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SPECIAL VALUES OF IBUKIYAMA–SAITO L-FUNCTIONS

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Abstract. Following the method of Arakawa, we express the special values of an L-function originally introduced by Arakawa and Hashimoto and later generalized by Ibukiyama and Saito at non-positive integers by finite sums of products of Bernoulli polynomials. As a corollary, we prove an infinite family of identities expressing finite sums of products of Bernoulli polynomials by generalized Bernoulli numbers.

1. Introduction

Fix an odd prime p and let ψ denote the Legendre symbol mod p: $\psi(a) = \left(\frac{a}{p}\right)$. We also put $\zeta = \exp(2\pi i/p)$.

We first define the L-function $L(s, \Lambda, \psi)$ originally introduced by Arakawa and Hashimoto and later generalized by Ibukiyama and Saito. Let V be the vector space of 2×2 rational symmetric matrices in $M_2(\mathbb{Q})$. Let $\Lambda \subset V$ be any lattice invariant under the following action of $\mathrm{SL}_2(\mathbb{Z}) : \Lambda \ni T \to {}^{\mathrm{t}}gTg \in \Lambda$ for $g \in \mathrm{SL}_2(\mathbb{Z})$. Ibukiyama and Saito [**3**] showed that any $\mathrm{SL}_2(\mathbb{Z})$-invariant lattice in V is a constant multiple of one of the following four lattices:

$$L_2^* = \text{the set of half-integral symmetric matrices},$$

$$L_2 = \text{the set of integral symmetric matrices},$$

$$M = \left\{ \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in L_2^* \,\middle|\, a \equiv b \equiv c \bmod 2 \right\},$$

$$N = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in L_2 \,\middle|\, a + b + c \equiv 0 \bmod 2 \right\}.$$

Thus we may assume $\Lambda \subset L_2^*$. For each $T \in \Lambda$, we define $\Psi(T)$ as follows: we put $\Psi(T) = 0$ unless $\det(T) \equiv 0(p)$, in which case ${}^{\mathrm{t}}gTg \equiv \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) \bmod p$ for some $g \in \mathrm{SL}_2(\mathbb{F}_p)$ and $a \in \mathbb{F}_p$, and we put $\Psi(T) = \psi(a)$. We denote by Λ_+ the set of all positive definite matrices of Λ. Following Ibukiyama and Saito [**3**], we define the L-function $L(s, \Lambda, \psi)$ by

$$L(s, \Lambda, \psi) = \sum_{T \in \Lambda_+/\sim} \frac{\Psi(T)}{\epsilon(T)(\det(T))^s},$$

where $\Lambda_+/\sim$ denotes the representatives of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes in Λ_+ and $\epsilon(T) = \#\{g \in \mathrm{SL}_2(\mathbb{Z}) \mid {}^{\mathrm{t}}gTg = T\}$.

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In this paper, we express the special values $L(1-m, \Lambda, \psi)$ $(m \in \mathbb{N})$ for $\Lambda \in \{L_2^*, L_2, M, N\}$ by finite sums of products of Bernoulli polynomials (Theorem 2.14). The shape of our result is a consequence of the Shintani method employed. Our result generalizes an earlier result of Arakawa [**1**] who followed the same method in calculating the special value $L(0, L_2^*, \psi)$.

On the other hand, by applying the arithmetic of orders of imaginary quadratic number fields, Ibukiyama and Saito [**3**] proved that $L(s, \Lambda, \psi)$ for $\Lambda \in \{L_2^*, L_2, M, N\}$ is up to elementary factors simply the Riemann zeta function $\zeta(2s-1)$ (Theorem 3.1). This gives an alternative evaluation of the special values $L(1-m, \Lambda, \psi)$ $(m \in \mathbb{N})$ for $\Lambda \in \{L_2^*, L_2, M, N\}$ of principally simpler shape than those obtained by the method of Shintani.

Comparing these two different evaluations of the special values $L(1-m, \Lambda, \psi)$ $(m \in \mathbb{N})$ for $\Lambda \in \{L_2^*, L_2, M, N\}$, we obtain, for each $m = 1, 2, \ldots,$ formulas (to be called Arakawa identities) expressing finite sums of products of Bernoulli polynomials by generalized Bernoulli numbers (Corollary 3.2). The question arises whether a direct and elementary proof of these identities is possible. That seems to be a very difficult problem, similar in spirit to giving an elementary proof of Dirichlet's class number formula for $h(-p) = \#$ of reduced quadratic irrationalities in the complex upper half-plane of discriminant $-p$, $p \equiv 3(4)$, $p \geq 7$.

We conclude this paper with an explicit formulation of the Arakawa identities for $m = 1$ (Theorem 3.3), the simplest case.

2. Special values of $L(s, \Lambda, \psi)$

Arakawa [**1**] worked primarily with the L-function $L(s, L_2^*, \psi)$, where he gave an explicit formula expressing $L(0, L_2^*, \psi)$ by finite sums. We shall follow the method of Arakawa and express all of the special values $L(1-m, \Lambda, \psi)$ $(m \in \mathbb{N})$ for $\Lambda \in \{L_2^*, L_2, M, N\}$ by finite sums of products of Bernoulli polynomials. While we follow the method of Arakawa for the most part, we also have to overcome additional difficulties not occurring in Arakawa's work. These difficulties are resolved using Carlitz's reciprocity theorem for generalized Dedekind–Rademacher sums (Lemma 2.7).

Remark. To aid the interested reader, we shall adopt as much of Arakawa's notation as possible.

2.1. *Expressing $L(s, \Lambda, \psi)$ in terms of partial zeta functions*

The aim of this section is to represent the L-function $L(s, \Lambda, \psi)$ for $\Lambda \in \{L_2^*, L_2, M, N\}$ as a finite linear combination of partial zeta functions. We need more notation. Let $V_{\mathbb{R}}^{(n)}$ be the $\mathbb{R}$-vector space of real symmetric matrices of size n, let $\mathscr{P}_n \subset V_{\mathbb{R}}^{(n)}$ be the set of positive definite symmetric matrices of size n, and let $\partial\mathscr{P}_n$ denote the boundary of the domain $\mathscr{P}_n$ in $V_{\mathbb{R}}^{(n)}$, that is, $\partial\mathscr{P}_n$ is the set of positive semi-definite symmetric matrices of size n. Let $\{W_1, W_2, \ldots, W_r\}$ be an r-tuple of elements in $\mathscr{P}_2 \cup \partial\mathscr{P}_2$ such that $W_1, W_2, \ldots, W_r$ are linearly independent over $\mathbb{R}$. Then, necessarily, $r \leq 3$. For any r-tuple $\xi = (\xi_1, \ldots, \xi_r)$ of positive numbers, we define a partial zeta function $\zeta(s;\{W_1,\dots,W_r\},\xi)$ as follows:

$$\zeta(s;\{W_1,\dots,W_r\},\xi)=\sum_{m_1,\dots,m_r=0}^{\infty}\det\left(\sum_{j=1}^{r}(\xi_j+m_j)W_j\right)^{-s}.$$

Let $C=C(W_1,\dots,W_r)$ be a simplicial cone spanned by $W_1,\dots,W_r$:

$$C=C(W_1,\dots,W_r)=\left\{\sum_{j=1}^{r}\lambda_jW_j \;\middle|\; \lambda_j>0\ \ (1\le j\le r)\right\}.$$

We assume that the cone $C=C(W_1,\dots,W_r)$ is contained in $\mathscr{P}_2$. Then it is easily shown that the zeta function $\zeta(s;\{W_1,\dots,W_r\},\xi)$ is absolutely convergent for $\mathrm{Re}(s)>r/2$. For any subset M of $V_{\mathbb{R}}^{(2)}$, the zeta function $\zeta(s;C,M)$, if it converges absolutely, is defined by

$$\zeta(s;C,M)=\sum_{T\in C\cap M}\det(T)^{-s}.$$

As a fundamental domain of $\mathscr{P}_n$ under the usual action of $\mathrm{GL}_n(\mathbb{Z})$, we take the Minkowski domain $\mathscr{R}_n$ of reduced matrices (see Maass [**5**]). In the case of $n=2$, the domain $\mathscr{R}_2$ has a simple form:

$$\mathscr{R}_2=\left\{\begin{pmatrix} y_1 & y_{12}\\ y_{12} & y_2\end{pmatrix}:\ 0\le 2y_{12}\le y_1\le y_2,\ 0<y_1\right\}.$$

We fix three special elements V_1, V_2 and V_3 in $\mathscr{P}_2\cup\partial\mathscr{P}_2$; put

$$V_1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad V_2=\begin{pmatrix}1&1/2\\1/2&1\end{pmatrix},\qquad V_3=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

For simplicity, we set

$$\begin{cases} C_{123}=C(V_1,V_2,V_3),\\ C_{ij}=C(V_i,V_j)\quad (1\le i<j\le 3),\\ C_j=C(V_j)\quad (j=1,2),\end{cases}\tag{2.1.1}$$

which are simplicial cones contained in $\mathscr{P}_2$.

Then the domain $\mathscr{R}_2$ has the decomposition

$$\mathscr{R}_2=C_{123}\cup C_{12}\cup C_{13}\cup C_{23}\cup C_1\cup C_2\quad\text{(disjoint union)}.\tag{2.1.2}$$

For each cone C in (2.1.1) and any $Y\in C$, observe that the order $\epsilon^*(Y)$ of the group $\{U\in\mathrm{GL}_2(\mathbb{Z})\mid {}^tUYU=Y\}$ takes the same value independent of Y belonging to C, and one can put

$$\epsilon^*(C)=\epsilon^*(Y)\quad (Y\in C).$$

It is easily verified that

$$\epsilon^*(C_{123})=2,\quad \epsilon^*(C_{ij})=4\quad (1\le i<j\le 3),\quad \epsilon^*(C_1)=8,\quad \epsilon^*(C_2)=12.\tag{2.1.3}$$

For any $x\in\mathbb{R}$, we denote by $\langle x\rangle$ the unique real number which satisfies $0<\langle x\rangle\le 1$ and $x-\langle x\rangle\in\mathbb{Z}$. Let p be an odd prime. Let

$$\mathscr{M}(p)=\{(\alpha,\gamma)\in\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}\mid(\alpha,\gamma)\neq(0,0)\bmod p\}.$$

To study the special values of $L(s, \Lambda, \psi)$ for $\Lambda \in \{L_2^*, L_2, M, N\}$, we must consider the six special sets of edge vectors:

$$X^i = \{X_1^i, X_2^i, X_3^i\} = \begin{cases} \{V_1, V_2, V_3\}, & \text{if } i = 1, \\ \{V_1, 2V_2, V_3\}, & \text{if } i = 2, \\ \{2V_1, V_2, 2V_3\}, & \text{if } i = 3, \\ \{V_1, 2V_2, 2V_3\}, & \text{if } i = 4, \\ \{V_1, 4V_2, V_3\}, & \text{if } i = 5, \\ \{V_1, 4V_2, 2V_3\}, & \text{if } i = 6. \end{cases}$$

Let $X = \{X_1, X_2, X_3\} = \{a_1V_1, a_2V_2, a_3V_3\}$ for natural numbers a_1, a_2, a_3 relatively prime to p. Let $\Lambda \in \{L_2^*, L_2, M, N\}$. For any integer μ prime to p, let $\Lambda(\mu)$ be the set consisting of all elements $T \in \Lambda$ satisfying $\Psi(T) = \psi(\mu)$. Then it immediately follows that

$$\Lambda(\mu) = \left\{ T \in \Lambda \,\middle|\, T \equiv \mu \begin{pmatrix} \alpha^2 & \alpha\gamma \\ \alpha\gamma & \gamma^2 \end{pmatrix} \bmod p \text{ for some } (\alpha, \gamma) \in \mathscr{M}(p) \right\}$$

and that $\Lambda(\mu l^2) = \Lambda(\mu)$ for any integer l prime to p. For each $(\alpha, \gamma) \in \mathscr{M}(p)$ and for each integer μ prime to p, we put

$$\xi_{\alpha,\gamma,\mu}(X) = (\langle a_1^{-1}\mu(\alpha^2 - 2\alpha\gamma)/p\rangle,\ \langle a_2^{-1}2\mu\alpha\gamma/p\rangle,\ \langle a_3^{-1}\mu(\gamma^2 - \alpha^2)/p\rangle), \tag{2.1.4}$$

where a_i^{-1} denotes the inverse of a_i mod p.

Let $\Xi_\mu(X)$ be the set of all triples $\xi_{\alpha,\gamma,\mu}(X) : \Xi_\mu(X) = \{\xi_{\alpha,\gamma,\mu}(X) \mid (\alpha, \gamma) \in \mathscr{M}(p)\}$. Observe that $\mathscr{M}(p)/\{\pm 1\}$ corresponds to $\Xi_\mu(X)$ bijectively by $\pm(\alpha, \gamma) \to \xi_{\alpha,\gamma,\mu}(X)$ $(= \xi_{-\alpha,-\gamma,\mu}(X))$. For any integers i, j with $1 \le i < j \le 3$, we set

$$\Xi_\mu^{(i,j)}(X) = \{\xi = (\xi_1, \xi_2, \xi_3) \in \Xi_\mu(X) \mid \xi_k = 1\},$$

where k is the unique integer of 1, 2, 3 satisfying $\{i, j, k\} = \{1, 2, 3\}$. We notice that

$$\begin{cases} \Xi_\mu^{(1,2)}(X) = \{\xi_{\alpha,\gamma,\mu}(X) \mid (\alpha, \gamma) \in \mathscr{M}(p),\ \alpha^2 \equiv \gamma^2(p)\}, \\ \Xi_\mu^{(1,3)}(X) = \{\xi_{\alpha,\gamma,\mu}(X) \mid (\alpha, \gamma) \in \mathscr{M}(p),\ \alpha\gamma \equiv 0(p)\}, \\ \Xi_\mu^{(2,3)}(X) = \{\xi_{\alpha,\gamma,\mu}(X) \mid (\alpha, \gamma) \in \mathscr{M}(p),\ \alpha^2 \equiv 2\alpha\gamma(p)\}. \end{cases}$$

PROPOSITION 2.1. *The following expressions hold for the zeta functions $\zeta(s;\, C,\, \Lambda(\mu))$:*

$$\zeta(s;\, C_{123},\, \Lambda(\mu)) = \begin{cases} p^{-2s} \displaystyle\sum_{\xi\in\Xi_\mu(X^1)} \zeta(s;\, \{X_1^1,\, X_2^1,\, X_3^1\},\, \xi), & \text{if } \Lambda = L_2^*, \\ p^{-2s} \displaystyle\sum_{\xi\in\Xi_\mu(X^2)} \zeta(s;\, \{X_1^2,\, X_2^2,\, X_3^2\},\, \xi), & \text{if } \Lambda = L_2, \\ p^{-2s} \displaystyle\sum_{\xi\in\Xi_\mu(X^3)} \zeta(s;\, \{X_1^3,\, X_2^3,\, X_3^3\},\, \xi), & \text{if } \Lambda = M, \\ p^{-2s}\Big\{ \displaystyle\sum_{\xi\in\Xi_\mu(X^2)} \zeta(s;\, \{X_1^2,\, X_2^2,\, X_3^2\},\, \xi) & \\ \quad - \displaystyle\sum_{\xi\in\Xi_\mu(X^4)} \zeta(s;\, \{X_1^4,\, X_2^4,\, X_3^4\},\, \xi) & \\ \quad - \displaystyle\sum_{\xi\in\Xi_\mu(X^5)} \zeta(s;\, \{X_1^5,\, X_2^5,\, X_3^5\},\, \xi) & \\ \quad + 2 \displaystyle\sum_{\xi\in\Xi_\mu(X^6)} \zeta(s;\, \{X_1^6,\, X_2^6,\, X_3^6\},\, \xi)\Big\}, & \text{if } \Lambda = N \end{cases}$$

$$(\xi = (\xi_1,\, \xi_2,\, \xi_3));$$

$$\zeta(s;\, C_{ij},\, \Lambda(\mu)) = \begin{cases} p^{-2s} \displaystyle\sum_{\xi\in\Xi_\mu^{(i,j)}(X^1)} \zeta(s;\, \{X_i^1,\, X_j^1\},\, (\xi_i,\, \xi_j)), & \text{if } \Lambda = L_2^*, \\ p^{-2s} \displaystyle\sum_{\xi\in\Xi_\mu^{(i,j)}(X^2)} \zeta(s;\, \{X_i^2,\, X_j^2\},\, (\xi_i,\, \xi_j)), & \text{if } \Lambda = L_2, \\ p^{-2s} \displaystyle\sum_{\xi\in\Xi_\mu^{(i,j)}(X_3)} \zeta(s;\, \{X_i^3,\, X_j^3\},\, (\xi_i,\, \xi_j)), & \text{if } \Lambda = M, \\ p^{-2s}\Big\{ \displaystyle\sum_{\xi\in\Xi_\mu^{(i,j)}(X^2)} \zeta(s;\, \{X_i^2,\, X_j^2\},\, (\xi_i,\, \xi_j)) & \\ \quad - \displaystyle\sum_{\xi\in\Xi_\mu^{(i,j)}(X^4)} \zeta(s;\, \{X_i^4,\, X_j^4\},\, (\xi_i,\, \xi_j)) & \\ \quad - \displaystyle\sum_{\xi\in\Xi_\mu^{(i,j)}(X^5)} \zeta(s;\, \{X_i^5,\, X_j^5\},\, (\xi_i,\, \xi_j)) & \\ \quad + 2 \displaystyle\sum_{\xi\in\Xi_\mu^{(i,j)}(X^6)} \zeta(s;\, \{X_i^6,\, X_j^6\},\, (\xi_i,\, \xi_j))\Big\}, & \text{if } \Lambda = N \end{cases}$$

$$(1 \le i < j \le 3);$$

$$\zeta(s;\, C_1,\, \Lambda(\mu)) = 0;$$

$$\zeta(s;\, C_2,\, \Lambda_{(}\mu)) = \begin{cases} 0, & \textit{if } p > 3, \\ p^{-2s}\zeta(s;\, \{V_2\},\, \langle \mu/p\rangle), & \textit{if } p = 3,\ \Lambda = L_2^*, \\ p^{-2s}\zeta(s;\, \{2V_2\},\, \langle 2\mu/p\rangle), & \textit{if } p = 3,\ \Lambda = L_2, \\ p^{-2s}\zeta(s;\, \{V_2\},\, \langle \mu/p\rangle), & \textit{if } p = 3,\ \Lambda = M, \\ p^{-2s}\zeta(s;\, \{4V_2\},\, \langle \mu/p\rangle), & \textit{if } p = 3,\ \Lambda = N. \end{cases}$$

Proof. We only prove the case $\Lambda \in \{L_2^*,\, L_2,\, M\}$. The case $\Lambda = N$ follows similarly after an application of the inclusion–exclusion principle. Let

$$X = \begin{cases} X^1, & \text{if } \Lambda = L_2^*, \\ X^2, & \text{if } \Lambda = L_2, \\ X^3, & \text{if } \Lambda = M, \end{cases}$$

and write $X = \{X_1,\, X_2,\, X_3\} = \{a_1V_1,\, a_2V_2,\, a_3V_3\}$ for the appropriate $a_1, a_2, a_3 \in \mathbb{N}$. Take $T \in C_{123} \cap \Lambda(u)$ and write $T = \sum_{j=1}^{3} m_jX_j$ with all $m_j \in \mathbb{N}$. We take a pair $(\alpha,\, \gamma) \in \mathscr{M}(p)$ such that

$$T \equiv \mu \begin{pmatrix} \alpha^2 & \alpha\gamma \\ \alpha\gamma & \gamma^2 \end{pmatrix} \bmod p. \tag{2.1.5}$$

Then, the m_j must necessarily satisfy the following congruences:

$$\begin{cases} m_1 \equiv a_1^{-1}\mu(\alpha^2 - 2\alpha\gamma)\ (p), \\ m_2 \equiv a_2^{-1}2\mu\alpha\gamma\ (p), \\ m_3 \equiv a_3^{-1}\mu(\gamma^2 - \alpha^2)\ (p). \end{cases}$$

Therefore, there exists a triple $l = (l_1, l_2, l_3)$, l_j being non-negative integers, such that $(m_1, m_2, m_3) = p(\xi_{\alpha,\gamma,\mu}(X) + l)$. As each $T \in C_{123} \cap \Lambda(\mu)$ determines a triple l uniquely and also $(\alpha,\, \gamma) \in \mathscr{M}(p)$ uniquely up to (± 1)-multiplication, the first identity in Proposition 2.1 follows. Next, let $T \in C_{12} \cap \Lambda(\mu)$ and write $T = \sum_{j=1}^{2} m_jX_j$ $(m_j \in \mathbb{N})$. Taking a pair $(\alpha,\, \gamma) \in \mathscr{M}(p)$ as in (2.1.5), the congruences $m_1 \equiv a_1^{-1}\mu(\alpha^2 - 2\alpha\gamma)\ (p)$ and $m_2 \equiv a_2^{-1}2\mu\alpha\gamma\ (p)$ follow, and necessarily the relation $\alpha^2 \equiv \gamma^2\ (p)$ has to hold. Hence the identity for $\zeta(s;\, C_{12},\, L(\mu))$ immediately follows. The other identities are similarly verified, so the proofs are omitted. □

Let κ be a non-quadratic residue mod p so that we can write $\Lambda = \Lambda(1) \cup \Lambda(\kappa)$ (disjoint union). Let $\delta_{x,y}$ be the Kronecker delta symbol ($\delta_{x,y} = 1$ if $x = y$, and 0 otherwise).

PROPOSITION 2.2. *Let ψ be the unique non-trivial quadratic character mod p. Then, we have*

$$L(s,\, \Lambda,\, \psi) = \sum_{\mu} \psi(\mu) \left\{ \zeta(s;\, C_{123},\, \Lambda(\mu)) + \frac{1}{2}\sum_{i<j} \zeta(s;\, C_{ij},\, \Lambda(\mu)) + \frac{\delta_{p,3}}{6}\zeta(s;\, C_2,\, \Lambda(\mu)) \right\},$$

where μ is taken over 1 and κ, and the summation $\sum_{i<j}$ indicates that i, j run over all integers with $1 \le i < j \le 3$.

Proof. We introduce the L-function $\mathscr{L}(s, \Lambda, \psi)$ which is very similar to $L(s, \Lambda, \psi)$. We set

$$\mathscr{L}(s, \Lambda, \psi) = \sum_{T \in \Lambda_+/\sim} \frac{\psi_{H,p}(T)}{\epsilon^*(T)(\det(T))^s},$$

where $\Lambda_+/\sim$ denotes the representatives of $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes in Λ_+ and $\epsilon^*(T) = \#\{g \in \mathrm{GL}_2(\mathbb{Z}) \mid {}^t gTg = T\}$. An elementary observation shows that $L(s, \Lambda, \psi) = 2\mathscr{L}(s, \Lambda, \psi)$. In view of the decomposition (2.1.2) of $\mathscr{R}_2$, we may take a disjoint union $\bigcup_C (C \cap \Lambda)$, with C varying over all the simplicial cones in (2.1.1), as a complete set of $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of all elements in Λ_+. Thus we get, with the help of the decomposition $\Lambda = \Lambda(1) \cup \Lambda(\kappa)$ (disjoint union),

$$L(s, \Lambda, \psi) = 2 \sum_C \epsilon^*(C)^{-1} \sum_\mu \psi(\mu)\zeta(s; C, \Lambda(\mu)),$$

where C runs over all the simplicial cones in (2.1.1) and μ is taken over 1 and κ. This, together with (2.1.3) and Proposition 2.1, completes the proof of Proposition 2.2. □

2.2. *Integral representations of partial zeta functions*

The aim of this section is to obtain convenient integral representations for all of the partial zeta functions found in Proposition 2.1. Let $X = \{X_1, X_2, X_3\} = \{a_1 V_1, a_2 V_2, a_3 V_3\}$ with natural numbers a_1, a_2, a_3 relatively prime to p.

We put

$$\Gamma_2(s) = \pi^{1/2}\Gamma(s)\Gamma(s - 1/2),$$

where $\Gamma(s)$ is the gamma function.

We set, for $t \in \mathbb{C}$, $x \in \mathbb{R}$,

$$\phi(t; x) = \frac{e^{tx}}{e^t - 1},$$

which is the generating function of Bernoulli polynomials $B_k(x)$. Recall that the Laurent expansion at $t = 0$ of $\phi(t; x)$ is given by

$$\phi(t; x) = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^{k-1} \quad (|t| < 2\pi). \tag{2.2.1}$$

Taking the kth derivative of (2.2.1), one gets, if $|t| < 2\pi$,

$$\frac{\phi^{(k)}(t; a)}{k!} = (-1)^k t^{-k-1} + \sum_{n=k+1}^{\infty} \frac{B_n(a)}{n!} \binom{n-1}{k} t^{n-1-k}. \tag{2.2.2}$$

Let $\{W_1, W_2, \ldots, W_r\}$ be an r-tuple of elements in $\mathscr{P}_2 \cup \partial\mathscr{P}_2$ such that $W_1, W_2, \ldots, W_r$ are linearly independent over $\mathbb{R}$. Following Satake [**6**], for any r-tuple $\xi = (\xi_1, \xi_2, \xi_3)$ of positive numbers, we have the following integral representation for the partial zeta function $\zeta(s; \{W_1, \ldots, W_r\}, \xi)$:

$$\begin{aligned}
&\zeta(s; \{W_1, \ldots, W_r\}, \xi) \\
&= \frac{1}{\Gamma_2(s)} \int_0^\infty dt \int_0^1 du \int_0^\pi d\theta \cdot t^{2s-1} u^{s-3/2}(1-u) \qquad (2.2.3)\\
&\times \Phi((t, u, \theta); \{W_1, \ldots, W_r\}, \xi),
\end{aligned}$$

where we put

$$\Phi((t, u, \theta); \{W_1, \dots, W_r\}, \xi) = \prod_{j=1}^{r} \phi(t\lambda((u, \theta), W_j); 1 - \xi_j)$$

and

$$\lambda((u, \theta), W) = \mathrm{tr}\left(W k_\theta \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} {}^{\mathrm{t}}k_\theta \right) \quad \text{for any } W \in \mathscr{P}_2 \cup \partial\mathscr{P}_2.$$

If all of the edge vectors $\{W_1, \dots, W_r\}$ are contained in $\mathscr{P}_2$, then the integral representation in (2.2.3) can be evaluated by following the method of Satake and Kurihara [**6, 4**]. However, in view of Propositions 2.1 and 2.2, we have to consider the cases in which, with respect to an r-tuple $\{W_1, \dots, W_r\}$, the vectors $W_1, \dots, W_{r-1}$ are all in $\mathscr{P}_2$, and W_r coincides with the special vector $X_3 = a_3 V_3$ in $\partial\mathscr{P}_2$. Because of this, Satake and Kurihara's method cannot be applied directly to our situation, and consequently the integral representation given in (2.2.3) cannot be easily evaluated. To overcome this obstruction, we follow the method of Arakawa.

We set

$$\psi(t; x) = \phi(t; x) - \frac{1}{t},$$

which is a holomorphic function of t in the region $|t| < 2\pi$. Let $\{W_1, \dots, W_{r-1}, X_3\}$ ($r = 2$ or 3) be an r-tuple of vectors in $\mathscr{P}_2 \cup \partial\mathscr{P}_2$ such that $W_1, \dots, W_{r-1}$ are all in $\mathscr{P}_2$. We set, for an r-tuple $\xi = (\xi_1, \dots, \xi_r)$ of positive numbers,

$$\begin{aligned} &\Phi_P((t, u, \theta); \{W_1, \dots, W_{r-1}, X_3\}, \xi) \\ &\quad = \prod_{j=1}^{r-1} \phi(t\lambda((u, \theta), W_j); 1 - \xi_j)\, \psi(t\lambda((u, \theta), X_3); 1 - \xi_3), \end{aligned}$$

and, for an $(r-1)$-tuple $\xi' = (\xi_1, \dots, \xi_{r-1})$,

$$\Phi_S((t, u, \theta); \{W_1, \dots, W_{r-1}, X_3\}, \xi') = \frac{1}{t\lambda((u, \theta), X_3)} \prod_{j=1}^{r-1} \phi(t\lambda((u, \theta), W_j); 1 - \xi_j).$$

Next we define the principal and singular parts of $\zeta(s; \{W_1, \dots, W_{r-1}, X_3\}, \xi)$:

$$\begin{aligned} &\zeta_P(s; \{W_1, \dots, W_{r-1}, X_3\}, \xi) \\ &\quad = \frac{1}{\Gamma_2(s)} \int_0^\infty dt \int_0^1 du \int_0^\pi d\theta \cdot t^{2s-1} u^{s-3/2} (1-u) \\ &\qquad \times \Phi_P((t, u, \theta); \{W_1, \dots, W_{r-1}, X_3\}, \xi), \end{aligned} \tag{2.2.4}$$

$$\begin{aligned} &\zeta_S(s; \{W_1, \dots, W_{r-1}, X_3\}, \xi') \\ &\quad = \frac{1}{\Gamma_2(s)} \int_0^\infty dt \int_0^1 du \int_0^\pi d\theta \cdot t^{2s-1} u^{s-3/2} (1-u) \\ &\qquad \times \Phi_S((t, u, \theta); \{W_1, \dots, W_{r-1}, X_3\}, \xi'). \end{aligned} \tag{2.2.5}$$

The integrals in (2.2.4) and (2.2.5) are absolutely convergent for $\mathrm{Re}(s) > 3/2$, and obviously

$$\zeta(s; \{W_1, \dots, X_3\}, \xi) = \zeta_P(s; \{W_1, \dots, X_3\}, \xi) + \zeta_S(s; \{W_1, \dots, X_3\}, \xi').$$

While the principal part of these integral representations is easy to evaluate by following the method of Satake and Kurihara, the singular part is considerably more difficult. Fortunately for us, however, Arakawa in his work on $L(1-m, L_2^*, \psi)$ $(m \in \mathbb{N})$ came up with the ideas needed to evaluate these integral representations. Due to the highly technical and laborious nature of our work, we will only provide details for the results which do not follow directly from the work of Arakawa [**1**].

Corresponding to $\Lambda(\mu)$, we shall define the principal part $\zeta_P(s; C, \Lambda(\mu))$ and the singular part $\zeta_S(s; C, \Lambda(\mu))$ of the zeta function $\zeta(s; C, \Lambda(\mu))$, C being the simplicial cones C_{123} and C_{j3} $(j = 1, 2)$. In view of Proposition 2.1, we set

$$\zeta_P(s; C_{123}, \Lambda(\mu)) = \begin{cases} p^{-2s} \displaystyle\sum_{\xi \in \Xi_\mu(X^1)} \zeta_P(s; \{X_1^1, X_2^1, X_3^1\}, \xi), & \text{if } \Lambda = L_2^*, \\ p^{-2s} \displaystyle\sum_{\xi \in \Xi_\mu(X^2)} \zeta_P(s; \{X_1^2, X_2^2, X_3^2\}, \xi), & \text{if } \Lambda = L_2, \\ p^{-2s} \displaystyle\sum_{\xi \in \Xi_\mu(X^3)} \zeta_P(s; \{X_1^3, X_2^3, X_3^3\}, \xi), & \text{if } \Lambda = M, \\ p^{-2s} \Bigg\{ \displaystyle\sum_{\xi \in \Xi_\mu(X^2)} \zeta_P(s; \{X_1^2, X_2^2, X_3^2\}, \xi) & \\ \quad - \displaystyle\sum_{\xi \in \Xi_\mu(X^4)} \zeta_P(s; \{X_1^4, X_2^4, X_3^4\}, \xi) & \\ \quad - \displaystyle\sum_{\xi \in \Xi_\mu(X^5)} \zeta_P(s; \{X_1^5, X_2^5, X_3^5\}, \xi) & \\ \quad + 2 \displaystyle\sum_{\xi \in \Xi_\mu(X^6)} \zeta_P(s; \{X_1^6, X_2^6, X_3^6\}, \xi) \Bigg\}, & \text{if } \Lambda = N \end{cases}$$

$(\xi = (\xi_1, \xi_2, \xi_3))$;

$$\zeta_P(s;\, C_{j3}\Lambda(\mu)) = \begin{cases} p^{-2s} \displaystyle\sum_{\xi\in\Xi_\mu^{(j,3)}(X^1)} \zeta_P(s;\, \{X_j^1,\, X_3^1\},\, (\xi_j,\, \xi_3)), & \text{if } \Lambda = L_2^*, \\ p^{-2s} \displaystyle\sum_{\xi\in\Xi_\mu^{(j,3)}(X^2)} \zeta_P(s;\, \{X_j^2,\, X_3^2\},\, (\xi_j,\, \xi_3)), & \text{if } \Lambda = L_2, \\ p^{-2s} \displaystyle\sum_{\xi\in\Xi_\mu^{(j,3)}(X^3)} \zeta_P(s;\, \{X_j^3,\, X_3^3\},\, (\xi_j,\, \xi_3)), & \text{if } \Lambda = M, \\ p^{-2s}\Bigg\{ \displaystyle\sum_{\xi\in\Xi_\mu^{(j,3)}(X^2)} \zeta_P(s;\, \{X_j^2,\, X_3^2\},\, (\xi_j,\, \xi_3)) & \\ \quad - \displaystyle\sum_{\xi\in\Xi_\mu^{(j,3)}(X^4)} \zeta_P(s;\, \{X_j^4,\, X_3^4\},\, (\xi_j,\, \xi_3)) & \\ \quad - \displaystyle\sum_{\xi\in\Xi_\mu^{(j,3)}(X^5)} \zeta_P(s;\, \{X_j^5,\, X_3^5\},\, (\xi_j,\, \xi_3)) & \\ \quad + 2 \displaystyle\sum_{\xi\in\Xi_\mu^{(j,3)}(X^6)} \zeta_P(s;\, \{X_j^6,\, X_3^6\},\, (\xi_j,\, \xi_3))\Bigg\}, & \text{if } \Lambda = N \end{cases}$$

$$(j = 1,\, 2);$$

$$\zeta_S(s;\, C_{123},\, \Lambda(\mu)) = \begin{cases} p^{-2s} \displaystyle\sum_{\xi\in\Xi_\mu(X^1)} \zeta_S(s;\, \{X_1^1,\, X_2^1,\, X_3^1\},\, (\xi_1,\, \xi_2)), & \text{if } \Lambda = L_2^*, \\ p^{-2s} \displaystyle\sum_{\xi\in\Xi_\mu(X^2)} \zeta_S(s;\, \{X_1^2,\, X_2^2,\, X_3^2\},\, (\xi_1,\, \xi_2)), & \text{if } \Lambda = L_2, \\ p^{-2s} \displaystyle\sum_{\xi\in\Xi_\mu(X^3)} \zeta_S(s;\, \{X_1^3,\, X_2^3,\, X_3^3\},\, (\xi_1,\, \xi_2)), & \text{if } \Lambda = M, \\ p^{-2s}\Bigg\{ \displaystyle\sum_{\xi\in\Xi_\mu(X^2)} \zeta_S(s;\, \{X_1^2,\, X_2^2,\, X_3^2\},\, (\xi_1,\, \xi_2)) & \\ \quad - \displaystyle\sum_{\xi\in\Xi_\mu(X^4)} \zeta_S(s;\, \{X_1^4,\, X_2^4,\, X_3^4\},\, (\xi_1,\, \xi_2)) & \\ \quad - \displaystyle\sum_{\xi\in\Xi_\mu(X^5)} \zeta_S(s;\, \{X_1^5,\, X_2^5,\, X_3^5\},\, (\xi_1,\, \xi_2)) & \\ \quad + 2 \displaystyle\sum_{\xi\in\Xi_\mu(X^6)} \zeta_S(s;\, \{X_1^6,\, X_2^6,\, X_3^6\},\, (\xi_1,\, \xi_2))\Bigg\}, & \text{if } \Lambda = N; \end{cases}$$

$$\zeta_S(s;\, C_{j3}\Lambda(\mu)) = \begin{cases} p^{-2s} \displaystyle\sum_{\xi \in \Xi_\mu^{(j,3)}(X^1)} \zeta_S(s;\, \{X_j^1,\, X_3^1\},\, \xi_j), & \text{if } \Lambda = L_2^*, \\ p^{-2s} \displaystyle\sum_{\xi \in \Xi_\mu^{(j,3)}(X^2)} \zeta_S(s;\, \{X_j^2,\, X_3^2\},\, \xi_j), & \text{if } \Lambda = L_2, \\ p^{-2s} \displaystyle\sum_{\xi \in \Xi_\mu^{(j,3)}(X^3)} \zeta_S(s;\, \{X_j^3,\, X_3^3\},\, \xi_j), & \text{if } \Lambda = M, \\ p^{-2s} \Bigg\{ \displaystyle\sum_{\xi \in \Xi_\mu^{(j,3)}(X^2)} \zeta_S(s;\, \{X_j^2,\, X_3^2\},\, \xi_j) \\ \quad - \displaystyle\sum_{\xi \in \Xi_\mu^{(j,3)}(X^4)} \zeta_S(s;\, \{X_j^4,\, X_3^4\},\, \xi_j) \\ \quad - \displaystyle\sum_{\xi \in \Xi_\mu^{(j,3)}(X^5)} \zeta_S(s;\, \{X_j^5,\, X_3^5\},\, \xi_j) \\ \quad + 2 \displaystyle\sum_{\xi \in \Xi_\mu^{(j,3)}(X^6)} \zeta_S(s;\, \{X_j^6,\, X_3^6\},\, \xi_j) \Bigg\}, & \text{if } \Lambda = N \end{cases}$$

$$(j = 1,\, 2).$$

Proposition 2.2 then makes it possible to define the principal and singular parts of the L-function $L(s,\, \Lambda,\, \psi)$. We set

$$\begin{aligned} L_P(s,\, \Lambda,\, \psi) = \sum_\mu \psi(\mu) \Bigg\{ &\zeta_P(s;\, C_{123},\, \Lambda(\mu)) + \frac{1}{2} \sum_{j=1}^{2} \zeta_P(s;\, C_{j3},\, \Lambda(\mu)) \\ &+ \frac{1}{2} \zeta(s;\, C_{12},\, \Lambda(\mu)) + \frac{\delta_{p,3}}{6} \zeta(s;\, C_2,\, \Lambda(\mu)) \Bigg\}, \end{aligned} \tag{2.2.6}$$

$$L_S(s,\, \Lambda,\, \psi) = \sum_\mu \psi(\mu) \Bigg\{ \zeta_S(s;\, C_{123},\, \Lambda(\mu)) + \frac{1}{2} \sum_{j=1}^{2} \zeta_S(s;\, C_{j3},\, \Lambda(\mu)) \Bigg\},$$

where μ runs over 1 and κ (κ being a non-quadratic residue mod p). Thus we have the obvious identity

$$L(s,\, \Lambda,\, \psi) = L_P(s,\, \Lambda,\, \psi) + L_S(s,\, \Lambda,\, \psi). \tag{2.2.7}$$

2.3. *Determination of the special values of $L_P(s,\, \Lambda,\, \psi)$*

The aim of the next two sections is to express all of the special values $L(1-m,\, \Lambda,\, \psi)$ ($m \in \mathbb{N}$) for $\Lambda \in \{L_2^*,\, L_2,\, M,\, N\}$ by finite sums of products of periodic Bernoulli polynomials. This is accomplished by determining the special values $L_P(1-m,\, \Lambda,\, \psi)$ ($m \in \mathbb{N}$) and $L_S(1-m,\, \Lambda,\, \psi)$ ($m \in \mathbb{N}$) for $\Lambda \in \{L_2^*,\, L_2,\, M,\, N\}$. We now fix the notation. Let p be an odd prime and let ψ denote the Legendre symbol mod p: $\psi(a) = \left(\frac{a}{p}\right)$. Let $\delta_{x,y}$ be the Kronecker delta symbol. Let $X = \{X_1,\, X_2,\, X_3\} = \{a_1 V_1,\, a_2 V_2,\, a_3 V_3\}$ with natural numbers $a_1,\, a_2,\, a_3$ relatively prime to p.

We recall the periodic Bernoulli polynomials $P_k(x)$ given by

$$P_k(x) = \begin{cases} 0, & \text{if } k = 1,\, x \in \mathbb{Z}, \\ B_k(\{x\}), & \text{otherwise,} \end{cases}$$

where $\{x\}$ denotes the fractional part of x.

We also remind the reader of the generalized Bernoulli numbers $B_{k,\psi}$, given by

$$B_{k,\psi} = p^{k-1} \sum_{a(p)} P_k(a/p)\psi(a). \tag{2.3.1}$$

We set

$$I_{n,m} = \int_{\mathbb{R}} \frac{x^n}{(1+x^2)^m} dx \qquad \text{for } 0 \le n < 2m-1. \tag{2.3.2}$$

Partial integration yields the following recursive relations,

$$\begin{cases} I_{2i,2i+j+1} = I_{2i-2,2i+j} - I_{2i-2,2i+j+1}, \\ I_{0,j+1} = \pi \binom{2j}{j} \left(\frac{1}{4}\right)^j \quad (i,\, j \ge 0). \end{cases} \tag{2.3.3}$$

We set $Q(u) = 3u^2 + 10u + 3$. As the function $u^{-m+1}(1+u)^{2m-2-2j} Q(u)^j$ $(0 \le j \le m-1)$ is invariant under the transformation $u \to 1/u$, and is therefore a polynomial of $(u + 1/u)$ of degree $m-1$, we can write

$$u^{-m+1}(1+u)^{2m-2-2j} Q(u)^j = \sum_{k=0}^{m-1} b_{j,m,k}(u^k + u^{-k}) \quad (0 \le j \le m-1) \tag{2.3.4}$$

for some $b_{j,m,k} \in \mathbb{Q}$.

We define the sequence $\{\alpha_n\}$ satisfying the following recursive formula:

$$\begin{cases} \alpha_0 = 0; \quad \alpha_1 = -16/3; \\ 3(n-1/2)\alpha_n + 10(n-1)\alpha_{n-1} + 3(n-3/2)\alpha_{n-2} = -16 \quad (n \ge 2). \end{cases} \tag{2.3.5}$$

We put

$$R_{k_1,k_3}(u) = i(1+u)^{k_1-1} \operatorname*{Res}_{x=i} \{(1+u)x^2 + (1-u)x + (1+u)\}^{-1} \left(1 - \frac{1-u}{1+x^2}\right)^{k_3-1}. \tag{2.3.6}$$

We note that $R_{k_1,k_3}(u)$ is a polynomial of u with coefficients in the Gaussian field $\mathbb{Q}(i)$. Therefore, the real part of $R_{k_1,k_3}(u)$ is a polynomial of u with rational coefficients.

For a positive number ϵ, we let $I_\epsilon(\infty)$ (respectively $I_\epsilon(1)$) be the contour path consisting of the oriented half-line $(+\infty, \epsilon)$ (respectively $(1, \epsilon)$), a counterclockwise circle of radius ϵ around the origin, and the oriented half-line $(\epsilon, +\infty)$ (respectively $(\epsilon, 1)$). We also let Γ_ϵ denote a circle of radius ϵ around the origin oriented counterclockwise.

We state the following useful formula:

$$\int_{I_\epsilon(1)} u^{k-1/2}\, du = \frac{-4}{2k+1} \qquad \text{for any } k \in \mathbb{Z}. \tag{2.3.7}$$

For any integers k_1, k_2, k_3 with $k_1, k_2 \geq 0,\ k_3 \geq 1$, and $k_1 + k_2 + k_3 = 2m + 1$, we set

$$\Lambda_{(k_1,k_2,k_3)} = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{I_\epsilon(1)} u^{-m-1/2}(1-u)(1+u)^{k_1-1} \sum_{i=0}^{\lfloor (k_2-1)/2 \rfloor} \sum_{j=0}^{k_3-1} (-1)^j \binom{k_2-1}{2i} \binom{k_3-1}{j} \\ \quad \times (1+u)^{k_2-1-2i}(1-u)^{2i+j} \cdot I_{2i,2i+j+1}\, du, & \text{if } k_2 \geq 1, \\ 2^{1-k_3} \displaystyle\sum_{j=0}^{\lfloor (k_3-1)/2 \rfloor} (-1)^j \binom{k_3-1}{2j} \Big(-2b_{j,m,0}\,\alpha_1 + \sum_{k=1}^{m-1} b_{j,m,k}(\alpha_{k-1} - \alpha_{k+1})\Big) \\ \quad + \displaystyle\int_{I_\epsilon(1)} u^{-m-1/2}(1-u)\, \mathrm{Re}(R_{k_1,k_3}(u))\, du, & \text{if } k_2 = 0. \end{cases} \tag{2.3.8}$$

Remark. Our $\Lambda_{(k_1,k_2,k_3)}$ corresponds to Arakawa's $(1/\pi i)\Lambda_{(k_1-1,k_2-1,k_3-1)}$.

We state the following lemma which is used in Proposition 2.5.

LEMMA 2.3. *Let $m \in \mathbb{N}$. Then,*

$$\Lambda_{(1,1,2m-1)} = \begin{cases} 4, & \text{if } m = 1, \\ 0, & \text{if } m > 1. \end{cases}$$

Proof. The case of $m = 1$ is clear, so we consider the case $m > 1$. We show that $\Lambda_{(1,1,2n+1)} = 0$ for $n \in \mathbb{N}$. From (2.3.3), we have

$$\Lambda_{(1,1,2n+1)} = \frac{1}{2} \sum_{j=0}^{2n} (-1)^j \binom{2n}{j} \binom{2j}{j} \left(\frac{1}{4}\right)^j \int_{I_\epsilon(1)} u^{-n-3/2}(1-u)^{j+1}\, du.$$

Since

$$\binom{2j}{j}\left(\frac{1}{4}\right)^j = \frac{\Gamma(j+1/2)}{\Gamma(1/2)\Gamma(j+1)}$$

and

$$\int_{I_\epsilon(1)} u^{-n-3/2}(1-u)^{j+1}\, du = -2B(-n-1/2,\, j+2),$$

where $B(x,\, y)$ is the beta function, we get

$$\Lambda_{(1,1,2n+1)} = -\frac{\Gamma(-n-1/2)}{\Gamma(1/2)} \sum_{j=0}^{2n} (-1)^j \binom{2n}{j} (j+1) \frac{\Gamma(j+1/2)}{\Gamma(j-n+3/2)}.$$

Let $\left\{ {n \atop k} \right\}$ denote Stirling numbers of the second kind. Because

$$(j+1)\frac{\Gamma(j+1/2)}{\Gamma(j-n+3/2)}$$

is a polynomial of j of degree n, and

$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} j^k = (2n)! \left\{ {k \atop 2n} \right\},$$

we get

$$\Lambda_{(1,1,2n+1)} = \sum_{k=0}^{n} a_k \begin{Bmatrix} k \\ 2n \end{Bmatrix}$$

for some real numbers a_k $(0 \leq k \leq n)$. Since $\left\{ {k \atop 2n} \right\} = 0$ for $0 \leq k \leq n < 2n$ for $n \in \mathbb{N}$, the result follows. □

We set

$$C(m) = (2m-1)!/(2^{2m+2}\pi^2 i).$$

We follow the method of Arakawa to evaluate the integral representations for the principal part of the partial zeta functions given in Proposition 2.1.

PROPOSITION 2.4. *Let ξ_j $(j = 1, 2, 3)$ be positive numbers and $m \in \mathbb{N}$. Then the following expressions hold.*

(i) $\zeta_P(1-m; \{a_1V_1, a_2V_2, a_3V_3\}, (\xi_1, \xi_2, \xi_3))$

$$= -4\pi^2 i C(m) \sum_{k_1,k_2,k_3}{}' \frac{a_1^{k_1-1} a_2^{k_2-1} a_3^{k_3-1} B_{k_1}(\xi_1) B_{k_2}(\xi_2) B_{k_3}(\xi_3)}{k_1! k_2! k_3!} \Lambda_{(k_1,k_2,k_3)},$$

where k_1, k_2, k_3 run over all integers with $k_1, k_2 \geq 0$, $k_3 \geq 1$, and $k_1 + k_2 + k_3 = 2m+1$.

(ii) $\zeta_P(1-m; \{a_1V_1, a_2V_3\}, (\xi_1, \xi_3))$

$$= \frac{4\pi^2 i C(m)}{a_2} \sum_{k_1,k_3}{}' \frac{a_1^{k_1-1} a_3^{k_3-1} B_{k_1}(\xi_1) B_{k_3}(\xi_3)}{k_1! k_3!} \Lambda_{(k_1,0,k_3)},$$

where k_1, k_3 run over all integers with $k_1 \geq 0$, $k_3 \geq 1$, and $k_1 + k_3 = 2m+1$.

(iii) $\zeta_P(1-m; \{a_2V_2, a_3V_3\}, (\xi_2, \xi_3))$

$$= \frac{4\pi^2 i C(m)}{a_1} \sum_{k_2,k_3}{}' \frac{a_2^{k_2-1} a_3^{k_3-1} B_{k_2}(\xi_2) B_{k_3}(\xi_3)}{k_2! k_3!} \Lambda_{(0,k_2,k_3)},$$

where k_2, k_3 run over all integers with $k_2 \geq 0$, $k_3 \geq 1$, and $k_2 + k_3 = 2m+1$.

(iv) $\zeta(1-m; \{a_1V_1, a_2V_2\}, (\xi_1, \xi_2))$

$$= \frac{4\pi^2 i C(m)}{a_3} \sum_{k_1,k_2}{}' \frac{a_1^{k_1-1} a_2^{k_2-1} B_{k_1}(\xi_1) B_{k_2}(\xi_2)}{k_1! k_2!} \Lambda_{(k_1,k_2,0)},$$

where k_1, k_2 run over all integers with $k_1, k_2 \geq 0$, and $k_1 + k_2 = 2m+1$.

(v) $$\zeta(1-m; \{a_2V_2\}, \xi_2) = -4\pi^2 i C(m) a_2^{2m-2} \frac{B_{2m-1}(\xi_2)}{(2m-1)!} \Lambda_{(1,2m-1,1)}.$$

We omit the proof of Proposition 2.4, which can be proved in a similar manner as in the proof of Proposition 2.12 of [**1**].

Let $m \in \mathbb{N}$. Let k_1, k_2, k_3 be integers satisfying $k_1, k_2 \geq 0$, $k_3 \geq 1$ and $k_1 + k_2 + k_3 = 2m+1$. For any triple $\xi = (\xi_1, \xi_2, \xi_3)$ of positive numbers, we write, for convenience,

$$B(k_1, k_2, k_3; \xi) = \prod_{j=1}^{3} \frac{a_j^{k_j-1} B_{k_j}(\xi_j)}{k_j!}, \quad P(k_1, k_2, k_3; \xi) = \prod_{j=1}^{3} \frac{a_j^{k_j-1} P_{k_j}(\xi_j)}{k_j!}.$$

Let μ be any integer prime to p. In view of Proposition 2.4, we define the numbers $\mathscr{A}_{(k_1,k_2,k_3)}(X, \mu)$ as follows.

(i) If $k_1, k_2, k_3 \neq 1$, we set

$$\begin{aligned}\mathscr{A}_{(k_1,k_2,k_3)}(X, \mu) &= -4\pi^2 i C(m) \Lambda_{(k_1,k_2,k_3)} \sum_{\xi \in \Xi_\mu(X)} B(k_1, k_2, k_3; \xi) \\ &= -2\pi^2 i C(m) \Lambda_{(k_1,k_2,k_3)} \sum_{\alpha,\gamma(p)} P(k_1, k_2, k_3; \xi_{\alpha,\gamma,\mu}(X)).\end{aligned}$$

(ii) Let r be an integer with $1 \leq r \leq 3$. If $k_r = 1$ and the other $k_j \neq 1$, we set

$$\begin{aligned}\mathscr{A}_{(k_1,k_2,k_3)}(X, \mu) &= -4\pi^2 i C(m) \Lambda_{(k_1,k_2,k_3)} \sum_{\xi \in \Xi_\mu(X),\ \xi_r \neq 1} B(k_1, k_2, k_3; \xi) \\ &= -2\pi^2 i C(m) \Lambda_{(k_1,k_2,k_3)} \sum_{\alpha,\gamma(p)} P(k_1, k_2, k_3; \xi_{\alpha,\gamma,\mu}(X)).\end{aligned}$$

(iii) Let r, n be integers with $1 \leq r < n \leq 3$. If $k_r = k_n = 1$, and the remaining $k_j \neq 1$ (then, necessarily, $m > 1$), we set

$$\begin{aligned}\mathscr{A}_{(k_1,k_2,k_3)}(X, \mu) &= -4\pi^2 i C(m) \Lambda_{(k_1,k_2,k_3)} \\ &\qquad \times \left\{ \sum_{\xi \in \Xi_\mu(X),\ \xi_r, \xi_n \neq 1} B(k_1, k_2, k_3; \xi) - s(k_1, k_2, k_3; \mu) \right\} \\ &= -2\pi^2 i C(m) \Lambda_{(k_1,k_2,k_3)} \\ &\qquad \times \left\{ \sum_{\alpha,\gamma(p)} P(k_1, k_2, k_3; \xi_{\alpha,\gamma,\mu}(X)) - 2s(k_1, k_2, k_3; \mu) \right\},\end{aligned}$$

where

$$s(k_1, k_2, k_3; \mu) = \begin{cases} \dfrac{a_3^{2m-2}}{8} g_{2m-1}(a_3\mu) & (k_1, k_2, k_3) = (1, 1, 2m-1), \\ \dfrac{a_2^{2m-2}\delta_{p,3}}{24} g_{2m-1}(a_2\mu) & (k_1, k_2, k_3) = (1, 2m-1, 1), \\ 0 & (k_1, k_2, k_3) = (2m-1, 1, 1), \end{cases}$$

and

$$g_{2m-1}(\mu) = \frac{1}{(2m-1)!} \sum_{\substack{\alpha(p) \\ \alpha \not\equiv 0(p)}} B_{2m-1}(\langle \mu\alpha^2/p \rangle) = \frac{\psi(\mu)}{(2m-1)!\,p^{2m-2}} \cdot B_{2m-1,\psi}.$$

(iv) In the case of $(k_1, k_2, k_3) = (1, 1, 1)$, we set

$$\begin{aligned}\mathscr{A}_{(1,1,1)}(X, \mu) &= -4\pi^2 i C(1) \Lambda_{(1,1,1)} \\ &\quad \times \left\{ \sum_{\xi \in \Xi_\mu(X),\ \xi_j \neq 1\ (j=1,2,3)} B(1, 1, 1; \xi) - \frac{1}{8} g_1(a_3\mu) - \frac{\delta_{p,3}}{24} g_1(a_2\mu) \right\} \\ &= -2\pi^2 i C(1) \Lambda_{(1,1,1)} \\ &\quad \times \left\{ \sum_{\alpha,\gamma(p)} P(1, 1, 1; \xi_{\alpha,\gamma,\mu}(X)) - \frac{1}{4} g_1(a_3\mu) - \frac{\delta_{p,3}}{12} g_1(a_2\mu) \right\}.\end{aligned}$$

We note that $\mathscr{A}_{(k_1,k_2,k_3)}(X, d^2\mu) = \mathscr{A}_{(k_1,k_2,k_3)}(X, \mu)$ for any d prime to p.

We put

$$\begin{cases} c(k_1, k_2, k_3) = \dfrac{\Lambda_{(k_1,k_2,k_3)}}{k_1! k_2! k_3!}, \\ T(m) = \{(k_1, k_2, k_3) \in \mathbb{Z}^3 \mid k_2 \geq 0,\ k_1, k_3 \geq 1,\ k_1 + k_2 + k_3 = 2m + 1\}. \end{cases} \tag{2.3.9}$$

We define the Arakawa sums $A_i(k_1, k_2, k_3)$ $(1 \leq i \leq 6)$ by

$$\begin{cases} A_1(k_1, k_2, k_3) = \displaystyle\sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{\alpha^2 - 2\alpha\gamma}{p}\right) P_{k_2}\left(\frac{2\alpha\gamma}{p}\right) P_{k_3}\left(\frac{\gamma^2 - \alpha^2}{p}\right), \\ A_2(k_1, k_2, k_3) = \displaystyle\sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{\alpha^2 - 2\alpha\gamma}{p}\right) P_{k_2}\left(\frac{\alpha\gamma}{p}\right) P_{k_3}\left(\frac{\gamma^2 - \alpha^2}{p}\right), \\ A_3(k_1, k_2, k_3) = \displaystyle\sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{\alpha^2 - 2\alpha\gamma}{p}\right) P_{k_2}\left(\frac{4\alpha\gamma}{p}\right) P_{k_3}\left(\frac{\gamma^2 - \alpha^2}{p}\right), \\ A_4(k_1, k_2, k_3) = \displaystyle\sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{2(\alpha^2 - 2\alpha\gamma)}{p}\right) P_{k_2}\left(\frac{2\alpha\gamma}{p}\right) P_{k_3}\left(\frac{\gamma^2 - \alpha^2}{p}\right), \\ A_5(k_1, k_2, k_3) = \displaystyle\sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{2(\alpha^2 - 2\alpha\gamma)}{p}\right) P_{k_2}\left(\frac{\alpha\gamma}{p}\right) P_{k_3}\left(\frac{2(\gamma^2 - \alpha^2)}{p}\right), \\ A_6(k_1, k_2, k_3) = \displaystyle\sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{2(\alpha^2 - 2\alpha\gamma)}{p}\right) P_{k_2}\left(\frac{\alpha\gamma}{p}\right) P_{k_3}\left(\frac{\gamma^2 - \alpha^2}{p}\right). \end{cases} \tag{2.3.10}$$

The special values $L_P(1 - m, \Lambda, \psi)$ $(m \in \mathbb{N})$ for $\Lambda \in \{L_2^*, L_2, M, N\}$ can now be evaluated with the use of the numbers defined above.

PROPOSITION 2.5. *Let $m \in \mathbb{N}$. Then, we get the following:*

(i) $$L_P(1 - m, L_2^*, \psi) = -4\pi^2 i C(m) p^{2(m-1)} \sum_{(k_1,k_2,k_3) \in T(m)} c(k_1, k_2, k_3) A_1(k_1, k_2, k_3)$$

$$+ \frac{\pi^2 i C(m)}{3} (12\delta_{m,1} + c(1, 2m - 1, 1)\delta_{p,3}) B_{2m-1,\psi},$$

(ii) $L_P(1-m, L_2, \psi) =$

$$-4\pi^2 iC(m)p^{2(m-1)} \sum_{(k_1,k_2,k_3)\in T(m)} 2^{k_2-1}c(k_1, k_2, k_3)A_2(k_1, k_2, k_3)$$
$$+\frac{\pi^2 iC(m)}{3}(12\delta_{m,1} - 2^{2m-2}c(1, 2m-1, 1)\delta_{p,3})B_{2m-1,\psi},$$

(iii) $L_P(1-m, M, \psi) =$

$$-4\pi^2 iC(m)\psi(2)p^{2(m-1)} \sum_{(k_1,k_2,k_3)\in T(m)} 2^{k_1+k_3-2}c(k_1, k_2, k_3)A_3(k_1, k_2, k_3)$$
$$+\frac{\pi^2 iC(m)}{3}(12\psi(2)\delta_{m,1} + c(1, 2m-1, 1)\delta_{p,3})B_{2m-1,\psi},$$

(iv) $L_P(1-m, N, \psi) =$

$$L_P(1-m, L_2, \psi) - 4\pi^2 iC(m)\psi(2)p^{2(m-1)}$$
$$\times \sum_{(k_1,k_2,k_3)\in T(m)} c(k_1, k_2, k_3)\big\{-2^{k_2+k_3-2}A_4(k_1, k_2, k_3) - 2^{2k_2-2}A_5(k_1, k_2, k_3)$$
$$+ 2^{2k_2+k_3-2}A_6(k_1, k_2, k_3)\big\} + \frac{\pi^2 iC(m)}{3}(12(\psi(2)-1)\delta_{m,1}$$
$$+ (4^{2m-2} + 2^{2m-2})c(1, 2m-1, 1)\delta_{p,3})B_{2m-1,\psi}.$$

Proof. Let $X = \{X_1, X_2, X_3\} = \{a_1V_1, a_2V_2, a_3V_3\}$ for natural numbers a_1, a_2, a_3 relatively prime to p. Let $\xi_{\alpha,\gamma,\mu}(X)$ be the triple of $\Xi_\mu(X)$ given by (2.1.4). We notice that $B_1(1) = 1/2$, and moreover that

$$\begin{cases} \Xi_\mu^{(2,3)}(X) \cap \Xi_\mu^{(1,3)}(X) = \{\xi_{\alpha,\gamma,\mu}(X) \mid \alpha \equiv 0(p),\ \gamma \not\equiv 0(p)\}, \\ \Xi_\mu^{(2,3)}(X) \cap \Xi_\mu^{(1,2)}(X) = \begin{cases} \phi & (p > 3), \\ \{\xi_{\alpha,\gamma,\mu}(X) \mid \gamma \equiv -\alpha(p),\ \alpha \not\equiv 0(p)\} & (p = 3), \end{cases} \\ \Xi_\mu^{(1,3)}(X) \cap \Xi_\mu^{(1,2)}(X) = \phi. \end{cases} \tag{2.3.11}$$

Taking very carefully (2.2.6), Proposition 2.4, and (2.3.11) into account, we get

$$L_P(1-m, \Lambda, \psi) = p^{2(m-1)} \sum_{\mu} \sideset{}{'}\sum_{k_1,k_2,k_3} \psi(\mu)$$
$$\times \begin{cases} \mathscr{A}_{(k_1,k_2,k_3)}(X^1, \mu), & \text{if } \Lambda = L_2^*, \\ \mathscr{A}_{(k_1,k_2,k_3)}(X^2, \mu), & \text{if } \Lambda = L_2, \\ \mathscr{A}_{(k_1,k_2,k_3)}(X^3, \mu), & \text{if } \Lambda = M, \\ \mathscr{A}_{(k_1,k_2,k_3)}(X^2, \mu) - \mathscr{A}_{(k_1,k_2,k_3)}(X^4, \mu) & \\ \quad -\mathscr{A}_{(k_1,k_2,k_3)}(X^5, \mu) + 2\mathscr{A}_{(k_1,k_2,k_3)}(X^6, \mu) & \text{if } \Lambda = N, \end{cases} \tag{2.3.12}$$

where μ is over 1 and κ, and (k_1, k_2, k_3) runs over all triples of integers with $k_1, k_2 \geq 0$, $k_3 \geq 1$, and $k_1 + k_2 + k_3 = 2m+1$. Moreover, for any triple of integers (k_1, k_2, k_3) with

$k_1, k_2 \geq 0, k_3 \geq 1$ and $k_1 + k_2 + k_3 = 2m + 1$, we have

$$\sum_{\alpha,\gamma(p)} P(k_1, k_2, k_3; \xi_{\alpha,\gamma,\mu}(X^i)) = \frac{\psi(\mu)}{k_1!k_2!k_3!} \times \begin{cases} A_1(k_1, k_2, k_3), & \text{if } i = 1, \\ 2^{k_2-1} A_2(k_1, k_2, k_3), & \text{if } i = 2, \\ \psi(2)\, 2^{k_1+k_3-2} A_3(k_1, k_2, k_3), & \text{if } i = 3, \\ \psi(2)\, 2^{k_2+k_3-2} A_4(k_1, k_2, k_3), & \text{if } i = 4, \\ \psi(2)\, 2^{2k_2-2} A_5(k_1, k_2, k_3), & \text{if } i = 5, \\ \psi(2)\, 2^{2k_2+k_3-3} A_6(k_1, k_2, k_3), & \text{if } i = 6. \end{cases} \tag{2.3.13}$$

Since $A_i(0, k_2, k_3) = 0$ when $k_2 \geq 0, k_3 \geq 1$ with $k_2 + k_3 = 2m + 1$ for parity reasons, we have $\mathscr{A}_{(0,k_2,k_3)}(X^i, \mu) = 0$ for $k_2 \geq 0, k_3 \geq 1$ with $k_2 + k_3 = 2m + 1$. Consequently, in (2.3.12), we only have to consider the triples $(k_1, k_2, k_3) \in T(m)$. The expressions (i)–(iv) for $L_P(1 - m, \Lambda, \psi)$ for $\Lambda \in \{L_2^*, L_2, M, N\}$ then readily follow from (2.3.12), (2.3.13), and Lemma 2.3. □

To evaluate the special values $L_P(0, \Lambda, \psi)$ for $\Lambda \in \{L_2^*, L_2, M, N\}$, we need the following explicit values of $c(k_1, k_2, k_3)$.

PROPOSITION 2.6. *We have*
(i) $c(1, 0, 2) = 4/3$,
(ii) $c(1, 1, 1) = 4$,
(iii) $c(2, 0, 1) = 8/3$.

Proof. The assertion (ii) follows from Lemma 2.3. From (2.3.8), we get $\Lambda_{(2,0,1)} = 2\Lambda_{(1,0,2)} = -2b_{0,1,0}\alpha_1 = 16/3$, from which the assertions (i) and (iii) immediately follow. □

2.4. *Determination of the special values of $L_S(s, \Lambda, \psi)$*

The aim of this section is to express all of the special values $L_S(1 - m, \Lambda, \psi)$ $(m \in \mathbb{N})$ for $\Lambda \in \{L_2^*, L_2, M, N\}$ by generalized Bernoulli numbers. We keep the notation used previously. We remind the reader that $X = \{X_1, X_2, X_3\} = \{a_1V_1, a_2V_2, a_3V_3\}$ with natural numbers a_1, a_2, a_3 relatively prime to p.

For integers k with $k \geq 0$, we set

$$\mathscr{A}_{2k}(u) = 2i(1 - u)^{2k} \operatorname*{Res}_{x=i} \left(\frac{1}{x^2 + u} \left(\frac{x}{1 + x^2} \right)^{2k} \right). \tag{2.4.1}$$

We note that $\mathscr{A}_{2k}(u)$ is a polynomial of u with rational coefficients.

We put

$$\begin{cases} \mu_n(u) = \displaystyle\sum_{k=0}^{\kappa_n} \binom{n-1}{2k} (1 + u)^{n-1-2k} (-u)^k, \\ \nu_n(u) = \displaystyle\sum_{k=0}^{\kappa_n} \binom{n-1}{2k} (1 + u)^{n-1-2k} \mathscr{A}_{2k}(u) \quad \left(\kappa_n = \left[\frac{n-1}{2} \right],\ n \geq 1 \right), \end{cases} \tag{2.4.2}$$

and set

$$\begin{cases} \mu_0(u) = \dfrac{1+u}{1+3u+u^2}, \\ \nu_0(u) = \displaystyle\sum_{j=0}^{\infty} \dfrac{\mathscr{A}_{2j}(u)}{(1+u)^{2j+1}} \qquad \text{for } |u| \text{ sufficiently small.} \end{cases}$$

Let $m \in \mathbb{N}$. For integers k, n with $k, n \geq 0,\ k+n = 2m+1$, we define the numbers $\mathscr{M}_{(k-1,n-1)}$ and $\mathscr{N}_{(k-1,n-1)}$ by putting

$$\begin{cases} \mathscr{M}_{(k-1,n-1)} = \dfrac{1}{\pi i} \displaystyle\int_{I_\epsilon(1)} \log u \cdot u^{-m-1}(1-u)(1+u)^{k-1} \mu_n(u)\, du, \\ \mathscr{N}_{(k-1,n-1)} = \displaystyle\int_{I_\epsilon(1)} u^{-1/2-m}(1-u)(1+u)^{k-1} \nu_n(u)\, du. \end{cases} \tag{2.4.3}$$

The numbers $\mathscr{M}_{(k-1,n-1)}$ and $\mathscr{N}_{(k-1,n-1)}$ are independent of the choice of small ϵ.

Here, we state the following useful formula:

$$\frac{1}{\pi i} \int_{I_\epsilon(1)} \log u \cdot u^k\, du = \frac{2}{k+1} \qquad \text{for } k \in \mathbb{Z},\ k \neq -1. \tag{2.4.4}$$

For integers n with $0 \leq n \leq 2m+1$, we set

$$\mathscr{B}_{n,m} = (-1)^n \cdot \frac{\mathscr{M}_{(2m-n,n-1)} - \mathscr{N}_{(2m-n,n-1)}}{(2m+1-n)!n!}. \tag{2.4.5}$$

We define the Carlitz ϕ and ψ functions

$$\begin{cases} \phi_{r,s}(h, k; x, y) = \displaystyle\sum_{a(k)} P_r\left(h\frac{a+y}{k} + x\right) P_s\left(\frac{a+y}{k}\right), \\ \psi_{r,s}(h, k; x, y) = \displaystyle\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} h^{r-j} \phi_{j,r+s-j}(h, k; x, y). \end{cases}$$

We state the powerful reciprocity theorem given by Carlitz.

LEMMA 2.7. (Carlitz reciprocity [**2**]) *Let* $r, s \in \mathbb{Z}$ *with* $r, s \geq 0,\ h, k \in \mathbb{N}$ *with* $(h, k) = 1$, *and* $x, y \in \mathbb{R}$. *Then, we have*

$$\begin{aligned} &(s+1)k^s \psi_{r+1,s}(h, k; x, y) - (r+1)h^r \psi_{s+1,r}(k, h; y, x) \\ &\quad = (s+1)kP_{r+1}(x)P_s(y) - (r+1)hP_r(x)P_{s+1}(y) + (-1)^{r+1}k^s h^r \delta_{r+s,1}\delta_{x,0}\delta_{y,0}/2. \end{aligned}$$

We follow the method of Arakawa to evaluate the integral representations for the singular part of the partial zeta functions given in Proposition 2.1.

PROPOSITION 2.8. *Let* ξ, ξ_1, ξ_2 *be positive numbers and* $m \in \mathbb{N}$. *Then the following expressions hold.*

(i) $\zeta_S(1-m;\{a_1V_1,a_2V_2,a_3V_3\},(\xi_1,\xi_2))$

$$= -\frac{4\pi a_2^{2m}(-1)^m C(m)}{a_3(2m)!}\int_{I_\epsilon(\infty)}\log t\cdot\phi(a_1t;1-\xi_1)\phi^{(2m)}(a_2t;1-\xi_2)\,dt$$

$$+\frac{4\pi C(m)}{a_3}\sum_{j=0}^{m-1}\frac{(2m-2j-1)!a_2^{2j}(-1)^j}{\{(m-j)!\}^2(2j)!}\int_{\Gamma_\epsilon}t^{2(j-m)}$$

$$\cdot\phi(a_1t;1-\xi_1)\phi^{(2j)}(a_2t;1-\xi_2)\,dt$$

$$+\frac{2\pi^2 iC(m)}{a_3}\sum_{n=0}^{2m+1}(-1)^n a_1^{2m-n}a_2^{n-1}\mathscr{B}_{n,m}\cdot B_{2m+1-n}(\xi_1)B_n(\xi_2),$$

(ii) $\zeta_S(1-m;\{a_1V_1,a_3V_3\},\xi)=\dfrac{2\pi^2 ia_1^{2m-1}C(m)}{a_3}\left\{\dfrac{2}{m(m!)^2}+\mathscr{B}_{1,m}\right\}B_{2m}(\xi),$

(iii) $\zeta_S(1-m;\{a_2V_2,a_3V_3\},\xi)$

$$=\frac{2\pi^2 ia_2^{2m-1}C(m)}{a_3}\left\{\frac{2}{m}\sum_{j=0}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j)!}-\mathscr{B}_{2m,m}\right\}B_{2m}(\xi).$$

We omit the proof of Proposition 2.8, which can be proved in a similar manner as in the proofs of (2.3.25) and Propositions 2.22, 2.23, and 2.24 of [**1**].

This implies the following proposition.

PROPOSITION 2.9. *Let $m\in\mathbb{N}$. Then, we get the following:*

(i) $\displaystyle\sum_{\mu}\psi(\mu)\sum_{\xi\in\Xi_\mu(X)}\zeta_S(1-m;\{a_1V_1,a_2V_2,a_3V_3\},(\xi_1,\xi_2))$

$$=-\frac{4\pi\psi(a_1)a_2^{2m}(-1)^m C(m)}{a_3(2m)!}\sum_{x=1}^{p-1}\psi(x)$$

$$\times\int_{I_\epsilon(\infty)}\log t\cdot\sum_{u(p)}\phi(a_1t;1-\langle(x-a_2u)/p\rangle)\phi^{(2m)}(a_2t;1-\langle a_1u/p\rangle)\,dt$$

$$+\frac{4\pi\psi(a_1)C(m)}{a_3}\sum_{j=0}^{m-1}\frac{(2m-2j-1)!a_2^{2j}(-1)^j}{\{(m-j)!\}^2(2j)!}\sum_{x=1}^{p-1}\psi(x)$$

$$\times\int_{\Gamma_\epsilon}t^{2(j-m)}\cdot\sum_{u(p)}\phi(a_1t;1-\langle(x-a_2u)/p\rangle)\phi^{(2j)}(a_2t;1-\langle a_1u/p\rangle)\,dt$$

$$+\frac{2\pi^2 i\psi(a_1)C(m)}{a_3}\sum_{n=0}^{2m+1}a_1^{2m-n}a_2^{n-1}\mathscr{B}_{n,m}\sum_{x=1}^{p-1}\psi(x)$$

$$\cdot\phi_{2m+1-n,n}(a_1^{-1}a_2,p;x/p,0)$$

$$+\frac{\pi^2 iC(m)}{a_3p^{2m-1}}(-a_1^{2m-1}\psi(a_1)\mathscr{B}_{1,m}+a_2^{2m-1}\psi(a_2)\mathscr{B}_{2n,m}),$$

(ii) $$\sum_{\mu} \psi(\mu) \sum_{\xi \in \Xi_{\mu}^{(1,3)}(X)} \zeta_S(1-m; \{a_1V_1, a_3V_3\}, \xi_1)$$
$$= \frac{2\pi^2 i a_1^{2m-1} \psi(a_1) C(m)}{a_3 p^{2m-1}} \left\{ \frac{2}{m(m!)^2} + \mathscr{B}_{1,m} \right\} B_{2m,\psi},$$

(iii) $$\sum_{\mu} \psi(\mu) \sum_{\xi \in \Xi_{\mu}^{(2,3)}(X)} \zeta_S(1-m; \{a_2V_2, a_3V_3\}, \xi_2)$$
$$= \frac{2\pi^2 i a_2^{2m-1} \psi(a_2) C(m)}{a_3 p^{2m-1}} \left\{ \frac{2}{m} \sum_{j=0}^{m} \frac{(-1)^j}{\{(m-j)!\}^2 (2j)!} - \mathscr{B}_{2m,m} \right\} B_{2m,\psi},$$

where μ runs over 1 and κ.

Proof. Let $\xi_{\alpha,\gamma,\mu}^{(j)}(X)$ $(j = 1, 2, 3)$ be the j-component of $\xi_{\alpha,\gamma,\mu}(X) \in \Xi_{\mu}(X)$ (see (2.1.4)). We set $\mu\alpha^2 = x$ and $2\mu\alpha\gamma = u$. If (α, γ) runs over all elements of $\mathscr{M}(p)$ with $\alpha \not\equiv 0(p)$, and μ is over 1 and κ, then (x, u) just doubly covers all elements of $\mathscr{M}(p)$ with $x \not\equiv 0(p)$. If $\alpha \equiv 0(p)$, then $\xi_{\alpha,\gamma,\mu}^{(1)} = \xi_{\alpha,\gamma,\mu}^{(2)} = 1$. Hence, we get

$$\sum_{\mu} \psi(\mu) \sum_{\xi \in \Xi_{\mu}(X)} \zeta_S(1-m; \{a_1V_1, a_2V_2, a_3V_3\}, (\xi_1, \xi_2))$$
$$= \frac{1}{2} \sum_{\mu} \psi(\mu) \sum_{(\alpha,\gamma) \in \mathscr{M}(p)} \zeta_S(1-m; \{a_1V_1, a_2V_2, a_3V_3\}, (\xi_{\alpha,\gamma,\mu}^{(1)}(X), \xi_{\alpha,\gamma,\mu}^{(2)}(X)))$$
$$= \sum_{\substack{x(p) \\ x \not\equiv 0(p)}} \psi(x) \sum_{u(p)} \zeta_S(1-m; \{a_1V_1, a_2V_2, a_3V_3\}, (\langle a_1^{-1}(x-u)/p \rangle, \langle a_2^{-1}u/p \rangle)).$$

Thus the assertion (i) readily follows from replacing (x, u) by (a_1x, a_1a_2u) and applying Proposition 2.8. The other assertions are similarly verified, so the proofs are omitted. □

The following lemma plays a key role in evaluating the special values $L_S(1-m, \Lambda, \psi)$ $(m \in \mathbb{N})$ for $\Lambda \in \{L_2^*, L_2, M, N\}$.

LEMMA 2.10. *Let $j \in \mathbb{Z}$ with $j \geq 0$, and $x \in \mathbb{Z}$ with $1 \leq x \leq p-1$. Then, the following hold.*

(i) $$\sum_{u(p)} \phi(t; 1 - \langle (x-u)/p \rangle) \phi^{(2j)}(t; 1 - \langle u/p \rangle)$$
$$= \delta_{j,0} B_1 \cdot \phi(t, 1 - x/p) + \sum_{k=0}^{2j+1} \gamma_{k,j}^1 \cdot \phi^{(k)}(t; 1 - x/p)$$

with

$$\gamma_{k,j}^1 = \frac{(-1)^k \binom{2j+1}{k}}{(2j+1)p^{2j-k}} \cdot B_{2j+1-k} \quad (0 \leq k \leq 2j+1).$$

(ii) $\sum_{u(p)} \phi(t;\, 1-\langle (x-2u)/p\rangle)\phi^{(2j)}(2t;\, 1-\langle u/p\rangle)$

$$= \delta_{j,0} B_1 \cdot \phi(t,\, 1-x/p) + \sum_{k=0}^{2j+1} (\gamma_{k,j}^2(x)\phi^{(k)}(2t;\, 1-\{x/(2p)+1/2\}) + \gamma_{k,j}^3(x)\phi^{(k)}(2t;\, 1-x/(2p)))$$

with

$$\gamma_{k,j}^2(x) = \frac{(-1)^k\binom{2j+1}{k}}{(2j+1)p^{2j-k}} \cdot \begin{cases} B_{2j+1-k} & \text{if } x \text{ is odd,} \\ B_{2j+1-k}(1/2) & \text{if } x \text{ is even,} \end{cases}$$

$$\gamma_{k,j}^3(x) = \frac{(-1)^k\binom{2j+1}{k}}{(2j+1)p^{2j-k}} \cdot \begin{cases} B_{2j+1-k}(1/2) & \text{if } x \text{ is odd,} \\ B_{2j+1-k} & \text{if } x \text{ is even,} \end{cases}$$

$$(0 \le k \le 2j+1).$$

(iii) $\sum_{u(p)} \phi(2t;\, 1-\langle (x-u)/p\rangle)\phi^{(2j)}(t;\, 1-\langle 2u/p\rangle)$

$$= \delta_{j,0} B_1 \cdot \phi(2t;\, 1-x/p) + \sum_{k=0}^{2j+1} \gamma_{k,j}^4 \cdot \phi^{(k)}(t;\, 1-\langle 2x/p\rangle)$$

with

$$\gamma_{k,j}^4 = \frac{(-1)^k\binom{2j+1}{k}2^{2j-k}}{(2j+1)p^{2j-k}} \cdot B_{2j+1-k} \quad (0 \le k \le 2j+1).$$

Proof. Recalling the Laurent expansion (2.2.2) of $\phi^{(k)}(t;\, a)$, we compare the coefficients in the Laurent expansions at $t=0$ of both sides of the assertions. When the $\gamma_{k,j}^i(x)$ $(1 \le i \le 4)$ are given as in the lemma, the equality of Laurent coefficients can be verified by applying Carlitz reciprocity (Lemma 2.7). □

For integers n with $1 \le n \le 2m$, we set

$$\mathscr{C}_{n,m} = \sum_{j=1}^{n} \binom{2m+1-j}{n-j} \cdot \mathscr{B}_{j,m}, \tag{2.4.6}$$

where $\mathscr{B}_{j,m}$ is given by (2.4.5). We give a lemma for Proposition 2.12 which will help simplify the correction terms resulting from Carlitz reciprocity.

LEMMA 2.11. *Let $m \in \mathbb{N}$ and $\mathscr{C}_{n,m}$ be given by (2.4.6). Then,*

$$\mathscr{C}_{2n,m} = \frac{m+1-n}{n} \cdot \mathscr{C}_{2n-1,m} \quad (1 \le n \le m).$$

Proof. This follows from the following recursive relations:

$$\begin{cases} \mu_{2n}(u) = -\dfrac{1}{2n}\displaystyle\sum_{j=1}^{2n-1}(-1)^j j\binom{2n}{j}(1+u)^{2n-j}\mu_j(u), \\ \nu_{2n}(u) = -\dfrac{1}{2n}\displaystyle\sum_{j=1}^{2n-1}(-1)^j j\binom{2n}{j}(1+u)^{2n-j}\nu_j(u) \qquad (1 \le n \le m). \end{cases} \qquad \square$$

We define the numbers $\mathscr{D}_{k,m}$ $(0 \le k \le m)$ by

$$\mathscr{D}_{k,m} = \left\{ \frac{4}{(2m+1-2k)(2k)!}\sum_{j=k}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j+1-2k)!} + \sum_{n=\max(1,2k-1)}^{2m}\frac{(-1)^{n+1}\binom{n+1}{2k}(2m+1-n)\mathscr{C}_{n,m}}{n+1} \right\} B_{2k}. \tag{2.4.7}$$

We now evaluate $L_S(1-m, \Lambda, \psi)$ $(m \in \mathbb{N})$ for $\Lambda \in \{L_2^*, L_2, M, N\}$.

PROPOSITION 2.12. *Let $m \in \mathbb{N}$. Then, we get the following:*

(i) $\quad L_S(1-m, L_2^*, \psi) = \dfrac{2\pi^2 i C(m)}{p}\displaystyle\sum_{k=0}^{m}\mathscr{D}_{k,m}B_{2m+1-2k,\psi}$,

(ii) $\quad L_S(1-m, L_2, \psi)$

$$= \frac{\pi^2 i C(m)}{p}\sum_{k=0}^{m}(2^{2m+1-2k}\psi(2)(2^{2k}-1) + (2-2^{2k}))\mathscr{D}_{k,m}B_{2m+1-2k,\psi},$$

(iii) $\quad L_S(1-m, M, \psi) = \dfrac{\pi^2 i C(m)}{2p}\displaystyle\sum_{k=0}^{m}2^{2k}\mathscr{D}_{k,m}B_{2m+1-2k,\psi}$,

(iv) $\quad L_S(1-m, N, \psi) = \frac{1}{2}L_S(1-m, L_2, \psi)$.

Proof. By (2.2.6) and Proposition 2.9, we need to evaluate the two integrals and sum of Carlitz ϕ functions given in expression (i) of Proposition 2.9. The first integral can be evaluated by applying Lemma 2.10 and noting that

$$\int_{I_\epsilon(\infty)} \log t \cdot \phi^{(k)}(at; 1-\xi)\, dt = -2\pi i\frac{(-1)^k P_k(\xi)}{ak} \quad (k \in \mathbb{N},\ a \in \mathbb{N},\ 0 < \xi < 1).$$

The second integral can be evaluated by applying Lemma 2.10 and noting that the coefficient of the term $t^{2(m-j)-1}$ $(0 \le j \le m-1)$ in the Laurent expansion at $t = 0$ of $\phi^{(k)}(at; 1-\xi)$ for $t < |2\pi/a|$ is

$$\frac{(-1)^k\binom{k+2m-2j-1}{k}k!a^{2m-2j-1}P_{k+2m-2j}(\xi)}{(k+2m-2j)!} \qquad (k \ge 0,\ a \in \mathbb{N},\ 0 < \xi < 1).$$

To evaluate the sums of Carlitz ϕ functions, we first express them as sums of Carlitz ψ functions as follows:

$$\sum_{n=1}^{2m} \mathscr{B}_{n,m} \cdot \phi_{2m+1-n,n}(1, p; x/p, 0) = \sum_{n=1}^{2m} \mathscr{C}_{n,m} \cdot \psi_{2m+1-n,n}(1, p; x/p, 0),$$

$$\sum_{n=1}^{2m} 2^{n-1}\mathscr{B}_{n,m} \cdot \phi_{2m+1-n,n}(2, p; x/p, 0) = \sum_{n=1}^{2m} 2^{n-1}\mathscr{C}_{n,m} \cdot \psi_{2m+1-n,n}(2, p; x/p, 0),$$

$$\sum_{n=1}^{2m} 2^{2m-n}\mathscr{B}_{n,m} \cdot \phi_{2m+1-n,n}(2^{-1}, p; x/p, 0)$$

$$= 2^{2m-1}\sum_{n=1}^{2m} \mathscr{C}_{n,m}(\psi_{2m+1-n,n}(1, p; x/p, 0) + \psi_{2m+1-n,n}(1, p; x/p + 1/2, p/2)).$$

These sums can then be evaluated by applying Carlitz reciprocity (Lemma 2.7). We note that all of the correction terms resulting from Carlitz reciprocity vanish by Lemma 2.11.

The assertions of the proposition then readily follow from (2.3.1). □

To evaluate the special values $L_S(0, \Lambda, \psi)$ for $\Lambda \in \{L_2^*, L_2, M, N\}$, we need the following explicit values of $\mathscr{D}_{k,m}$.

PROPOSITION 2.13. *We have*

(i) $\mathscr{D}_{0,1} = 22/9$,

(ii) $\mathscr{D}_{1,1} = -1/3$.

Proof. From (2.4.1)–(2.4.4), we get

$$\mathscr{M}_{(0,1)} = \mathscr{M}_{(1,0)} = \frac{1}{\pi i}\int_{I_\epsilon(1)} \log u \cdot u^{-2}(1-u^2)\, du = -4$$

and $\mathscr{N}_{(0,1)} = \mathscr{N}_{(1,0)} = 0$. Thus the assertions follow from (2.4.5)–(2.4.7). □

2.5. *Main theorem*

In this section, we express all of the special values $L(1-m, \Lambda, \psi)$ ($m \in \mathbb{N}$) for $\Lambda \in \{L_2^*, L_2, M, N\}$ in terms of Arakawa sums and generalized Bernoulli numbers. We remind the reader of the notation.

Let p be an odd prime and let ψ denote the Legendre symbol mod p: $\psi(a) = \left(\frac{a}{p}\right)$. Let $\delta_{x,y}$ be the Kronecker delta symbol. We put

$$T(m) = \{(k_1, k_2, k_3) \in \mathbb{Z}^3 \mid k_2 \geq 0,\ k_1, k_3 \geq 1,\ k_1 + k_2 + k_3 = 2m+1\}.$$

Let $A_i(k_1, k_2, k_3)$ $(1 \le i \le 6)$ be the Arakawa sums given by

$$A_1(k_1, k_2, k_3) = \sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{\alpha^2 - 2\alpha\gamma}{p}\right) P_{k_2}\left(\frac{2\alpha\gamma}{p}\right) P_{k_3}\left(\frac{\gamma^2 - \alpha^2}{p}\right),$$

$$A_2(k_1, k_2, k_3) = \sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{\alpha^2 - 2\alpha\gamma}{p}\right) P_{k_2}\left(\frac{\alpha\gamma}{p}\right) P_{k_3}\left(\frac{\gamma^2 - \alpha^2}{p}\right),$$

$$A_3(k_1, k_2, k_3) = \sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{\alpha^2 - 2\alpha\gamma}{p}\right) P_{k_2}\left(\frac{4\alpha\gamma}{p}\right) P_{k_3}\left(\frac{\gamma^2 - \alpha^2}{p}\right),$$

$$A_4(k_1, k_2, k_3) = \sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{2(\alpha^2 - 2\alpha\gamma)}{p}\right) P_{k_2}\left(\frac{2\alpha\gamma}{p}\right) P_{k_3}\left(\frac{\gamma^2 - \alpha^2}{p}\right),$$

$$A_5(k_1, k_2, k_3) = \sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{2(\alpha^2 - 2\alpha\gamma)}{p}\right) P_{k_2}\left(\frac{\alpha\gamma}{p}\right) P_{k_3}\left(\frac{2(\gamma^2 - \alpha^2)}{p}\right),$$

$$A_6(k_1, k_2, k_3) = \sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{2(\alpha^2 - 2\alpha\gamma)}{p}\right) P_{k_2}\left(\frac{\alpha\gamma}{p}\right) P_{k_3}\left(\frac{\gamma^2 - \alpha^2}{p}\right).$$

Let $c(k_1, k_2, k_3)$ and $\mathscr{D}_{k,m}$ be the numbers given by (2.3.9) and (2.4.7).

We now state the main theorem of this paper.

THEOREM 2.14. *Let $m \in \mathbb{N}$. Then, we have the following:*

(i) $L(1 - m, L_2^*, \psi)$

$$= \frac{(2m-1)!}{2^{2m}} \left\{ -p^{2(m-1)} \sum_{(k_1,k_2,k_3)\in T(m)} c(k_1, k_2, k_3) A_1(k_1, k_2, k_3) + \frac{1}{2p} \sum_{k=0}^{m} \mathscr{D}_{k,m} B_{2m+1-2k,\psi} + \frac{1}{12}(12\delta_{m,1} + c(1, 2m-1, 1)\delta_{p,3}) B_{2m-1,\psi} \right\},$$

(ii) $L(1 - m, L_2, \psi)$

$$= \frac{(2m-1)!}{2^{2m}} \left\{ -p^{2(m-1)} \sum_{(k_1,k_2,k_3)\in T(m)} 2^{k_2-1} c(k_1, k_2, k_3) A_2(k_1, k_2, k_3) + \frac{1}{4p} \sum_{k=0}^{m} (2^{2m+1-2k}\psi(2)(2^{2k}-1) + (2 - 2^{2k})) \mathscr{D}_{k,m} B_{2m+1-2k,\psi} + \frac{1}{12}(12\delta_{m,1} - 2^{2m-2} c(1, 2m-1, 1)\delta_{p,3}) B_{2m-1,\psi} \right\},$$

(iii) $L(1-m, M, \psi)$

$$= \frac{(2m-1)!}{2^{2m}} \Bigg\{ -\psi(2) p^{2(m-1)} \sum_{(k_1,k_2,k_3)\in T(m)} 2^{k_1+k_3-2} c(k_1, k_2, k_3) A_3(k_1, k_2, k_3)$$
$$+ \frac{1}{8p} \sum_{k=0}^{m} 2^{2k} \mathscr{D}_{k,m} B_{2m+1-2k,\psi}$$
$$+ \frac{1}{12}(12\psi(2)\delta_{m,1} + c(1, 2m-1, 1)\delta_{p,3}) B_{2m-1,\psi} \Bigg\},$$

(iv) $L(1-m, N, \psi)$

$$= L(1-m, L_2, \psi) + \frac{(2m-1)!}{2^{2m}}$$
$$\times \Bigg\{ \psi(2) p^{2(m-1)} \sum_{(k_1,k_2,k_3)\in T(m)} c(k_1, k_2, k_3)(2^{k_2+k_3-2} A_4(k_1, k_2, k_3)$$
$$+ 2^{2k_2-2} A_5(k_1, k_2, k_3) - 2^{2k_2+k_3-2} A_6(k_1, k_2, k_3))$$
$$- \frac{1}{8p} \sum_{k=0}^{m} (2^{2m+1-2k} \psi(2)(2^{2k}-1) + (2-2^{2k})) \mathscr{D}_{k,m} B_{2m+1-2k,\psi}$$
$$+ \frac{1}{12}(12(\psi(2)-1)\delta_{m,1} + (4^{2m-2} + 2^{2m-2}) c(1, 2m-1, 1)\delta_{p,3}) B_{2m-1,\psi} \Bigg\}.$$

Proof. This follows from (2.2.7), Proposition 2.5, and Proposition 2.12. □

Remark. The special values $L(1-m, \Lambda, \psi)$ ($m \in \mathbb{N}$) for $\Lambda \in \{L_2^*, L_2, M, N\}$ vanish in the case of $p \equiv 1(4)$ as the Arakawa sums and generalized Bernoulli numbers vanish for parity reasons.

The only obstruction in calculating the special values $L(1-m, \Lambda, \psi)$ ($m \in \mathbb{N}$) for $\Lambda \in \{L_2^*, L_2, M, N\}$ is determining the numbers $c(k_1, k_2, k_3)$ and $\mathscr{D}_{k,m}$. However, we can methodically compute $c(k_1, k_2, k_3)$ for $(k_1, k_2, k_3) \in T(m)$ by (2.3.2)–(2.3.9), and we can methodically compute $\mathscr{D}_{k,m}$ $(0 \le k \le m)$ by (2.3.7) and (2.4.1)–(2.4.7). Thus we can overcome this obstruction. For reference, in Table 1 we give the numbers $c(k_1, k_2, k_3)$ and $\mathscr{D}_{k,m}$ needed to explicitly express the special values $L(1-m, \Lambda, \psi)$ ($m = 1, 2, 3$) for $\Lambda \in \{L_2^*, L_2, M, N\}$ in terms of Arakawa sums and generalized Bernoulli numbers.

As a special case of the above theorem, we calculate the special values $L(0, \Lambda, \psi)$ for $\Lambda \in \{L_2^*, L_2, M, N\}$. We first prepare a lemma.

For convenience, we write

$$\begin{cases} A_i = A_i(1, 1, 1) \quad (1 \le i \le 6), \\ a = A_1(1, 0, 2). \end{cases} \tag{2.5.1}$$

LEMMA 2.15. *We have*

(i) $A_i(2, 0, 1) = -A_i(1, 0, 2) = -a,\ i \in \{1, 2, 3\}$,

(ii) $A_5(2, 0, 1) = -A_5(1, 0, 2) = -\psi(2)a$,

(iii) $A_4(i, 0, j) = A_6(i, 0, j),\ i, j \in \mathbb{N}$.

TABLE 1. $c(k_1, k_2, k_3)$ $((k_1, k_2, k_3) \in T(m))$ and $\mathscr{D}_{k,m}$ $(0 \le k \le m)$ for $m = 1, 2, 3$.

$m = 1$		$m = 3$	
$c(1, 0, 2) = 4/3$	$\mathscr{D}_{0,1} = 22/9$	$c(1, 0, 6) = 64/18225$	$\mathscr{D}_{0,3} = 1759/44100$
$c(1, 1, 1) = 4$	$\mathscr{D}_{1,1} = -1/3$	$c(1, 1, 5) = 0$	$\mathscr{D}_{1,3} = -1/200$
$c(2, 0, 1) = 8/3$		$c(1, 2, 4) = 0$	$\mathscr{D}_{2,3} = -7/16200$
		$c(1, 3, 3) = 8/135$	$\mathscr{D}_{3,3} = -1/7560$
		$c(1, 4, 2) = 1/10$	
$m = 2$		$c(1, 5, 1) = 3/50$	
$c(1, 0, 4) = -8/81$	$\mathscr{D}_{0,2} = 203/450$	$c(2, 0, 5) = 32/6075$	
$c(1, 1, 3) = 0$	$\mathscr{D}_{1,2} = -1/18$	$c(2, 1, 4) = 0$	
$c(1, 2, 2) = 2/3$	$\mathscr{D}_{2,2} = -1/180$	$c(2, 2, 3) = 4/45$	
$c(1, 3, 1) = 2/3$		$c(2, 3, 2) = 11/45$	
$c(2, 0, 3) = -8/81$		$c(2, 4, 1) = 1/5$	
$c(2, 1, 2) = 2/3$		$c(3, 0, 4) = 28/3645$	
$c(2, 2, 1) = 4/3$		$c(3, 1, 3) = 8/135$	
$c(3, 0, 2) = 8/81$		$c(3, 2, 2) = 4/15$	
$c(3, 1, 1) = 8/9$		$c(3, 3, 1) = 44/135$	
$c(4, 0, 1) = 8/81$		$c(4, 0, 3) = 16/729$	
		$c(4, 1, 2) = 2/15$	
		$c(4, 2, 1) = 4/15$	
		$c(5, 0, 2) = 184/6075$	
		$c(5, 1, 1) = 8/75$	
		$c(6, 0, 1) = 368/18225$	

Proof. The assertion (iii) is obvious. The assertion (i) follows from replacing (α, γ) by $(\gamma - \alpha, \gamma)$ in the sums $A_i(2, 0, 1)$ for $i \in \{1, 2, 3\}$. The first equality in the assertion (ii) follows similarly. To prove the second equality, we treat the cases of $p \equiv 1(4)$ and $p \equiv 3(4)$ separately. If $p \equiv 1(4)$, there exists a c such that $c^2 \equiv -1(p)$. Replacing (α, γ) by $(c\alpha, c\gamma)$ in the sums $A_5(1, 0, 2)$ and a, we see that both sums vanish for parity reasons. If $p \equiv 3(4)$, there exists a c such that $c^2 \equiv \psi(2)2^{-1}(p)$, and replacing (α, γ) by $(c\alpha, c\gamma)$ in the sum $A_5(1, 0, 2)$ and using the parity condition for P_1, we get $A_5(1, 0, 2) = \psi(2)a$. Hence the assertion (ii) is established. □

The following is a generalization of Arakawa's theorem 1 in [**1**].

THEOREM 2.16. *We have the following:*

(i) *the special values* $L(1 - m, \Lambda, \psi)$ $(m \in \mathbb{N})$ *for* $\Lambda \in \{L_2^*, L_2, M, N\}$ *are rational numbers,*

(ii) $$L(0, L_2^*, \psi) = -\left(A_1 - \frac{1}{3}a\right) + \frac{6p - 1 + 2p\,\delta_{p,3}}{24p} B_{1,\psi} + \frac{11}{36p} B_{3,\psi},$$

(iii) $$L(0, L_2, \psi) = -\left(A_2 - \frac{1}{6}a\right) + \frac{6p + 1 - 3\psi(2) - 2p\,\delta_{p,3}}{24p} B_{1,\psi} + \frac{11}{72p} B_{3,\psi},$$

(iv) $$L(0, M, \psi) = -\psi(2)\left(A_3 - \frac{2}{3}a\right) + \frac{6p\psi(2) - 1 + 2p\,\delta_{p,3}}{24p} B_{1,\psi} + \frac{11}{144p} B_{3,\psi},$$

(v) $$L(0, N, \psi) = L(0, L_2, \psi) + \psi(2)\left(A_4 + A_5 - 2A_6 - \frac{\psi(2)}{12}a\right) + \frac{12p(\psi(2)-1) - 1 + 3\psi(2) + 8p\,\delta_{p,3}}{48p}B_{1,\psi} - \frac{11}{144p}B_{3,\psi}.$$

Proof. Let $m \in \mathbb{N}$. The rationality of the constants $c(k_1, k_2, k_3)$ $((k_1, k_2, k_3) \in T(m))$ and $\mathscr{D}_{k,m}$ $(0 \le k \le m)$ can be deduced from (2.3.2)–(2.3.9) and (2.3.7), (2.4.1)–(2.4.7), respectively. Thus the assertion (i) follows from Theorem 2.14.

By Proposition 2.6 and Proposition 2.13, we have $c(1, 0, 2) = 4/3$, $c(1, 1, 1) = 4$, $c(2, 0, 1) = 8/3$, and $\mathscr{D}_{0,1} = 22/9$, $\mathscr{D}_{1,1} = -1/3$. By Lemma 2.15, we also have

$$A_i(2, 0, 1) = -A_i(1, 0, 2) = -a, \quad i \in \{1, 2, 3\},$$

$$A_5(2, 0, 1) = -A_5(1, 0, 2) = -\psi(2)a,$$

$$A_4(i, 0, j) = A_6(i, 0, j), \quad i, j \in \mathbb{N}.$$

Thus the assertions (ii)–(v) follow from Theorem 2.14. □

Remark. The rationality of the special values $L(1-m, L_2^*, \psi)$ $(m \in \mathbb{N})$ and the assertion (ii) were proved by Arakawa in Theorem 1 in [**1**]. The rationality of all the special values $L(1-m, \Lambda, \psi)$ $(m \in \mathbb{N})$ for $\Lambda \in \{L_2^*, L_2, M, N\}$ was established by Ibukiyama and Saito from their explicit evaluation of $L(s, \Lambda, \psi)$ given by Theorem 1 in [**3**] (Theorem 3.1). The remaining assertions (iii)–(v) are new.

3. Arakawa identities

Fix an odd prime p. We keep the notation used previously. In Theorem 2.14, we express the special values $L(1-m, \Lambda, \psi)$ $(m \in \mathbb{N})$ for $\Lambda \in \{L_2^*, L_2, M, N\}$ in terms of Arakawa sums and generalized Bernoulli numbers. On the other hand, Ibukiyama and Saito proved the following result.

THEOREM 3.1. [**3**, Theorem 1] *For every $s \in \mathbb{C}$, we have*

(i) $$L(s, L_2^*, \psi) = -\frac{2^{2s-1}B_{1,\psi}}{p^s}\zeta(2s-1),$$

(ii) $$L(s, L_2, \psi) = -\frac{B_{1,\psi}}{p^s}\zeta(2s-1),$$

(iii) $$L(s, M, \psi) = -\frac{B_{1,\psi}}{2p^s}\zeta(2s-1)\{(2^{2s-1}-1) + \psi(2)(2 - 2^{2s-1})\},$$

(iv) $$L(s, N, \psi) = -\frac{B_{1,\psi}}{2p^s}\zeta(2s-1)\{(2^{2-2s}-1)\psi(2) + (2 - 2^{2-2s})\},$$

where $\zeta(s)$ is the Riemann zeta function.

Noting that $\zeta(1-2m) = -B_{2m}/(2m)$ $(m \in \mathbb{N})$, this gives an alternative evaluation of the special values $L(1-m, \Lambda, \psi)$ $(m \in \mathbb{N})$ for $\Lambda \in \{L_2^*, L_2, M, N\}$. Comparing these two different evaluations, we obtain, for each $m = 1, 2, \ldots,$ formulas to be called Arakawa identities expressing Arakawa sums in terms of generalized Bernoulli numbers. We remind the reader of the notation.

We denote by ψ the Legendre symbol mod p: $\psi(a) = \left(\frac{a}{p}\right)$. Let $\delta_{x,y}$ be the Kronecker delta symbol. We put

$$T(m) = \{(k_1, k_2, k_3) \in \mathbb{Z}^3 \mid k_2 \geq 0,\ k_1, k_3 \geq 1,\ k_1 + k_2 + k_3 = 2m+1\}.$$

Let $A_i(k_1, k_2, k_3)$ be the Arakawa sums given by (2.3.10), and let $c(k_1, k_2, k_3)$ and $\mathscr{D}_{k,m}$ be the numbers given by (2.3.9) and (2.4.7).

Corresponding to the special values $L(1-m, \Lambda, \psi)$ ($m \in \mathbb{N}$) for $\Lambda \in \{L_2^*, L_2, M, N\}$, respectively, we have the Arakawa identities.

COROLLARY 3.2. (Arakawa identities) *Let $m \in \mathbb{N}$. Then, the following hold:*

(i)
$$\sum_{(k_1,k_2,k_3)\in T(m)} c(k_1, k_2, k_3) A_1(k_1, k_2, k_3)$$
$$= \frac{1}{2p^{2m-1}} \sum_{k=0}^{m} \mathscr{D}_{k,m} B_{2m+1-2k,\psi}$$
$$+ \frac{1}{12p^{2m-2}}(12\delta_{m,1} + c(1, 2m-1, 1)\delta_{p,3}) B_{2m-1,\psi} - \frac{2B_{2m}}{(2m)!\,p^{m-1}} \cdot B_{1,\psi},$$

(ii)
$$\sum_{(k_1,k_2,k_3)\in T(m)} 2^{k_2-1} c(k_1, k_2, k_3) A_2(k_1, k_2, k_3)$$
$$= \frac{1}{4p^{2m-1}} \sum_{k=0}^{m} (2^{2m+1-2k}\psi(2)(2^{2k}-1) + (2-2^{2k}))\mathscr{D}_{k,m} B_{2m+1-2k,\psi}$$
$$+ \frac{1}{12p^{2m-2}}(12\delta_{m,1} - 2^{2m-2}c(1, 2m-1, 1)\delta_{p,3}) B_{2m-1,\psi} - \frac{2^{2m} B_{2m}}{(2m)!\,p^{m-1}} \cdot B_{1,\psi},$$

(iii)
$$\sum_{(k_1,k_2,k_3)\in T(m)} 2^{k_1+k_3-2} c(k_1, k_2, k_3) A_3(k_1, k_2, k_3)$$
$$= \frac{\psi(2)}{8p^{2m-1}} \sum_{k=0}^{m} 2^{2k} \mathscr{D}_{k,m} B_{2m+1-2k,\psi}$$
$$+ \frac{1}{12p^{2m-2}}(12\delta_{m,1} - c(1, 2m-1, 1)\delta_{p,3}) B_{2m-1,\psi}$$
$$- \frac{B_{2m}}{(2m)!\,p^{m-1}}((1-2^{2m-1})\psi(2) + (2^{2m}-1)) B_{1,\psi},$$

(iv)
$$\sum_{(k_1,k_2,k_3)\in T(m)} c(k_1, k_2, k_3)$$
$$\times \{2^{k_2+k_3-2} A_4(k_1, k_2, k_3) + 2^{2k_2-2} A_5(k_1, k_2, k_3) - 2^{2k_2+k_3-2} A_6(k_1, k_2, k_3)\}$$
$$= \frac{\psi(2)}{8p^{2m-1}} \sum_{k=0}^{m} (2^{2m+1-2k}\psi(2)(2^{2k}-1) + (2-2^{2k}))\mathscr{D}_{k,m} B_{2m+1-2k,\psi}$$

$$+ \frac{1}{12p^{2m-2}}(12(\psi(2)-1)\delta_{m,1} + (4^{2m-2}+2^{2m-2})c(1, 2m-1, 1)\delta_{p,3})B_{2m-1,\psi}$$
$$- \frac{2^{2m-1}B_{2m}}{(2m)!p^{m-1}}(2^{2m}\psi(2) + (1-2^{2m}))B_{1,\psi}.$$

Proof. This follows immediately from Theorem 2.14 and Theorem 3.1. □

Remark. Both sides of the Arakawa identities vanish in the case of $p \equiv 1(4)$ as the Arakawa sums and generalized Bernoulli numbers vanish for parity reasons.

We now give an explicit formulation of the Arakawa identities for $m = 1$.

THEOREM 3.3. *Let A_i and a be the Arakawa sums given by (2.5.1). Then, we have the following:*

(i) $$A_1 - \frac{1}{3}a = \frac{5p-1+2p\,\delta_{p,3}}{24p}B_{1,\psi} + \frac{11}{36p}B_{3,\psi},$$

(ii) $$A_2 - \frac{1}{6}a = \frac{4p+1-3\psi(2)-2p\,\delta_{p,3}}{24p}B_{1,\psi} + \frac{11}{72p}B_{3,\psi},$$

(iii) $$A_3 - \frac{2}{3}a = \frac{(9+\psi(2))p-2\psi(2)-4p\,\delta_{p,3}}{48p}B_{1,\psi} + \frac{11\psi(2)}{144p}B_{3,\psi},$$

(iv) $$A_4 + A_5 - 2A_6 - \frac{\psi(2)}{12}a = \frac{2(2\psi(2)-3)p-3+\psi(2)+8p\,\delta_{p,3}}{48p}B_{1,\psi} + \frac{11\psi(2)}{144p}B_{3,\psi}.$$

Proof. This follows immediately from Theorem 2.16 and Theorem 3.1. □

Remark. The assertion (i) in Theorem 3.3 corresponding to the special value $L(0, L_2^*, \psi)$ was established by Ibukiyama and Saito [**3**] from the work of Arakawa [**1**]. The remaining assertions are new.

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REFERENCES

[1] T. Arakawa. Special values of L-functions associated with the space of quadratic forms and the representation of $Sp(2n, \mathbb{F}_p)$ in the space of Siegel cusp forms. Automorphic Forms and Arithmetic Varieties (Advanced Studies in Pure Mathematics, 15). Kinokuniya and Academic Press, 1989, pp. 99–169.

[2] L. Carlitz. Some theorems on generalized Dedekind–Rademacher sums. Pacific J. Math. **75** (1978), 347–358.

[3] T. Ibukiyama and H. Saito. On L-functions of ternary zero forms and exponential sums of Lee and Weintraub. J. Number Theory **48**(2) (1994), 252–257.

[4] A. Kurihara. On the values at non-positive integers of Siegel's zeta functions of $\mathbb{Q}$-anisotropic quadratic forms with signature $(1, n-1)$. J. Fac. Sci., Univ. Tokyo, Sect. IA Math. **28** (1982), 567–584.

[5] H. Maass. Siegel's Modular Forms and Dirichlet Series (Lecture Notes in Mathematics, 216). Springer, Berlin, 1971.

[6] I. Satake. Special values of zeta functions associated with self-dual homogeneous cones. Manifolds and Lie Groups (Progress in Mathematics, 14). Birkhäuser, Basel, 1981, pp. 359–384.

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