as subspace $A$.

<table>
<thead>
<tr>
<th>Recall:</th>
<th>Products of Spectral Sequences.</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>There is a product of spectral sequences, i.e. the category of spectral sequences is monoidal. A spectral sequence is a monoid if and only if it is multiplicative. If $E$ and $E'$ are spectral sequences, then the product is given by $(E \otimes E')^k_{p,q} = \oplus_{i,j} E^k_{i,j} \otimes E'^k_{(p-i),(q-j)}$ and $d(e \otimes e') = d(e) \otimes e' + (-1)^{i+j} e \otimes d(e')$.</td>
</tr>
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</table>

The following theorem is again due to Meier ([16], [17]).

**Theorem B.** Let $M$ be an $h_*$ oriented $d$-dimensional manifold. Then the Serre spectral sequence $S(\Omega M \to LM \to M)$ when shifted by $(-d,0)$ has the structure of a multiplicative spectral sequence - with product converging to the Chas Sullivan product on $h_*(LM)$. The induced product on the $E^2$ term is the intersection product on $M$ with coefficients in the local system $h_*(\Omega M)$ using the Pontryagin product.

In the geometric homology description the picture is even clearer. Looking directly at the $E^1$ page, we show

\textsuperscript{2}This is the same as the Transverse Intersection map or the Umkehr map
Lemma 8.1. The loop homology spectral sequence filters the Chas-Sullivan product.

Proof. As we have done before, we consider the cross product map followed by transverse intersection with the diagonal followed by the loop concatenation just as in section 6, except that this time we construct a map of spectral sequences.

Let us take a closer look, reading right to left.

The final map is loop concatenation. It is induced by the map on the space level and it is this map that induces the Pontryagin product on \( h_* (\Omega M) \).

The middle map is the intersection on the base with \( \Delta (M) \) and it is the basis for the intersection product on \( h_* (M) \). This map has degree \((-d, 0)\).

I want to focus on the first map, for the moment in the unshifted case.

So

\[
E^1_{p,q} = h_{p+q} \left( LM^{(p)}, LM^{(p-1)} \right).
\]

By the product formula we have

\[
(S \otimes S)^1_{p,q} = \bigoplus h_{i+j} (LM^{(i)}, LM^{(i-1)}) \otimes h_{p-i+q-j} (LM^{(p-i)}, LM^{(p-i-1)}).
\]

The fact that the cross product is a map of spectral sequences, means that we land in

\[
h_{p+q} \left( (LM \times LM)^{(p)}, (LM \times LM)^{(p-1)} \right).
\]
This in turn is equal to

\[ \bigoplus_{i=0}^{p} h_{p+q} \left( LM^{(i)} \times LM^{(p-i)}, LM^{(i-1)} \times LM^{(p-i)} \right) \cup LM^{(i)} \times LM^{(p-i-1)} \]

which is more commonly written as

\[ \bigoplus_{i=0}^{p} h_{p+q} \left( \left( LM^{(i)}, LM^{(i-1)} \right) \times \left( LM^{(p-i)}, LM^{(p-i-1)} \right) \right) \]

The leftmost map above, the product map, is the map from the relative Künneth theorem.

**Recall:** Relative Künneth Theorem

There is a short exact sequence

\[
0 \to (h_*(X, A) \otimes h_*(Y, B))_{p+q} \to h_{p+q} ((X, A) \times (Y, B))) \to \ldots \\
\text{Tor} (h_*(X, A), h_*(Y, B))_{p+q-1} \to 0.
\]

Nominally, this map has degree \((0,0)\), However when we shift our spectral sequence by \((-d,0)\), then this map has a degree of \((d,0)\). When we postcompose with the transverse intersection of the diagonal in the base, the resulting map has degree \((0,0)\).

In detail, including all signs, what happens to \([P_1, a_1, f_1] \otimes [P_2, a_2, f_2] \) in the above chain of morphisms?

\[
[P_1] \otimes [P_2] \mapsto (-1)^{P_1a_2} [P_1 \times P_2, a_1 \times a_2, f_1 \times f_2] \mapsto \zeta_1 [P_{12}, a_{12}, f_{12}] \mapsto \zeta_2 [P_{12}, a_{12}, \gamma f_{12}]
\]

or

\[
[P_1] \otimes [P_2] \longrightarrow [P_1] \times [P_2] \longrightarrow [P_1] * [P_2] \longrightarrow [P_1] \cdot [P_2].
\]

59
So we see already on $E^1$ that the product is the Chas-Sullivan product; albeit on the constituent pieces $LM^{(\alpha)}$ as opposed to $LM$ itself. Thus the spectral sequence filters the Chas-Sullivan product.

8.2 Convergence of the Serre and LHSS

We examine the convergence of the Serre and the Loop Homology spectral sequences, described in the context of geometric homology theory. It turns out that geometric homology allows for a lovely narrative for convergence in which a lot of the technicalities involved in the chain level description (iterated sub-quotients) fall away and a clear geometric picture emerges to answer the question of when a class survives and also why the spectral sequence converges to the associated graded as opposed to the homology of the total space itself.

Let us begin with a fibration $F \to T \to B$ over a CW space $B$; we obtain the Serre Spectral Sequence in homology by using the filtration of $B$ by skeleta $B^p$. We filter the total space $T$ by $T^{(p)} := \pi^{-1}(B^p)$

$$\emptyset = T^{(-1)} \subset T^{(0)} \subset T^{(1)} \subset \ldots \subset T^{(n)} = T.$$  

What are classes on the $E^1$ page?

The $E^1$ page of the spectral sequence is

$$E_{p,q}^1 = h_{p+q}(T^{(p)}, T^{(p-1)}).$$

In other words, classes on the $E^1$ page are given by geometric cycles whose boundaries fall in (at least) one lower filtration - see figure 1.
Since the same geometric cycle can represent classes on different pages of the spectral sequence, we need to keep track of our current context. Thus, when we regard classes in the $E^r$ page of the spectral sequence, we shall denote this by a subscript; so for example, classes in $E^1_{p,q}$ are written as $[a]_1$, where $[a] = \langle P_a, x_a, f_a \rangle$ with

$$f_a : (P_a, \partial P_a) \to (T(p), T(p-1)).$$

**What does it mean for the class to survive to $E^2$?**

In order to survive, it must be a cycle with respect to the differential $d_1$. The differential $d_1$
is given by

\[ h_{p+q}(T^{(p)}, T^{(p-1)}) \]

\[ \xrightarrow{\partial} \]

\[ h_{p+q-1}(T^{(p-2)}) \]

\[ \xrightarrow{i_*} \]

\[ h_{p+q-1}(T^{(p-1)}) \]

\[ \xrightarrow{j_*} \]

\[ h_{p+q-1}(T^{(p-1)}, T^{(p-2)}) \]

\[ \xrightarrow{\ldots} \]

and in order for \([a]_1\) to survive to \([a]_2\) on the \(E^2\) page, we need \(d_1 [a]_1 = 0\).

This means that \(\partial [a] \in im(i_*)\) i.e. there exists a class \([b] \in H_{p+q-1}(T^{(p-2)})\) s.t. \(i_* [b] = \partial [a]\).

We note that \(\partial [a]\) is given by \((-1)^{x_*}\partial P_a, x_a|_{\partial P_a}, f_a|_{\partial P_a}\). Geometrically, this tells us that \([a]_1\) survives to the \(E^2\) page iff \(\langle \partial a \rangle\) is homologous to a geometric cycle \(\langle b \rangle\) mapping to \(T^{(p-2)}\). After possible bundle-modification, such a homology can be realised as a bordism \(\langle a' \rangle\) as indicated in Figure 2.

Note that \([a] = [a + a'] \in H_{p+q}(T^{(p)}, T^{(p-1)})\) and \(\partial [a] = \partial [a + a']\).

Geometrically, glueing \(\langle a \rangle\) to \(\langle a' \rangle\) along \(\langle \partial a \rangle\) allows us to construct a geometric cycle – call it \(\langle a_2 \rangle\) – with the property that \([a] = [a_2] \in H_{p+q}(T^{(p)}, T^{(p-1)})\) (i.e. \([a]_1 = [a_2]_1\)), and \(\langle a_2 \rangle\) represents a class in \(H_{p+q}(T^{(p)}, T^{(p-2)})\).
In short: A class survives from $E^1_{p,q}$ to $E^2_{p,q}$ if it has a representative whose boundary maps to two filtrations lower – in this case $T^{(p-2)}$.

Lastly, we note that the choices of $[b]$, $[a']$ and $[a_2]$ were not unique, so while we can say that $[a]_1$ survives to $E^2$ if it has a representative $[a_2]$ that represents a class in $h_{p+q}(T^{(p)}, T^{(p-2)})$, we do not have a well defined class in $h_{p+q}(T^{(p)}, T^{(p-2)})$. Choices are allowed to differ below $T^{(p)}$ (i.e. in $T^{(p-1)}$) and this is expressed in the fact that the spectral sequence does not converge to $h_*(T)$ itself, but only to the associated graded, and it is part of the reason we need to distinguish between $[a_2]$, $[a_2]_1$ and $[a_2]_2$.

**What does it mean for a class to survive to $E^3$ and higher?**

For $[a]_1$ to survive to the $E^3$ page, we need $d_2 [a_2]_2 = 0.$
The boundary map $d_2$, just as $d_1$, is essentially just a restriction to boundary, only in this case we end up two filtrations lower. Indeed this is true for all $n$; the boundary map $d_n$ restricts to boundary $n$-filtrations below and includes into the next relative homology group. For our class $[a_2]_2$ on the $E^2$ page we have representatives $\langle a_2 \rangle$ in $h_{p+q}(T^{(p)}, T^{(p-2)})$ on which $d_2$ is given by

$$
\cdots \xrightarrow{\partial} h_{p+q-1}(T^{(p-3)}) \xrightarrow{i_*} h_{p+q-1}(T^{(p-2)}) \xrightarrow{j_*} h_{p+q-1}(T^{(p-2)}, T^{(p-3)}) \xrightarrow{d_2} \cdots
$$

Focus on the bottom row; the long exact sequence

$$
\cdots \rightarrow h_{p+q-1}(T^{(p-3)}) \xrightarrow{i_*} h_{p+q-1}(T^{(p-2)}) \xrightarrow{j_*} h_{p+q-1}(T^{(p-2)}, T^{(p-3)}) \rightarrow \cdots
$$

There exists $c \in H_{p+q-1}(T^{(p-3)})$ s.t. $i_*(c) = [b]$ and $j_*(c) = [a]$ survoves through to the $E^3$ page. We ask whether or not an element $\langle c \rangle$ exists as indicated. There are two cases to consider.

**Case 1**

If $j_* [b] = 0$, then $[b] = i_* [c]$ and $[a]$, $a_1$ survives through to the $E^3$ page.

**Case 2** (Illustrated in Figure 3)

If $j_* [b] \neq 0$, but $j_* [b] \in \text{Im } d_1$, then $[j_* [b]]_2 = 0$ on $E^2$ page. This means that we can modify $\langle a_2 \rangle$ by an $h : (P_h, \partial P_h) \rightarrow (T^{(p-1)}, T^{(p-2)})$ in such a way that $d_1 [h] = j_* [\partial h] = j_* [b]$. So $j_* [\partial h] = j_* [b]$ and so $j_* [b - \partial h] = 0$ and so we are back in Case 1 using $[b - \partial h]$ instead of just $[b]$. Thus $\exists [c] \in H_{p+q-1}(T^{(p-3)})$ s.t. $i_* [c] = [b - \partial h]$. And in the relative homology $h_{p+q}(T^{(p)}, T^{(p-1)})$ we have $[a] = [a + a'] = [a + a' - h]$ and we have $\partial [a] = \partial [a + a'] = \partial [a + a' - h] = [b - \partial h] = i_* [c]$.

After a possible bundle modification we can realise these equalities by bordisms – let’s call them $h$ and $a''$ — and this allows us to construct a cycle $\langle a_3 \rangle$ with boundary in $T^{(p-3)}$. The
cycle $\langle a_3 \rangle$ represents a class in $h_{p+q} (T^{(p)}, T^{(p-3)})$, but due to the choices involved the class is not well defined. However, the fact that we can make these choices implies that $\langle a \rangle_1$ survives to the $E^3$ page as $\langle a_3 \rangle_3$.

![Diagram](image.png)

**Figure 3:** $\langle a_3 \rangle$ surviving to $E^3$

Finally, we can see that a relative homology class in $E^{1}_{p,q}$ survives all the way to $E^\infty$ if it can be extended to have boundary in $T^{(-1)} = \emptyset$ i.e. if it has a representative that is closed in $T^{(p)}$. The choices involved will dictate that we do not necessarily have a well defined class in $h_* (T)$ but only in the associated graded.
8.3 Convergence of the Chas Sullivan product in the LHSS

The discussion above applies equally to the homological Serre Spectral Sequence as to the Loop Homology Spectral Sequence; the latter essentially being just a special case of the former. The grading shift we introduced in the Loop Homology Spectral Sequence only starts to play a role when we look at the multiplicative structure.

Let the dimension of our manifold $M$ be $d$. Then the Chas Sullivan product takes the form

$$h_i(\mathcal{L}M) \otimes h_j(\mathcal{L}M) \rightarrow h_{i+j-d}(\mathcal{L}M)$$

and for this reason we introduced the

**Loop Homology Grading.** It is traditional to write $\mathbb{H}_s$ for $h_{s+d}$. Thus for example $\mathbb{H}_k(\mathcal{L}M)$ is $h_{k+d}(\mathcal{L}M)$ and the product preserves the grading:

$$\bullet : \mathbb{H}_k(\mathcal{L}M) \otimes \mathbb{H}_l(\mathcal{L}M) \rightarrow \mathbb{H}_{k+l}(\mathcal{L}M).$$

In the spectral sequence we have an additional complication given by the filtration, namely we have

$$\bullet : \mathbb{H}_k(\mathcal{L}M^{(s)}, \mathcal{L}M^{(s-r)}) \otimes \mathbb{H}_l(\mathcal{L}M^{(s')}, \mathcal{L}M^{(s'-r)}) \rightarrow \mathbb{H}_{k+l}(\mathcal{L}M^{(s+s'-d)}, \mathcal{L}M^{(s+s'-r-d)}).$$

We thus introduce another grading shift in our notation by introducing the

**Loop Space Filtration.** We write $\mathbb{L}^{(p)}$ for $\mathcal{L}M^{(p+d)}$ and also write $\mathbb{L}$ to denote $\mathcal{L}M$ in this context.
Using these conventions, the Chas Sullivan product

\[ \bullet : h_{p+d+q} \left( LM^{(p+d)}, LM^{(p+d-r)} \right) \times h_{p'+d+q'} \left( LM^{(p'+d)}, LM^{(p'+d-r)} \right) \to h_{p+p'+d+q+q'} \left( LM^{(p+p'+d)}, LM^{(p+p'+d-r)} \right) \]

now simply becomes

\[ \bullet : \mathbb{H}_{p+q} \left( L^{(p)}, L^{(p-r)} \right) \times \mathbb{H}_{p'+q'} \left( L^{(p')}, L^{(p'-r)} \right) \to \mathbb{H}_{p+p'+q+q'} \left( L^{(p+p')}, L^{(p+p'-r)} \right). \]

The Loop Homology Spectral Sequence is a Serre Spectral Sequence with a shift of bidegree \((-d, 0)\). Thus we complete our notation by introducing the

**Loop Homology Spectral Sequence grading.** We write \( E \) in the loop homology spectral sequence and \( E \) in the Serre spectral sequence so that \( \mathbb{E}_{p,q} = E_{p+d,q}^r \).

Using this notation and grading, the spectral sequence is multiplicative with the product

\[ \bullet : E_{p+d,q}^r \times E_{p'+d,q'}^r \to E_{p+p'+d,q+q'}^r \]

is now written as

\[ \bullet : \mathbb{E}_{p,q}^r \times \mathbb{E}_{p',q'}^r \to \mathbb{E}_{p+p',q+q'}^r. \]

Theorem 1 from [3] tells us that the product on the \( \mathbb{E}_r \) page converges to the Chas Sullivan product. However, this is not the product on \( \mathbb{H}_*(LM) \) itself, but only on the associated graded.

Recall, the filtration on \( L \) induces a filtration on the Loop Homology:

\[ \mathbb{F}^p \coloneqq F^p \mathbb{H}_*(L) = im \left( \mathbb{H}_*(L^{(p)}) \to \mathbb{H}_*(L) \right), \]
giving the associated graded $\oplus_{p} \mathbb{F}_p$ where

$$\mathbb{F}_p := \mathbb{F}^p / \mathbb{F}^{p-1} = \mathbb{F}^p \mathbb{H}^* (L) / \mathbb{F}^{p-1} \mathbb{H}^* (L).$$

If we have (non-zero) classes $\alpha$ and $\beta$ in $\mathbb{H}^i(LM)$ and $\mathbb{H}^j(LM)$, then they are said to “live” in filtration $p_a$ and $p_b$ if they project to non-zero classes in the associated graded as

\[ \bar{\alpha} \in \mathbb{F}_{p_a} \text{ and } \bar{\beta} \in \mathbb{F}_{p_b}. \]

If $\alpha$ is represented by a geometric cycle $\langle a \rangle = \langle P_a, x_a, f_a \rangle$ and $\beta$ is represented by $\langle b \rangle = \langle P_b, x_b, f_b \rangle$, then $\langle a \rangle$ and $\langle b \rangle$ also represent classes $[a]_1$ and $[b]_1$ on $E^1_{p_a, q_a}$ and $E^1_{p_b, q_b}$ respectively. The classes $[a]_1$ and $[b]_1$ converge to $\bar{\alpha}$ and $\bar{\beta}$. The product $[a]_1 \cdot [b]_1$ converges to $\bar{\alpha} \cdot \bar{\beta}$.

The issue we need to be careful about is that of “products jumping filtration”, also known as “the extension problem”.

It can occur that $\bar{\alpha} \cdot \bar{\beta} \neq \alpha \cdot \beta$. More concretely, it can happen that $[a]_r \cdot [b]_r$ converges to zero even though $\alpha \cdot \beta \neq 0$ because $[a]_r \cdot [b]_r$ converges to an element in $\mathbb{F}_{p_a + p_b}$ but the actual product $\alpha \cdot \beta$ may live in filtration $\mathbb{F}_{p_a + p_b - n}$ for some $n > 0$ and is thus zero when viewed in $\mathbb{F}_{p_a + p_b}$. This phenomenon is illustrated in figure 4.
While this phenomenon occurs in any multiplicative spectral sequence, it is especially important in our context, because we must be able to identify when a product is zero in order to be in a situation in which Chas-Sullivan-Massey products can even be defined. The issue of Chas-Sullivan-Massey products in the spectral sequence will be discussed below in section 9.

8.4 Example: using the LHSS

In this section we recall calculations made by Seeliger in [21]. She calculates the string topology ring for spheres and projective spaces and we make extensive use of these results later.
on in section 10. Below we will recall Seeliger’s calculation for odd dimensional spheres and also recall her results for even spheres and complex projective spaces. Then, in the following section (8.5) we illustrate the geometry behind this calculation in the case of spheres. This geometric narrative will be used again later in section 10.

**Proposition A. (Seeliger)** For \( n > 1 \) and \( n \) odd, there exists an isomorphism of algebras

\[
\mathbb{H}(LS^n; \mathbb{Z}) \cong \mathbb{Z}[y] \otimes E(x) \quad \text{with } |x| = -n, \quad |y| = n - 1
\]

where \( E(x) \) denotes the exterior algebra on \( x \).

**Proof.** We first look at the fibration \( \Omega S^n \to LS^n \to S^n \). We know that \( h^*(S^n) \cong E(x) \) with \( |x| = n \) and \( h_*(\Omega S^n) \cong \mathbb{Z}[y] \) with \( |y| = n - 1 \). We also know \( h^*(\Omega S^n) \cong \Gamma(\gamma) \) where \( |\gamma| = n - i \) and \( \Gamma \) is the divided powers algebra.

The \( E^2 \) term of the loop homology spectral sequence (a regraded homological Serre spectral sequence) is isomorphic to

\[
\mathbb{E}^2_{p,q} \cong h^p(S^n; h_q(\Omega S^n))
\]

and the \( E_2 \) term of the cohomological Serre spectral sequence is isomorphic to

\[
E_2^{p,q} \cong h^p(S^n, h^q(\Omega S^n)).
\]

The first possibly non-trivial differentials are \( d_n \) and \( d^n \) respectively.
Since the fibration has a section $\sigma : M \rightarrow LM; m \mapsto \text{const}_m$, the elements in the base of the Serre spectral sequence must survive to $E^\infty$. This is because $\pi \circ \sigma = \text{id}_M$ and so $\sigma^* \circ \pi^* = \text{id}_{H^*(LM)}$; and so we cannot kill the classes coming from the base. Therefore, in the Serre cohomology spectral sequence (pictured above, right) we know that the class $x$ will not be killed and thus $d^n(\gamma_1) = 0$. This in turn tells us that the cohomology of $LM$ is torsion free through degree $n - 1$ and by the universal coefficient theorem, the homology of $LM$ must also be torsion free in this range. We conclude that $d_n(y) = 0$ in the loop homology spectral sequence (pictured above, left). Since the differentials $d_n$ are derivations of the product, the spectral sequence collapses.

Similarly, from [21] we have for even spheres

**Proposition B.** For $n \geq 2$ and $n$ even, the loop homology spectral sequence for $S^n$ looks like this:
and there exists an isomorphims of algebras

\[ H(LS^n; \mathbb{Z}) \cong E(z) \otimes \mathbb{Z}[x,y] / (x^2, x \cdot z, 2 \cdot x \cdot y) \]

with \( |x| = -n, |y| = 2n - 2, |z| = -1 \),

and for complex projective space

**Proposition C.** The loop homology spectral sequence for \( \mathbb{C}P^n \) looks like this:

\[
\begin{array}{cccccc|c}
  x^n y z & \cdots & x^2 y z & x y z & y z & 2n + 1 \\
  x^n y & \cdots & x^2 y & x y & y & 2n \\
  x^n z & \cdots & x^2 z & x z & z & 1 \\
  x^n & \cdots & x^2 & x & 1 & 0 \\
  -2n & \cdots & -4 & -2 & 0 & \\
\end{array}
\]

where \( d_{2n}(z) = (n + 1)x^ny \). We obtain an isomorphims of algebras

\[ H(\mathbb{C}P^n; \mathbb{Z}) \cong E(w) \otimes \mathbb{Z}[x,y] / (x^{n+1}, (n + 1)x^n \cdot y, w \cdot x^n) \]

with \( |x| = -2, |y| = 2n, |w| = -1 \).
8.5 Example: using the LHSS with Geometric Homology

In this section we look more closely at the geometry of the homology of the free loop space of a sphere. This will illustrate what one of the boundary maps $d_n$ in the LHSS looks like geometrically. We write generic homology classes as $[a]$ and cycles as $<a>$ and only occasionally sketch how to import the ideas into full geometric homology language.

This section is based off of a talk Nancy Hingston gave at a recent workshop in Copenhagen, ([6] – see also [7]). She presented a nice geometric description of the homology of $LS^n$ using geodesics. In particular she states:

The based loop space

The homology of the based loop space of a sphere $h_*(\Omega S^n)$ is generated by

- a class $[b]$ of degree 0 – where $<b>$ is given by the 0-cycle “constant-loop-at-basepoint”
  - and it is the unit with respect to the Pontryagin (Понтр´ ягин) product,

- a class $[y_b]$ of degree $n - 1$ – where $<y_b>$ is the $n-1$ cycle given by “all-circles-at-basepoint-with-initial-tangent-vector-$\vec{v}$” and the vector $\vec{v}$ is arbitrary, but fixed, (this cycle is illustrated in Figure 5),

- a class $[\nu_b]$ of degree $2(n - 1)$ – where $<\nu_b>$ is given by “all-circles-at-the-basepoint”;

but it turns out, that $[\nu_b] = [y_b] \cdot [y_b]$ (using the Pontryagin product). Indeed $h_*(\Omega S^n) \cong \mathbb{Z}[y_b]$.

Let us look at these classes in detail:
Figure 5: The cycle \( < y_p > \) over \( \Omega S^n \) situated at a point \( p \in S^n \)

The fact that \([b]\) is the unit class is obvious.

The cycle \( < y_b > \) (as illustrated in figure 5) we can think of as the intersection of the sphere with all planes which cut through point \( b \) in direction of \( \bar{v} \). The degree of \( < y_b > \) comes about because the initial velocity \( \bar{v} \) is given and the only ambiguity left is where to cross the meridian. In geometric homology we can represent this by \( \langle S^{n-1},1,f_b \rangle \) in which we identify the \( S^{n-1} \) with the meridian (defined as the great circle through \( b \) orthogonal to \( \bar{v} \)), and the map \( f_b(m) \) picks out the circle/loop, based at \( b \) crossing the meridian at the point \( m \).

Finally, we have the \( 2(n-1) \)-cycle \( < \nu_b > \) of “all-circles-at-basepoint” in \( \Omega S^n \). This cycle is the union of \( < y_b > \) for all choices of \( \bar{v} \). This it has \( n-1 \) dimensions for the choice of initial tangent vector and another \( n-1 \) dimensions for the choice of where to cross the corresponding meridian; and we could realise this as some \( [S^{n-1} \times S^{n-1},1,h] \). On closer analysis, it turns
out that this class is the same as \([y^2_b]\) or \([y_b] \cdot [y_b]\) - where we are composing loops using the Pontryagin product. This is because in the cycle \(< y_b >\) of all-circles-with-initial-tangent-\(\vec{v}\), we can relax the condition on \(\vec{v}\): by homotopy invariance, instead of fixing one direction \(\vec{v}\), all we really need to do is to exclude one direction - because \(S^{n-1} - pt\) is contractible; and since \(< y^2_b >\) is the composition of one \(< y_b >\) (using \(\vec{v}\)) with another \(< y_b >\) (using \(\vec{v}'\)), we must be allowing one more direction in \(< y^2_b > = < y_b > \cdot < y_b >\) than in \(< y_b >\); therefore we must allow all initial directions. So \([y^2_b] = [\nu_b]\).

The free loop space

Every cycle over \(\Omega_b(S^n)\) can be pushed forward along the inclusion \(\Omega_b(S^n) \hookrightarrow LS^n\) where it may or may not be a boundary. This can be done using any point in \(S^n\) as the new basepoint. Thus the homology of the free loop space of a sphere \(h_\ast (LS^n)\) contains

- the class \([b]\) of dimension 0 just as in the based case; though we wish to note that over \(LS^n\) we can have the cycle sit at any point: we can form the cycle \(< p >\) of the constant loop based at \(p\) for every point \(p \in S^n\); (but of course \([p] = [\bar{b}]\)).

- the class \([y_b]\) of dimension \(n - 1\) just as in the based case; though again we note that we can have \(< y_p >\) sitting at any point \(p\) and not just the basepoint (but of course \([y_p] = [y_b]\)).

Some of these cycles can easily be extended to the free loop space \(LS^n\) while others can not. What we are asking, when we talk of extending a cycle \(\omega_b\) from the based to the free loop space, is, if there is a cycle \(\omega\) over \(LS^n\) which restricts to \(\omega_b\) at the basepoint and to the
corresponding $\omega_p$ for any point $p$. The geometry involved is quite interesting.

Cycles in $LS^n$ extended from the based loop space include

- a class $[\epsilon]$ of dimension $n$, where $<\epsilon>$ is given by the cycle of “all constant loops”; it is the extension of $[b]$ and it is the unit with respect to the Chas-Sullivan product,

- a class $[\nu]$ of dimension $2(n-1)$ which is the extension of $[\nu_b]$ (i.e. it restricts to $[\nu_p]$ at all points $p \in S^n$).

The interesting question that comes up with respect to this last point is that of extending $<y_b>$ itself to get $<y>$ and not just $<\nu_b>=<y^2_b>$ to get $<\nu>$. It turns out that there is not always a class $[y]$ restricting to $[y_p]$ for all $p$, i.e. the free loop space may have a $<y^2>$ but not $<y>^2$.

Geometrically, when we study the existence of cycle $<y>$, we are studying how the cycle $<y_b>$ spreads around the sphere $S^n$. We ask the question: “Can we have a a cycle $<y>$ on $LS^n$ whose restriction to $b$ is $<y_b>$ and indeed whose restriction to each point $p$ is the corresponding $<y_p>$?”. In short, is there a $[y]$ for which $<y>>p>_<yp>$ for all $p$?

This question has a lovely geometric reinterpretation. The question can be reduced to the following: “Is there a nowhere vanishing vector field on $S^n$?”. If we had such a vector field $\xi$, (as we do when $n$ is odd), then we can start with $<y_x>$ at some point and translate it, using the vector field. Thus, at every point $p$ we take all circles at $p$ with initial tangent vector $\xi(p)$. Taken together, we’d have our $n(n-1)$-cycle $<y>$ over $LS^n$, or, using geometric homology, we’d have $[y]$ realised as $[S^n \times S^{n-1},1,g]$ where we might write $g(p,m) = f_p(m)$.
with \( f_p \) using initial vector \( \xi(p) \). The upshot is

\[
\text{when } n \text{ is odd: } h_\ast(LS^n) \cong \mathbb{Z}[y] \otimes E([e])
\]

as indicated in Proposition A.

If on the other hand \( n \) is even, then our vector field must have a zero/singularity at some point \( p \). At that point \( p \) we would have lost all information of initial direction; at such a point our translated \( [y_b] \) would necessarily restrict to (some multiple of) \( [\nu_p] \). (In fact, it would restrict to \(-k[\nu_p]\) where \( k \) is the index of \( \xi \) at \( p \)).

The even sphere serves as an excellent example to shed light on the geometry behind the boundary map of the loop homology spectral sequence. We take a decomposition of \( S^n \) into two cells with the south pole ‘\( S \)’ as 0-cell. In the free loop fibration, above (the interior of) the \( n \)-cell the total space is isomorphic to \( \Omega_N(S^n) \times \mathbb{R}^n \) - with loops based at the north pole \( N \). The boundary map \( d_n \) in the spectral sequence tells us how to glue this to \( \Omega_S(S^n) \) - loops based at the south pole. Note also that \( \Omega_S(S^n) \cong \Omega_N(S^n) \cong \Omega_p(S^n) \) for any point \( p \) and that we just include the subscript to aid understanding of the geometry involved.

By definition of the loop homology spectral sequence

\[
\mathbb{E}_p^n, q = h_{p+q}(LS^n, \Omega_S(S^n))
\]

In our two-cell decomposition we have only \( \mathbb{E}_0^n, q \) for the \( n \)-cell and \( \mathbb{E}_{-n,q} \) for the 0-cell.

In \( \mathbb{E}_{0, n-1} \) we have \([y_N]\) spread over the \( n \)-cell; which can be seen by following the isomorphims:

\[
\mathbb{E}_{0,q} = h_{n+q}(LS^n, \Omega_S(S^n))
\]
after fattening up the south pole and using excision we get

\[ \mathbb{E}^1_{0,q} \cong h_{n+q}(\Omega_N(S^n) \times D^n, \Omega_S(S^n) \times S^{n-1}) \]

\[ \cong h_{n+q}(\Omega_N(S^n) \times D^n, \Omega_N(S^n) \times S^{n-1}) \]

\[ \cong h_q(\Omega_N(S^n)). \]

While in \( \mathbb{E}^1_{-n,n-1} \) we have \([y_S]\) at the south pole:

\[ E^1_{-n,q} = h_q(\Omega_S(S^n)). \]

The question is: What does the boundary map \( d_n \) do to \([y_N]\)? To answer this, regard the following commutative square
For the odd sphere, we have a vector field that can be extended to all of $S^n$. The geometry tells us that the boundary should be $[y_S]$. This makes $d_n([y_N]) = 0$ for degree reasons. However, when $n$ is even, the vector field over the top $n$ cell cannot be extended to the south pole (it has to have a pole at the pole! see figure 6); then spreading $<y_N>$ around the entire sphere using this vector field gives us at least $<\nu_S>$-at-the-south-pole (i.e. class $[S][\nu]$). How many copies of $[\nu_S]$ do we get? Well that all depends exactly on the level of ambiguity of initial direction that the vector field imposes on us at the south pole. It is given by the index. Intriguingly, $[\nu_S]$ is not a boundary and it ends up representing an element of order 2.

Comparing again to the results of Seeliger [21], we have (using $x$ as south pole):

<table>
<thead>
<tr>
<th>LHSS for $S^n$; $n$ - odd</th>
<th>LHSS for $S^n$; $n$ - even</th>
</tr>
</thead>
<tbody>
<tr>
<td>$xy^3$</td>
<td>$y^3$</td>
</tr>
<tr>
<td>$d_n = 0$</td>
<td>$d_n = 0$</td>
</tr>
<tr>
<td>$3(n - 1)$</td>
<td>$3(n - 1)$</td>
</tr>
<tr>
<td>$xy^2$</td>
<td>$y^2$</td>
</tr>
<tr>
<td>$d_n = 0$</td>
<td>$d_n = 0$</td>
</tr>
<tr>
<td>$2(n - 1)$</td>
<td>$2(n - 1)$</td>
</tr>
<tr>
<td>$xy$</td>
<td>$y$</td>
</tr>
<tr>
<td>$d_n = 0$</td>
<td>$d_n = 0$</td>
</tr>
<tr>
<td>$n - 1$</td>
<td>$n - 1$</td>
</tr>
<tr>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>$y_n$...</td>
<td>$y_n$...</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

The loop homology of an even sphere is given as:

$$\mathbb{H}(LM) \cong \mathbb{Z}[[b, [\nu]] \otimes E([y_b])/([b]^2, [b][y_b], 2[b][\nu])$$

$$\cong \mathbb{Z}[[x, [y^2]] \otimes E([xy])/([x]^2, 2[x][y^2])]$$
where

\[
|b| = |x| = -n \\
|y| = n - 1 \\
|\nu| = |y^2| = 2(n - 1) \\
|xy^2| = n - 2 \\
|y_b| = |xy| = -1.
\]

9 The Chas-Sullivan-Massey Product in the LHSS

In this section we discuss convergence of Chas-Sullivan-Massey products in the Loop Homology Spectral Sequence and use the result to calculate some examples of non-zero Chas-Sullivan-Massey product in the case of even spheres and of complex projective space.

9.1 Convergence in the LHSS

Throughout this section, assume we have a closed, oriented manifold \( M \) of dimension \( d \) and we have classes \( \alpha \in H_i(LM) \), \( \beta \in H_j(LM) \) and \( \gamma \in H_k(LM) \) with \( \alpha \cdot \beta = 0 \) and \( \beta \cdot \gamma = 0 \), so that we have Chas-Sullivan-Massey products \( \langle \alpha, \beta, \gamma \rangle \in H_{i+j+k+1}(LM) \).

Recall that the Loop Homology Spectral Sequence converges to the associated graded of \( H_*(LM) \) where the filtration on \( H_*(LM) \) is induced by the skeletal filtration of \( M \). Let us denote by \( \bar{\alpha} \), \( \bar{\beta} \) and \( \bar{\gamma} \) the classes in the associated graded. Since \( \alpha \cdot \beta = 0 \), we also have \( \bar{\alpha} \cdot \bar{\beta} = 0 \) (likewise \( \bar{\beta} \cdot \bar{\gamma} = 0 \))\(^{3}\).

\(^{3}\text{Remember that products can jump filtration, leading to } \bar{x} \cdot \bar{y} = 0 \text{ but } x \cdot y \neq 0 \text{ (and thus also } \bar{x} \cdot \bar{y} \neq 0); \text{ but here we are dealing with the reverse. We know } \alpha \cdot \beta = 0 \text{ and thus the same must be true for } \bar{\alpha} \cdot \bar{\beta}.\)
In this section we are trying to answer the following

**Question:** If \( \langle \alpha, \beta, \gamma \rangle \) is defined, and we have classes \( a, b, c \) in the spectral sequence converging to \( \bar{\alpha}, \bar{\beta} \) and \( \gamma \), does the product \( \langle a, b, c \rangle \) converge to \( \langle \alpha, \beta, \gamma \rangle \)?

Let us review our **Notation:**

- Loop Homology Grading - we write \( \mathbb{H} \) for \( h_{*+d} \). Thus for example \( \mathbb{H}_i(LM) = H_{*+d}(LM) \).
- Loop Space Filtration - we write \( L(p) \) for \( LM(p+d) \) and also use \( L \) to denote \( LM \).
- Spectral Sequence Grading - we write \( E \) in the loop homology spectral sequence and \( E \) for the Serre spectral sequence so that \( E_{p,q} = E_{p+d,q} \).
- Loop Homology Filtration - we write \( F_p = F_p^\ast \mathbb{H} = \text{im} (H^\ast(L_p) \to \mathbb{H}(L)) \) and we write \( F_p = F_p/F_p-1 \).

Using this notation and grading, the spectral sequence is multiplicative with the product

\[
E_{p,q}^r \times E_{p',q'}^r \to E_{p+p',q+q'}^r,
\]

or, more explicitly,

\[
\mathbb{H}_{p+q}(L^{(p)}, L^{(p'-r)}) \times \mathbb{H}_{p'+q'}(L^{(p')}, L^{(p'-r)}) \to \mathbb{H}_{p+p'+q+q'}(L^{(p+p')}, L^{(p+p'-r)}).
\]

Each class in \( \mathbb{H} \) lives in some filtration. Let \( \alpha \) live in filtration \( p_a \), \( \beta \) in filtration \( p_b \) and \( \gamma \) in filtration \( p_c \). We can represent \( \alpha \) by a geometric cycle \( \langle Q_{\alpha}, x_\alpha, f_\alpha \rangle \) where the image of the map \( f_\alpha \) lies in \( L^{(p_a)} \); similarly, \( \beta \) represented by \( \langle Q_{\beta}, x_\beta, f_\beta \rangle \) with image in \( L^{(p_b)} \) and \( \gamma \) by \( \langle Q_{\gamma}, x_\gamma, f_\gamma \rangle \) with image in \( L^{(p_c)} \). We can also assume that transversality conditions are
met for our choice of these cycles, i.e. $f_\alpha$ transverse to $f_\beta$ and $f_\beta$ transverse to $f_\gamma$. In the associated graded these classes become $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$. For example,

$$\bar{\alpha} \in F_{p_a} = \frac{F_{p_a} \mathbb{H}(L)}{F_{p_a-1} \mathbb{H}(L)}.$$  

Before we state the theorem we make one more simplifying notation, writing

$$p_{ab} = p_a + p_b \quad \text{and} \quad p_{bc} = p_b + p_c$$

so that the product in the spectral sequence takes the form

$$E^r_{p_a,q_a} \times E^r_{p_b,q_b} \rightarrow E^r_{p_{ab},q_{ab}}.$$

**Theorem 9.1.** Let $M$ be a closed, oriented manifold of dimension $d$. Let $\alpha \in \mathbb{H}_i(LM)$, $\beta \in \mathbb{H}_j(LM)$ and $\gamma \in \mathbb{H}_k(LM)$ with $\alpha \cdot \beta = 0$ and $\beta \cdot \gamma = 0$. Suppose we have classes $a \in E^1_{p_a,q_a}$, $b \in E^1_{p_b,q_b}$ and $c \in E^1_{p_c,q_c}$ converging to $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ respectively and suppose that $\langle a, b, c \rangle$ is defined on the $r+1$ page of the Loop Homology Spectral Sequence$^4$. Then, if condition (*) is satisfied, any representative of $\langle a, b, c \rangle$ converges to a representative of $\langle \alpha, \beta, \gamma \rangle$ in the associated graded.  

(*) Let $(-1)^{i+j} a \cdot X + (-1)^{i+1} Y \cdot z$ be any element in $\langle a, b, c \rangle$. Then, for $[X]$, $[Y]$ we have a bigrading $(m, n)$ (from the $E^1$ page). For each pair $(m, n)$ we require

$$E^r_{m+u, n-u} \subset E^r_{m+u+1, \infty}$$

for all $u \geq 0$ - where the superscript $\infty$ in $E^r_{m+u+1, \infty}$ indicates all classes in $E^r_{m+u+1}$ surviving to $\infty$.

$^4$For the product $\langle a, b, c \rangle$ to be defined, we need $a \cdot b = 0$ and $b \cdot c = 0$. This supposition says that these products are zero (at least) from the $r+1$ page onward.
Admittedly, condition (*) looks cryptic right now, but its meaning will become clear in the proof. Essentially what we are excluding, is the case that a product $a \bullet b$ is zero in the spectral sequence because it jumps filtration (say from $p_{ab}$ to $p_{ab} - s$) and that it is not geometrically zero within $F_{pab}$, but only in some higher filtration. That is to say $a \bullet b$ is zero when viewed in all of $L$, but only appears to be zero in $F_{pab}$ because it jumps filtration. Later on (Section 11.2) we will prove a direct generalisation of this theorem (Theorem 11.4) and the proofs are very similar. It is however very instructive to follow the proof here as it illuminates the process clearly.

**Proof.** Our proof is based on May’s proof of Theorem 4.1 in [15], however since we do not start with a differential graded associative algebra, our proof will necessarily have a much more geometric flavour. We will show that we may choose the exact same geometric cycle to represent both $<a, b, c>$ and $<\alpha, \beta, \gamma>$.

The classes $a, b$ and $c$ (converging to $\bar{\alpha}, \bar{\beta}$ and $\bar{\gamma}$) start out on the $E^1$ page as

$$a \in E^1_{p_a, q_a}; b \in E^1_{p_b, q_b}; c \in E^1_{p_c, q_c}.$$  

Nominally, since this is the $E^1$ page, they would be represented by a relative cycles of the form

$$(N, \partial N) \rightarrow (L^{(p_a)}, L^{(p_a - 1)}).$$

However, knowing that the classes survive to the $E^r$ page, means that we can find representatives whose boundary maps to $L^{(p_a - r)}$. Further still, since we know that in our case the classes survive all the way to $E^\infty$, we may choose closed geometric cycles $\langle P_a, x_a, f_a \rangle$, $\langle P_b, x_b, f_b \rangle$ and $\langle P_c, x_c, f_c \rangle$ to represent them. Indeed let us choose the same cycles we chose...
(a) The product $a \bullet b$ is killed by a $d_r$

(b) $\partial Y - a \bullet b = e$

Figure 7

for $\alpha$, $\beta$ and $\gamma$, eg.

$$\langle P_a, x_a, f_a \rangle = \langle Q_a, x_a, f_a \rangle.$$  

The crux is to decide whether we can choose the same $X$ and $Y$ for $<a,b,c>$ and $<\alpha,\beta,\gamma>$. Let us study the case for $Y$.

As noted, $\alpha \bullet \beta = 0$ implies that $\bar{\alpha} \bullet \bar{\beta} = 0$. This in turn tells us that $a \bullet b$ must get killed on some page in the spectral sequence. By assumption $<a,b,c>$ is defined on the $r + 1$ page so $a \bullet b$ is zero (at least) from the $r + 1$ page onward. Suppose\(^5\) that $a \bullet b$ is zero precisely from the $r + 1$ page onward, i.e. it is hit by a $d_r$. Thus there exists a relative cycle $\langle Y, y, \rho \rangle$.

---

\(^5\) This does not represent a loss of generality for the proof, but it does simplify the narrative by reducing the number of indices required. The geometry of the argument is identical when $a \bullet b$ is killed by a $d_{r-k}$ for some $k > 0$ as it is for $k = 0$. 

84
representing a class $[Y]_r$ on the $E^r$ page with $d_r [Y]_r = [a \cdot b]_r$ (see Figure 7a).

$$\rho : (Y, \partial Y) \to \left( \mathbb{L}^{(p_{ab}+r)}, \mathbb{L}^{(p_{ab})} \right) \text{ and } (-1)^y [\partial Y, \partial Y, \rho] \equiv [a \cdot b]_r.$$  

However, even though the class $[Y]$ kills class $a \cdot b = [P_{ab}]$ on the $E^{r+1}$ page, it is possible that, geometrically, $\partial Y$ contains more than just $P_{ab}$; it is possible that $\rho$ maps other “stuff” to $\mathbb{L}^{(p_{ab}-1)}$. Let us refer to this extra “$\langle \partial Y \rangle - \langle P_{ab} \rangle$” by $e$ (see Figure 7b)

There are two cases to consider.

(i) If $e$ bounds in filtration $\mathbb{L}^{(p_{ab}+r-1)}$ then we just made a bad choice for representing $[Y]_r$, and we can “close off” the “extra boundary” without leaving the equivalence class $[Y]_r$.

---

6Here, and in the following, we are suppressing sphere-bundle-modification. Whenever we make the claim that the boundary of something is something else, the reader may insert the phrase “after a potential sphere-bundle modification we can say...”
In other words there is a choice of representative for \( \langle Y \rangle \) with boundary equal to only \( P_{ab} \). This is illustrated in Figure 8(i). This choice of \( \langle Y \rangle \) kills \( a \cdot b \) in the spectral sequence and also, this same \( \langle Y \rangle \) kills \( \alpha \cdot \beta \) in \( \mathbb{H}_*(L) \).

(ii) If \( e \) does not bound in filtration \( L(p_{ab}+r-1) \), we cannot “close off” that part of the boundary of \( Y \) we can only “push it down as far as it will go”. More precisely, let us assume, that in filtration \( L(p_{ab}+r-1) \), \( e \) is homologous to a cycle \( e' \) in \( L(p_{ab}-s) \) but not to a cycle in \( L(p_{ab}-s-1) \) for some \( s > 0 \). This is illustrated in Figure 8(ii). Please note: although it is not visible as pictured in Figure 8(ii), the bordism drawn from \( e \) to \( e' \) maps to \( L(p_{ab}+r-1) \), and not just \( L(p_{ab}-1) \).

This looks exactly like the situation where the product \( a \cdot b \) jumps filtration \( p_{ab} \) to filtration \( p_{ab}-s \) and so seems to vanish when viewed in the spectral sequence; except in this instance it actually does vanish because \( e' \) is a boundary when viewed in all of \( L \) due to \( a \cdot b = 0 \).

So \( e' \) lives in \( L(p_{ab}-s) \) and is not yet a boundary within \( L(p_{ab}+r-1) \) (survives to \( r+s \) page); it is a class of degree \( p_{ab} + q_{ab} \) and as such it represents a non-zero class in \( E^{r+s}_{p_{ab}-s,q_{ab}+s} \). However, since \( a \cdot b \) is zero at \( E^\infty \), \( e' \) must be killed on some page, \( r+s+1 \) or above.

Let’s say \( e' \) gets killed on \( E^{r+s+1+u} \) (for some \( u \geq 0 \)). This means that \( e' \) gets hit by a \( d_{r+s+u} \); ie, it is the boundary of some \( \langle Z,z;\varphi \rangle \) with

\[
\varphi : (Z,\partial Z) \to \left( L(p_{ab}+r+u), L(p_{ab}-s) \right).
\]

This is illustrated in Figure 9.

However, this means that \( [Z]_{r+s+u} \) in \( E^{r+s+u}_{p_{ab}+r+u,q_{ab}+u} \) supports a differential \( d_{r+s+u} \)
Figure 9: \(e'\) gets killed by a \(d_{r+s+u}\) contradicting (*) and therefore does not survive to \(E^\infty\). In other words

\[
E^{r+s+u}_{p_{ab}+r+u, q_{ab}-r+1-u} \not\subseteq E^{r+s+u, \infty}_{E^r}.
\]

This gives us a contradiction for condition (*) for \(Y\) (i.e. \(m = p_{ab}+r\) and \(n = q_{ab}-r+1\)).

The upshot is that there is no such \(Z\) and we are always in case (i) and we can choose the same geometric cycle \(\llbracket Y \rrbracket\) to kill \(a \bullet b\) as well as \(a \bullet \beta\). The same arguments apply for \(\llbracket X \rrbracket\); namely, if condition (*) holds, we can choose the same \(\llbracket X \rrbracket\) to kill \(b \bullet c\) and \(\beta \bullet \gamma\). Thus we have a representative for \(<a, b, c>\) given by

\[
\left[\left((-1)^{i+j} \llbracket P_a \rrbracket \bullet \llbracket X \rrbracket + (-1)^{j+1} \llbracket Y \rrbracket \bullet \llbracket P_c \rrbracket \right]\right]_{r+1}
\]

\(^7\)All we know about \(s\) is that \(s > 0\). For \(s > 1\), the condition that is violated above [using \(r+s+u\)] is actually weaker than (*) [using \(r+1+u\)]. In any event, when viewed on the \(r+1+u\) page, the existence of \([Z]_{r+1+u}\) violates (*) verbatim.
and we have a representative for $\langle \alpha, \beta, \gamma \rangle$ given by

$$\left[ (-1)^{i+j} \langle Q_\alpha \rangle \cdot \langle X \rangle + (-1)^{j+1} \langle Y \rangle \cdot \langle Q_\gamma \rangle \right].$$

Both give the same product in the associated graded $E^\infty$ because both are represented by the exact same geometric cycle!

With Theorem 9.1 in hand, we now have a chance of actually calculating some Chas-Sullivan-Massey products. After verifying that condition (\*) holds, we are allowed to calculate them on any page of the loop homology spectral sequence. In the following section will do just that and calculate two examples of non-trivial Chas-Sullivan-Massey product for the free loop space of an even sphere (one of them a torsion element, the other one is still non-zero over $\mathbb{Q}$); and we calculate an example for complex projective space $\mathbb{C}P^n$ (again surviving to $\mathbb{Q}$ coefficients).

10 Examples of non-zero Chas-Sullivan-Massey products

In this section we present calculations of a non-zero Chas-Sullivan-Massey product using the Loop Homology Spectral Sequence (Section 8) and Theorem 9.1. We find examples of non-zero Chas-Sullivan-Massey products for even spheres and for complex projective space even when using rational coefficients. This is noteworthy, because the spaces $S^n$ and $\mathbb{C}P^n$ are formal, and for a formal space all ordinary Massey products vanish (indeed they vanish coherently); and yet, here we find that with loops attached, they can survive as non-zero Chas-Sullivan-Massey products.
10.1 Example 1: The even Sphere

For this example we make use of our previous discussions in sections 8.4 and 8.5 and begin by recalling the results.

Let \( M \) be \( S^n \) for \( n \) even. In the loop homology \( \mathbb{H}_*(LS^n) \) we have (c.f. section 8.5)

- \( b \) - [constant loop at basepoint \( b \)] \( \in \mathbb{H}_{-n} \)
- \( y_b \) - [circles at \( b \) with specified initial tangency] \( \in \mathbb{H}_{-1} \)
- \( \nu_b \) - [all circles at \( b \)] \( \in \mathbb{H}_{n-2} \). (Note: \( \nu_b = y_b^2 \) and \( \nu_b = \nu \cdot b \))
- \( e \) - [all constant loops] \( \in \mathbb{H}_0 \)
- \( \nu \) - [all circles] \( \in \mathbb{H}_{2(n-1)} \).

Recall also the LHSS example from section 8.4, rewritten as:

\[
\begin{array}{cccc}
\nu_b^3 & y^3 & 3(n-1) \\
\nu_b & \nu & 2(n-1) \\
y_b & y & n-1 \\
b & 1 & 0 \\
-\nu_b & \cdots & 0 \\
\end{array}
\]

and recall the result:

\[
\mathbb{H}(LM) \cong \mathbb{Z}[b, \nu] \otimes E(y_b) / (b^2, b \cdot y_b, 2 \cdot b \cdot \nu).
\]

**Goal:** we wish to calculate \( \langle 2\nu, b, y_b \rangle \) and show that it is non trivial.
Are the prerequisites satisfied?

In order to have a Chas-Sullivan-Massey product, we need the pairwise Chas-Sullivan products to be zero.

- The second pair $b \bullet y_b$ is clearly zero because the basepoint-intersection product $b \ast y_b$ is already zero - the transverse intersection of a point with a point is empty in $S^n$.
- The first pair is also zero. This is the upshot of the discussion on page 79 about vector fields on $S^n$ and extending $y_N$ to the whole sphere, and is reflected in the map $d_n$ above.

What are the components?

In this example we have $a = 2\nu \in \mathbb{E}_{0,2(n-1)}^1$, $b \in \mathbb{E}_{-n,0}^1$ and $c = y_b \in \mathbb{E}_{-n,n-1}^1$ as described in section 8.5. Killing $a \bullet b = 2\nu_b$ is the class of “$< y_N >$ spread over the $n$-cell” – this is our $Y \in \mathbb{E}_{0,n-1}^1$ (marked $y$ in the spectral sequence above). Finally, $X$ is zero.

Is condition (*) satisfied?

The product is defined on the $n + 1$ page. For the condition to be violated, we would need to have some class in $\mathbb{E}_{p+n,a,q-u}^{n+1+u}$ not surviving to $\mathbb{E}^\infty$. In our example we have only zeroes since the spectral sequence collapses after $\mathbb{E}^n$. More explicitly for example we have $r = n$, bigrading $(0,n-1)$ for $Y$ and $\mathbb{E}_{u,n-1-u}^{n+1+u}$ contains only 0 as so codition (*) is satisfied.

The Chas-Sullivan-Massey product

With $X = 0$ and $c = y_b$, the product is given by

\[ Y \bullet y_b \in \langle 2\nu , b , y_b \rangle. \]
The product $Y \cdot y_b$ is simply $\nu_b$ or $y_b^2$ which is non-zero.

**Indeterminacies?**

We have

$$\langle 2\nu, b, y_b \rangle \in H_{n-2}(LM)/(2\nu) \cdot H_{-n}(LM) + H_{n-1}(LM) \cdot y_b$$

but in this case $(2\nu) \cdot H_{-n}(LM)$ contains only even multiples of $\nu_b = y_b^2$, while $H_{n-1}(LM) \cdot y_b$ is zero because the transverse intersection of a point with a point is empty in $S^n$. Therefore we have indeed found a non-zero Chas-Sullivan-Massey product:

$$\langle 2\nu, b, y_b \rangle = \nu_b \neq 0 \in H_{n-2}(LM)/(2\nu) \cdot H_{-n}(LM) + H_{n-1}(LM) \cdot y_b.$$

### 10.2 Example 2: The even Sphere - over $\mathbb{Q}$.

Example 1 found a non-zero Chas-Sullivan-Massey product in for an even dimensional sphere in the form of $\nu_b$. However this is a torsion element in that $2\nu_b = 0$. In the present example we will describe a non-zero Chas-Sullivan-Massey product that survives to rational coefficients (i.e. not a torsion element).

**Goal:** we wish to calculate $\langle \nu, 2b, b \rangle$ and show that it is non trivial over $\mathbb{Q}$.

**Are the prerequisites satisfied?**

In order to have a Chas-Sullivan-Massey product, we need the pairwise Chas-Sullivan products to be zero.

- The second pair $2b \cdot b$ is clearly zero because the transverse intersection of a point with a point is empty in $S^n$. 

91
• The first pair $\nu \cdot 2b = 2\nu b$ is also zero. This is exhibited by the map $d_n$ as before.

**What are the components?**

In the role of $a, b$ and $c$ in this example we have $\nu \in E^1_{0,2(n-1)}$, $2b \in E^1_{-n,0}$ and $b \in E^1_{-n,0}$. Killing $a \cdot b = 2\nu b$ is the class of "$y_N$ spread over the $n$-cell" – this is our $Y \in E^1_{0,n-1}$ (marked $y$ in the spectral sequence above). Finally, $X$ is zero.

**Is condition (*) satisfied?**

The product is defined on the $n + 1$ page. For the condition to be violated, we would need to have some class in $E^{n+1+u}_{p+u,q-u}$ not surviving to $E^\infty$. In our example we have only zeroes since the spectral sequence collapses after $E^n$. More explicitly for example we have $r = n$, bigrading $(0, n - 1)$ for $Y$ and $E^{n+1+u}_{u,n-1-u}$ contains only 0, so condition (*) is satisfied.

**The Chas-Sullivan-Massey product**

With $X = 0$, and $c = b$ we are left with

$$Y \cdot b \in <\nu, 2b, b>.$$  

The product $Y \cdot b$ is simply $y_b$ which is non-zero - even when we are taking rational coefficients.

**Indeterminacies?**

We have

$$<\nu, 2b, b> \in H_{-1}(LM) / \nu \cdot H_{-2n}(LM) + H_{n-2}(LM) \cdot b$$  

so we see that $y_b$ is not in the indeterminacy as it is neither divisible by $\nu$ nor by $b$. Thus

$$<\nu, 2b, b> = y_b \neq 0 \in H_{-1}(LM) / \nu \cdot H_{-2n}(LM) + H_{n-2}(LM) \cdot b.$$
Note: Example 1 and 2 are closely related. We would like to say that they differ only be a factor of $y$; and that would be true, but for the fact that in $\mathbb{H}(L^nS^n)$ there is no $y$ (for $n$ even).

10.3 Example 3: Complex Projective Space

The string topology of $\mathbb{CP}^n$ is calculated in [21] and [22]. For $n > 1$ it is given by

$$\mathbb{H}(L\mathbb{CP}^n) \cong E(w) \otimes \mathbb{Z}[x,y]/x^{n+1}, (n+1)x^n \cdot y, w \cdot x^n$$

with $\text{deg}(x) = -2$, $\text{deg}(y) = 2n$ and $\text{deg}(w) = -1$.

We wish to calculate the product $\langle (n+1)y, x^n, x \rangle$.

Are the prerequisites satisfied?

Clearly both $(n+1)yx^n = 0$ and $x^n x = 0$ because both of these terms appear in the “denominator” of the quotient in equation (10.1).

What are the components?

In the role of $a, b$ and $c$ are $(n+1)y \in E^1_{0,2n}$, $x^n \in E^1_{-2n,0}$ and $x \in E^1_{-2,0}$. In the role of $Y$, killing $a \cdot b$ is $z \in E^1_{0,1}$ via a $d_{2n}$. Finally $X$ is zero.

Is condition (*) satisfied?

The product is defined on the $2n + 1$ page. For the condition to be violated, we would need to have some class in $E^{2n+1+u}_{p+u,q-u}$ not surviving to $E^\infty$. In our example we have only zeroes since the spectral sequence collapses after $E^{2n+1}$. More explicitly for example we have $r = 2n$, bigrading $(0,1)$ for $Y$ and $E^{2n+1+u}_{u,1-u}$ contains only 0 so condition (*) is satisfied.
The Chas-Sullivan-Massey product

With $X = 0$, $Y = z$ and $c = x$ we are left with

$$z \ast x \in \langle (n + 1)^y, x^n, x \rangle.$$ 

The product $z \ast x$ is $w$ which is non-zero.

**Indeterminacies?**

Nominally, we have

$$\langle (n + 1)y, x^n, x \rangle \in \mathbb{H}_1(LM)/( (n + 1)y) \ast \mathbb{H}_{-2n-1}(LM) + \mathbb{H}_1(LM) \ast x$$

but in this case $((n+1)y)\ast\mathbb{H}_{-2n-1}(LM)$ contains only classes divisible by $(n+1)y$, and $w$ is not such a class, while $\mathbb{H}_1(LM) \ast x$ contains only classes divisible by $x$, and $w$ is not. Therefore we have indeed found another a non-zero Chas-Sullivan-Massey product that survives to $\mathbb{Q}$-coefficients:

$$\langle (n + 1)y, x^n, x \rangle = w \notin \mathbb{H}_1(LM)/( (n + 1)y) \ast \mathbb{H}_{-2n-1}(LM) + \mathbb{H}_1(LM) \ast x.$$ 

**Is there a geometric picture**

The obvious next question is, if there is a similar geometric picture just as in Example 1. The answer is that there is, but is decidedly more complicated. We sketch the elements below:

The homology of $\mathbb{C}\mathbb{P}^n$ has a class $x$ representing the “equatorial” $\mathbb{C}\mathbb{P}^{n-2}$ inside it. Consequently $x^2$ is $\mathbb{C}\mathbb{P}^{n-4}$ and so on until $x^n$, which is a point.
As argued in [21],[22], the based loop space of $\mathbb{C}P^n$ is homotopy equivalent (as an $h$-space) to a circle crossed with loops on an odd sphere:

$$\Omega\mathbb{C}P^n \cong S^1 \times \Omega(S^{2n+1}). \quad (10.2)$$

Therefore in its homology we have:

- a class $z$ of degree 1 representing $S^1$ with $z^2 = 0$, and we have
- the homology of the based loops of an odd sphere $S^{2n+1}$.

We discussed the based loops of an odd sphere in detail in Section 8.5. It contains the the basepoint class $b$ of degree 0 and the class $y_p$ of “all-circles-with-initial-tangency-$\vec{v}$” of degree $2n$.

Following our previous approach in Section 8.5 we ask:

**Q:** Is there a class $z_n$, extending over the entire base manifold $\mathbb{C}P^n$ such that when restricted to $\mathbb{C}P^{n-1}$ it gives the corresponding $z_{n-1}$? In other words, is there a $z_n$ such that $z_n \ast x = z_{n-1}$ and in particular $z_n \ast x^n = z$? Such a class would live in $\mathbb{H}_1(\mathcal{L}\mathbb{C}P^n)$.

**A:** No. $z_n$ does not exist. However we can spread out $z$ to “$z$-on-all-but-a-point” and so “$z_n \ast x$” does exist\(^8\). So while we don’t have $z_n$ we do have $z_{n-1}$ and this is the class we call $w$.

**Q:** Is there a class $y_n$ extending $y_p$ such that $x y_n = y_{n-1}$?

**A:** There is such a class $y_n$. It lives in $\mathbb{H}_{2n}(\mathcal{L}\mathbb{C}P^n)$ and is the class we have been calling $y$.

---

\(^8\)This may be thought of as the analog to saying that for an even sphere $y$ does not exist, but $y \ast b = y_b$ does – notation from Section 8.5
When restricted to $\mathbb{CP}^{n-1}$ we get $y_{n-1}$ (i.e. $x \ast y_n = y_{n-1}$) and when restricted to a point we get $y_p$ (i.e. $x^n y_n = y_p$).

There is again a description of $y_p$ using initial tangent conditions, by Nancy Hingston [8]. And just as we had "$2y_p = 0$" in the case of even spheres ($2 = \chi(S^{2n})$), we have $(n + 1)y_p = 0$ here ($n + 1 = \chi(\mathbb{CP}^n)$). Yet the geometry of this argument is subtly different. For one thing we do have a global class $y_n$ that we did not have for $S^{2n}$, and then, relatedly, $(n + 1)y_p$ is killed not by $y_n$ but by the class $z$.

Regarding our triple product $\langle (n + 1)y_n, x^n, x \rangle$, we see that the first product is zero by $d_{2n}(z)$ and the second is zero because the intersection is empty. Thus the triple product is the intersection of "$z$-on-all-but-a-point" with $x$ – and this is just $w$. 

96
IIIGeneralisations & Outlook

11 Higher Chas-Sullivan-Massey Products

So far we have only been dealing with triple Massey products and its derivatives. It is
natural to ask how we can extend our discussion to include general $k$-fold Massey products
([18], [19], [12], [13]) or even “Matric Massey Products” ([15]). We will not discuss matric
products, but we sketch below how we might proceed for $k$-fold products.

11.1 Defining higher Chas-Sullivan-Massey products

Let $[P_1], [P_2] \ldots [P_n]$ be given classes in $\mathbb{H}_*(LM)$.

**Definition 11.1.** We say that a family of cycles $A = \{a_{i,j}\}$, $1 \leq i \leq j \leq n$, $(i,j) \neq (1,n)$ is
a **defining system** for $\langle [P_1], [P_2] \ldots [P_n] \rangle$ if

- $\langle a_{i,i} \rangle = [P_i]$ - a class representing $[P_i]$ in $\mathbb{H}(LM)$.

- Products $\langle a_{i,k} \rangle \cdot \langle a_{k+1,j} \rangle$ are defined. (In particular appropriate transversality condi-
tions are satisfied)

- $\partial \langle a_{i,j} \rangle$ is equivalent to $\sum_{k=1}^{j-1} (-1)^{[a_{i,k}]} \langle a_{i,k} \rangle \cdot \langle a_{k+1,j} \rangle$.

Note: The sign $(-1)^{[a_{i,k}]}$ above takes the homological degree of $[a_{i,j}]$ in $\mathbb{H}$, not the dimension
of the manifold carrying it. Note also that the sum above appears so frequently we define a
shorthand notation:

\[ \langle A_{i,j} \rangle = \sum_{k=i}^{j-1} (-1)^{[a_{i,k}]} \langle a_{i,k} \rangle \bullet \langle a_{k+1,j} \rangle. \]

This leads to the definition of the \( n \)-fold generalised Chas-Sullivan-Massey product:

**Definition 11.2.** Given \([P_1], [P_2] \ldots [P_n]\), we say that their higher Chas-Sullivan-Massey product \( \langle [P_1], [P_2] \ldots [P_n] \rangle \) exists if there exists a defining system for it. Given such a defining system \( A = \{ \langle a_{i,j} \rangle \} \), we write

\[ \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} A_{1,n} \end{bmatrix} = \sum_{k=1}^{n-1} (-1)^{[a_{i,k}]} \langle a_{i,k} \rangle \bullet \langle a_{k+1,n} \rangle \]

and declare the product to be given by \( \begin{bmatrix} A \end{bmatrix} \). The \( A_{i,j} \) are sub-products of length \( j - i \).

The indeterminacies of higher Massey products are notoriously complicated. This is partly because each \( n \)-fold product consists of lower length sub-products which in turn consist of sub-products, each with their own indeterminacies. It is therefore common to restrict attention to strictly defined \( n \)-fold products. We say an \( n \)-fold product is **strictly defined** if all sub-products have no indeterminacy. For example a 3-fold product is always strictly defined because the intersection product has no indeterminacy, while a 4-fold product \( \langle a, b, c, d \rangle \) is only strictly defined if \( \langle a, b, c \rangle \) and \( \langle b, c, d \rangle \) have no indeterminacy. We will be relying on strictly defined products when we revisit Theorem 9.1 below in Theorem 11.4.

**Note 11.3.** With the signs given as in definitions 11.1 and 11.2, the Massey products agree with those of [18] and [19] and [12] while they differ from [15] and [13]. The signs agree with those in section 3 even if this is not immediately obvious. For a triple product \( \langle [P_1], [P_2], [P_3] \rangle \) Definition 3.1 called for

\[ (-1)^{i+j} [P_1] [X] + (-1)^{j+1} [Y] [P_3] \]
where
\[ \partial [X] = [P_2] [P_3] \] and \[ \partial [Y] = [P_1] [P_2] \]

while Definition 11.2 calls for
\[ (-1)^i [P_1] [a_{2,3}] + (-1)^{i+j+1} [a_{1,2}] [P_3] \]

where
\[ \partial [a_{2,3}] = (-1)^j [P_2] [P_3] \] and \[ \partial [a_{1,2}] = (-1)^i [P_1] [P_2] . \]

These are however equivalent via \( \langle X \rangle = (-1)^i \langle a_{2,3} \rangle \) and \( \langle Y \rangle = (-1)^i \langle a_{1,2} \rangle \).

11.2 Convergence of higher products in the LHSS

In this section we present the proof that the higher products of definition 11.2 can be calculated using the LHSS. This will be a direct generalisation of theorem 9.1 and the proof will be very similar.

**Theorem 11.4.** Let \( M \) be a closed, \( h_* \)-oriented manifold of dimension \( d \).

Suppose that \( \langle [P_1], [P_2], \ldots, [P_n] \rangle \) is strictly defined in \( \mathbb{H}(LM) \). Suppose we have \( \langle a_i \rangle \) with \( [a_i] \in \mathbb{E}_{p_i,q_i}^1 \), \( i = 1, \ldots, n \), converging to \( [P_i]_\infty \), and suppose that \( \langle [a_1]_{r+1}, [a_2]_{r+1}, \ldots, [a_n]_{r+1} \rangle \) is defined on the \( r+1 \) page of \( \mathbb{E} \).

Then, if condition (*) is satisfied, given any defining system \( \{ \langle a_{i,j} \rangle \} \) for \( \langle [a_1], [a_2], \ldots, [a_n] \rangle \), then \( \langle \bar{A} \rangle \), i.e. the class of the product cycle

\[ \sum_{r=1}^{n-1} (-1)^{[a_{1,k}]} \langle a_{1,k} \rangle \cdot \langle a_{k+1,n} \rangle, \]

converges to an element of \( \langle [P_1], [P_2], \ldots, [P_n] \rangle \) in \( \mathbb{E}^\infty \).
From any defining system $A$, we get a bigrading $(p,q)$ for the constituent pieces $[a_{i,j}]$ (from the $E^1$ page). For each pair $(p,q)$ we require

$$E^{r+u+1}_{p+u,q-u} \subset E^{r+u+1,\infty}$$

for all $u \geq 0$.

**Proof.** The situation is as follows:

- The $n$-fold product $<[P_1], \ldots, [P_n]>$ is strictly defined in $\mathbb{H}_*(LM)$

- while $<[a_1]_{r+1}, \ldots, [a_n]_{r+1}>$ defined on $E^{r+1}$.

- We also know that the $[a_i]_{r+1}$ converge to $[P_i]_{\infty}$ on $E^\infty$.

Let $A = \{<a_{i,j}>\}$ be a defining system for $<[a_1]_{r+1}, \ldots, [a_n]_{r+1}>$. We wish to construct a defining system $B = \{<b_{i,j}>\}$ by modifying $A$ in such a way that both

- $[a_{i,j}]_r = [b_{i,j}]_r$ - so that $B$ is still a defining system for $<[a_1]_{r+1}, \ldots, [a_n]_{r+1}>$ and $[A]_{r+1} = [B]_{r+1}$

- $B$ is also a defining system for $<[P_1], \ldots, [P_n]>$.

This way we have the same geometric cycle $\langle B \rangle$ representing both $n$-fold products, proving that they are equal (in $E^\infty$).

Let $j - i = 0$. Since, in the defining system $A$ we have $[a_{i,i}]_{r+1} = [a_i]_{r+1} = [P_i]_{r+1}$ we can simply choose our $<b_{i,i}>$ to be given by

$$<b_{i,i}> = <P_i>.$$
Suppose we have constructed $\langle b_{i,j} \rangle$ for $i - j < s$. So, for a fixed pair $i, j$ with $j - i = s$, coming from $A$ we have

$$\partial \langle a_{i,j} \rangle \sim \langle \tilde{A}_{i,j} \rangle = \Sigma_{k=i}^{j-1} \langle a_{1,k} \rangle \bullet \langle a_{k+1,j} \rangle$$

Which, on the $E^r$ page, looks like $[a_{i,j}]_r \in E^{p+r,*}_r$ with $d_r[a_{i,j}]_r = [\tilde{A}_{i,j}]_r$.

By what we have constructed so far (induction hypothesis) $[\tilde{B}_{i,j}]_r = [\tilde{A}_{i,j}]_r$ i.e.

$$\Sigma_{k=i}^{j-1} \pm [a_{1,k}]_r \bullet [a_{k+1,j}]_r = \Sigma_{k=i}^{j-1} \pm [b_{1,k}]_r \bullet [b_{k+1,j}]_r.$$  

We are looking for $\langle b_{i,j} \rangle$ such that $[a_{i,j}]_r = [b_{i,j}]_r$ and $\partial [b_{i,j}] = [\tilde{B}_{i,j}]$. What we have is the ability to choose $\langle b_{i,j} \rangle$ such that $[b_{i,j}]_r = [a_{i,j}]_r$ and $d_r[b_{i,j}]_r = [\tilde{B}_{i,j}]_r = [\tilde{A}_{i,j}]_r$.

As before let us denote by $\langle e \rangle$ the difference $\langle \tilde{A}_{i,j} - \tilde{B}_{i,j} \rangle$.

**case (i)** If $\langle e \rangle$ is homologically trivial in filtration $(p + r - 1)$, then (up to bundle modification) it is a boundary in $(p + r - 1)$ and so, after gluing, we can have a cycle $\langle b_{i,j} \rangle$ in $L^{(p+r)}$ such that $[b_{i,j}]_r = [a_{i,j}]_r$ and $\partial [b_{i,j}] = [B_{i,j}]$.

**case (ii)** If on the other hand $\langle e \rangle$ is not homologically trivial in $(p + r - 1)$ then there is an $s > 0$ such that it is homologous to $\langle e' \rangle$ in filtration $(p - s)$ but not homologous to a cycle in $(p - s - 1)$. However, since we assumed that the product is strictly defined, the difference $\langle \tilde{A}_{i,j} - \tilde{B}_{i,j} \rangle$ must converge to zero, and so we know that $\langle e' \rangle$ must get killed on some page $E^{r+s+1+u}$ for some $u \geq 0$. This means that there exists a class $[z]_{r+s+u}$ with $d_{r+s+u} [z]_{r+s+u} = [e']_{r+s+u}$. This means that $[z]_{r+s+u} \in E^{r+s+u}_{p+r+u,q-r-u+1}$ supports a differential in contradiction to condition (*).
12 Generalised Homology Theories

The geometric homology of Jakob ([9],[10]), allows us to describe generalised homology the-
tories too. Most of our constructions generalise to other homology theories almost verbatim.
When dealing with a generalised homology theory $h_*$, we look at triples $\langle P,a,f \rangle$ where $P$ is
a manifold orientable in $h_*$ and $a$ is a class in the corresponding cohomology theory $h^*$. We
also require all bordisms to be $h_*$ orientable. The only qualitative difference we must make
is in the definition of bundle modification, or more precisely, the definition of umkehr maps
contained therein. In Section 2 we appealed to Poincaré duality to define maps $\sigma^!$ and now
we must amend that to use a Thom-isomorphism definition instead. The following is from
[9]:

Given a map $f : P \to Q$ of $h_*$-orientable manifolds of dimensions $i$ and $j$, we wish to define
\[
\sigma : h^n(P) \to h^{n+j-i}(Q).
\]

We embed $Q$ into $R^k$ for large $k$ giving unit disk bundle $D \xrightarrow{D} Q$ and unit sphere bundle
$S \xrightarrow{S} Q$. Then we embed $P$ into the open unit disk bundle on $Q$ such that
\[
P \xrightarrow{i} \hat{D} = D^k \times Q
\]

\[
f \xrightarrow{\text{proj}}
\]

commutes. Let $r$ be the codimension of this embedding, and let $\nu_f$ be the normal bundle -
which we identify with a tubular neighbourhood $N_f \subset D$. The unit disk and sphere bundles
of $\nu_f$ we write as $D' \xrightarrow{D'} P$ and $S' \xrightarrow{S'} P$. Now, since $P$ and $Q$ are $h_*$-orientable, we have a
Thom class $\tau_{\nu_g}$ in $h'(D',S')$ and Thom and suspension isomorphisms:

\[ t_P : h^n(P) \rightarrow h^{n+r}(D',S') \quad \text{and} \quad t_Q : h^n(Q) \rightarrow h^{n+k}(D,S). \]

Then $f^!$ is defined as the composition

\[ f^! : h^n(P) \xrightarrow{t_P} h^{n+r}(D',S') \xrightarrow{\sim} h^{n+r}(D,D-D') \xrightarrow{\text{res}} h^{n+r}(D,S) \xrightarrow{t_Q^{-1}} h^{n+r-k}(Q). \]

Then we can (re)-define bundle modification as follows:

Given $\{P,a,f\}$, embed $P$ into Euclidian space with an $h_*$-orientable normal bundle $E \rightarrow P$. Let $\sigma$ be a section of the unit sphere bundle of $E \oplus 1$ i.e.

\[ \sigma : P \rightarrow S = S(E \oplus 1) \quad \text{such that} \quad \pi \circ \sigma = \text{id}_P. \]

Then we declare

\[ \{P,a,f\} \sim \{S,\sigma^!(a),f \circ \pi\} \]

to be equivalent by bundle modification.

For example, we could now repeat chapters I and II using spin manifolds and spin bordisms and $K^*$ classes to get Chas-Sullivan-Massey products in the $K_*$ homology of $LM$. We would use triples $[P,a,f]$ with $P$ being a spin manifold, $a$ a $K^*$ class on $P$. All bordisms and bundles would also be required to be orientable in $K_*$ (i.e. spin manifolds). We would define the intersection product as in the ordinary case, noting that the transverse intersection of two spin manifolds $P_1$ and $P_2$ within a spin manifold $M$ is again spin. Thus we can define the intersection product, the Massey product, the Chas-Sullivan Product and the Chas-Sullivan-Massey product in perfect analogy to the ordinary case. This is interesting in as much as we...
can define Massey type products even though $K_*$ is not based on a chain complex and classically we only talk about Massey-type products coming from differential graded algebras. Similar statements hold for any other theory from oriented bordism to stable homotopy; the challenge is not in the definition, but in calculating examples.

**Note 12.1.** *Theorem 11.4 applies equally for generalised homology theories and the same proof applies verbatim.***

### 13 Beyond the Free Loop Space

The Chas-Sullivan product of Section 5 (Definition 5.2) combines the Intersection Product in the homology of a manifold (Definition 2.10) with the Pontryagin product on the based loop space.

The Pontryagin product exists because $S^1$ is a co-h-space. A co-h-space $C$ is a space which comes with a map $\varphi : C \to C \vee C$ such that $\pi_i \circ \varphi$ are homotopic to $id_C$ ($i = 1, 2$ and $\pi_i$ are the projections). Given a co-h-space structure on $C$, the pointed mapping space $Map_*(C, M)$ inherits a binary operation which is unital up to homotopy:

$$\gamma : Map_*(C, M) \vee Map_*(C, M) \to Map_*(C, M)$$

$$f_1 \vee f_2 \mapsto (f_1 \vee f_2) \circ \varphi.$$

Another feature of the circle $S^1$ is, that it is a particularly nice co-h-space: it is a co-group. This means that $\gamma$ is associative and unital and has inverses (up to homotopy) and $\gamma$ induces a group structure up to homotopy on $Map_*(C, M)$.

In any event, it seems that we could *define* such a Chas-Sullivan type product for any fibration.
where \( C \) is finite dimensional smooth manifold with a co-h-space structure. The only mani-
folds with such a structure are the spheres \( S^n \) and they are all co-groups: that is, \( \gamma \) induces
a group structure up to homotopy on \( \text{Map}_*(S^n, M) \).

It is thus natural to make the following generalisation of Definition 5.2:

**Definition 13.1.** Let \( S^n M \) denote the mapping space of maps \( S^n \to M \) and let \( \Omega^n M \) denote
the space of all pointed maps \( S^n \to M \). Let \([P_1,a_1,f_1]\) and \([P_2,a_2,f_2]\) be geometric homology
classes on \( S^n M \) of degrees \( i \) and \( j \) and let \([P_{12},a_{12},f_{12}]\) be the class given by transverse
intersection along basepoints just as in section 5.1. The we define a first **generalised Chas-
Sullivan Product** by:

\[
\star_n : h_i(S^n M) \otimes h_j(S^n M) \longrightarrow h_{i+j-d}(S^n M)
\]

\([P_1,a_1,f_1] \otimes [P_2,a_2,f_2] \longrightarrow \zeta_2^1 [P_{12},a_{12},\gamma \circ f_{12}].
\]

At this point we may say that we could try to go further still and that there is no need
to restrict to mapping spaces. Indeed we can at least imagine proceeding with any locally
trivial fibration over a manifold in which the fiber has an h-space or an h-group structure.
However, we cannot neglect the other ingredient to the Chas-Sullivan product; **transversality.**

If we are given just any locally trivial fibration

\[
\begin{align*}
F & \longrightarrow E \\
\downarrow & \\
M &
\end{align*}
\]
over a smooth, compact, oriented, $d$-manifold $M$, where the fibre $F$ has an h-space structure $h : F \vee F \to F$, then we may not be able to define a product the way we hope. The Chas-Sullivan product works because mapping spaces are well behaved; in particular $LM$ (resp. $S^nM$) is a Hilbert Manifold, and

$$LM \times_M LM \to LM \times LM$$

is a pull-back diagram of Hilbert Manifolds and $LM \times_M LM \to LM \times LM$ is an embedding of a codimension-$d$ submanifold. For a general fibration, intersecting with the pull-back along the diagonal may not be as well behaved geometrically as it is for $LM$ (c.f. [23], [2]).

Avoiding such issues, we can extend our definition to cover the case where $F$ is a finite monoid or a finite dimensional h-space manifold. However, interesting examples are hard to come by. In general a fibre bundle $G \to E \to M$ does not qualify because the fibres are only $G$-torsors. It is only if the fibre bundle has a section, that we can identify a group structure on the fibres and define a Chas-Sullivan type product; and any principal bundle with a section is trivial.

Regardless of the context though, in every case where we can define an associative product “$\bullet$” of this type, we can also construct the corresponding Massey products too.

**Note 13.2.** Let $S^n$ be a sphere and let $S^nM$ denote the mapping space $\text{Map}(S^n, M)$. Let $[P_i] \in \mathbb{H}_*(SM)$, $i = 1, \ldots, n$. Let $E$ denote the Serre Spectral sequence of the fibration (13.1) shifted by $(-d, 0)$. Then theorem 11.4 applies using $\bullet_n$ as the product; using $S^nM$ for $LM$.

The same proof applies (almost) verbatim.
Appendices

A Signs and Orientations in the Intersection Product

A.1 Orienting the transverse intersection of submanifolds

Let \( M \) be a smooth oriented manifold of dimension \( d \). In section 2.3.1 we discussed the case of two oriented submanifolds \( P \) and \( Q \) of \( M \) and found in (2.11) (Lemma 2.6) that

\[
P \cap Q \cong (-1)^{d(d-Q)} (P \times Q) \cap \Delta.
\]

Notably \( P \cap Q \) is oriented as a submanifold of \( M \) whereas \( (P \times Q) \cap \Delta \) is oriented as a submanifold of \( M \times M \).

**Proposition A.1.** Viewing the intersection of \( P \) and \( Q \) as a submanifold of \( M \) gives us the same orientation as viewing it as a submanifold of \( P \). We call this the **standard orientation**.

**Corollary A.2.** The induced orientation from \( M \times M \) is the same as that from \( P \times Q \).

**Proof.** We introduce new notation for normal bundles that includes context: we write \( \nu^M P \) to denote the normal bundle of \( P \) in \( M \). All isomorphisms below are understood to be orientation preserving.

Recall, when \( P \cap Q \) is viewed as a submanifold of \( M \) we orient \( \nu^M P \) and \( \nu^M Q \) by

\[
TP \oplus \nu^M P \cong TM \quad \text{and} \quad TQ \oplus \nu^M Q \cong TM.
\]  

(A.1)
We then orient \( P \pitchfork Q \) by declaring
\[
T(P \pitchfork Q) \oplus \nu^M Q \oplus \nu^M P \cong TM. \tag{A.2}
\]
If we now view \( P \pitchfork Q \) as a submanifold of \( P \), we get the \textit{same} induced orientation. The orientation condition is
\[
T(P \pitchfork Q) \oplus \nu^P(P \pitchfork Q) \cong TP.
\]
At a point in the intersection \( P \pitchfork Q \), being tangent to \( P \) and normal to \( P \pitchfork Q \) is the same as being a tangent vector to \( M \) that is normal to \( Q \) by virtue of transversality. In other words we have
\[
\nu^P(P \pitchfork Q) = \nu^M|_{P \pitchfork Q} \tag{A.3}
\]
whence
\[
T(P \pitchfork Q) \oplus \nu^M(Q) \cong T(P \pitchfork Q) \oplus \nu^P(P \pitchfork Q) \cong TP \\
T(P \pitchfork Q) \oplus \nu^M(Q) \oplus \nu^M(P) \cong TP \oplus \nu^M P \cong TM.
\]

Proposition A.3. Viewing \( P \pitchfork Q \) as a submanifold of \( Q \) changes the orientation of \( P \pitchfork Q \) by \((-1)^{(d-P)(d-Q)}\) relative to the standard.

Corollary A.4. The induced orientation of \( P \pitchfork Q \) viewed as a submanifold of \( \Delta \) differs by \((-1)^{d(P+Q)}\) from the orientation coming from \( P \times Q \) or \( M \times M \). (Note: the sign comes about as \((-1)^{(2d-(P+Q))(2d-d)}\).)

Proof. The orientation of the normal bundle coming from \( T(P \pitchfork Q) \oplus \nu^Q(P \pitchfork Q) \cong TQ \) is incompatible with the converse of formula (A.3):
\[
\nu^Q(P \pitchfork Q) = \nu^M|_{P \pitchfork Q}. \tag{A.4}
\]

108
What we get is

\[ T(P \parallel Q) \oplus \nu^M(P) \cong T(P \parallel Q) \oplus \nu^Q(P \parallel Q) \cong TQ \]

\[ T(P \parallel Q) \oplus \nu^M(P) \oplus \nu^M(Q) \cong TQ \oplus \nu^M(Q) \cong TM \]

but the default orientation from formula (A.2) tells us

\[ (-1)^{(d-P)(d-Q)} T(P \parallel Q) \oplus \nu^M(P) \oplus \nu^M(Q) \cong T(P \parallel Q) \oplus \nu^M Q \oplus \nu^M(P) \cong TM \]

where the sign comes from switching factors.

The results of Propositions A.1, A.3, Corollaries A.2, A.4 and Lemma 2.6 are summarised in the following diagram.

Note for example that there is a difference between viewing \( P \parallel Q \) as a submanifold of \( M \) as opposed to \( \Delta \).

### A.2 Orienting the transverse intersection of singular manifolds

In general, given a geometric cycles \( \langle P, a, f \rangle \) and \( \langle Q, b, g \rangle \) we cannot assume that \( f \) and \( g \) are embeddings. Recall (c.f. [14] chapter 13) that the transversality condition in this case is

\[ df(T_x P) \oplus dg(T_y Q) = T_z(M) \]

where \( f(x) = g(y) = z \).

We distinguish between two cases:
A.2.1 The case where \( g \) is an embedding

Let us assume that \( g \) is still an embedding even though \( f \) may not be; so \( Q \subset M \). We can then naturally view \( P \downarrow Q \) as a submanifold of \( P \):

\[
P \downarrow Q = \{ x \in P | f(x) \in Q \}.
\]

The standard orientation will be the one induced from \( P \) just as in Proposition A.1 above. This means that \( P \downarrow Q \) will be oriented by

\[
T(P \downarrow Q) \oplus \nu^P(P \downarrow Q) \cong TP.
\]

What we are missing is an orientation of the normal bundle \( \nu^P(P \downarrow Q) \). The convention here is to elevate equation (A.3) to a definition

\[
\nu^P(P \downarrow Q) \cong f^*(\nu^M Q).
\]

Thus we have defined an orientation of \( P \downarrow Q \) which coincides with the standard orientation in the case where \( P \) is also a submanifold of \( M \).

**Note:** Since \( \Delta \) is a submanifold of \( M \times M \), we can always view \( P \downarrow Q \) as a submanifold of \( P \times Q \) (as indeed we did in section 2.3 on page 11) and we may use this method to give it an orientation; however the induced orientation differs from the standard orientation by \((-1)^{d(d-Q)}\). The calculation for this sign is similar to the one in section 2.3.1.

A.2.2 The case where neither \( f \) nor \( g \) are embeddings

We follow \[14\] in defining the standard orientation in this case. If \( g \) is not an embedding, we choose an embedding \( \varphi : Q \hookrightarrow \mathbb{R}^N \). We then regard the pair of maps

\[
f \times 1 : P \times \mathbb{R}^N \to M \times \mathbb{R}^n \quad \text{and} \quad (g, \varphi) : Q \hookrightarrow M \times \mathbb{R}^N.
\]
Now \( f \times 1 \) is transverse to the embedding \((g, \varphi)\) in \(M \times \mathbb{R}^N\), and their intersection is canonically diffeomorphic to \( P_\downarrow Q \). We can use the method in section A.2.1 to give it an orientation. This is the standard orientation of \( P_\downarrow Q \) and it depends neither on \( N \) nor on \( \varphi \) and it coincides with the standard orientation for submanifolds (Proposition A.1).

**B  Associativity of the Intersection Product**

The standard orientation of transverse intersections is associative. This implies that the intersection product defined in Section 2.3.3 is associative. Indeed

\[
([P_1][P_2])[P_3] = (-1)^{P_1 a_2 + da_2} [P_{12}][P_3]
\]

\[
= (-1)^{P_1 a_2 + da_2} (-1)^{P_{2a_3} + da_3} [P_{123}]
\]

\[
= (-1)^{P_1 a_2 + da_2 + P_1 a_3 + P_2 a_3} [P_{123}]
\]

\[
= (-1)^{P_1 a_2 + P_1 a_3 + da_2 + da_3} (-1)^{P_{2a_3} + da_3} [P_{123}]
\]

\[
= (-1)^{P_1 a_2 + da_2} (-1)^{P_{2a_3} + da_3} [P_{123}]
\]

\[
= (-1)^{P_{2a_3} + da_3} [P_{12}][P_{23}]
\]

\[
= [P_1]([P_2][P_3]).
\]

On the other hand, the orientations induced from the diagonal are not associative. Indeed regard the following application of calculation (2.14):

\[
\Delta_! \left( \Delta_! (\xi \times \eta) \times \rho \right) = \Delta^* \left( \left( \Delta^* (\bar{\xi} \times \bar{\eta}) \cap [M] \right) \times \rho \right) \cap [M]
\]

\[
= (-1)^{d(d-\rho)} \Delta^* \left( \Delta^* (\bar{\xi} \times \bar{\eta}) \times \rho \right) \cap [M] \quad (B.1)
\]

\[
= (-1)^{d(d-\rho)} (-1)^{d(d-\eta)} \Delta^* \left( \Delta^* (\bar{\xi} \times \bar{\eta}) \times \rho \right) \cap [M]
\]
while

\[
\Delta_1 \left( \xi \times \Delta_1 (\eta \times \rho) \right) = \Delta^* \left( \left( \xi \times \Delta^* (\eta \times \rho) \right) \cap \left[ M \right] \right) \cap [M] \\
= (-1)^d(-\eta+\rho-d) \Delta^* \left( \left( \xi \times \Delta^* (\eta \times \rho) \right) \cap \left[ M \right] \right) \\
= (-1)^d(-\eta+\rho-d)(-1)^d(-\eta) \Delta^* \left( \left( \xi \times \Delta^* (\eta \times \rho) \right) \cap \left[ M \right] \right). 
\]

This justifies equation (4.5) on page 29.
Bibliography and References

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http://bit.ly/1Es09MY


