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In this paper, we express three different, yet related, character sums in terms of generalized Bernoulli numbers. Two of these sums are generalizations of sums introduced and studied by Berndt and Arakawa–Ibukiyama–Kaneko in the context of the theory of modular forms. A third sum generalizes a sum already studied by Ramanujan in the context of theta function identities. Our methods are elementary, relying only on basic facts from algebra and number theory.

Keywords: Character sums; cotangent sums; exponential sums; generalized Bernoulli numbers; trigonometric sums.

Mathematics Subject Classification 2020: 11L03

1. Introduction

Let χ be a primitive Dirichlet character with conductor $f > 1$. Let h be a positive integer prime to f. Let m, n be integers such that $m, n \geq 0$. We put $\zeta = \exp(2\pi i/f)$. In this paper, we obtain formulas expressing the closely related character sums

$$M_{m,n}(h,\chi) = \sum_{a=1}^{f-1} \frac{\chi(a)}{(\zeta^{ha}-1)^m(\zeta^a-1)^n},$$

$$S_{m,n}(h,\chi;e_1,\ldots,e_{m+n}) = \sum_{a_1,\ldots,a_{m+n}=1}^{f-1} \chi\left(h\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} a_{m+j}\right) a_1^{e_1}\ldots a_{m+n}^{e_{m+n}},$$

$$c_{m,n}(h,\chi) = \sum_{a=1}^{f-1} \cot^m\left(\frac{ha}{f}\right)\cot^n\left(\frac{a}{f}\right)\chi(a),$$

by generalized Bernoulli numbers using only elementary methods from algebra and number theory. Observe that $M_{1,1}(h,\chi)$ without the character is the classical

Dedekind sum $s(h,f)$ up to trivial constants. For this reason, we refer to the sum $M_{m,n}(h,\chi)$ as a twisted generalization of the classical Dedekind sum.

The sums $M_{m,n}(h,\chi)$ and $S_{m,n}(h,\chi;e_1,\dots,e_{m+n})$ are natural generalizations of sums introduced and studied by Berndt [6] and Arakawa–Ibukiyama–Kaneko[3], where formulas for $M_{m,n}(h,\chi)$ and $S_{m,n}(h,\chi;e_1,\dots,e_{m+n})$ in the case of $h=1$ were obtained. In [15], the present author obtained formulas for $M_{m,1}(h,\chi)$ and $S_{m,1}(h,\chi;e_1,\dots,e_{m+n})$, and in this paper, we treat the general case. These types of sums are of interest due to their connection with the action of the symplectic group $Sp(4,\mathbb{F}_p)$ on the space of Siegel cusp forms of degree 2 belonging to the principal congruence subgroup $\Gamma(p)$. While studying this action by the holomorphic Lefshetz fixed-point theorem of Atiyah–Singer, traces of this action give rise to certain exponential sums which are variations of the sums $M_{m,n}(h,\chi)$ and $S_{m,n}(h,\chi;e_1,\dots,e_{m+n})$. For examples, we refer the interested reader to [2, 3, 9–16, 19, 20].

The sum $c_{m,n}(h,\chi)$ is a variation of trigonometric sums first investigated by Ramanujan in connection to certain theta function identities. Similar sums were later studied by Berndt *et al.* [4, 8] using complex contour integration and Beck–Halloran [5] using discrete Fourier analysis.

Before stating our results, we need more notation. Let $\tau(\chi)$ denote the Gaussian sum $\tau(\chi)=\sum_{a=0}^{f-1}\chi(a)\zeta^a$. For $x\in\mathbb{Q}$ with denominator prime to f, we extend the definition of χ by multiplicativity. Following Snyder [18], we define the generalized Bernoulli functions $B_{k,\chi}(x)$ by the generating function

$$\sum_{a=0}^{f-1}\frac{\chi(a+x)te^{(a+x)t}}{e^{ft}-1}=\sum_{k=0}^{\infty}B_{k,\chi}(x)\frac{t^k}{k!},$$

where $B_{k,\chi}=B_{k,\chi}(0)$ are the generalized Bernoulli numbers. Let $\overline{B}_{k,\chi}(x)$ be the periodic generalized Bernoulli functions given by $\overline{B}_{k,\chi}(x)=B_{k,\chi}(x-[x])$, where $[x]$ denotes the greatest integer not exceeding x.

We first express the sum $M_{m,n}(h,\chi)$ by periodic generalized Bernoulli functions $\overline{B}_{j,\overline{\chi}}(x)$.

Theorem 3.3. *Let χ,f,h,m,n be as above. We have*

$$M_{m,n}(h,\chi)=\frac{(-1)^{m+n-1}\overline{\chi}(h)\tau(\chi)}{(m+n-1)!}\sum_{j=1}^{m+n}\sum_{c=0}^{n(h-1)}C_h(n,c)$$

$$\times\left(\sum_{r=0}^{m+n-j}\frac{(-1)^r}{r+j}\begin{bmatrix}m+n\\ r+j\end{bmatrix}\binom{r+j}{j}\left(\frac{c}{h}\right)^r\right)\overline{B}_{j,\overline{\chi}}\left(\frac{c}{h}\right),$$

where $\left[{n \atop m}\right]$ *are Stirling numbers of the first kind* (*Stirling's cycle numbers*), *and*

$$C_h(n,c) = \begin{cases} 1 & \text{if } n = 0, \\ \displaystyle\sum_{j=0}^{[c/h]} (-1)^j \binom{n}{j} \binom{n-1+c-hj}{n-1} & \text{if } n \geq 1. \end{cases}$$

For any positive integer u with $u \mid h$, denote by h_u the u-primary part of h, that is, the maximum integer which divides h and is prime to u. We denote by $Y(u)$ the set of primitive Dirichlet characters with conductor u. Let ϕ denote the Euler phi function. The next theorem expresses periodic generalized Bernoulli functions $\overline{B}_{k,\chi}(c/h)$ by generalized Bernoulli numbers. This was proved in [15] and is a generalization of [10, Theorem 2].

Theorem 1.1 ([15]). *Let* χ, f, h *be as above. For any integer* c *prime to* h, *we have*

$$\overline{B}_{k,\chi}\left(\frac{c}{h}\right) = \frac{\overline{\chi}(h)}{\phi(h)h^{k-1}} \sum_{u|h} \sum_{\delta \in Y(u)} \left(\bar{\delta}(c) B_{k,\delta\chi} \prod_{\substack{q|h_u \\ q \text{ prime}}} (1 - q^{k-1}\chi(q)\delta(q)) \right).$$

The relationship between the sums $S_{m,n}(h,\chi;e_1,\ldots,e_{m+n})$ and $M_{m,n}(h,\chi)$ can be seen by the following theorem.

Theorem 4.1. *Let* χ, f, h, m, n *be as above. We have*

$$\begin{aligned} S_{m,n}(h,\chi;e_1,\ldots,e_{m+n}) = \frac{(-1)^{e_1+\cdots+e_{m+n}}}{\tau(\overline{\chi})} \\ \times \sum_{\substack{1\leq m_j \leq e_j \\ 1 \leq l_j \leq e_j - m_j + 1 \\ 1 \leq j \leq m+n}} \left(\prod_{i=1}^{m+n} (-f)^{l_i} \binom{e_i}{l_i} \left\{ {e_i - l_i + 1 \atop m_i} \right\} (m_i - 1) \right) \\ \times M_{m_1+\cdots+m_m, m_{m+1}+\cdots+m_{m+n}}(h, \overline{\chi}), \end{aligned}$$

where $\left\{{n \atop m}\right\}$ *are Stirling numbers of the second kind* (*Stirling's subset numbers*).

The relationship between the sums $c_{m,n}(h,\chi)$ and $M_{m,n}(h,\chi)$ can be seen by the following theorem.

Theorem 5.1. *Let* χ, f, h, m, n *be as above. We have*

$$c_{m,n}(h,\chi) = i^{m+n} \sum_{j=0}^{m} \sum_{k=0}^{n} 2^{j+k} \binom{m}{j} \binom{n}{k} M_{j,k}(h,\chi).$$

By virtue of the above theorems, we can express the sums $M_{m,n}(h,\chi)$, $S_{m,n}(h,\chi;e_1,\ldots,e_{m+n})$ and $c_{m,n}(h,\chi)$ by generalized Bernoulli numbers. As a corollary, we obtain a formula for cotangent power sums considered by Apostol [1].

Corollary 5.2. *Let χ, f, m be as above. We have*

$$\sum_{a=1}^{f-1} \cot^m \left(\frac{\pi a}{f}\right) \chi(a) = -i^m \tau(\chi) \sum_{k=1}^{m} \frac{1}{k} \left(\sum_{j=k}^{m} \frac{(-2)^j \binom{m}{j} \left[{j \atop k}\right]}{(j-1)!} \right) B_{k,\overline{\chi}}.$$

Corollary 5.2 resolves the computational problem of having to evaluate cotangent power sums recursively, as discussed in [17].

2. Preliminaries

In this section, we discuss all relevant background information needed for the proofs of the theorems given in Sec 1. We omit several proofs which can be found in the literature.

The Bernoulli polynomials $B_k(x)$ are defined by using the generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$

Corresponding to the Bernoulli polynomials $B_k(x)$, we have two types of periodic Bernoulli functions $\overline{B}_k(x), P_k(x)$ given by

$$\overline{B}_k(x) = B_k(\{x\}),$$

$$P_k(x) = \begin{cases} 0 & \text{if } k = 1,\ x \in \mathbb{Z}, \\ \overline{B}_k(x) & \text{otherwise,} \end{cases}$$

where $\{x\}$ denotes the fractional part of x. We note that $\overline{B}_k(x) = P_k(x)$ unless $k = 1$ and $x \in \mathbb{Z}$, in which case $\overline{B}_k(x) = -1/2$ and $P_k(x) = 0$.

Let χ be a primitive Dirichlet character with conductor f. For integers a, b with b prime to f, we extend the definition of χ by multiplicativity by defining $\chi(a/b) = \chi(ab^{-1})$. For $x \in \mathbb{Q}$ with denominator prime to f, we follow Snyder [18] and define the generalized Bernoulli functions $B_{k,\chi}(x)$ by the generating function

$$\sum_{a=0}^{f-1} \frac{\chi(a+x)te^{(a+x)t}}{e^{ft} - 1} = \sum_{k=0}^{\infty} B_{k,\chi}(x) \frac{t^k}{k!}.$$

Note that if χ is trivial, then $B_{k,\chi}(x)$ reduces to the ordinary Bernoulli polynomials $B_k(x)$. This definition leads to the formula

$$B_{k,\chi}(x) = f^{k-1} \sum_{a=0}^{f-1} B_k \left(\frac{a+x}{f}\right) \chi(a+x),$$

or equivalently, for any positive integer h prime to f and any integer c, we have

$$B_{k,\chi}\left(\frac{c}{h}\right) = \overline{\chi}(h) f^{k-1} \sum_{a=0}^{f-1} B_k \left(\frac{ha+c}{hf}\right) \chi(ha+c).$$

Corresponding to the generalized Bernoulli functions $B_{k,\chi}(x)$, we have the periodic generalized Bernoulli functions $\overline{B}_{k,\chi}(x)$ given by $\overline{B}_{k,\chi}(x) = B_{k,\chi}(\{x\})$.

Remark. The more common formulation of generalized Bernoulli functions is the one given by Berndt [7] which does not contain shifted character values, namely,

$$B_{k,\chi}(x) = f^{k-1} \sum_{a=0}^{f-1} B_k\left(\frac{a+x}{f}\right)\overline{\chi}(a).$$

For the purposes of this paper, we will use Snyder's definition.

We define the generalized Bernoulli numbers by $B_{k,\chi} = B_{k,\chi}(0)$, or equivalently,

$$B_{k,\chi} = f^{k-1} \sum_{a=0}^{f-1} B_k\left(\frac{a}{f}\right)\chi(a).$$

We state the following without proof which appears as Lemma 8.11 in [3].

Proposition 2.1 ([3]). Let e be a natural number and b be an integer not divisible by f. Then, we have

$$\sum_{a=1}^{f-1} a^e \zeta^{ab} = (-1)^e \sum_{m=1}^{e} \frac{(m-1)!}{(\zeta^b - 1)^m} \sum_{l=1}^{e-m+1} (-1)^l f^l \binom{e}{l} \left\{ \begin{matrix} e-l+1 \\ m \end{matrix} \right\}.$$

We prepare two propositions which are used in Lemma 3.2.

Proposition 2.2. *Let x be a rational number with denominator relatively prime to f. Then, we have*

$$B_{j,\chi}(x) = \begin{cases} \overline{B}_{j,\chi}(x) + j \displaystyle\sum_{k=0}^{[x]-1} (k+\{x\})^{j-1}\chi(k+\{x\}) & \text{if } x \geq 0, \\ \overline{B}_{j,\chi}(x) - j \displaystyle\sum_{k=1}^{-[x]} (-k+\{x\})^{j-1}\chi(-k+\{x\}) & \text{if } x < 0. \end{cases}$$

Proof. We only prove the case where $x \geq 0$ as the case where $x < 0$ can be proven similarly. Suppose $x \geq 0$. Writing $B_{j,\chi}(x)$ as a telescoping sum, we have

$$B_{j,\chi}(x) = f^{j-1} \sum_{a=0}^{f-1} B_j\left(\frac{a+\{x\}}{f}\right)\chi(a+\{x\},\chi)$$

$$+ f^{j-1} \sum_{k=0}^{[x]-1} \left(B_j\left(\frac{f+k+\{x\}}{f}\right) - B_j\left(\frac{k+\{x\}}{f}\right)\right)\chi(k+\{x\}).$$

Since $B_j(x+1) - B_j(x) = jx^{j-1}$, we get

$$B_{j,\chi}(x) = \overline{B}_{j,\chi}(x) + j \sum_{k=0}^{[x]-1} (k+\{x\})^{j-1}\chi(k+\{x\}). \qquad \square$$

Proposition 2.3. *Let h, k be positive integers with $k \geq 2$. Let c_0 be any integer with $0 \leq c_0 \leq h-1$. Let s be an integer with $1 \leq s \leq [k(h-1)/h]$ and l be an integer such that $0 \leq l \leq s-1$. Then, we have*

$$\sum_{j=1}^{k}\sum_{r=0}^{k-j}(-1)^r \begin{bmatrix} k \\ r+j \end{bmatrix} \binom{r+j-1}{j-1} \left(\frac{c_0}{h}+s\right)^r \left(\frac{c_0}{h}+l\right)^{j-1} = 0.$$

Proof. Let S denote the left-hand side of the assertion of the lemma. Interchanging the order of summation, taking $u = j+r$, and interchanging the order of summation again, we get

$$S = \sum_{u=1}^{k} \begin{bmatrix} k \\ u \end{bmatrix} (l-s)^{u-1} = \frac{(l-s)^{(k)}}{l-s},$$

where $(x)^{(k)}$ is the rising factorial given by $(x)^{(k)} = x(x+1)\cdots(x+k-1)$. Clearly $(l-s)^{(k)} = 0$, and the lemma is proved. □

3. The Evaluation of $M_{m,n}(h, \chi)$

In this section, we obtain formulas expressing $M_{m,n}(h, \chi)$ by generalized Bernoulli numbers. All of the results in this section will be obtained using only elementary methods from algebra and number theory.

Let χ be a primitive Dirichlet character with conductor $f > 1$. Let h be a positive integer prime to f. Let m, n be integers such that $m, n \geq 0$. We put $\zeta = \exp(2\pi i/f)$. Let $\tau(\chi)$ denote the Gaussian sum $\tau(\chi) = \sum_{a=0}^{f-1} \chi(a)\zeta^a$. We recall the definition of $M_{m,n}(h, \chi)$:

$$M_{m,n}(h, \chi) = \sum_{a=1}^{f-1} \frac{\chi(a)}{(\zeta^{ha}-1)^m(\zeta^a-1)^n}.$$

We begin by expressing the sum $M_{m,n}(h, \chi)$ by generalized Bernoulli functions $\overline{B}_{k,\chi}(x)$. We first prepare a lemma.

For $k, s \in \mathbb{Z}$ with $k \geq 1$, we define the character sum function $P_{k,s,\chi}(t)$ by

$$P_{k,s,\chi}(t) = \sum_{a=0}^{f-1} \frac{\chi(a)\zeta^{as}}{(\zeta^a e^t - 1)^k}.$$

In [15], we proved the following.

Proposition 3.1 ([15]). *Let χ, h, k be as above. For any integer c, we have*

$$P_{k,ch^{-1},\chi}(0) = \frac{(-1)^{k-1}\tau(\chi)}{(k-1)!} \sum_{j=1}^{k} \left(\sum_{r=0}^{k-j} \frac{(-1)^r}{r+j} \begin{bmatrix} k \\ r+j \end{bmatrix} \binom{r+j}{j} \left(\frac{c}{h}\right)^r \right) B_{j,\overline{\chi}}\left(\frac{c}{h}\right),$$

where h^{-1} is regarded as an element of $(\mathbb{Z}/f\mathbb{Z})^*$.

In Proposition 3.1, it turns out that we can replace $B_{j,\chi}(c/h)$ by $\overline{B}_{j,\chi}(c/h)$ for any integer c such that $0 \le c \le k(h-1)$.

Lemma 3.2. *Let χ, h, k be as above. For any integer c such that $0 \le c \le k(h-1)$, we have*

$$P_{k,ch^{-1},\chi}(0) = \frac{(-1)^{k-1}\tau(\chi)}{(k-1)!}\sum_{j=1}^{k}\left(\sum_{r=0}^{k-j}\frac{(-1)^r}{r+j}\begin{bmatrix} k \\ r+j\end{bmatrix}\binom{r+j}{j}\left(\frac{c}{h}\right)^r\right)\overline{B}_{j,\overline{\chi}}\left(\frac{c}{h}\right).$$

Proof. Since $B_{j,\overline{\chi}}(c/h) = \overline{B}_{j,\overline{\chi}}(c/h)$ for $0 \le c \le h-1$, the assertion follows immediately from Proposition 3.1 in this case. Thus we assume $h \le c \le k(h-1)$ with $k \ge 2$. We write $c = c_0 + sh$ for integers c_0, s with $0 \le c_0 \le h-1$ and $1 \le s \le [k(h-1)/h]$. By Propositions 3.1 and 2.2, it remains to show

$$\frac{(-1)^{k-1}\tau(\chi)}{(k-1)!}\sum_{j=1}^{k}\left(\sum_{r=0}^{k-j}\frac{(-1)^r}{r+j}\begin{bmatrix} k \\ r+j\end{bmatrix}\binom{r+j}{j}\left(\frac{c_0}{h}+s\right)^r\right)$$

$$\times j\sum_{l=0}^{s-1}\left(l+\frac{c_0}{h}\right)^{j-1}\overline{\chi}\left(l+\frac{c_0}{h}\right) = 0,$$

but this follows immediately from Proposition 2.3. □

Theorem 3.3. *Let χ, f, h, m, n be as above. We have*

$$M_{m,n}(h,\chi) = \frac{(-1)^{m+n-1}\overline{\chi}(h)\tau(\chi)}{(m+n-1)!}\sum_{j=1}^{m+n}\sum_{c=0}^{n(h-1)} C_h(n,c)$$

$$\times\left(\sum_{r=0}^{m+n-j}\frac{(-1)^r}{r+j}\begin{bmatrix} m+n \\ r+j\end{bmatrix}\binom{r+j}{j}\left(\frac{c}{h}\right)^r\right)\overline{B}_{j,\overline{\chi}}\left(\frac{c}{h}\right),$$

where

$$C_h(n,c) = \begin{cases} 1 & \text{if } n = 0, \\ \displaystyle\sum_{j=0}^{[c/h]}(-1)^j\binom{n}{j}\binom{n-1+c-hj}{n-1} & \text{if } n \ge 1. \end{cases}$$

Proof. The theorem is clear in the case of $m, n = 0$ as both sides are identically zero. Thus we assume that $m+n > 0$. Letting $C_h(n,c)$ be defined as in the theorem, we have

$$(1+\zeta^a+\cdots+\zeta^{(h-1)a})^n = \sum_{c=0}^{n(h-1)} C_h(n,c)\zeta^{ca}.$$

For $f \nmid a$, we have

$$\frac{\zeta^{ha}-1}{\zeta^a-1} = \sum_{c=0}^{h-1} \zeta^{ca}.$$

Hence,

$$\begin{aligned} M_{m,n}(h,\chi) &= \sum_{a=1}^{f-1} \frac{\chi(a)}{(\zeta^{ha}-1)^m(\zeta^a-1)^n} \\ &= \sum_{a=1}^{f-1} \frac{\chi(a)\left(\sum_{c=0}^{h-1}\zeta^{ca}\right)^n}{(\zeta^{ha}-1)^{m+n}} \\ &= \sum_{c=0}^{n(h-1)} C_h(n,c) \sum_{a=1}^{f-1} \frac{\chi(a)\zeta^{ca}}{(\zeta^{ha}-1)^{m+n}} \\ &= \overline{\chi}(h) \sum_{c=0}^{n(h-1)} C_h(n,c) P_{m+n,ch^{-1},\chi}(0). \end{aligned}$$

The assertion of the theorem then follows by Lemma 3.2. □

We now give examples expressing $M_{m,n}(h,\chi)$ by generalized Bernoulli numbers.

Proposition 3.4. *Let $f > 1$ be a natural number prime to 4 and let χ be a primitive Dirichlet character with conductor f. We denote by δ the unique primitive Dirichlet character modulo 4. We also put $\zeta = exp(2\pi i/f)$. Then, we get*

(i)

$$M_{0,1}(4,\chi) = \chi(4)M_{1,0}(4,\chi) = \sum_{a=1}^{f-1} \frac{\chi(a)}{(\zeta^a-1)} = \tau(\chi)B_{1,\overline{\chi}},$$

(ii)

$$M_{0,2}(4,\chi) = \chi(4)M_{2,0}(4,\chi) = \sum_{a=1}^{f-1} \frac{\chi(a)}{(\zeta^a-1)^2} = -\tau(\chi)\left\{B_{1,\overline{\chi}} + \frac{1}{2}B_{2,\overline{\chi}}\right\},$$

(iii)

$$\begin{aligned} M_{1,1}(4,\chi) &= \sum_{a=1}^{f-1} \frac{\chi(a)}{(\zeta^{4a}-1)(\zeta^a-1)} \\ &= -\tau(\chi)\left\{\frac{1+\overline{\chi}(4)}{2}B_{1,\overline{\chi}} + \frac{1}{4}B_{1,\delta\overline{\chi}} + \frac{1}{8}B_{2,\overline{\chi}}\right\}, \end{aligned}$$

(iv)

$$
\begin{aligned}
M_{2,1}(4,\chi) &= \sum_{a=1}^{f-1} \frac{\chi(a)}{(\zeta^{4a}-1)^2(\zeta^a-1)} \\
&= \tau(\chi)\left\{ \frac{13-\overline{\chi}(2)+20\overline{\chi}(4)}{32} B_{1,\overline{\chi}} + \frac{1}{4} B_{1,\delta\overline{\chi}} + \frac{1+2\overline{\chi}(4)}{8} B_{2,\overline{\chi}} \right. \\
&\quad \left. + \frac{1}{32} B_{2,\delta\overline{\chi}} + \frac{1}{96} B_{3,\overline{\chi}} \right\},
\end{aligned}
$$

(v)

$$
\begin{aligned}
M_{1,2}(4,\chi) &= \sum_{a=1}^{f-1} \frac{\chi(a)}{(\zeta^{4a}-1)(\zeta^a-1)^2} \\
&= \tau(\chi)\left\{ \frac{7+\overline{\chi}(2)}{8} B_{1,\overline{\chi}} + \frac{1}{4} B_{1,\delta\overline{\chi}} + \frac{3}{8} B_{2,\overline{\chi}} + \frac{1}{24} B_{3,\overline{\chi}} \right\}.
\end{aligned}
$$

Proof. Clearly $B_{j,\overline{\chi}}(0/4) = B_{j,\overline{\chi}}$, and by Theorem 1.1, we have

$$
\begin{aligned}
B_{j,\overline{\chi}}(1/4) &= \frac{\chi(4)}{2\cdot 4^{j-1}}((1-2^{j-1}\bar{\chi}(2))B_{j,\bar{\chi}} + B_{j,\delta\bar{\chi}}), \\
B_{j,\overline{\chi}}(2/4) &= \overline{B}_{j,\overline{\chi}}(1/2) = \frac{\chi(2)}{2^{j-1}}(1-2^{j-1}\bar{\chi}(2))B_{j,\bar{\chi}}, \\
B_{j,\overline{\chi}}(3/4) &= \frac{\chi(4)}{2\cdot 4^{j-1}}((1-2^{j-1}\bar{\chi}(2))B_{j,\bar{\chi}} - B_{j,\delta\bar{\chi}}) \quad (j\in\mathbb{Z},\ j\geq 0).
\end{aligned}
$$

Thus the assertions (i)–(v) follow from Theorem 3.3. □

4. The Evaluation of $S_{m,n}(h,\chi;e_1,\ldots,e_{m+n})$

In this section, we obtain formulas expressing $S_{m,n}(h,\chi;e_1,\ldots,e_{m+n})$ by generalized Bernoulli numbers. We keep the notation used previously. We begin by establishing the relationship between the sums $S_{m,n}(h,\chi;e_1,\ldots,e_{m+n})$ and $M_{m,n}(h,\chi)$.

Theorem 4.1. *Let χ, f, h, m, n be as above. We have*

$$
\begin{aligned}
S_{m,n}(h,\chi;e_1,\ldots,e_{m+n}) &= \frac{(-1)^{e_1+\cdots+e_{m+n}}}{\tau(\overline{\chi})} \\
&\quad \times \sum_{\substack{1\leq m_j\leq e_j \\ 1\leq l_j\leq e_j-m_j+1 \\ 1\leq j\leq m+n}} \left(\prod_{i=1}^{m+n} (-f)^{l_i} \binom{e_i}{l_i} \left\{ {e_i-l_i+1 \atop m_i} \right\} (m_i-1)! \right) \\
&\quad \times M_{m_1+\cdots+m_m,\, m_{m+1}+\cdots+m_{m+n}}(h,\overline{\chi}),
\end{aligned}
$$

where $\left\{ {n \atop m} \right\}$ are Stirling numbers of the second kind (Stirling's subset numbers).

Proof. Since

$$\chi\left(h\sum_{i=1}^{m}a_i+\sum_{j=1}^{n}a_{m+j}\right)=\frac{1}{\tau(\overline{\chi})}\sum_{b=1}^{f-1}\overline{\chi}(b)\zeta^{\left(h\sum_{i=1}^{m}a_i+\sum_{j=1}^{n}a_{m+j}\right)b},$$

we have

$$S_{m,n}(h,\chi;e_1,\ldots,e_{m+n})$$

$$=\frac{1}{\tau(\overline{\chi})}\sum_{b=1}^{m-1}\overline{\chi}(b)\sum_{a_1,\ldots,a_{m+n}=1}^{m-1}a_1^{e_1}\cdots a_{m+n}^{e_{m+n}}\zeta^{b\left(h\sum_{i=1}^{m}a_i+\sum_{j=1}^{n}a_{m+j}\right)}.$$

The assertion of the theorem then follows by applying Proposition 2.1 to each of the sums over a_j ($j=1,\ldots,m+n$). □

Remark. In the case where $e_j=1$ for $j=1,\ldots,m+n$, we get

$$S_{m,n}(h,\chi;1,\ldots,1)=\frac{f^{m+n}}{\tau(\overline{\chi})}M_{m,n}(h,\overline{\chi}).$$

We now give examples expressing $S_{m,n}(h,\chi;e_1,\ldots,e_{m+n})$ by generalized Bernoulli numbers.

Proposition 4.2. *Let $f>1$ be a natural number prime to 4 and let χ be a primitive Dirichlet character with conductor f. We denote by δ the unique primitive Dirichlet character modulo 4. Then, we get*

(i)

$$S_{1,1}(4,\chi;1,1)=\sum_{a,b=1}^{f-1}\chi(4a+b)ab=-f^2\left\{\frac{1+\chi(4)}{2}B_{1,\chi}+\frac{1}{4}B_{1,\delta\chi}+\frac{1}{8}B_{2,\chi}\right\},$$

(ii)

$$S_{1,1}(4,\chi;2,1)=\sum_{a,b=1}^{f-1}\chi(4a+b)a^2b$$

$$=-f^2\left\{\frac{8f-3-\chi(2)+(8f+4)\chi(4)}{16}B_{1,\chi}+\frac{f}{4}B_{1,\delta\chi}\right.$$

$$\left.+\frac{f+4\chi(4)}{8}B_{2,\chi}+\frac{1}{16}B_{2,\delta\chi}+\frac{1}{48}B_{3,\chi}\right\},$$

(iii)

$$
\begin{aligned}
S_{1,1}(4,\chi;1,2) &= \sum_{a,b=1}^{f-1} \chi(4a+b)ab^2 \\
&= -f^2\left\{\frac{2f+3+\chi(2)+(2f-4)\chi(4)}{4}B_{1,\chi}+\frac{f}{4}B_{1,\delta\chi}\right. \\
&\qquad \left. +\frac{f+4}{8}B_{2,\chi}+\frac{1}{12}B_{3,\chi}\right\}.
\end{aligned}
$$

Proof. By Theorem 4.1, we have

$$
\begin{aligned}
S_{1,1}(4,\chi;1,1) &= \frac{f^2}{\tau(\overline{\chi})}M_{1,1}(4,\overline{\chi}), \\
S_{1,1}(4,\chi;2,1) &= \frac{f^2}{\tau(\overline{\chi})}((f-2)M_{1,1}(4,\overline{\chi})-2M_{2,1}(4,\overline{\chi})), \\
S_{1,1}(4,\chi;1,2) &= \frac{f^2}{\tau(\overline{\chi})}((f-2)M_{1,1}(4,\overline{\chi})-2M_{1,2}(4,\overline{\chi})).
\end{aligned}
$$

Thus the assertions (i)–(iii) follow from Proposition 3.4. □

5. The Evaluation of $c_{m,n}(h,\chi)$

In this section, we obtain formulas expressing $c_{m,n}(h,\chi)$ by generalized Bernoulli numbers. We keep the notation used previously. We begin by establishing the relationship between the sums $c_{m,n}(h,\chi)$ and $M_{m,n}(h,\chi)$.

Theorem 5.1. *Let χ, f, h, m, n be as above. We have*

$$
c_{m,n}(h,\chi) = i^{m+n}\sum_{j=0}^{m}\sum_{k=0}^{n} 2^{j+k}\binom{m}{j}\binom{n}{k}M_{j,k}(h,\chi).
$$

Proof. For $m, n = 0$, the assertion is clear since both sides are identically zero. Thus we assume $m+n>0$. Without loss of generality, we further assume $n>0$. We recall the discrete Fourier transform of $P_1(a/f)\,(a\in\mathbb{Z})$,

$$
P_1\left(\frac{a}{f}\right) = -\frac{1}{2if}\sum_{b=1}^{f-1}\cot\left(\frac{\pi b}{f}\right)\zeta^{ba}. \tag{5.1}
$$

We put

$$
T_{m,n}(h,\chi) = \sum_{a_1,\ldots,a_{m+n}(f)} P_1\left(\frac{a_1}{f}\right)\cdots P_1\left(\frac{a_{m+n}}{f}\right)\chi\left(h\sum_{i=1}^{m}a_i+\sum_{j=1}^{n}a_{m+j}\right),
$$

where $a_1, a_2, \ldots, a_{m+n}$ each run over a complete residue system modulo f.

We begin by expressing $M_{m,n}(h,\chi)$ by $T_{j,k}(h,\chi)$. Noting that $a/f = P_1(a/f) + 1/2$ for $1 \le a \le f-1$, we have

$$\begin{aligned} M_{m,n}(h,\chi) &= \frac{\tau(\chi)}{f^{m+n}} S_{m,n}(h,\overline{\chi};1,\ldots,1) \\ &= \tau(\chi) \sum_{a_1,\ldots,a_{m+n}=1}^{f-1} \left(P_1\left(\frac{a_1}{f}\right)+\frac{1}{2}\right)\cdots\left(P_1\left(\frac{a_{m+n}}{f}\right)+\frac{1}{2}\right) \\ &\quad \times \overline{\chi}\left(h\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} a_{m+j}\right) \\ &= \tau(\chi)\sum_{k=0}^{m+n}\left(\frac{-1}{2}\right)^k \sum_{j=0}^{k}\binom{m}{j}\binom{n}{k-j} T_{m-j,n-k+j}(h,\overline{\chi}). \end{aligned}$$

Interchanging the order of summation and taking $r = m-j$, $s = n-k+j$, we get

$$M_{m,n}(h,\chi) = \tau(\chi)\left(\frac{-1}{2}\right)^{m+n}\sum_{r=0}^{m}\sum_{s=0}^{n}(-2)^{r+s}\binom{m}{r}\binom{n}{s}T_{r,s}(h,\overline{\chi}). \tag{5.2}$$

We next express the sum $T_{m,n}(h,\overline{\chi})$ by $c_{m,n}(h,\chi)$. Taking the discrete Fourier transform (5.1) of each of the P_1's in $T_{m,n}(h,\overline{\chi})$, we have

$$\begin{aligned} T_{m,n}(h,\chi) &= \left(\frac{-1}{2if}\right)^{m+n}\sum_{b_1,\ldots,b_{m+n}=1}^{f-1}\cot\left(\frac{\pi b_1}{f}\right)\cdots\cot\left(\frac{\pi b_{m+n}}{f}\right) \\ &\quad \times \sum_{a_1,\ldots,a_{m+n}(f)} \zeta^{b_1a_1+\cdots+b_{m+n}a_{m+n}}\overline{\chi}\left(h\sum_{i=1}^{m}a_i + \sum_{j=1}^{n}a_{m+j}\right). \end{aligned}$$

Replacing a_{m+n} by $a_{m+n} - \left(h\sum_{i=1}^{m} a_i + \sum_{j=1}^{n-1} a_{m+j}\right)$ and evaluating the Gaussian sum over a_{m+n}, we get

$$\begin{aligned} T_{m,n}(h,\chi) &= \left(\frac{-1}{2if}\right)^{m+n}\tau(\overline{\chi})\sum_{b_1,\ldots,b_{m+n}=1}^{f-1}\cot\left(\frac{\pi b_1}{f}\right)\cdots\cot\left(\frac{\pi b_{m+n}}{f}\right)\chi(b_{m+n}) \\ &\quad \times \sum_{a_1,\ldots,a_{m+n-1}(f)} \zeta^{\sum_{i=1}^{m} a_i(b_i - hb_{m+n}) + \sum_{j=1}^{n-1} a_{m+j}(b_{m+j}-b_{m+n})}. \end{aligned}$$

Since

$$\sum_{a(f)} \zeta^{an} = \begin{cases} f & \text{if } n \equiv 0(f), \\ 0 & \text{if } n \not\equiv 0(f), \end{cases}$$

we obtain

$$T_{m,n}(h,\overline{\chi}) = \left(\frac{-1}{2if}\right)^{m+n} f^{m+n-1}\tau(\overline{\chi})c_{m,n}(h,\chi). \tag{5.3}$$

From (5.2) and (5.3), noting that $\tau(\chi)\tau(\overline{\chi}) = \chi(-1)f$, we get

$$M_{m,n}(h,\chi) = \left(\frac{-1}{2}\right)^{m+n} \chi(-1)\sum_{j=0}^{m}\sum_{k=0}^{n}(-i)^{j+k}\binom{m}{j}\binom{n}{k}c_{j,k}(h,\chi).$$

Since $c_{m,n}(h,\chi) = (-1)^{m+n}\chi(-1)c_{m,n}(h,\chi)$ for parity reasons, this simplifies to

$$M_{m,n}(h,\chi) = \left(\frac{-1}{2}\right)^{m+n} \sum_{j=0}^{m}\sum_{k=0}^{n} i^{j+k}\binom{m}{j}\binom{n}{k}c_{j,k}(h,\chi).$$

Let S denote the right-hand side of the assertion of the theorem. Plugging this expression of $M_{m,n}(h,\chi)$ into S, we get

$$S = i^{m+n}\sum_{j=0}^{m}\sum_{k=0}^{n}(-1)^{j+k}\binom{m}{j}\binom{n}{k}\sum_{r=0}^{j}\sum_{s=0}^{k} i^{r+s}\binom{j}{r}\binom{k}{s}c_{r,s}(h,\chi).$$

Interchanging the order of summations and taking $t = j-r$, $u = k-s$, we have

$$S = i^{m+n}\sum_{r=0}^{m}\sum_{s=0}^{n}(-i)^{r+s}\binom{m}{r}\binom{n}{s}c_{r,s}(h,\chi)\sum_{t=0}^{m-r}\sum_{u=0}^{n-k}(-1)^{t+u}\binom{m-r}{t}\binom{n-s}{u}.$$

The inner two sums vanish unless $r = m$, $k = n$, in which case we have $S = c_{m,n}(h,\chi)$, as desired. □

As a corollary to Theorem 5.1, we obtain a formula for cotangent power sums considered by Apostol [1].

Corollary 5.2. *Let χ, f, m be as above. We have*

$$\sum_{a=1}^{f-1}\cot^m\left(\frac{\pi a}{f}\right)\chi(a) = -i^m\tau(\chi)\sum_{k=1}^{m}\frac{1}{k}\left(\sum_{j=k}^{m}\frac{(-2)^j\binom{m}{j}\left[{j \atop k}\right]}{(j-1)!}\right)B_{k,\overline{\chi}}.$$

Proof. By Theorem 5.1, we have

$$c_{m,0}(1,\chi) = i^m\sum_{j=0}^{m}2^j\binom{m}{j}M_{j,0}(1,\chi), \tag{5.4}$$

where from Theorem 3.3, we have

$$M_{j,0}(1,\chi) = \frac{(-1)^{j-1}\tau(\chi)}{(j-1)!}\sum_{k=1}^{j}\frac{1}{k}\begin{bmatrix} j \\ k \end{bmatrix} B_{k,\overline{\chi}}.$$

Plugging these values of $M_{j,0}(1,\chi)$ into (5.4) and interchanging the order of summation, we get the assertion of the corollary. □

Corollary 5.2 resolves the computational problem of having to evaluate cotangent power sums recursively, as discussed in [17].

Remark. A consequence of Corollary 5.2 is that

$$\sum_{j=k}^{m}\frac{(-2)^j\binom{m}{j}\begin{bmatrix} j \\ k \end{bmatrix}}{(j-1)!} = 0$$

whenever k and m are of opposite parity.

We now give examples expressing $c_{m,n}(h,\chi)$ by generalized Bernoulli numbers.

Proposition 5.3. *Let $f > 1$ be a natural number prime to 4 and let χ be a primitive Dirichlet character with conductor f. We denote by δ the unique primitive Dirichlet character modulo 4. Then, we get*

(i)

$$c_{1,1}(4,\chi) = \sum_{j=1}^{f-1}\cot\left(\frac{4\pi j}{f}\right)\cot\left(\frac{\pi j}{f}\right)\chi(j) = \tau(\chi)\left\{B_{1,\delta\overline{\chi}} + \frac{1}{2}B_{2,\overline{\chi}}\right\},$$

(ii)

$$c_{2,1}(4,\chi) = \sum_{j=1}^{f-1}\cot^2\left(\frac{4\pi j}{f}\right)\cot\left(\frac{\pi j}{f}\right)\chi(j)$$

$$= -i\tau(\chi)\left\{\frac{5-\overline{\chi}(2)+4\overline{\chi}(4)}{4}B_{1,\overline{\chi}} + \frac{1}{4}B_{2,\delta\overline{\chi}} + \frac{1}{12}B_{3,\overline{\chi}}\right\},$$

(iii)

$$c_{1,2}(4,\chi) = \sum_{j=1}^{f-1}\cot\left(\frac{4\pi j}{f}\right)\cot^2\left(\frac{\pi j}{f}\right)\chi(j)$$

$$= -i\tau(\chi)\left\{(5+\overline{\chi}(2)-2\overline{\chi}(4))B_{1,\overline{\chi}} + \frac{1}{3}B_{3,\overline{\chi}}\right\}.$$

Proof. From Theorem 5.1, we have

$$c_{1,1}(4,\chi) = -2\{M_{1,0}(4,\chi) + M_{0,1}(4,\chi) + 2M_{1,1}(4,\chi)\},$$

$$c_{2,1}(4,\chi) = -2i\{M_{0,1}(4,\chi) + 2M_{1,0}(4,\chi) + 2M_{0,2}(4,\chi) + 4M_{1,1}(4,\chi) + 2M_{2,0}(4,\chi) + 4M_{2,1}(4,\chi)\},$$

$$c_{1,2}(4,\chi) = -2i\{2M_{0,1}(4,\chi) + M_{1,0}(4,\chi) + 2M_{0,2}(4,\chi) + 4M_{1,1}(4,\chi) + 2M_{2,0}(4,\chi) + 4M_{1,2}(4,\chi)\}.$$

Thus the assertions (i)–(iii) follow from Proposition 3.4. □

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