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Problems and Solutions

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PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman**, and **Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Stan Wagon, Lawrence Washington, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by April 30, 2021, via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12216. *Proposed by Zachary Franco, Houston, TX.* A regular icosahedron with volume 1 is rotated about an axis connecting opposite vertices. What is the volume of the resulting solid?

12217. *Proposed by Giuseppe Fera, Vicenza, Italy.* Let I be the incenter and G be the centroid of a triangle ABC. Prove

$$\frac{3}{2} < \frac{AI}{AG} + \frac{BI}{BG} + \frac{CI}{CG} \le 3.$$

12218. *Proposed by Richard Stong, Center for Communications Research, La Jolla, CA, and Stan Wagon, Macalester College, St. Paul, MN.* For which positive integers n does there exist an ordering of all permutations of $\{1, \ldots, n\}$ so that their composition in that order is the identity?

12219. *Proposed by Brad Isaacson, New York City College of Technology, New York, NY.* Let k and m be positive integers with $k < m$. Let $c(m, k)$ be the number of permutations of $\{1, \ldots, m\}$ consisting of k cycles. (The numbers $c(m, k)$ are known as unsigned Stirling numbers of the first kind.) Prove

$$\sum_{j=k}^{m} \frac{(-2)^j \binom{m}{j} c(j,k)}{(j-1)!} = 0$$

whenever m and k have opposite parity.

12220. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Let $a_n = \sum_{k=1}^{n} 1/k^2$ and $b_n = \sum_{k=1}^{n} 1/(2k-1)^2$. Prove

$$\lim_{n\to\infty} n\left(\frac{b_n}{a_n} - \frac{3}{4}\right) = \frac{3}{\pi^2}.$$

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12221. *Proposed by Necdet Batır, Nevşehir Hacı Bektaş Veli University, Nevşehir, Turkey.* Prove

$$\int_0^1 \frac{\log(x^6+1)}{x^2+1}\,dx = \frac{\pi}{2}\log 6 - 3G,$$

where G is Catalan's constant $\sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$.

12222. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* Prove

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sum_{n=k}^{\infty}\frac{1}{n2^n} = -\frac{13\,\zeta(3)}{24},$$

where $\zeta(3)$ is Apéry's constant $\sum_{k=1}^{\infty} 1/k^3$.

SOLUTIONS

Polynomial Roots Lead to a Triangle

12105 [2019, 370]. *Proposed by Gary Brookfield, California State University, Los Angeles, CA.* Let r be a real number, and let $f(x) = x^3 + 2rx^2 + (r^2-1)x - 2r$. Suppose that f has real roots a, b, and c. Prove $a, b, c \in [-1, 1]$ and $|\arcsin a| + |\arcsin b| + |\arcsin c| = \pi$.

Solution by O. P. Lossers, Eindhoven University of Technology, Netherlands. If $r = 0$, then the roots of f are -1, 0, and 1, and the conclusion clearly follows. So we may assume from now on that $r \neq 0$, and therefore $f(0) \neq 0$.

If x is a zero of f, then $r^2x + 2r(x^2-1) + x^3 - x = 0$ and $x \neq 0$. It follows that $r^2 + 2r(x - 1/x) + x^2 - 1 = 0$. Thus $0 \le (r + x - 1/x)^2 = (x - 1/x)^2 - x^2 + 1 = 1/x^2 - 1$, and hence $x^2 \le 1$, proving $x \in [-1, 1]$.

Next, let $\alpha = |\arcsin a|$, $\beta = |\arcsin b|$, and $\gamma = |\arcsin c|$, so that $a = \pm\sin\alpha$, $b = \pm\sin\beta$, $c = \pm\sin\gamma$, and $0 < \alpha, \beta, \gamma \le \pi/2$. It suffices to show that there is a triangle with angles α, β, and γ. When $\alpha + \beta + \gamma = \pi$, one can use the formula for the cosine of a sum to show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma = 1, \tag{1}$$

and the converse holds also for acute angles. Now $\cos\alpha = \sqrt{1-a^2}$, $\cos\beta = \sqrt{1-b^2}$, and $\cos\gamma = \sqrt{1-c^2}$, so (1) is equivalent to

$$3 - (a^2+b^2+c^2) + 2\sqrt{(1-a^2)(1-b^2)(1-c^2)} = 1. \tag{2}$$

By Vieta's formulas for the coefficients of f, we have $a + b + c = -2r$, $ab + bc + ca = r^2 - 1$, and $abc = 2r$. Hence

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = (-2r)^2 - 2(r^2-1) = 2r^2 + 2.$$

Also

$$\begin{aligned} a^2b^2 + b^2c^2 + c^2a^2 &= (ab+bc+ca)^2 - 2abc(a+b+c) \\ &= (r^2-1)^2 - 2(2r)(-2r) = r^4 + 6r^2 + 1 \end{aligned}$$

and

$$a^2b^2c^2 = (abc)^2 = (2r)^2 = 4r^2.$$

Using these three computations, we find that (2) indeed holds.

Also solved by A. J. Bevelacqua, E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), R. Boukharfane (Saudi Arabia), P. Bracken, G. Fera (Italy), L. Giugiuc (Romania), J.-P. Grivaux (France), J. C. Kieffer, O. Kouba (Syria), J. H. Lindsey II, X. Liu (China), M. Omarjee (France), T. Pedapolu, C. R. Pranesacher (India), M. A. Prasad (India), N. Singer, R. Stong, R. Tauraso (Italy), F. Visescu (Romania), L. Zhou, FAU Problem Solving Group, GCHQ Problem Solving Group (UK), and the proposer.

A Reciprocal Summation Identity

12106 [2019, 370]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For any positive integer n, prove

$$\sum_{k=1}^{n}\binom{n}{k}\sum_{1\le i\le j\le k}\frac{1}{ij}=\sum_{1\le i\le j\le n}\frac{2^n-2^{n-i}}{ij}.$$

Solution by Robin Chapman, University of Exeter, Exeter, UK. We first obtain a general recurrence for certain binomial sums. Given a sequence $\langle C\rangle$ with $C_0=0$, let $R_n=\sum_{k=0}^{n}\binom{n}{k}B_k$, where $B_0=0$ and $B_k=\sum_{j=1}^{k}C_j/j$ when $k\ge 1$. For $n\ge 1$, we compute

$$\begin{aligned}R_n &= \sum_{k=0}^{n}\binom{n}{k}B_k = \sum_{k=0}^{n-1}\binom{n-1}{k}B_k+\sum_{k=1}^{n}\binom{n-1}{k-1}B_k = R_{n-1}+\sum_{k=0}^{n-1}\binom{n-1}{k}B_{k+1}\\ &= 2R_{n-1}+\sum_{k=0}^{n-1}\binom{n-1}{k}(B_{k+1}-B_k) = 2R_{n-1}+\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{C_{k+1}}{k+1}\\ &= 2R_{n-1}+\frac{1}{n}\sum_{k=0}^{n-1}\binom{n}{k+1}C_{k+1} = 2R_{n-1}+\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}C_k. \qquad (*)\end{aligned}$$

Let S_n and T_n be the left and right sides, respectively, of the desired identity, and set $S_0=T_0=0$. We have $S_n=\sum_{k=0}^{n}\binom{n}{k}A_k$, where $A_0=0$ and $A_k=\sum_{1\le i\le j\le k}1/(ij)$ when $k\ge 1$. Also $A_k=\sum_{j=1}^{k}H_j/j$, where H_j is the jth harmonic number $\sum_{i=1}^{j}1/i$. With $H_0=0$, $(*)$ yields $S_n=2S_{n-1}+\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}H_k$ for $n\ge 1$. Also,

$$T_n = 2^n\sum_{1\le i\le j\le n}\frac{1-2^{-i}}{ij} = 2T_{n-1}+\frac{2^n}{n}\sum_{i=1}^{n}\frac{1-2^{-i}}{i}.$$

Thus $S_n=2S_{n-1}+U_n/n$ and $T_n=2T_{n-1}+V_n/n$ for $n\ge 1$, where $U_n=\sum_{k=0}^{n}\binom{n}{k}H_k$ and $V_n=2^n\sum_{i=1}^{n}(1-2^{-i})/i$. Since $S_0=T_0=0$, it suffices to show $U_n=V_n$ for $n\ge 1$.

Since $U_0=V_0=0$, it suffices to show that U_n and V_n satisfy the same recurrence. Again $(*)$ applies to yield

$$U_n=2U_{n-1}+\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}\cdot 1=2U_{n-1}+\frac{2^n-1}{n}.$$

Also

$$V_n=2V_{n-1}+2^n\frac{1-2^{-n}}{n}=2V_{n-1}+\frac{2^n-1}{n},$$

and the proof is complete.

Editorial comment. With $r\in\mathbb{N}$ and $P(r,m)$ denoting the set of integer r-tuples $(i_1,\dots,i_r)$ with $1\le i_1\le\cdots\le i_r\le m$, the proposer and Pierre Lalonde gave the generalization

$$\sum_{k=1}^{n}\binom{n}{k}\sum_{P(r,k)}\frac{1}{i_1\cdots i_r}=\sum_{P(r,n)}\frac{2^n-2^{n-i_1}}{i_1\cdots i_r}.$$

Also solved by H. Chen, K. Gatesman, N. Grivaux (France), N. Hodges, O. Kouba (Syria), P. Lalonde (Canada), K.-W. Lau (China), J. H. Lindsey II, X. Liu (China), O. P. Lossers (Netherlands), V. H. Moll & T. Amdeberhan, M. Omarjee (France), A. Pathak, M. Prasad (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), T. Wiandt, M. Wildon (UK), ARC Problem Solving Group (India), GCHQ Problem Solving Group (UK), and the proposer.

Integration by Trigonometric Substitution

12107 [2019, 370]. *Proposed by Cornel Ioan Vălean, Teremia Mare, Romania.* Prove

$$\int_0^1 \int_0^1 \frac{1}{\sqrt{1+x^2}\sqrt{1+y^2}(1-x^2y^2)}\,dx\,dy = G,$$

where G is Catalan's constant $\sum_{n=1}^{\infty}(-1)^{n-1}/(2n-1)^2$.

Solution by FAU Problem Solving Group, Boca Raton, FL. Changing variables with $x = \tan\theta$ and $y = \tan\varphi$, applying the symmetry of the integrand, and then evaluating the inner integral yields

$$\begin{aligned}
\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{1+x^2}\sqrt{1+y^2}(1-x^2y^2)} &= \int_0^{\pi/4}\int_0^{\pi/4} \frac{\sec^2\theta\sec^2\varphi\,d\theta\,d\varphi}{\sec\theta\sec\varphi(1-\tan^2\theta\tan^2\varphi)} \\
&= \int_0^{\pi/4}\int_0^{\pi/4} \frac{\cos\theta\cos\varphi\,d\theta\,d\varphi}{\cos^2\theta\cos^2\varphi-\sin^2\theta\sin^2\varphi} \\
&= \int_0^{\pi/4}\int_0^{\pi/4} \frac{\cos\theta\cos\varphi\,d\theta\,d\varphi}{\cos(\theta+\varphi)\cos(\theta-\varphi)} \\
&= \frac{1}{2}\int_0^{\pi/4}\int_0^{\pi/4}\left(\frac{1}{\cos(\theta+\varphi)}+\frac{1}{\cos(\theta-\varphi)}\right)d\theta\,d\varphi \\
&= \int_0^{\pi/4}\int_0^{\theta}\big(\sec(\theta+\varphi)+\sec(\theta-\varphi)\big)\,d\varphi\,d\theta \\
&= \int_0^{\pi/4} \log\big(\sec(\theta+\varphi)+\tan(\theta+\varphi)\big)-\log\big(\sec(\theta-\varphi)+\tan(\theta-\varphi)\big)\Big|_{\varphi=0}^{\theta}\,d\theta \\
&= \int_0^{\pi/4}\log\big(\sec(2\theta)+\tan(2\theta)\big)\,d\theta = \frac{1}{2}\int_0^{\pi/2}\log(\sec\theta+\tan\theta)\,d\theta \\
&= \frac{1}{2}\left(\int_0^{\pi/2}\log(1+\sin\theta)\,d\theta - \int_0^{\pi/2}\log(\cos\theta)\,d\theta\right). \qquad (*)
\end{aligned}$$

The two integrals in (∗) are evaluated in J. Jameson and N. Lord (2017), Integrals evaluated in terms of Catalan's constant, *Math. Gazette*, 101: 38–49, as follows:

$$\int_0^{\pi/2}\log(1+\sin\theta)\,d\theta = 2G - \frac{\pi}{2}\log 2 \quad\text{and}\quad \int_0^{\pi/2}\log(\cos\theta)\,d\theta = -\frac{\pi}{2}\log 2.$$

Combining these with (∗) gives the desired conclusion.

Also solved by A. A. Ayinde (Nigeria), M. Bataille (France), K. C. Betal, R. Boukharfane (Saudi Arabia), P. Bracken, B. Bradie, H. Chen, B. E. Davis, G. Fera (Italy), P. Fulop (Hungary), K. Gatesman, L. Glasser, H. Grandmontagne (France), J. E. Kampmeyer, O. Kouba (Syria), S. S. Kumar, R. McCrorie, V. H. Moll & T. Amdeberhan, M. Omarjee (France), M. A. Prasad (India), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), M. Wildon (UK), L. Zhou, GCHQ Problem Solving Group (UK), and the proposer.

An Irreducible Determinant of Indeterminates

12108 [2019, 370]. *Proposed by Yifei Pan and William D. Weakley, Purdue University Fort Wayne, Fort Wayne, IN.* Let n be a positive integer, and let $\beta_1, \ldots, \beta_n$ be indeterminates over a field F. Let M be the n-by-n matrix whose i, j-entry m_{ij} is given by $m_{ij} = \beta_i$ when $i = j$ and $m_{ij} = 1$ when $i \neq j$. Show that the polynomial $\det(M)$ is irreducible over F.

Solution by Koopa Tak Lun Koo, Chinese STEAM Academy, Hong Kong, China. The claim is trivial for $n = 1$. For $n \geq 2$, let $\alpha_i = \beta_i - 1$ for $1 \leq i \leq n$. We first prove

$$\det(M) = \prod_{k=1}^{n} \alpha_k + \sum_{k=1}^{n} \prod_{j \neq k} \alpha_j.$$

Let $\mathbf{u}$ be the all-1 column vector in $\mathbb{R}^n$. Let D be the diagonal matrix with diagonal $\alpha_1, \ldots, \alpha_n$. Now $M = D + \mathbf{u}\mathbf{u}^T$. We use $\det(I_p + AB) = \det(I_q + BA)$, where I_k is the k-by-k identity matrix and both A and B^T are p-by-q matrices. Applying this with $(p, q) = (n, 1)$ yields

$$\det(M) = \det(D + \mathbf{u}\mathbf{u}^T) = \det\left((I_n + \mathbf{u}\mathbf{u}^T D^{-1})D\right) = \det(1 + \mathbf{u}^T D^{-1}\mathbf{u}) \det(D)$$

$$= \left(1 + \sum_{k=1}^{n} \frac{1}{\alpha_k}\right) \prod_{k=1}^{n} \alpha_k = \prod_{k=1}^{n} \alpha_k + \sum_{k=1}^{n} \prod_{j \neq k} \alpha_j.$$

To show that $\det(M)$ is irreducible as a polynomial in $F[\beta_1, \ldots, \beta_n]$, it suffices to show that $\det(M)$ is irreducible as a polynomial in $F[\alpha_1, \ldots, \alpha_n]$. Note that $\det(M) = \alpha_n(s_{n-1} + s_{n-2}) + s_{n-1}$, where s_k is the kth elementary symmetric polynomial in the $n-1$ variables $\alpha_1, \ldots, \alpha_{n-1}$. Since $\det(M)$ has degree 1 as a polynomial of α_n, it is irreducible in $F[\alpha_1, \ldots, \alpha_{n-1}][\alpha_n]$. Thus, the only possible nonunit factor of $\det(M)$ is a nonunit common factor of $s_{n-1} + s_{n-2}$ and s_{n-1} in $F[\alpha_1, \ldots, \alpha_{n-1}]$. We complete the proof by observing $\gcd(s_{n-1} + s_{n-2}, s_{n-1}) = \gcd(s_{n-1}, s_{n-2}) = 1$.

Editorial comment. The statement remains true over an integral domain R when the off-diagonal entries of M are replaced by nonzero elements of R. For a proof, expand $\det(M)$ along one column and then use an argument like that of the last paragraph above.

Also solved by P. Budney, K. Gatesman, N. Grivaux (France), J. C. Kieffer, J. H. Lindsey II, O. P. Lossers (Netherlands), R. Stong, M. Wildon (UK), T. N. Xuan (Vietnam), Missouri State University Problem Solving Group, and the proposer.

Integration by the Beta and Gamma Functions

12109 [2019, 370]. *Proposed by George Stoica, Saint John, NB, Canada.* Let f be a function on $[0, \infty)$ that is nonnegative, bounded, and continuous. For $a > 0$ and $x \geq 0$, let $g(x) = \exp\left(\int_0^a \log(1 + xf(s))\, ds\right)$. For $0 < p < 1$, prove

$$\int_0^a f^p(s)\, ds = \frac{p \sin(p\pi)}{\pi} \int_0^\infty \frac{\log g(x)}{x^{p+1}}\, dx.$$

Solution by Robin Chapman, University of Exeter, Exeter, UK. Observe that

$$I = \int_0^\infty \frac{\log g(x)}{x^{p+1}}\, dx = \int_0^\infty \frac{1}{x^{p+1}} \int_0^a \log(1 + xf(s))\, ds\, dx.$$

The integrand of this double integral is nonnegative, so we may reverse the order of integration to get

$$I = \int_0^a \int_0^\infty \frac{\log(1 + xf(s))}{x^{p+1}}\, dx\, ds.$$

For $t > 0$, the substitution $y = xt$ gives

$$\int_0^\infty \frac{\log(1+xt)}{x^{p+1}}\,dx = t^p \int_0^\infty \frac{\log(1+y)}{y^{p+1}}\,dy.$$

This formula remains valid for $t = 0$, since both sides then vanish. Therefore

$$I = \int_0^a f(s)^p \int_0^\infty \frac{\log(1+y)}{y^{p+1}}\,dy\,ds = J(p)\int_0^a f(s)^p\,ds,$$

where

$$J(p) = \int_0^\infty \frac{\log(1+y)}{y^{p+1}}\,dy.$$

To evaluate $J(p)$, first integrate by parts:

$$J(p) = \frac{1}{p}\left[-\frac{\log(1+y)}{y^p}\right]_{y=0}^\infty + \frac{1}{p}\int_0^\infty \frac{dy}{y^p(1+y)} = \frac{1}{p}\int_0^\infty \frac{dy}{y^p(1+y)}.$$

This calculation uses $p < 1$ and $\log(1+y) = O(y)$ as $y \to 0$.

The beta function B is well known to satisfy

$$B(u,v) = \int_0^\infty \frac{y^{u-1}}{(1+y)^{u+v}}\,dy$$

for $u, v > 0$. Letting $(u,v) = (1-p, p)$ yields

$$B(1-p,p) = \int_0^\infty \frac{dy}{y^p(1+y)}\,dy.$$

Expressing the beta function in terms of the gamma function gives

$$J(p) = \frac{B(1-p,p)}{p} = \frac{\Gamma(1-p)\Gamma(p)}{p\Gamma(1)} = \frac{\Gamma(p)\Gamma(1-p)}{p}.$$

The reflection formula for the gamma function implies $J(p) = \dfrac{\pi}{p\sin(p\pi)}$. Therefore

$$I = \frac{\pi}{p\sin(p\pi)}\int_0^a f(s)^p\,ds,$$

which is equivalent to the desired formula.

Also solved by K. F. Andersen (Canada), P. Bracken, H. Chen, G. H. Chung, K. Gatesman, N. Hodges, F. Holland (Ireland), O. Kouba (Syria), O. P. Lossers (Netherlands), M. A. Prasad (India), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, T. Wiandt, FAU Problem Solving Group, and the proposer.

The Limit of an Infinite Product

12110 [2019, 371]. *Proposed by Pedro Jesús Rodríguez de Rivera (student) and Ángel Plaza, University of Las Palmas de Gran Canaria, Las Palmas, Spain.* Let $\alpha_k = (k+\sqrt{k^2+4})/2$. Evaluate

$$\lim_{k\to\infty}\prod_{n=1}^\infty\left(1-\frac{k}{\alpha_k^n+\alpha_k}\right).$$

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. The limit is $1/2$. Since $\alpha_k - k = \alpha_k^{-1}$, the product equals

$$\prod_{n=1}^\infty \frac{\alpha_k^n+\alpha_k^{-1}}{\alpha_k^n+\alpha_k},$$

which in turn is

$$\prod_{n=1}^{\infty} \frac{1+\alpha_k^{-n-1}}{1+\alpha_k^{-n+1}}.$$

This product telescopes, so it equals $(1+\alpha_k^0)^{-1} \cdot (1+\alpha_k^{-1})^{-1}$. Since $\alpha_k^0 = 1$ and $\alpha_k^{-1} \to 0$, it follows that the limit is $1/2$.

Editorial comment. Some solvers noted that the product for $k = 1$ appears in *The Fibonacci Quarterly* as Elementary Problem B-1237 [2018, 366] by Hideyuki Ohtsuka.

Also solved by K. F. Andersen (Canada), F. R. Ataev (Uzbekistan), M. Bataille (France), P. Bracken, B. Bradie, N. Caro (Brazil), R. Chapman (UK), G. Fera (Italy), D. Fleischman, K. Gatesman, R. Guadalupe (Philippines), E. A. Herman, F. Holland (Ireland), M. Kaplan & M. Goldenberg, O. Kouba (Syria), P. Lalonde (Canada), J. H. Lindsey II, X. Liu (China), R. Molinari, A. Natian, H. Ohtsuka (Japan), M. Omarjee (France), A. Pathak, M. A. Prasad (India), T. de Souza Leao, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), D. Terr, T. Wiandt, J. Zacharias, L. Zhou, GCHQ Problem Solving Group (UK), and the proposer.

Harmony on the n-Cube

12111 [2019, 468]. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* A line segment AB can be oriented in two ways, which we denote (AB) and (BA). A square $ABCD$ can be oriented in two ways, which we denote $(ABCD)$ (the same as $(BCDA)$, $(CDAB)$, and $(DABC)$) and $(DCBA)$ (the same as $(CBAD)$, $(BADC)$, and $(ADCB)$). We say that the orientation $(ABCD)$ of a square *agrees with* the orientations (AB), (BC), (CD), and (DA) of its sides. Suppose that each edge and 2-dimensional face of an n-dimensional cube is given an orientation.
(a) What is the largest possible number of 2-dimensional faces whose orientation agrees with the orientations of its four sides?
(b) What is the largest possible number of edges whose orientation agrees with the orientations of all 2-dimensional faces containing the edge?

Solution by GCHQ Problem Solving Group, Cheltenham, UK. Denote by N_F the answer to part (a) and by N_E the answer to part (b). We prove $N_F = 2^{n-2}\lfloor n^2/4 \rfloor$ and $N_E = 2^n$.

(a) Choose a vertex A of the n-cube and consider all n edges and all $\binom{n}{2}$ 2-dimensional faces containing A. For a face $ABCD$ to agree with the orientations of its four sides, the edges AB and AD must have opposite orientations (one toward A and one away from A). If exactly t of the edges containing A are oriented away from A, at most $t(n-t)$ faces containing A can agree with the orientations of their four sides. It follows that the number of faces containing A whose orientations agree with the orientations of the four sides is at most $\lfloor n^2/4 \rfloor$, the largest possible value of $t(n-t)$. Summing over all vertices, we have

$$4N_F \le 2^n \left\lfloor \frac{n^2}{4} \right\rfloor,$$

since the left side counts each face four times.

The following construction attains this bound. Let $B_1, \ldots, B_n$ be the vertices adjacent to A, and let $t = \lfloor n/2 \rfloor$. For $i \in \{1, \ldots, t\}$ orient the edges as AB_i, and for $j \in \{t+1, \ldots, n\}$ orient the edges as B_jA. Let C_{ij} be the remaining vertex in the face containing the sides AB_i and AB_j. Orient that face as $AB_iC_{ij}B_j$. (The remaining faces containing A may be oriented arbitrarily.)

Every edge PQ not containing A is parallel to exactly one edge AB_r, for some $r \in \{1, \ldots, n\}$. Let d be the Hamming distance from the edge PQ to AB_r. Give PQ the orientation parallel to AB_r if d is even, antiparallel to AB_r if d is odd.

A face is now agreeably oriented if and only if the parallel face containing A is agreeably oriented, and each equivalence class of parallel faces has size 2^{n-2}. The bound $2^{n-2}\lfloor n^2/4\rfloor$ is attained.

(b) Let B and D be two vertices adjacent to vertex A. If the orientations of the edges AB and AD agree with the orientation of all (2-dimensional) faces containing them, then they must be "oppositely" oriented (such as (AB) and (DA)). It follows that no more than *two* edges containing A can agree with the orientations of all their faces. Since each such edge is counted at both endpoints, summing over all vertices yields $2N_E \le 2\cdot 2^n$.

This bound can be attained by choosing B and D adjacent to A, orienting the edges as (AB) and (DA), and orienting each face containing one or both of these edges such that it agrees with them. Orient the remaining faces and edges containing A arbitrarily.

Extend this to an orientation of the faces and edges not containing A in the same manner as in (a). An edge is now agreeably oriented if and only if it is parallel to AB or AD, and thus $N_E = 2^n$.

Also solved by R. Ehrenborg, M. Meyerson, J. Smith, R. Stong, L. Zhou, and the proposers.

A Parallel to the Simson Line

12112 [2019, 468]. *Proposed by Dao Thanh Oai, Thai Binh, Vietnam.* Let ABC be a triangle with circumcenter O and nine-point center N. Let P be a point on its circumcircle and let D, E, and F be the circumcenters of triangles AOP, BOP, and COP, respectively. Let A', B', and C' be the feet of perpendiculars from D, E, and F onto the lines BC, CA, and AB, respectively. Prove that A', B', C', and N are collinear.

Solution I by Oliver Geupel, Brühl, Germany. We work in the complex plane, with lowercase letters denoting the complex representations of points designated by the corresponding uppercase letters. We may assume that the circumcircle of triangle ABC is the unit circle, so that $a\overline{a} = b\overline{b} = c\overline{c} = p\overline{p} = 1$, and that its chords AP, BP, and CP are not diameters. The point D is on the perpendicular bisector of the line segment OP, so

$$\overline{p}\left(d - \frac{p}{2}\right) + p\left(\overline{d} - \frac{\overline{p}}{2}\right) = 0;$$

that is, $d + p^2\overline{d} = p$. Similarly, because D is on the perpendicular bisector of OA, we have $d + a^2\overline{d} = a$. Solving these two equations in the unknowns d and $\overline{d}$ yields $d = ap/(a+p)$ and $\overline{d} = 1/(a+p)$. Now it is well known and easy to check that $a' = \left(b + c + d - bc\overline{d}\right)/2$. The nine-point center N lies at the midpoint between O and the orthocenter H, where $h = a + b + c$. Thus $n = (a+b+c)/2$. Therefore

$$n - a' = \frac{a+b+c}{2} - \frac{b+c+d-bc\overline{d}}{2} = \frac{a}{2} - \frac{ap}{2(a+p)} + \frac{bc}{2(a+p)} = \frac{a^2+bc}{2(a+p)}.$$

It follows that

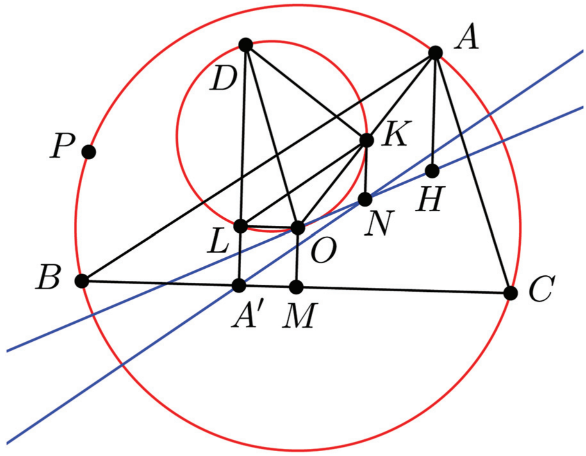
$$\frac{n-a'}{\overline{n}-\overline{a'}} = \frac{a^2+bc}{2(a+p)} \cdot \frac{2(\overline{a}+\overline{p})}{\overline{a}^2+\overline{b}\overline{c}} = \frac{abc}{p}.$$

By symmetry of notation, we have analogously

$$\frac{n-b'}{\overline{n}-\overline{b'}} = \frac{n-c'}{\overline{n}-\overline{c'}} = \frac{abc}{p}.$$

We conclude that the points A', B', C', and N are collinear.

Solution II by Li Zhou, Polk State College, Winter Haven, FL. Let $\angle(UV, XY)$ denote the



signed (or directed) angle from vector $\overrightarrow{UV}$ to vector $\overrightarrow{XY}$ (positive for counterclockwise and negative for clockwise). We also take all equivalences below modulo π. Letting $2\alpha = \angle(OA, OP)$, $2\beta = \angle(OB, OP)$, and $2\gamma = \angle(OC, OP)$, we have $\alpha - \beta \equiv \angle(CA, CB)$, $\beta - \gamma \equiv \angle(AB, AC)$, and $\gamma - \alpha \equiv \angle(BC, BA)$. Let H be the orthocenter of $\triangle ABC$, M the midpoint of BC, K the midpoint of OA, and L the foot of perpendicular from O onto $A'D$ (see figure). Since $LA' \parallel AH \parallel KN$ and $LA' = OM = (1/2)AH = KN$, we have $A'N \parallel LK$. By definition, D is the intersection of the perpendicular bisector of OA and the bisector of $\angle(OA, OP)$. Since $\angle OKD$ and $\angle OLD$ are right angles, $LOKD$ is cyclic and

$$\angle(A'N, HA) \equiv \angle(A'N, A'D) \equiv \angle(LK, LD) \equiv \angle(OK, OD) \equiv \alpha.$$

Hence, $\angle(A'N, OH) \equiv \angle(A'N, HA) + \angle(HA, OH) \equiv \alpha + \angle(HA, OH)$. Similarly, $\angle(B'N, OH) \equiv \beta + \angle(HB, OH)$ and $\angle(C'N, OH) \equiv \gamma + \angle(HC, OH)$. Therefore,

$$\begin{aligned}\angle(A'N, OH) - \angle(B'N, OH) &\equiv \alpha - \beta + \angle(HA, OH) + \angle(OH, HB)\\ &\equiv \angle(CA, CB) + \angle(HA, HB) \equiv 0.\end{aligned}$$

Likewise, $\angle(B'N, OH) - \angle(C'N, OH) \equiv 0$, completing the proof.

Editorial comment. Most solutions were similar to Solution I. Several solvers remarked that the line through A', B', C', and N is parallel to the Simson line ℓ of P with respect to ABC. Giuseppe Fera proved this using the fact that ℓ has the equation $pz - abc\bar{z} = k$, where k depends on a, b, c, and p. (See page 80 of L. Hahn (1994), *Complex Numbers and Geometry*, Washington, DC: Mathematical Association of America.) Indeed, Solution I above indicates $p\left(n - a'\right) = abc\left(\overline{n} - \overline{a'}\right)$, that is,

$$pa' - abc\overline{a'} = pn - abc\overline{n} = \frac{p(a+b+c) - (ab+bc+ca)}{2}.$$

Also solved by H. Bailey, M. Bataille (France), R. Chapman (UK), G. Fera (Italy), L. Giugiuc (Romania), J.-P. Grivaux (France), W. Janous (Austria), M. Kaplan & M. Goldenberg, K. T. L. Koo (China), O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, C. R. Pranesachar (India), A. Stadler (Switzerland), R. Stong, T. Wiandt, GCHQ Problem Solving Group (UK), and the proposer.