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Brad Isaacson
CUNY New York City College of Technology

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THE TSUKANO CONJECTURES ON EXPONENTIAL SUMS

Brad ISAACSON

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Abstract

We prove three conjectures of Tsukano about exponential sums stated in his Master's thesis written at Osaka University. These conjectures are variations of earlier conjectures made by Lee and Weintraub which were first proved by Ibukiyama and Saito.

1. Introduction

This paper is a continuation of a previous paper [6] on exponential sums originally introduced by Lee-Weintraub. The present paper is dedicated to the proof of conjectures stated by Tsukano, a student of Ibukiyama who came up with these conjectures in his Master's thesis [8] written at Osaka University.

Fix an odd prime p and let ψ denote the Legendre symbol mod p: $\psi(a) = \left(\frac{a}{p}\right)$. We put $\zeta = exp(2\pi i/p)$. Let $\tau(\psi)$ denote the Gaussian sum $\tau(\psi) = \sum_{n=1}^{p-1} \psi(n)\zeta^n$. Let $P(x), Q(x), R(x)$ be polynomials with integer coefficients. Following Lee-Weintraub[7], we define the Lee-Weintraub sum $S[P, Q, R]$ by

$$S[P,Q,R] = -\sum_{k,j(p)}{}' \frac{\psi(k)}{(\zeta^{kP(j)}-1)(\zeta^{kQ(j)}-1)(\zeta^{kR(j)}-1)},$$

where the summation is over a complete residue system modulo p, and the prime on the summation sign means that the meaningless terms are to be excluded. We note that $S[cP, cQ, cR] = \psi(c)S[P, Q, R]$ for any integer c. As discussed in [6], these exotic sums are related to special values of L-functions which are related to the theory of modular forms.

Let h be an integer prime to p. We put

$$S_h = \{(a, b, c) \in \mathbb{Z}^3 : 1 \leq a, b, c \leq p-1, ab + bc + hca \equiv 0(p)\}.$$

Following Tsukano[8], we define $I(h, p)$ and $J(h, p)$ as follows.

$$I(h,p) = \sum_{(a,b,c)\in S_h} \frac{\psi(abc)}{(1-\zeta^a)(1-\zeta^b)(1-\zeta^c)},$$

$$J(h,p) = \sum_{(a,b,c)\in S_h} \psi(abc)abc.$$

The sums $I(h, p)$ and $J(h, p)$ are closely related to the Lee-Weintraub sums $S[-x, x+1, hx(x+1)]$ and $S[1, hx^2, h(x+1)^2]$, respectively. Indeed, we have $S[-x, x+1, hx(x+1)] = \psi(-h)I(h, p)$, and by Proposition 4.2, we have

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$$S[1, hx^2, h(x+1)^2] = \tau(\psi)\left(-\frac{\psi(-1)}{p^2}J(h,p) + \frac{p-1}{p^2}D_h - \frac{\psi(h)(p-1)}{2}B_{1,\psi} - \frac{\psi(h)(p-1)}{4}B_{2,\psi}\right),$$

where

$$D_h = \sum_{a,b=1}^{p-1} ab\,\psi(a+hb).$$

Remark. The sums D_h are easily expressible by generalized Bernoulli numbers from the work in [5].

In [8], Tsukano made five conjectures (modulo equivalence) expressing the character sums $I(h,p)$ and $J(h,p)$ as linear combinations of generalized Bernoulli numbers. They are the following.

Theorem 5.4 (Tsukano Conjectures [8]). *When $p \equiv 3(4)$, we have*

$$\text{(i)} \quad I(2,p)/\tau(\psi) = -\frac{(1+\psi(2))(p+1)}{12}B_{1,\psi} - \frac{\psi(2)+4}{18}B_{3,\psi},$$

$$\text{(ii)} \quad I(2^{-1},p)/\tau(\psi) = \frac{p-2-\psi(2)(5p+2)}{24}B_{1,\psi} - \frac{1+16\psi(2)}{72}B_{3,\psi},$$

$$\text{(iii)} \quad J(1,p)/p^2 = \frac{p+1}{2}B_{1,\psi} - \frac{1}{6}B_{3,\psi},$$

$$\text{(iv)} \quad J(2,p)/p^2 = \frac{4p+1+\psi(2)(4p+7)}{12}B_{1,\psi} - \frac{1+4\psi(2)}{36}B_{3,\psi},$$

$$\text{(v)} \quad J(2^{-1},p)/p^2 = \frac{p+2+\psi(2)(2p+1)}{4}B_{1,\psi} - \frac{\psi(2)}{12}B_{3,\psi},$$

where 2^{-1} is regarded as an element of $(\mathbb{Z}/p\mathbb{Z})^*$, and the generalized Bernoulli numbers $B_{k,\psi}$ are given by (2.2).

The statement (i) was also conjectured by Lee-Weintraub[7] and (iii) by Ibukiyama-Kaneko[2]; both are established in [6]. (Note that $I(2,p) = \psi(-1)S[-x, x+2, x(x+2)]$.) In this paper, we prove the three remaining Tsukano Conjectures.

By virtue of the relationship between the sums $I(h,p)$, $J(h,p)$ and $S[-x, x+1, hx(x+1)]$, $S[1, hx^2, h(x+1)^2]$, respectively, noting that $S[-x, x+1, h^{-1}x(x+1)] = \psi(h)S[-hx, h(x+1), x(x+1)]$ and $S[1, h^{-1}x^2, h^{-1}(x+1)^2] = \psi(h)S[h, x^2, (x+1)^2]$, the following three theorems imply the Tsukano Conjectures (ii), (iv), and (v), respectively.

Theorem 5.1. *For any odd prime number p, we have*

$$S[-2x, 2(x+1), x(x+1)] = \tau(\psi)\left(\frac{-p+2+\psi(2)(5p+2)}{24}B_{1,\psi} + \frac{1+6\psi(2)}{4}B_{2,\psi} + \frac{1+16\psi(2)}{72}B_{3,\psi}\right).$$

Theorem 5.2. *For any odd prime number p, we have*

$$S[1, 2x^2, 2(x+1)^2] = \tau(\psi)\left(\frac{-2p+7-\psi(2)(8p-19)}{12}B_{1,\psi} + \frac{\psi(2)+1}{4}B_{2,\psi} - \frac{4\psi(2)+1}{36}B_{3,\psi}\right).$$

Theorem 5.3. *For any odd prime number p, we have*

$$S[2, x^2, (x+1)^2] = \tau(\psi)\left(\frac{-2p+5-\psi(2)(p-4)}{12}B_{1,\psi} + \frac{1}{2}B_{2,\psi} - \frac{1}{12}B_{3,\psi}\right).$$

The proofs of the above theorems are given roughly as follows. We first rewrite the exponential sum as a sum of triple products of P_1 (defined in (2.1)) evaluated on various parameters. By the work in [4], several such sums are known to be a linear combination of generalized Bernoulli numbers (Theorem 3.1), but our present sums are not the same as those. So we rewrite the sums in question, up to sums of some products of P_1 and P_2, to a linear combination of several standard sums of triple products. Since both parts are known, we are done.

2. Preliminaries

In this section, we discuss all relevant background information needed to evaluate the Lee-Weintraub sums given in the introduction. We omit the proofs which can be found in the literature.

We fix the notation. Let p be an odd prime. We denote by ψ the Legendre symbol mod p: $\psi(a) = \left(\frac{a}{p}\right)$. We put $\zeta = exp(2\pi i/p)$. Let $\tau(\psi)$ denote the Gaussian sum $\tau(\psi) = \sum_{n=1}^{p-1}\psi(n)\zeta^n$. Let $\delta_{x,y}$ be the Kronecker delta symbol ($\delta_{x,y} = 1$ if $x = y$, and 0 otherwise). We define the kth Bernoulli polynomial $B_k(x)$ by the generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}$$

and the kth periodic Bernoulli function $P_k(x)$ by

$$P_k(x) = \begin{cases} 0, & \text{if } k = 1,\ x \in \mathbb{Z}, \\ B_k(\{x\}), & \text{otherwise}, \end{cases} \tag{2.1}$$

where $\{x\}$ denotes the fractional part of x. We note that $P_k(-x) = (-1)^k P_k(x)$, based on $B_k(1-x) = (-1)^k B_k(x)$. We henceforth will apply this parity condition without mention. We denote by $B_{k,\psi}$ the kth generalized Bernoulli number attached to ψ:

$$B_{k,\psi} = p^{k-1}\sum_{a(p)} P_k\left(\frac{a}{p}\right)\psi(a), \tag{2.2}$$

where the summation is over a complete residue system modulo p.

We state the well-known multiplication formula for the periodic Bernoulli functions.

Lemma 2.1. *For any* $k \in \mathbb{Z},\ k \geq 0,\ n \in \mathbb{N},\ x \in \mathbb{R}$, *we have*

$$\frac{1}{n^{k-1}}P_k(nx) = \sum_{a(n)} P_k\left(x + \frac{a}{n}\right),$$

where the summation is over a complete residue system modulo n.

We next give an addition formula for the periodic Bernoulli functions.

Lemma 2.2. *Let $x, y \in \mathbb{R}$. Then, we have*

$$P_1(x)P_1(y) - P_1(x)P_1(x+y) - P_1(y)P_1(x+y)$$
$$= -\frac{1}{2}\Big(P_2(x) + P_2(y) + P_2(x+y)\Big) + \frac{1}{4}\delta(x,y),$$

where

$$\delta(x,y) = \begin{cases} 1, & \text{if } x, y \in \mathbb{Z}, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. The lemma can be checked by elementary algebra, assuming that $x, y \in [0,1)$ and distinguishing the cases where none, one, or both of x, y are equal to 0. □

Most of the sums we encounter will contain periodic Bernoulli functions whose arguments are homogeneous polynomials of degree 2. Thus we introduce the following lemma.

Lemma 2.3. *Let $n \in \mathbb{N}$, $\{k_j\}_{j=1}^n$ be a set of positive integers with $\sum_{j=1}^n k_j \in 2\mathbb{Z}+1$, and $\{q_j(k,t)\}_{j=1}^n$ be homogeneous polynomials of degree 2. Then for any $a \in \mathbb{Z}$, we have*

$$\sum_{k,t(p)} \left(\prod_{j=1}^{n} P_{k_j}(aq_j(k,t)/p) \right) = \psi(a) \sum_{k,t(p)} \left(\prod_{j=1}^{n} P_{k_j}(q_j(k,t)/p) \right).$$

Proof. In the case $p \equiv 1(4)$, there exists a c such that $c^2 \equiv -1(p)$. Replacing (k,t) by (ck, ct), both sides of the lemma vanish for parity reasons. In the case $p \equiv 3(4)$, there exists a c such that $c^2 \equiv \psi(a)a^{-1}(p)$. Replacing (k,t) by (ck, ct), we obtain the assertion of the lemma. □

3. Review of Arakawa Sums and Arakawa Identities

In this section, we state the relevant Arakawa sums and Arakawa Identities obtained in [4]. They play a vital role in evaluating Lee-Weintraub sums.

Fix an odd prime p. We keep the notation used previously. The Arakawa sums $A_i(k_1,k_2,k_3)\,(1 \le i \le 6)$ are defined by

$$(3.1)\qquad \begin{aligned}
A_1(k_1,k_2,k_3) &= \sum_{k,t(p)} P_{k_1}\left(\frac{k^2-2kt}{p}\right) P_{k_2}\left(\frac{2kt}{p}\right) P_{k_3}\left(\frac{t^2-k^2}{p}\right),\\
A_2(k_1,k_2,k_3) &= \sum_{k,t(p)} P_{k_1}\left(\frac{k^2-2kt}{p}\right) P_{k_2}\left(\frac{kt}{p}\right) P_{k_3}\left(\frac{t^2-k^2}{p}\right),\\
A_3(k_1,k_2,k_3) &= \sum_{k,t(p)} P_{k_1}\left(\frac{k^2-2kt}{p}\right) P_{k_2}\left(\frac{4kt}{p}\right) P_{k_3}\left(\frac{t^2-k^2}{p}\right),\\
A_4(k_1,k_2,k_3) &= \sum_{k,t(p)} P_{k_1}\left(\frac{2(k^2-2kt)}{p}\right) P_{k_2}\left(\frac{2kt}{p}\right) P_{k_3}\left(\frac{t^2-k^2}{p}\right),\\
A_5(k_1,k_2,k_3) &= \sum_{k,t(p)} P_{k_1}\left(\frac{2(k^2-2kt)}{p}\right) P_{k_2}\left(\frac{kt}{p}\right) P_{k_3}\left(\frac{2(t^2-k^2)}{p}\right),\\
A_6(k_1,k_2,k_3) &= \sum_{k,t(p)} P_{k_1}\left(\frac{2(k^2-2kt)}{p}\right) P_{k_2}\left(\frac{kt}{p}\right) P_{k_3}\left(\frac{t^2-k^2}{p}\right).
\end{aligned}$$

For convenience, we write

$$\begin{cases} A_i = A_i(1,1,1) \quad (1 \le i \le 6), \\ a = A_1(1,0,2). \end{cases} \tag{3.2}$$

The following are the Arakawa Identities pertinent to the evaluation of Lee-Weintraub sums.

Theorem 3.1 (Theorem 3.3 in [4]). *We have*

(i) $$A_1 - \frac{1}{3}a = \frac{5p - 1 + 2p\,\delta_{p,3}}{24p}B_{1,\psi} + \frac{11}{36p}B_{3,\psi},$$

(ii) $$A_2 - \frac{1}{6}a = \frac{4p + 1 - 3\psi(2) - 2p\,\delta_{p,3}}{24p}B_{1,\psi} + \frac{11}{72p}B_{3,\psi},$$

(iii) $$A_3 - \frac{2}{3}a = \frac{(9 + \psi(2))p - 2\psi(2) - 4p\,\delta_{p,3}}{48p}B_{1,\psi} + \frac{11\psi(2)}{144p}B_{3,\psi},$$

(iv) $$A_4 + A_5 - 2A_6 - \frac{\psi(2)}{12}a = \frac{2(2\psi(2) - 3)p - 3 + \psi(2) + 8p\,\delta_{p,3}}{48p}B_{1,\psi} + \frac{11\psi(2)}{144p}B_{3,\psi}.$$

REMARK. The assertion (i) in Theorem 3.1 was established by Ibukiyama-Saito[3] from the work of Arakawa[1]. The remaining assertions are new.

The question arises whether a direct and elementary proof of these identities is possible. This seems to be a very difficult problem.

4. Auxiliary Sums

In this section, we obtain formulas for all of the auxiliary sums needed in Section 5. We omit the proofs which can be found in the literature. To fix our standpoint, we are satisfied if an exponential sum or a character sum can be expressed in terms of generalized Bernoulli numbers.

Fix an odd prime p. We keep the notation used previously. Let h be an integer prime to p. We put

$$S_h = \{(a,b,c) \in \mathbb{Z}^3 : 1 \le a,b,c \le p-1, ab + bc + hca \equiv 0(p)\},$$
$$T_h = \{(a,b,c) \in \mathbb{Z}^3 : 1 \le a,b,c \le p-1, (-a + b + hc)^2 - 4hbc \equiv 0(p)\},$$

and

$$J(h,p) = \sum_{(a,b,c)\in S_h} abc\,\psi(abc),$$
$$K(h,p) = \sum_{(a,b,c)\in T_h} abc\,\psi(c).$$

We also define the following character sums.

$$D_h = \sum_{a,b=1}^{p-1} ab\,\psi(a + hb),$$
$$\alpha_h = \sum_{k(p)} P_1\left(\frac{k}{p}\right)P_1\left(\frac{hk}{p}\right)\psi(k),$$

$$\beta_h = \sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_1\left(\frac{hk(t+1)}{p}\right)\psi(k),$$

$$K_h = \sum_{k,t(p)} P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{ht^2}{p}\right) P_1\left(\frac{h(k+t)^2}{p}\right),$$

$$J_h = \sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_1\left(\frac{h(k^2+kt)}{p}\right) P_1\left(\frac{t^2+kt}{p}\right).$$

We note that in the sums K_h, J_h, it doesn't matter which P_1 factors contain the parameter h. Indeed, replacing (k,t) by (t,k) and $(-k, t+k)$ in the sums K_h, J_h, we get

$$\begin{aligned} K_h &= \sum_{k,t(p)} P_1\left(\frac{hk^2}{p}\right) P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{h(k+t)^2}{p}\right) \\ &= \sum_{k,t(p)} P_1\left(\frac{hk^2}{p}\right) P_1\left(\frac{ht^2}{p}\right) P_1\left(\frac{(k+t)^2}{p}\right), \end{aligned}$$

$$\begin{aligned} J_h &= \sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_1\left(\frac{k^2+kt}{p}\right) P_1\left(\frac{h(t^2+kt)}{p}\right) \\ &= \sum_{k,t(p)} P_1\left(\frac{hkt}{p}\right) P_1\left(\frac{k^2+kt}{p}\right) P_1\left(\frac{t^2+kt}{p}\right). \end{aligned}$$

We prepare a lemma to be used in Proposition 4.2 and Proposition 4.3.

Lemma 4.1. *For any odd prime number p, we get*

(i) $$D_1 = \sum_{a,b=1}^{p-1} ab\,\psi(a+b) = -\frac{p^2}{2}(2B_{1,\psi} + B_{2,\psi}),$$

(ii) $$\sum_{k,t(p)} P_1\left(\frac{k}{p}\right) P_1\left(\frac{hkt^2}{p}\right) P_1\left(\frac{hk(t+1)^2}{p}\right)\psi(k) = K_h,$$

(iii) $$\sum_{k,t(p)} P_1\left(\frac{k}{p}\right) P_1\left(\frac{hkt^2}{p}\right)\psi(k) = \sum_{k,t(p)} P_1\left(\frac{k}{p}\right) P_1\left(\frac{hk(t+1)^2}{p}\right)\psi(k) = 0,$$

(iv) $$\sum_{k,t(p)} P_1\left(\frac{hkt^2}{p}\right) P_1\left(\frac{hk(t+1)^2}{p}\right)\psi(k) = -\frac{\psi(h)}{p}B_{2,\psi},$$

(v) $$\sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_1\left(\frac{hk(t+1)}{p}\right) P_1\left(\frac{kt(t+1)}{p}\right)\psi(k) = J_h,$$

(vi) $$\sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_1\left(\frac{kt(t+1)}{p}\right)\psi(k) = \frac{1}{2}B_{2,\psi},$$

(vii) $$\sum_{k,t(p)} P_1\left(\frac{hk(t+1)}{p}\right) P_1\left(\frac{kt(t+1)}{p}\right)\psi(k) = -\beta_h.$$

Proof. The assertion (i) was proved in (iv) of Proposition 4.7 in [6], so we omit the proof. Let $S = \sum_{k,t(p)} P_1\left(\frac{k}{p}\right) P_1\left(\frac{hkt^2}{p}\right) P_1\left(\frac{hk(t+1)^2}{p}\right)\psi(k)$. Since the sum S without the character vanishes for parity reasons, we have

$$S = \sum_{k,t(p)} P_1\left(\frac{k}{p}\right) P_1\left(\frac{hkt^2}{p}\right) P_1\left(\frac{hk(t+1)^2}{p}\right)(\psi(k)+1)$$
$$= \sum_{k,t(p)} P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{hk^2t^2}{p}\right) P_1\left(\frac{hk^2(t+1)^2}{p}\right).$$

Replacing (k, t) by $(k, k^{-1}t)$ whenever $k \not\equiv 0(p)$, we see that $S = K_h$. Thus the assertion (ii) is established. The assertion (iii) follows from $\sum_{t(p)} P_1(hkt^2/p) = \psi(hk)B_{1,\psi}$ and $\sum_{k(p)} P_1(k/p) = 0$. Let $S = \sum_{k,t(p)} P_1\left(\frac{hkt^2}{p}\right) P_1\left(\frac{hk(t+1)^2}{p}\right)\psi(k)$. Replacing (k, t) by $(h^{-1}kt^2, t^{-1})$ in the sum S whenever $t \not\equiv 0(p)$, we get

$$S = \psi(h) \sum_{k,t(p)} P_1\left(\frac{k}{p}\right) P_1\left(\frac{k(t+1)^2}{p}\right)\psi(k) - \psi(h) \sum_{k(p)} P_1^2\left(\frac{k}{p}\right)\psi(k).$$

The first sum vanishes by (iii) and the second sum is simply $B_{2,\psi}/p$. Thus we get the assertion (iv). The assertion (v) can be proved in a manner similar to (ii), so we omit the proof. The assertion (vi) was proved in (xi) of Proposition 4.8 in [6], so we omit the proof. Let $S = \sum_{k,t(p)} P_1\left(\frac{hk(t+1)}{p}\right) P_1\left(\frac{kt(t+1)}{p}\right)\psi(k)$. Replacing (k, t) by $(h^{-1}k(hk^{-1}t + 1)^{-1}, hk^{-1}t)$ in the sum S, we get

$$S = \psi(h) \sum_{k,t(p)} P_1\left(\frac{k}{p}\right) P_1\left(\frac{t}{p}\right)\psi(ht + k).$$

On the other hand, replacing $(h^{-1}k + t, -(h^{-1}k + t)^{-1}t)$ in the sum β_h, we have

$$\beta_h = -\psi(h) \sum_{k,t(p)} P_1\left(\frac{t}{p}\right) P_1\left(\frac{k}{p}\right)\psi(k + ht).$$

Thus $S = -\beta_h$ and the assertion (vii) is established. □

The next two propositions allow us to express the Lee-Weintraub sums $S[-x, x+1, hx(x+1)]$, $S[1, hx^2, h(x+1)^2]$ by a sum of triple products of P_1 functions.

Proposition 4.2. *For any odd prime number p, we get*

(i) $$S[-x, x+1, hx(x+1)] = \tau(\psi)\left(-\frac{\psi(h)}{p^2}K(h, p) + \frac{(p-1)^2(2+\psi(h))}{4}B_{1,\psi}\right),$$

(ii) $$S[1, hx^2, h(x+1)^2]$$
$$= \tau(\psi)\left(-\frac{\psi(-1)}{p^2}J(h, p) + \frac{p-1}{p^2}D_h - \frac{\psi(h)(p-1)}{2}B_{1,\psi} - \frac{\psi(h)(p-1)}{4}B_{2,\psi}\right).$$

Proof. We first prove the assertion (i). Let S denote the sum $S[-x, x+1, hx(x+1)]$. Then, we have

$$S = -\sum_{k,j(p)}{}' \frac{\psi(k)}{(\zeta^{-kj} - 1)(\zeta^{k(j+1)} - 1)(\zeta^{hkj(j+1)} - 1)}.$$

Since $\frac{1}{\zeta^n - 1} = \frac{1}{p}\sum_{a=1}^{p-1} a\zeta^{an}$ for any n with $(n, p) = 1$, we get

$$S = -\frac{1}{p^3}\sum_{a,b,c=1}^{p-1} abc \sum_{\substack{j(p)\\ j\not\equiv 0,-1(p)}} \sum_{\substack{k(p)\\ k\not\equiv 0(p)}} \zeta^{k\{hcj^2+(-a+b+hc)j+b\}}\psi(k)$$

$$= -\frac{\tau(\psi)}{p^3}\sum_{a,b,c=1}^{p-1} abc\left\{\left(\sum_{j(p)}\psi(hcj^2+(-a+b+hc)j+b)\right)-\psi(b)-\psi(a)\right\}$$

$$= \tau(\psi)\left(\frac{(p-1)^2(2+\psi(h))}{4}B_{1,\psi}-\frac{\psi(h)}{p^2}K(h,p)\right).$$

We next prove the assertion (ii). Let S denote the sum $S[1, hx^2, h(x+1)^2]$. Proceeding as before, with the help of (i) in Lemma 4.1, we have

$$S = -\frac{1}{p^3}\sum_{a,b,c=1}^{p-1} abc \sum_{\substack{j(p)\\ j\not\equiv 0,-1(p)}} \sum_{\substack{k(p)\\ k\not\equiv 0(p)}} \zeta^{k\{(hb+hc)j^2+2hcj+(a+hc)\}}\psi(k)$$

$$= -\frac{\tau(\psi)}{p^3}\sum_{a,b,c=1}^{p-1} abc\left\{\left(\sum_{j(p)}\psi((hb+hc)j^2+2hcj+a+hc)\right)-\psi(a+hc)-\psi(a+hb)\right\}$$

$$= \tau(\psi)\left(\frac{p-1}{p^2}D_h-\frac{\psi(h)(p-1)}{2}B_{1,\psi}-\frac{\psi(h)(p-1)}{4}B_{2,\psi}-\frac{\psi(h)}{p^2}\sum_{(a,b,c)\in S_h} abc\,\psi(a+c)\right).$$

Since $\psi(a+c) = \psi(-h)\psi(abc)$ for every triple $(a,b,c) \in S_h$, we obtain the assertion (ii). □

Proposition 4.3. *For any odd prime number p, we get*

(i) $\quad K(h,p)/p^3 = K_h - \alpha_h + \dfrac{(p-2)(1+2\psi(h))}{4}B_{1,\psi} - \dfrac{\psi(h)}{2p}B_{2,\psi},$

(ii) $\quad J(h,p)/p^3 = -\psi(-h)\left(J_h + \beta_h + \dfrac{2+\psi(h)}{4}B_{1,\psi} + \dfrac{1}{4}B_{2,\psi}\right).$

Proof. We first prove the assertion (i). By the 1-to-1 correspondence between the sets T_h and $\{(hk(t+1)^2, hkt^2, k) \in (\mathbb{F}_p^\times)^3 \mid 1 \le k, t \le p-1\}$, we have

$$K(h,p) = \sum_{\substack{k,t(p)\\ kt(t+1)\not\equiv 0(p)}} [k][hkt^2][hk(t+1)^2]\,\psi(k),$$

where $[n]$ denotes the least nonnegative residue of n mod p. Since $[x]/p = P_1(x/p) + 1/2$ if $x \notin \mathbb{Z}$ and $P_1(x/p)$ otherwise, we express $K(h,p)$ by sums of products of P_1 functions and take full advantage of their periodicity. Thus, we have

$$K(h,p)/p^3$$
$$= \sum_{\substack{k,t(p)\\ kt(t+1)\not\equiv 0(p)}} \left(P_1\left(\frac{k}{p}\right)+\frac{1}{2}\right)\left(P_1\left(\frac{hkt^2}{p}\right)+\frac{1}{2}\right)\left(P_1\left(\frac{hk(t+1)^2}{p}\right)+\frac{1}{2}\right)\psi(k).$$

Upon simplifying, with the help of (ii)-(iv) in Lemma 4.1, we obtain the assertion (i).

We next prove the assertion (ii). By the 1-to-1 correspondence between the sets S_h and $\{(kt(t+1), hk(t+1), -kt) \in (\mathbb{F}_p^\times)^3 \mid 1 \le k, t \le p-1\}$, we have

$$J(h,p) = \sum_{\substack{k,t(p) \\ kt(t+1)\not\equiv 0(p)}} [-kt][hk(t+1)][kt(t+1)]\,\psi(-hk^3t^2(t+1)^2).$$

Proceeding as before, expressing $J(h,p)$ by sums of products of P_1 functions and simplifying with the help of (v)-(vii) in Lemma 4.1, we obtain the assertion (ii). □

In order to evaluate the Lee-Weintraub sums given in the introduction, we need the following proposition.

Proposition 4.4. *For any prime number p, we get*

$$\text{(i)} \qquad D_2 = \psi(2)D_{2^{-1}} = -\frac{p^2}{4}\left(2(1+\psi(2))B_{1,\psi} + B_{2,\psi}\right),$$

$$\text{(ii)} \qquad \alpha_2 = \psi(2)\alpha_{2^{-1}} = \frac{4+\psi(2)}{4p}B_{2,\psi},$$

$$\text{(iii)} \quad \beta_2 = \psi(2)\beta_{2^{-1}} = \frac{\psi(2)}{4}B_{2,\psi}.$$

Proof. In the notation of [5], we have $D_h = S_1(h,\psi;1)$. Therefore, by Theorems 4.5, 4.4, and 3.1 in [5], we get $D_2 = -\frac{p^2}{4}\left(2(1+\psi(2))B_{1,\psi} + B_{2,\psi}\right)$. Since $D_{2^{-1}} = \psi(2)D_2$ is clear, we have the assertion (i).

By a standard calculation as seen in Proposition 4.8 of [6], we get $\alpha_2 = \frac{4+\psi(2)}{4p}B_{2,\psi}$. Replacing k by hk in the sum $\alpha_{h^{-1}}$, we get $\alpha_{h^{-1}} = \psi(h)\alpha_h$. Thus the assertion (ii) follows.

Replacing (k,t) by $(h^{-1}k+t, -(h^{-1}k+t)^{-1}t)$ in the sum β_h whenever $h^{-1}k+t\not\equiv 0$, we get

$$\beta_h = -\psi(h)\sum_{k,t(p)} P_1\left(\frac{t}{p}\right)P_1\left(\frac{k}{p}\right)\psi(k+ht) = -\psi(h)\left(\frac{1}{p^2}D_h + \frac{\psi(h)+1}{2}B_{1,\psi}\right).$$

Thus the assertion (iii) follows from (i). □

REMARK. The sum D_h has been studied in greater generality in [5].

We now introduce some new notation. Let $q_j(k,t)\,(j=1,2,3)$ be polynomial expressions of k and t. We write $S(q_1(k,t), q_2(k,t), q_3(k,t))$ and $D(q_1(k,t), q_2(k,t))$ to denote the following sums of products of periodic Bernoulli functions.

$$S(q_1(k,t), q_2(k,t), q_3(k,t)) = \sum_{k,t(p)} P_1(q_1(k,t)/p)P_1(q_2(k,t)/p)P_1(q_3(k,t)/p),$$

$$D(q_1(k,t), q_2(k,t)) = \sum_{k,t(p)} P_1(q_1(k,t)/p)P_2(q_2(k,t)/p).$$

We note that the sum $S(q_1,q_2,q_3)$ is invariant under permutations of its arguments, whereas the sum $D(q_1,q_2)$ is not.

We next give a proposition that is used only in Theorem 4.6.

Proposition 4.5 ([6])**.** *Let $r, s \in \mathbb{Z}$. We have*

$$\text{(i)} \qquad D(rkt, s(t^2-k^2)) = 0,$$

$$\text{(ii)} \qquad D(r(t^2+kt), s(k^2+2kt)) = D(r(k^2+2kt), s(t^2+kt)) = 0,$$

(iii) $D(k^2+kt,kt) = D(k^2+2kt,2kt) = D(t^2+kt,k^2+kt)$
$$= \psi(2)D(2(t^2+kt),2kt) = \psi(2)D(2(t^2+kt),2(k^2+kt)) = -\frac{1}{3p}B_{3,\psi},$$

(iv) $D(k^2+kt,2kt) = D(t^2+kt,2(k^2+kt)) = \dfrac{1-\psi(2)}{2p}B_{1,\psi} - \dfrac{\psi(2)}{6p}B_{3,\psi},$

(v) $D(k^2+2kt,kt) = \psi(2)D(2(k^2+2kt),2kt) = \dfrac{1-\psi(2)}{4p}B_{1,\psi} - \dfrac{1}{12p}B_{3,\psi},$

(vi) $D(kt,k^2+kt) = D(2kt,k^2+2kt) = \dfrac{1}{3p}B_{3,\psi},$

(vii) $D(kt,k^2+2kt) = \psi(2)D(2kt,2(k^2+2kt)) = \dfrac{1-\psi(2)}{2p}B_{1,\psi} + \dfrac{1}{6p}B_{3,\psi},$

(viii) $D(2kt,k^2+kt) = -D(2(t^2+kt),k^2+kt) = -D(2(t^2+kt),kt)$
$$= \frac{1-\psi(2)}{4p}B_{1,\psi} + \frac{\psi(2)}{12p}B_{3,\psi},$$

(ix) $D(k^2+2kt,t^2+2kt) = \psi(2)D(2(k^2+2kt),2(t^2+2kt)) = -a,$

(x) $D(k^2+2kt,2(t^2+2kt)) = -\dfrac{4+\psi(2)}{3}\,a,$

(xi) $D(2(k^2+2kt),t^2+2kt) = -A_4(1,0,2).$

Since the assertions of Proposition 4.5 can be found in Propositions 4.3, 4.5, and 4.6 in [6], we omit the proof.

Set $\epsilon = p/2$. We define the following sums which frequently arise in the evaluation of Lee-Weintraub sums.

$$\begin{aligned}
X_1 &= S(k^2,t^2,(k+t)^2),\\
X_2 &= S(k^2+\epsilon,t^2,(k+t)^2),\\
X_3 &= S(k^2+\epsilon,t^2+\epsilon,(k+t)^2),\\
X_4 &= S(k^2+\epsilon,t^2+\epsilon,(k+t)^2+\epsilon),\\
Y_1 &= S(kt,k^2+kt,t^2+kt),\\
Y_2 &= S(kt+\epsilon,k^2+kt,t^2+kt),\\
Y_3 &= S(kt+\epsilon,k^2+kt+\epsilon,t^2+kt),\\
Y_4 &= S(kt+\epsilon,k^2+kt+\epsilon,t^2+kt+\epsilon),\\
\gamma &= D(kt,2(k^2+2kt)).
\end{aligned} \tag{4.1}$$

REMARK. For the Lee-Weintraub sums given in [6], we encountered the sums X_1, X_2, X_3, and Y_1, and for the Lee-Weintraub sums given in this paper, we encounter the sums X_1, X_2, Y_1, Y_2, and Y_3.

We note that in the sums $X_i, Y_i\,(i=2,3)$, it doesn't matter which P_1 factors contain the parameter ϵ. Indeed, replacing (k,t) by (t,k) and $(-k-t,t)$ in the sum X_2, by $(k,-k-t)$ and $(-k-t,t)$ in the sum X_3, by $(-k,k+t)$ and $(-t,k+t)$ in the sum Y_2, and by (t,k) and $(-k-t,t)$ in the sum Y_3, we get

$$X_2 = S(k^2, t^2 + \epsilon, (k+t)^2) = S(k^2, t^2, (k+t)^2 + \epsilon),$$
$$X_3 = S(k^2 + \epsilon, t^2, (k+t)^2 + \epsilon) = S(k^2, t^2 + \epsilon, (k+t)^2 + \epsilon),$$
$$Y_2 = S(kt, k^2 + kt + \epsilon, t^2 + kt) = S(kt, k^2 + kt, t^2 + kt + \epsilon),$$
$$Y_3 = S(kt + \epsilon, k^2 + kt, t^2 + kt + \epsilon) = S(kt, k^2 + kt + \epsilon, t^2 + kt + \epsilon).$$

We will routinely make use of the multiplication formula Lemma 2.1 to express periodic Bernoulli functions shifted by $1/2$ as follows.

$$(4.2) \qquad P_k(x + 1/2) = 1/2^{k-1} P_k(2x) - P_k(x) \quad (k \in \mathbb{Z},\ k \geq 0,\ x \in \mathbb{R}).$$

We will also regularly call upon Lemma 2.3 to "factor out" a $\psi(2)$ from a sum of products of periodic Bernoulli functions.

In the next theorem (Theorem 4.6), we will frequently simplify sums of triple products of periodic Bernoulli functions by applying the addition formula Lemma 2.2 to two of the terms, then multiplying throughout by the remaining term and carefully summing. Thus we write $[a, b;\ c;\ (d), (e)]$ to mean "applying Lemma 2.2 with $x = a/p$, $y = b/p$, multiplying throughout by $P_1(c/p)$ and summing over $k, t(p)$ with the help of $(d), (e)$ in Proposition 4.5". Through this process, we express our original sum of triple products as two new sums of triple products plus three sums of double products and a correction sum involving just one P_1. That is, we get

$$S(a, b, c) = S(a, a+b, c) + S(b, a+b, c) - \frac{1}{2}(D(c, a) + D(c, b) + D(c, a+b)) + \frac{1}{4} \sum_{k,t(p)} P_1(c/p)\delta(a/p, b/p),$$

where $\delta(x, y) = 1$ if $x, y \in \mathbb{Z}$, and 0 otherwise. All of the correction sums involving one P_1 are easy to evaluate and require no explanation. (In fact, all of the correction sums involving one P_1 will vanish with the exception of $\sum_{k,t(p)} P_1(kt/p + 1/2)\delta((k^2 + 2kt)/p, (t^2 + 2kt)/p))$ which equals $\delta_{p,3}(\psi(2) - 1)B_{1,\psi}$.) All of the sums of double products have already been evaluated in Proposition 4.5. Thus, we will only explain manipulations of sums of triple products of periodic Bernoulli functions. We will apply the same sequence of steps to each new sum of triple products arising. In this way, we will obtain a tree of sums. In the end, we will condense the tree and express our original sum of triple products by Arakawa sums, from which we can evaluate by virtue of the Arakawa Identities given by Theorem 3.1.

We remind the reader that $A_i(k_1, k_2, k_3)$ are the Arakawa sums given in (3.1), and that A_i, a are the Arakawa sums given in (3.2). We now express the sums $X_i,\ Y_i\ (1 \leq i \leq 4)$ in terms of γ and Arakawa sums $A_i\ (1 \leq i \leq 6),\ a$. We note that only the sums $Y_2,\ Y_3$ involve the sum γ.

Theorem 4.6. *For any odd prime number p, we get*

(i) $$X_1 = 6\left(A_1 - \frac{1}{3}a\right) - \frac{6p - 3 + 2p\,\delta_{p,3}}{4p} B_{1,\psi} - \frac{2}{p} B_{3,\psi},$$

(ii) $$X_2 = 2(1 + \psi(2))\left(A_1 - \frac{1}{3}a\right) - 2\psi(2)\left(A_2 - \frac{1}{6}a\right) - 2\left(A_3 - \frac{2}{3}a\right) + \frac{\psi(2) - 1}{4p} B_{1,\psi} - \frac{4 + \psi(2)}{6p} B_{3,\psi},$$

(iii) $X_3 = -2\left(A_1 - \frac{1}{3}a\right) + 2\psi(2)\left(A_2 - \frac{1}{6}a\right) + \frac{1-\psi(2)}{2}B_{1,\psi} + \frac{2-\psi(2)}{3p}B_{3,\psi},$

(iv) $X_4 = -6\left(A_1 - \frac{1}{3}a\right) + 6\left(A_3 - \frac{2}{3}a\right) + \frac{1-\psi(2)}{2}\delta_{p,3}B_{1,\psi} + \frac{4-\psi(2)}{2p}B_{3,\psi},$

(v) $Y_1 = -6\left(A_2 - \frac{1}{6}a\right) + \frac{3p+3-3\psi(2)-2p\,\delta_{p,3}}{4p}B_{1,\psi} + \frac{3}{4p}B_{3,\psi},$

(vi) $$Y_2 = 2\left(A_1 - \frac{1}{3}a\right) - 2\left(A_2 - \frac{1}{6}a\right) - 2\left(A_4 + A_5 - 2A_6 - \frac{\psi(2)}{12}a\right) - \frac{1}{4}\gamma + \frac{(\psi(2)-1)(4p-3)}{16p}B_{1,\psi} + \frac{7\psi(2)-12}{48p}B_{3,\psi},$$

(vii) $$Y_3 = 2(1-\psi(2))\left(A_2 - \frac{1}{6}a\right) + 2\left(A_4 + A_5 - 2A_6 - \frac{\psi(2)}{12}a\right) - \frac{1}{2}\gamma + \frac{\psi(2)-1}{4p}B_{1,\psi} + \frac{2\psi(2)-3}{12p}B_{3,\psi},$$

(viii) $Y_4 = -6\left(A_1 - \frac{1}{3}a\right) + 6\left(A_2 - \frac{1}{6}a\right) + \frac{3\psi(2)-3+4p\,\delta_{p,3}}{4p}B_{1,\psi} + \frac{3}{4p}B_{3,\psi}.$

Proof. The assertions (i)-(iii) were proved in Theorem 5.1 in [6], so we omit the proofs. We now prove the assertion (iv). We set

$$I = S(2k^2, 2t^2, 2(k+t)^2).$$

By Lemma 2.3, we have $I = \psi(2)X_1$. On the other hand, by the multiplication formula Lemma 2.1, we get $I = X_1 + 3X_2 + 3X_3 + X_4$. Thus $X_4 = (\psi(2)-1)X_1 - 3X_2 - 3X_3$ and the assertion (iv) follows from (i)-(iii).

The assertion (v) was proved in Theorem 5.3 in [6], so we omit the proof. For the remaining assertions (vi)-(viii), we apply a sequence of transformations similarly used in the proof of (v). We now prove the assertion (vi) by creating a tree of sums for Y_2 of depth 3. By $[kt+\epsilon, k^2+kt;\ t^2+kt;\ \text{(ii)-(iv)}]$, we have

$$Y_2 = S(kt+\epsilon, k^2+2kt+\epsilon, t^2+kt) + S(k^2+kt, k^2+2kt+\epsilon, t^2+kt) + \frac{\psi(2)-1}{8p}B_{1,\psi} + \frac{\psi(2)}{24p}B_{3,p}.$$

Replacing (k,t) by $(-k, k+t)$ in the second sum on the right, we get

$$Y_2 = S(kt+\epsilon, k^2+2kt+\epsilon, t^2+kt) + S(kt, k^2+2kt+\epsilon, t^2+kt) + \frac{\psi(2)-1}{8p}B_{1,\psi} + \frac{\psi(2)}{24p}B_{3,p}. \tag{4.3}$$

We now evaluate the first sum on the right of (4.3). By $[kt+\epsilon, t^2+kt;\ k^2+2kt+\epsilon;\ \text{(ii), (iii), (v), (ix)-(xi)}]$, we get

$$\begin{aligned} &S(kt+\epsilon, k^2+2kt+\epsilon, t^2+kt) \\ &\quad= S(kt+\epsilon, k^2+2kt+\epsilon, t^2+2kt+\epsilon) + S(t^2+kt, k^2+2kt+\epsilon, t^2+2kt+\epsilon) \\ &\quad+ \frac{\psi(2)+1}{6}a - \frac{1}{2}A_4(1,0,2) + \frac{1}{2}D(2(k^2+2kt), kt) + \frac{\psi(2)-1}{16p}B_{1,\psi} + \frac{\psi(2)-2}{48p}B_{3,p}. \end{aligned}$$

Replacing (k, t) by $(t - k, k)$ in the second sum on the right and applying (4.2), we obtain

$$\begin{aligned} &S(kt + \epsilon, k^2 + 2kt + \epsilon, t^2 + kt) \\ &\quad = S(kt + \epsilon, k^2 + 2kt + \epsilon, t^2 + 2kt + \epsilon) + S(kt, 2(t^2 - k^2), k^2 - 2kt) \\ &\quad - A_2 - A_5 + A_6 + \frac{\psi(2) + 1}{6} a - \frac{1}{2} A_4(1, 0, 2) + \frac{1}{2} D(2(k^2 + 2kt), kt) \\ &\quad + \frac{\psi(2) - 1}{16p} B_{1,\psi} + \frac{\psi(2) - 2}{48p} B_{3,p}. \end{aligned} \tag{4.4}$$

We next evaluate the first sum on the right. By $[-(k^2 + 2kt) + \epsilon, t^2 + 2kt + \epsilon;\ kt + \epsilon;\ (\mathrm{i}), (\mathrm{vi}), (\mathrm{vii})]$, we get

$$\begin{aligned} &S(kt + \epsilon, k^2 + 2kt + \epsilon, t^2 + 2kt + \epsilon) \\ &\quad = S(kt + \epsilon, k^2 + 2kt + \epsilon, t^2 - k^2) - S(kt + \epsilon, t^2 + 2kt + \epsilon, t^2 - k^2) - \frac{1}{2}\gamma \\ &\quad + \frac{1 - \psi(2)}{4p} B_{1,\psi} + \frac{\psi(2) - 2}{12p} B_{3,p}. \end{aligned}$$

Replacing (k, t) by $(k, -t)$ in the first sum and (k, t) by $(t, -k)$ in the second sum and applying (4.2), we get

$$\begin{aligned} &S(kt + \epsilon, k^2 + 2kt + \epsilon, t^2 + 2kt) \\ &\quad = 2A_1 - 2A_2 - 2A_4 + 2A_6 - \frac{1}{2}\gamma + \frac{1 - \psi(2)}{4p} B_{1,\psi} + \frac{\psi(2) - 2}{12p} B_{3,p}. \end{aligned} \tag{4.5}$$

We now evaluate the second sum on the right hand side of (4.3). By $[kt, t^2 + kt;\ k^2 + 2kt + \epsilon;\ (\mathrm{ii}), (\mathrm{v}), (\mathrm{ix}), (\mathrm{xi})]$, we get

$$\begin{aligned} &S(kt, k^2 + 2kt + \epsilon, t^2 + kt) \\ &\quad = S(kt, k^2 + 2kt + \epsilon, t^2 + 2kt) + S(t^2 + kt, k^2 + 2kt + \epsilon, t^2 + 2kt) - \frac{1}{2} a \\ &\quad + \frac{1}{2} A_4(1, 0, 2) - \frac{1}{2} D(2(k^2 + 2kt), kt) + \frac{(1 - 2p)(1 - \psi(2))}{8p} B_{1,\psi} - \frac{1}{24p} B_{3,\psi}. \end{aligned}$$

Replacing (k, t) by $(t - k, k)$ in the second sum on the right and applying (4.2), we get

$$\begin{aligned} &S(kt, k^2 + 2kt + \epsilon, t^2 + kt) \\ &\quad = S(kt, k^2 + 2kt + \epsilon, t^2 + 2kt) - S(kt, 2(t^2 - k^2), k^2 - 2kt) \\ &\quad + A_2 - \frac{1}{2} a + \frac{1}{2} A_4(1, 0, 2) - \frac{1}{2} D(2(k^2 + 2kt), kt) \\ &\quad + \frac{(1 - 2p)(1 - \psi(2))}{8p} B_{1,\psi} - \frac{1}{24p} B_{3,\psi}. \end{aligned} \tag{4.6}$$

We next evaluate the first sum on the right. By $[-(k^2 + 2kt) + \epsilon, t^2 + 2kt;\ kt;\ (\mathrm{i}), (\mathrm{vii})]$, we get

$$\begin{aligned} &S(kt, k^2 + 2kt + \epsilon, t^2 + 2kt) \\ &\quad = S(kt, k^2 + 2kt + \epsilon, t^2 - k^2 + \epsilon) - S(kt, t^2 + 2kt, t^2 - k^2 + \epsilon) + \frac{1}{4}\gamma. \end{aligned}$$

Replacing (k,t) by $(k,-t)$ in the first sum and (k,t) by $(t,-k)$ in the second sum and applying (4.2), we get

$$S(kt, k^2+2kt+\epsilon, t^2+2kt) = -A_5 + A_6 + \frac{1}{4}\gamma. \tag{4.7}$$

Then, from (4.3)-(4.7), we have

$$\begin{aligned} Y_2 = {} & 2\left(A_1 - \frac{1}{3}a\right) - 2\left(A_2 - \frac{1}{6}a\right) - 2\left(A_4 + A_5 - 2A_6 - \frac{\psi(2)}{12}a\right) \\ & - \frac{1}{4}\gamma + \frac{(\psi(2)-1)(4p-3)}{16p}B_{1,\psi} + \frac{7\psi(2)-12}{48p}B_{3,\psi}. \end{aligned}$$

Thus the assertion (vi) is established.

We next prove the assertion (vii) by applying a similar sequence of transformations to Y_3 to obtain a tree of sums of depth 3. By $[kt+\epsilon, k^2+kt+\epsilon;\ t^2+kt;\ \text{(ii)-(iv)}]$, we have

$$\begin{aligned} Y_3 = {} & S(kt+\epsilon, k^2+2kt, t^2+kt) + S(k^2+kt+\epsilon, k^2+2kt, t^2+kt) \\ & + \frac{\psi(2)-1}{4p}B_{1,\psi} + \frac{\psi(2)-4}{12p}B_{3,\psi}. \end{aligned}$$

Replacing (k,t) by $(-k, k+t)$ in the second sum, we get

$$Y_3 = 2S(kt+\epsilon, k^2+2kt, t^2+kt) + \frac{\psi(2)-1}{4p}B_{1,\psi} + \frac{\psi(2)-4}{12p}B_{3,\psi}. \tag{4.8}$$

By $[kt+\epsilon, t^2+kt;\ k^2+2kt;\ \text{(ii), (iii), (ix), (x)}]$, we get

$$\begin{aligned} & S(kt+\epsilon, k^2+2kt, t^2+kt) \\ & \quad = S(kt+\epsilon, k^2+2kt, t^2+2kt+\epsilon) + S(t^2+kt, k^2+2kt, t^2+2kt+\epsilon) \\ & \quad + \frac{\psi(2)-2}{12}a + \frac{1-\psi(2)}{8p}B_{1,\psi} + \frac{1}{24p}B_{3,\psi}. \end{aligned}$$

Replacing (k,t) by $(t-k, k)$ in the second sum on the right and applying (4.2), we obtain

$$\begin{aligned} & S(kt+\epsilon, k^2+2kt, t^2+kt) \\ & \quad = S(kt+\epsilon, k^2+2kt, t^2+2kt+\epsilon) + A_2 - A_6 + \frac{\psi(2)-2}{12}a \\ & \quad + \frac{1-\psi(2)}{8p}B_{1,\psi} + \frac{1}{24p}B_{3,\psi}. \end{aligned} \tag{4.9}$$

By $[-(k^2+2kt), t^2+2kt+\epsilon;\ kt+\epsilon;\ \text{(i), (vi), (vii)}]$, we get

$$\begin{aligned} & S(kt+\epsilon, k^2+2kt, t^2+2kt+\epsilon) \\ & \quad = S(kt+\epsilon, k^2+2kt, t^2-k^2+\epsilon) - S(kt+\epsilon, t^2+2kt+\epsilon, t^2-k^2+\epsilon) - \frac{1}{4}\gamma \\ & \quad + \frac{\psi(2)-1}{8p}B_{1,\psi} + \frac{\psi(2)}{24p}B_{3,\psi}. \end{aligned}$$

Replacing (k,t) by $(k,-t)$ in the first sum and (k,t) by $(t,-k)$ in the second sum and applying (4.2) together with Lemma 2.3, we get

(4.10)
$$\begin{aligned}&S(kt+\epsilon, k^2+2kt, t^2+2kt+\epsilon)\\&= -\psi(2)A_2 + A_4 + A_5 - A_6 - \frac{1}{4}\gamma + \frac{\psi(2)-1}{8p}B_{1,\psi} + \frac{\psi(2)}{24p}B_{3,\psi}.\end{aligned}$$

Then, from (4.8)-(4.10), we have

$$\begin{aligned}Y_3 &= 2(1-\psi(2))\left(A_2 - \frac{1}{6}a\right) + 2\left(A_4 + A_5 - 2A_6 - \frac{\psi(2)}{12}a\right)\\&\quad - \frac{1}{2}\gamma + \frac{\psi(2)-1}{4p}B_{1,\psi} + \frac{2\psi(2)-3}{12p}B_{3,\psi}.\end{aligned}$$

Thus the assertion (vii) is established.

We next prove the assertion (viii) by applying a similar sequence of transformations to Y_4 to obtain a tree of sums of depth 3. By $[kt+\epsilon, k^2+kt+\epsilon;\ t^2+kt+\epsilon;$ (ii)-(iv), (viii)], we have

$$Y_4 = S(kt+\epsilon, k^2+2kt, t^2+kt+\epsilon) + S(k^2+kt+\epsilon, k^2+2kt, t^2+kt+\epsilon) + \frac{1}{3p}B_{3,\psi}.$$

Replacing (k,t) by $(-k, k+t)$ in the second sum, we get

(4.11)
$$Y_4 = 2S(kt+\epsilon, k^2+2kt, t^2+kt+\epsilon) + \frac{1}{3p}B_{3,\psi}.$$

By $[kt+\epsilon, t^2+kt+\epsilon;\ k^2+2kt;$ (ii), (iii), (v), (ix)], we have

$$\begin{aligned}&S(kt+\epsilon, k^2+2kt, t^2+kt+\epsilon)\\&= S(kt+\epsilon, k^2+2kt, t^2+2kt) + S(t^2+kt+\epsilon, k^2+2kt, t^2+2kt) + \frac{1}{2}a\\&\quad + \frac{1-\psi(2)}{8p}B_{1,\psi} + \frac{1}{24p}B_{3,\psi}.\end{aligned}$$

Replacing (k,t) by $(t-k, k)$ in the second sum, we get $-A_1 + A_2$. Thus, we have

(4.12)
$$\begin{aligned}&S(kt+\epsilon, k^2+2kt, t^2+kt+\epsilon)\\&= S(kt+\epsilon, k^2+2kt, t^2+2kt) - A_1 + A_2 + \frac{1}{2}a + \frac{1-\psi(2)}{8p}B_{1,\psi}\\&\quad + \frac{1}{24p}B_{3,\psi}.\end{aligned}$$

By $[-(k^2+2kt), t^2+2kt;\ kt+\epsilon;$ (i), (vi), (vii)], we get

$$\begin{aligned}S(kt+\epsilon, k^2+2kt, t^2+2kt) &= S(kt+\epsilon, k^2+2kt, t^2-k^2) - S(kt+\epsilon, t^2+2kt, t^2-k^2)\\&\quad + \frac{\psi(2)-1+p\,\delta_{p,3}}{2p}B_{1,\psi} + \frac{1}{6p}B_{3,\psi}.\end{aligned}$$

Replacing (k,t) by $(k,-t)$ in the first sum and (k,t) by $(t,-k)$ in the second sum, we get $-A_1 + A_2$ and $A_1 - A_2$, respectively. Thus, we have

(4.13)
$$S(kt+\epsilon, k^2+2kt, t^2+2kt) = -2A_1 + 2A_2 + \frac{\psi(2)-1+p\,\delta_{p,3}}{2p}B_{1,\psi} + \frac{1}{6p}B_{3,\psi}.$$

Then, from (4.11)-(4.13), we get

(4.14)
$$Y_4 = -6\left(A_1 - \frac{1}{3}a\right) + 6\left(A_2 - \frac{1}{6}a\right) + \frac{3\psi(2)-3+4p\,\delta_{p,3}}{4p}B_{1,\psi} + \frac{3}{4p}B_{3,\psi}.$$

Thus the assertion (viii) is established. □

By virtue of the Arakawa Identities (Theorem 3.1), we express the sums $X_i,\ Y_i\,(1 \le i \le 4)$ and γ as linear combinations of generalized Bernoulli numbers.

Theorem 4.7. *For any odd prime number p, we get*

$$
\begin{aligned}
&\text{(i)} && X_1 = -\frac{p-2}{4p}B_{1,\psi} - \frac{1}{6p}B_{3,\psi},\\
&\text{(ii)} && X_2 = \frac{p-2+\psi(2)(p+4)}{24p}B_{1,\psi} - \frac{4+\psi(2)}{72p}B_{3,\psi},\\
&\text{(iii)} && X_3 = \frac{p-2+\psi(2)(1-2p)}{12p}B_{1,\psi} + \frac{2-\psi(2)}{36p}B_{3,\psi},\\
&\text{(iv)} && X_4 = \frac{(p-2)(\psi(2)-1)}{8p}B_{1,\psi} + \frac{4-\psi(2)}{24p}B_{3,\psi},\\
&\text{(v)} && Y_1 = -\frac{p-2}{4p}B_{1,\psi} - \frac{1}{6p}B_{3,\psi},\\
&\text{(vi)} && Y_2 = \frac{(p+1)(1+\psi(2))}{12p}B_{1,\psi} + \frac{2-\psi(2)}{36p}B_{3,\psi},\\
&\text{(vii)} && Y_3 = \frac{p-2+\psi(2)(1-2p)}{12p}B_{1,\psi} + \frac{2-\psi(2)}{36p}B_{3,\psi},\\
&\text{(viii)} && Y_4 = -\frac{p+1}{4p}B_{1,\psi} - \frac{1}{6p}B_{3,\psi},\\
&\text{(ix)} && \gamma = \frac{1-\psi(2)}{4p}B_{1,\psi} + \frac{\psi(2)}{12p}B_{3,\psi}.
\end{aligned}
$$

Proof. By Theorem 4.6 and Theorem 3.1, we immediately get the assertions (i)-(v), (viii). Thus it remains to prove the assertions (vi), (vii), and (ix). By (vi), (vii) of Theorem 4.6 and Theorem 3.1, we get

$$
\begin{aligned}
Y_2 &= -\frac{1}{4}\gamma + \frac{4p+7+\psi(2)(4p+1)}{48p}B_{1,\psi} + \frac{8-\psi(2)}{144p}B_{3,\psi},\\
Y_3 &= -\frac{1}{2}\gamma + \frac{2p-1-\psi(2)(4p+1)}{24p}B_{1,\psi} + \frac{\psi(2)+4}{72p}B_{3,\psi}.
\end{aligned}
\tag{4.15}
$$

We set

$$I = S(2kt, 2(k^2+kt), 2(t^2+kt)).$$

By Lemma 2.3, we have $I = \psi(2)Y_1$. On the other hand, by the multiplication formula Lemma 2.1, we get $I = Y_1 + 3Y_2 + 3Y_3 + Y_4$. Thus $\psi(2)Y_1 = Y_1 + 3Y_2 + 3Y_3 + Y_4$ and the assertion (ix) follows from (4.15) and (v), (viii). The assertions (vi), (vii) then immediately follow from (4.15) and (ix). □

5. Evaluation of related Lee-Weintraub sums

In this section, we evaluate the Lee-Weintraub sums given in the introduction, thus establishing the Tsukano Conjectures (ii), (iv), and (v). We remind the reader of the notation.

Fix an odd prime p and let ψ denote the Legendre symbol mod p: $\psi(a) = \left(\frac{a}{p}\right)$. We put

$\zeta = exp(2\pi i/p)$. Let $\tau(\psi)$ denote the Gaussian sum $\tau(\psi) = \sum_{n=1}^{p-1} \psi(n)\zeta^n$. We remind the reader that X_i, Y_i $(1 \le i \le 4)$ are the sums given in (4.1).

We now evaluate the Lee-Weintraub sums given in the introduction.

Theorem 5.1. *For any odd prime number p, we have*

$$S[-2x, 2(x+1), x(x+1)] = \tau(\psi)\left(\frac{-p+2+\psi(2)(5p+2)}{24}B_{1,\psi} + \frac{1+6\psi(2)}{4}B_{2,\psi} + \frac{1+16\psi(2)}{72}B_{3,\psi}\right).$$

Proof. Let S denote the sum $S[-2x, 2(x+1), x(x+1)]$. By Proposition 4.2, noting that $S = \psi(2)S[-x, x+1, 2^{-1}(x+1)]$, we get

$$S = \sqrt{\psi(-1)p}\left(-\frac{1}{p^2}K(2^{-1}, p) + \frac{(p-1)^2(2\psi(2)+1)}{4}B_{1,\psi}\right), \tag{5.1}$$

where by Proposition 4.3 together with Proposition 4.4, we get

$$K(2^{-1}, p)/p^3 = K_{2^{-1}} + \frac{(p-2)(1+2\psi(2))}{4}B_{1,\psi} - \frac{6\psi(2)+1}{4p}B_{2,\psi}. \tag{5.2}$$

By Lemma 2.3, and then by the multiplication formula Lemma 2.1, we get

$$K_{2^{-1}} = \psi(2)\sum_{k,t(p)} P_1\left(\frac{2k^2}{p}\right)P_1\left(\frac{t^2}{p}\right)P_1\left(\frac{(k+t)^2}{p}\right) = \psi(2)(X_1+X_2).$$

Thus, by Theorem 4.7, we have

$$K_{2^{-1}} = \frac{p+4-5\psi(2)(p-2)}{24p}B_{1,\psi} - \frac{16\psi(2)+1}{72p}B_{3,\psi}. \tag{5.3}$$

Theorem 5.1 then immediately follows from (5.1)-(5.3). □

Theorem 5.2. *For any odd prime number p, we have*

$$S[1, 2x^2, 2(x+1)^2] = \tau(\psi)\left(\frac{-2p+7-\psi(2)(8p-19)}{12}B_{1,\psi} + \frac{\psi(2)+1}{4}B_{2,\psi} - \frac{4\psi(2)+1}{36}B_{3,\psi}\right).$$

Proof. Let S denote the sum $S[1, 2x^2, 2(x+1)^2]$. By Proposition 4.2 together with (i) in Proposition 4.4, we have

$$S = \tau(\psi)\left(-\frac{\psi(-1)}{p^2}J(2,p) - \frac{(p-1)(2\psi(2)+1)}{2}B_{1,\psi} - \frac{(p-1)(\psi(2)+1)}{4}B_{2,\psi}\right), \tag{5.4}$$

where by Proposition 4.3 together with Proposition 4.4, we get

$$J(2,p)/p^3 = -\psi(-2)\left(J_2 + \frac{2+\psi(2)}{4}B_{1,\psi} + \frac{1+\psi(2)}{4}B_{2,\psi}\right). \tag{5.5}$$

By the multiplication formula Lemma 2.1, we have

$$J_2 = \sum_{k,t(p)} P_1\left(\frac{2kt}{p}\right)P_1\left(\frac{k^2+kt}{p}\right)P_1\left(\frac{t^2+kt}{p}\right) = Y_1 + Y_2.$$

Hence, by Theorem 4.7, we get

$$J_2 = \frac{-2p + 7 + \psi(2)(p+1)}{12p} B_{1,\psi} - \frac{4 + \psi(2)}{36p} B_{3,\psi}. \tag{5.6}$$

Theorem 5.2 then immediately follows from (5.4)-(5.6). □

Theorem 5.3. *For any odd prime number p, we have*

$$S[2, x^2, (x+1)^2] = \tau(\psi)\left(\frac{-2p + 5 - \psi(2)(p-4)}{12} B_{1,\psi} + \frac{1}{2} B_{2,\psi} - \frac{1}{12} B_{3,\psi}\right).$$

Proof. Let S denote the sum $S[2, x^2, (x+1)^2]$. By Proposition 4.2 together with (i) in Proposition 4.4, noting that $S = \psi(2)S[1, 2^{-1}x^2, 2^{-1}(x+1)^2]$, we get

$$S = \tau(\psi)\left(-\frac{\psi(-2)}{p^2} J(2^{-1}, p) - \frac{(p-1)(2+\psi(2))}{2} B_{1,\psi} - \frac{p-1}{2} B_{2,\psi}\right), \tag{5.7}$$

where by Proposition 4.3 together with Proposition 4.4, we have

$$J(2^{-1}, p)/p^3 = -\psi(-2)\left(J_{2^{-1}} + \frac{2 + \psi(2)}{4} B_{1,\psi} + \frac{1}{2} B_{2,\psi}\right). \tag{5.8}$$

By Lemma 2.3, and then by the multiplication formula Lemma 2.1, we get

$$J_{2^{-1}} = \psi(2) \sum_{k,t(p)} P_1\left(\frac{2kt}{p}\right) P_1\left(\frac{2(k^2 + kt)}{p}\right) P_1\left(\frac{t^2 + kt}{p}\right) = \psi(2)(Y_1 + 2Y_2 + Y_3).$$

Hence, by Theorem 4.7, we get

$$J_{2^{-1}} = \frac{2\psi(2) + 1}{4p} B_{1,\psi} - \frac{1}{12p} B_{3,\psi}. \tag{5.9}$$

Theorem 5.3 then immediately follows from (5.7)-(5.9). □

We are now in position to prove the Tsukano Conjectures.

Theorem 5.4 (Tsukano Conjectures [8])**.** *When $p \equiv 3(4)$, we have*

(i) $I(2, p)/\tau(\psi) = -\dfrac{(1 + \psi(2))(p+1)}{12} B_{1,\psi} - \dfrac{\psi(2) + 4}{18} B_{3,\psi},$

(ii) $I(2^{-1}, p)/\tau(\psi) = \dfrac{p - 2 - \psi(2)(5p+2)}{24} B_{1,\psi} - \dfrac{1 + 16\psi(2)}{72} B_{3,\psi},$

(iii) $J(1, p)/p^2 = \dfrac{p+1}{2} B_{1,\psi} - \dfrac{1}{6} B_{3,\psi},$

(iv) $J(2, p)/p^2 = \dfrac{4p + 1 + \psi(2)(4p+7)}{12} B_{1,\psi} - \dfrac{1 + 4\psi(2)}{36} B_{3,\psi},$

(v) $J(2^{-1}, p)/p^2 = \dfrac{p + 2 + \psi(2)(2p+1)}{4} B_{1,\psi} - \dfrac{\psi(2)}{12} B_{3,\psi},$

where 2^{-1} is regarded as an element of $(\mathbb{Z}/p\mathbb{Z})^*$.

Proof. We note that $B_{2,\psi}$ vanishes in the case of $p \equiv 3(4)$. Since $I(2, p) = \psi(-1)S[-x, x + 2, x(x+2)]$, the Tsukano Conjecture (i) follows from Theorem 5.1 in [6]. Since $I(2^{-1}, p) = \psi(-2)S[-x, x+1, 2^{-1}x(x+1)] = \psi(-1)S[-2x, 2(x+1), x(x+1)]$, the Tsukano Conjecture (ii) follows from Theorem 5.1. The Tsukano Conjecture (iii) follows from Corollary 5.4 in [6]. The Tsukano Conjecture (iv) follows from (5.5), (5.6). The Tsukano Conjecture (v) follows from (5.8), (5.9). □

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Department of Mathematics
New York City College of Technology
300 Jay St., Brooklyn, NY 11201
USA
e-mail: bisaacson@citytech.cuny.edu