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AMM Problem #12279

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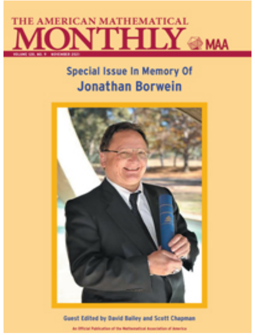

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Problems and Solutions

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PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman**, and **Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Stan Wagon, Lawrence Washington, and Li Zhou.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by March 31, 2022, via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12279. *Proposed by Brad Isaacson, Brooklyn, NY.* Let $S(m,k)$ denote the number of partitions of a set with m elements into k nonempty blocks. (These are the *Stirling numbers of the second kind*.) Let j and n be positive integers of opposite parity with $j<n$. Prove

$$\sum_{r=j}^{n} \frac{(-1)^r (r-1)!\binom{r}{j} S(n,r)}{2^r} = 0.$$

12280. *Proposed by Nguyen Duc Toan, Da Nang, Vietnam.* Let ABC be an acute scalene triangle with circumcenter O and orthocenter H. Let M and R be the midpoints of segments BC and OH, respectively, let S be the reflection across BC of the circumcenter of triangle BOC, and let T be the second point of intersection of the circumcircle of triangle BHC and line OH. Prove that M, R, S, and T are concyclic.

12281. *Proposed by Paolo Perfetti, Università di Roma "Tor Vergata," Rome, Italy.* Evaluate

$$\int_0^\infty \left(\frac{\cosh x}{\sinh^2 x} - \frac{1}{x^2}\right)(\ln x)^2\, dx.$$

12282. *Proposed by George Stoica, Saint John, NB, Canada.* Prove that the multiplicative group generated by $\{\lfloor \sqrt{2}n \rfloor / n : n \in \mathbb{Z}^+\}$ is the group of positive rational numbers.

12283. *Proposed by Yongge Tian, Shanghai Business School, Shanghai, China.* Let A and B be two n-by-n matrices that are orthogonal projections, that is, $A^2 = A = A^*$ and $B^2 = B = B^*$. Let $\sqrt{A+B}$ denote the positive semidefinite square root of $A+B$. Prove

$$\begin{aligned} \operatorname{trace}(A+B) - (2-\sqrt{2})\operatorname{rank}(AB) &\le \operatorname{trace}\sqrt{A+B} \\ &\le (\sqrt{2}-1)\operatorname{trace}(A+B) + (2-\sqrt{2})\operatorname{rank}(A+B), \end{aligned}$$

and show that equality holds simultaneously if and only if $AB = BA$.

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12284. *Proposed by Zachary Franco, Houston, TX.* Let ABC be a triangle with circumcenter O, incenter I, orthocenter H, sides of integer length, and perimeter 2021. Suppose that the perpendicular bisector of OH contains A and I. Find the length of BC.

12285. *Proposed by Atul Dixit, Indian Institute of Technology, Gandhinagar, India.* Prove

$$\sum_{m=1}^{\infty}\int_0^{\infty}\frac{t\cos t}{t^2+m^2u^2}\,dt=\int_0^{\infty}\left(-\frac{\pi}{2u}\cos t+\sum_{m=1}^{\infty}\frac{t\cos t}{t^2+m^2u^2}\right)dt$$

for $u>0$.

SOLUTIONS

Strengthening the Cauchy–Schwarz Inequality

12163 [2020, 179]. *Proposed by Thomas Speckhofer, Attnang-Puchheim, Austria.* Let $\mathbb{R}^n$ have the usual dot product and norm. When $v=(x_1,\dots,x_n)\in\mathbb{R}^n$, let $\Sigma v=x_1+\cdots+x_n$. Prove

$$\|v\|^2\|w\|^2\ge(v\cdot w)^2+\frac{1}{n}(\|v\|\,|\Sigma w|-\|w\|\,|\Sigma v|)^2$$

for all $v, w\in\mathbb{R}^n$.

Solution by the Davis Problem Solving Group, Davis, CA. If either $v=0$ or $w=0$ then both sides of the requested inequality are zero, so we may assume $v\neq 0$ and $w\neq 0$.

First assume $\Sigma v\neq 0$ and $\Sigma w\neq 0$. By homogeneity, we may assume $\Sigma v=\Sigma w=1$. We have $(v\cdot w)^2=\|v\|^2\|w\|^2\cos^2\theta$, where θ is the angle between the vectors v and w. Thus we must prove $\|v\|^2\|w\|^2\ge\|v\|^2\|w\|^2\cos^2\theta+(1/n)(\|v\|-\|w\|)^2$, or

$$\|v\|^2\|w\|^2\sin^2\theta\ge\frac{1}{n}(\|v\|-\|w\|)^2. \tag{1}$$

Let S denote the area of the triangle whose vertices are the origin, v, and w. If h is the altitude of the triangle from the origin, then $h\ge 1/\sqrt{n}$, since $1/\sqrt{n}$ is the minimum distance from the origin to a point in the hyperplane $x_1+\cdots+x_n=1$. Thus

$$\|v\|\|w\|\sin\theta=2S=h\|v-w\|\ge\frac{1}{\sqrt{n}}\|v-w\|,$$

and squaring yields

$$\|v\|^2\|w\|^2\sin^2\theta\ge\frac{1}{n}\|v-w\|^2\ge\frac{1}{n}(\|v\|-\|w\|)^2,$$

where the final inequality is a consequence of the triangle inequality. This establishes (1). Equality holds if and only if v and w are linearly dependent.

If $\Sigma v=0$ and $\Sigma w=0$, then the inequality reduces to the Cauchy–Schwarz inequality, and once again equality holds if and only if v and w are linearly dependent. Finally, assume that one of Σv or Σw is zero and the other is nonzero. It suffices to consider the case where $\Sigma w=0$ and $\Sigma v\neq 0$, and again we may assume $\sum v=1$. As before, if θ is the angle between v and w then the inequality to be proved reduces to $\|v\|^2\|w\|^2\sin^2\theta\ge(1/n)\|w\|^2$, and since we have assumed $w\neq 0$, this is equivalent to

$$\|v\|\sin\theta\ge\frac{1}{\sqrt{n}}. \tag{2}$$

The left side of (2) is the distance from v, which is in the hyperplane $x_1 + \cdots + x_n = 1$, to a point in the hyperplane $x_1 + \cdots + x_n = 0$. This distance must be at least $1/\sqrt{n}$, the distance between the two parallel hyperplanes, showing that (2) is true. In this case, equality is attained if and only if $\lambda v = \mu w + (1/n, \ldots, 1/n)$ for some real λ and μ; that is, if and only if $(1, \ldots, 1)$ is in the span of v and w.

Also solved by R. A. Agnew, K. F. Andersen (Canada), J. N. Caro Montoya (Brazil), R. Chapman (UK), L. Giugiuc (Romania), L. Han, E. A. Herman, W. Janous (Austria), K. T. L. Koo (China), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Mansuy (France), M. Omarjee (France), E. Schmeichel, A. Stadler (Switzerland), G. Stoica (Canada), R. Stong, Florida Atlantic University Problem Solving Group, and the proposer.

A Pell-Type Diophantine Equation

12164 [2020, 179]. *Proposed by Nikolai Osipov, Siberian Federal University, Krasnoyarsk, Russia.* Characterize the positive integers d such that $(d^2 + d)x^2 - y^2 = d^2 - 1$ has a solution in positive integers x and y.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. There are solutions exactly when $d + 1$ is a square. Write $d + 1 = gm^2$, where g is squarefree. If $g = 1$, then $d = m^2 - 1$, and $(x, y) = (1, m)$ is a solution. We show that there is no solution (x, y) when $g \geq 2$.

Fix a squarefree g with $g \geq 2$. Let d be minimal such that the specified equation has a solution (x, y) when d has the form $gm^2 - 1$. Note that $d > 1$, since when $d = 1$ the equation is $2x^2 - y^2 = 0$, which famously has no positive integer solutions. With d minimized, we reduce to checking finitely many cases by first showing $x < 2\sqrt{d}$ for the solution with smallest positive x and then showing $d < 14$.

It is convenient to work in the ring $\mathbb{Z}[\sqrt{D}]$, where $D = d(d + 1)$, which is the set of real numbers of the form $a + b\sqrt{D}$, where $a, b \in \mathbb{Z}$ and elements multiply as real numbers. The *norm* of an element $a + b\sqrt{D}$ is defined to be

$$(a + b\sqrt{D})(a - b\sqrt{D}),$$

which equals $a^2 - b^2 D$. With this definition, it is easy to confirm that the norm of a product is the product of the norms of the factors.

A solution (u, v) to an equation of the form $u^2 - kv^2 = c$ corresponds to an element $u + v\sqrt{k}$ in $\mathbb{Z}[\sqrt{k}]$ with norm c. In particular, the Pell equation $u^2 - Dv^2 = 1$ has the solution $(u, v) = (2d + 1, 2)$, which corresponds to the number $2d + 1 + 2\sqrt{D}$ of norm 1. Let α be this number.

Now choose $\beta = y + x\sqrt{D}$ with $x, y > 0$ so that β is the smallest real number in $\mathbb{Z}[\sqrt{D}]$ having norm $1 - d^2$. Thus (x, y) is a solution to $y^2 - Dx^2 = 1 - d^2$ with minimal positive x and y.

Because the norm of α is 1, we have $\alpha^{-1} = 2d + 1 - 2\sqrt{D}$, and hence α^{-1} is in $\mathbb{Z}[\sqrt{D}]$ and has norm 1. For suitable integers x' and y', we have

$$\alpha^{-1}\beta = (2d + 1 - 2\sqrt{D})(y + x\sqrt{D}) = y' + x'\sqrt{D}.$$

By the multiplicativity of the norm, $\alpha^{-1}\beta$ has norm $1 - d^2$. Also $\alpha^{-1}\beta < \beta$, since $\alpha^{-1} < 1$. By the minimality of the positive coefficients in β, at least one of x' and y' is nonpositive. Furthermore, since $\alpha^{-1}\beta$ is a positive real number, x' or y' is positive. We compute

$$\alpha^{-1}\beta(-y' + x'\sqrt{D}) = (y' + x'\sqrt{D})(-y' + x'\sqrt{D}) = d^2 - 1,$$

where the final equality holds because the middle expression is the negative of the norm of $\alpha^{-1}\beta$. Thus

$$\frac{(d^2-1)\alpha}{\beta} = \frac{d^2-1}{\alpha^{-1}\beta} = -y' + x'\sqrt{D}.$$

Since $d^2 - 1$ and $\alpha^{-1}\beta$ are positive, so is $-y' + x'\sqrt{D}$. With the restrictions above on x' and y', we conclude $y' \le 0 < x'$. Since $(y')^2 - (x')^2 D = 1 - d^2$, setting $y' = 0$ would give $(x')^2 = (d^2-1)/(d(d+1)) = (d-1)/d < 1$; hence $y' < 0$.

Since $-y' + x'\sqrt{D}$ has norm $1 - d^2$ with $-y'$ and x' both positive, the minimality of β implies that $(d^2-1)\alpha/\beta$ is at least β, so

$$x\sqrt{d(d+1)} < \beta \le \sqrt{(d^2-1)\alpha} = \sqrt{d^2-1}(\sqrt{d} + \sqrt{d+1}).$$

Therefore,

$$x < \sqrt{d-1} + \sqrt{(d^2-1)/d} < 2\sqrt{d}.$$

Next we bound d. Write the original equation as

$$y^2 = (d+1)((x^2-1)d+1) = gm^2((x^2-1)d+1).$$

It follows that $(x^2-1)d + 1 = gn^2$ for some positive integer n. Since $g \ge 2$, we have $x \neq 1$ and

$$n^2 - (x^2-1)m^2 = \frac{(x^2-1)d + 1 - (x^2-1)(d+1)}{g} = \frac{2-x^2}{g}.$$

In the ring $\mathbb{Z}[\sqrt{x^2-1}]$, consider γ and δ given by

$$\gamma = x + \sqrt{x^2-1} \text{ and } \delta = n + m\sqrt{x^2-1},$$

with norms 1 and $(2-x^2)/g$, respectively. Let n_1 and m_1 be positive integers such that $n_1 + m_1\sqrt{x^2-1}$ has norm $(2-x^2)/g$ in this ring. Setting $(x, y) = (x, gm_1n_1)$ yields a solution to the original equation with $d + 1 = gm_1^2$. The minimality of d for this g implies that δ is minimal among all elements of $\mathbb{Z}[\sqrt{x^2-1}]$ having positive coefficients and norm $(2-x^2)/g$.

The same argument given earlier for $(d^2-1)\alpha/\beta$ shows that $(x^2-2)\gamma/(g\delta)$ has norm $(2-x^2)/g$ and can be written as $n' + m'\sqrt{x^2-1}$ with n' and m' being positive integers. The minimality of δ now implies

$$gm^2(x^2-1) < g\delta^2 \le (x^2-2)\gamma < 2x(x^2-1),$$

and hence

$$d + 1 = gm^2 < 2x < 4\sqrt{d}.$$

Treating this as an inequality in $\sqrt{d}$ and applying the quadratic formula yields

$$d < (2+\sqrt{3})^2 < 14.$$

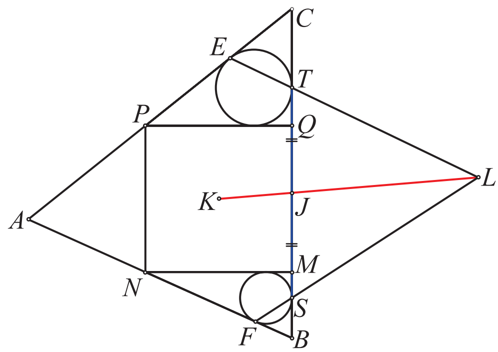
Since these minimal solutions require $d < 14$ and $x < 2\sqrt{d}$, there remain only finitely many cases to consider. The casework is streamlined by reducing the equation modulo $d - 1$, requiring $2x^2 \equiv y^2 \pmod{d-1}$. If $d - 1$ has as a factor any prime congruent to ± 3 modulo 8 (such as 3, 5, or 11), then x must also be a multiple of this factor, since 2 is not a square modulo any such number. Since $x < 2\sqrt{d}$, these possibilities are easily eliminated. For example, when $d - 1 = 9$, we need only consider 3 and 6 for x in the original equation, and neither $110 \cdot 9 - y^2 = 99$ nor $110 \cdot 36 - y^2 = 99$ has an integer solution. If $d - 1$ has 4 as a factor, then x must be even because 2 is not a square mod 4, and these possibilities can similarly be checked quickly.

For $d < 14$, in each case $d - 1$ is a multiple of some element of $\{3, 4, 5, 11\}$ except for the remaining cases where d is 2, 3, or 8. The last two of these have $g = 1$, so only $d = 2$ needs to be analyzed. In this case, the equation reads $6x^2 - y^2 = 3$, hence $y = 3z$ for some integer z, and $2x^2 - 3z^2 = 1$. Taking the equation modulo 3 shows that this also fails.

Also solved by the proposer.

An Unexpected Bisection

12165 [2020, 180]. *Proposed by Tran Quang Hung and Nguyen Minh Ha, Hanoi, Vietnam.* Let $MNPQ$ be a square with center K inscribed in triangle ABC with N and P lying on sides AB and AC, respectively, while M and Q lie on side BC. Let the incircle of $\triangle BMN$ touch side BM at S and side BN at F, and let the incircle of $\triangle CQP$ touch side CQ at T and side CP at E. Let L be the point of intersection of lines FS and ET. Prove that KL bisects the segment ST.



Solution I by Haoran Chen, Suzhou, China. Let G and H be the feet of the altitudes to BC from L and K, respectively. Let J be the intersection of KL and ST, and let I be the midpoint of ST. Our goal is to show that I and J are the same point.

Let s be the side length of the square $MNPQ$. Let $\alpha = \angle CTE = \angle STL$ and $\beta = \angle BSF = \angle TSL$. We establish formulas for $\cot\alpha$ and $\cot\beta$. To derive these formulas, let D be the foot of the perpendicular from E to CT, so that $\cot\alpha = DT/DE$. Let $x = QT$, $y = CT = CE$, and $z = PE$. This gives $x + z = PQ = s$. Since $\triangle CDE \sim \triangle CQP$, we have $DE/CE = QP/CP$, so

$$DE = CE \cdot QP/CP = y(x + z)/(y + z).$$

Similarly, $CD = y(x + y)/(y + z)$, so

$$DT = y - CD = y(z - x)/(y + z).$$

We conclude

$$\cot\alpha = \frac{DT}{DE} = \frac{y(z - x)/(y + z)}{y(x + z)/(y + z)} = 1 - \frac{2x}{x + z} = 1 - \frac{2x}{s}.$$

Similarly, if we let $u = MS$, then $\cot\beta = 1 - 2u/s$.

If $x = u$, then $\cot\alpha = \cot\beta$, so $\alpha = \beta$, and the desired conclusion follows by symmetry. Now assume without loss of generality that $x > u$, so $\cot\alpha = 1 - 2x/s < 1 - 2u/s = \cot\beta$. Letting $t = GL$, we have $GT = t\cot\alpha < t\cot\beta = GS$, so $GT < ST/2$. Also, $HT = x + s/2 > u + s/2 = HS$, so $HT > ST/2$. Thus G lies between H and T, and I lies between G and H. Clearly J is also between G and H, so to show that $I = J$ it suffices to prove $IG/IH = JG/JH$.

By similar triangles, we have

$$\frac{JG}{JH} = \frac{LG}{KH} = \frac{t}{s/2} = \frac{2t}{s}.$$

Also,

$$IG = \frac{ST}{2} - GT = \frac{GS+GT}{2} - GT = \frac{GS-GT}{2} = \frac{t(\cot\beta - \cot\alpha)}{2}$$

and

$$IH = IS - HS = \frac{x+s+u}{2} - \left(\frac{s}{2} + u\right) = \frac{x-u}{2}.$$

Therefore

$$\frac{IG}{IH} = \frac{t(\cot\beta - \cot\alpha)/2}{(x-u)/2} = \frac{t[(1-2u/s)-(1-2x/s)]}{x-u} = \frac{2t}{s},$$

which completes the proof.

Solution II by L. Richie King, Davidson, NC. Let the bisector of $MNPQ$ parallel to MQ and NP intersect line ET at U and line FS at V. We show that K is the midpoint of UV. The result follows from this, since LK is the median of $\triangle LUV$ from L, and so it bisects every section parallel to UV, including ST.

Let O be the center of the incircle of $\triangle PQC$. Note that QO bisects $\angle PQC$. Let P', E', and T' be the reflections of P, E, and T in QO. The line PQ is tangent to the incircle at T', and the lines $P'E'$ and $P'T$ are also tangent to the incircle.

We use some known results about polars. The *polar* of a point Z with respect to the incircle of $\triangle PQC$ is the line perpendicular to ZO that passes through the image of Z under inversion in the incircle. A fundamental fact about polars is that if the polar of Z passes through a point Y then the polar of Y passes through Z.

Since E is fixed under inversion in the incircle, the polar of E is PC, the line tangent to the incircle at E. Similarly, the polar of T' is PQ. Since the polars of both E and T' pass through P, the polar of P must pass through both E and T', so it must be the line ET'. Similarly, the polar of P' is $E'T$. Let X be the point of intersection of ET' and $E'T$. Then X lies on the polars of both P and P', so the polar of X is the line PP', which is perpendicular to QO.

The point X is one of the vertices of the *diagonal triangle* of the concyclic quadrilateral $ETT'E'$. The other two vertices are the point Y where the lines ET and $E'T'$ intersect, which lies on QO, and the point Z at infinity on the lines EE' and TT'. We now use one more known fact about polars: the polar of each vertex of the diagonal triangle of a concyclic quadrilateral is the line through the other two vertices (see H. S. M. Coxeter, (1998), *Non-Euclidean Geometry*, 6th ed., Washington, DC: Mathematical Association of America, p. 57). In particular, PP', which is the polar of X, passes through Y, and therefore Y is the intersection point of PP' and QO. We conclude that PQY is an isosceles right triangle, with right angle at Y. Therefore Y lies on the bisector of $MNPQ$ parallel to MQ and NP, so $U = Y$ and UK has length equal to the side length of the square. Similar reasoning shows that VK has the same length, which establishes our claim that K is the midpoint of UV.

Editorial comment. Marty Getz and Dixon Jones generalized the problem to a rectangle inscribed in a triangle, as did the Davis Problem Solving Group. Giuseppe Fera and Giorgio Tescaro generalized to an inscribed parallelogram.

Also solved by W. Burleson & C. Helms & L. Ide & A. Liendo & M. Thomas, W. Chang, P. De (India), G. Fera & G. Tescaro (Italy), M. Getz & D. Jones, O. Geupel (Germany), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), N. Hodges (UK), W. Hu (China), E.-Y. Jang (Korea), W. Janous (Austria), K. T. L. Koo (China), O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, C. R. Pranesachar (India), V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, T. Zvonaru (Romania), Davis Problem Solving Group, and the proposers.

Asymptotics of a Recursive Sequence

12166 [2020, 180]. *Proposed by Erik Vigren, Swedish Institute of Space Physics, Uppsala, Sweden.* Let $a_0 = 0$, and define a_k recursively by $a_k = e^{a_{k-1}-1}$ for $k \geq 1$.
(a) Prove $k/(k+2) < a_k < k/(k+1)$ for all $k \geq 1$.
(b) Is there a number c such that $a_k < (k+c)/(k+c+2)$ for all k?

Solution by Jean-Pierre Grivaux, Paris, France. We prove part **(a)** by induction on k. The base case $k = 1$ follows from $2 < e < 3$. For the induction step, the inductive hypothesis implies that

$$e^{-2/(k+2)} < a_{k+1} < e^{-1/(k+1)}.$$

Thus it suffices to show that

$$e^{-2/(k+2)} > \frac{k+1}{k+3} \quad \text{and} \quad e^{-1/(k+1)} < \frac{k+1}{k+2}.$$

The first of these is a rearrangement of the inequality

$$e^{2x} = 1 + 2x + 2x^2 + \cdots + \frac{2^n x^n}{n!} + \cdots < 1 + 2x + 2x^2 + 2x^3 + \cdots = \frac{1+x}{1-x}$$

for $0 < x < 1$ applied at $x = 1/(k+2)$, and the second is a rearrangement of the inequality $e^x > 1 + x$ for $x \neq 0$ applied at $x = 1/(k+1)$.

The answer to part **(b)** is no. To establish this, we first study the asymptotics of a_k more carefully. Let $v_k = a_k - 1 = e^{v_{k-1}} - 1$. From part **(a)** we conclude that v_k tends to 0 as $k \to \infty$. Thus we compute

$$\frac{1/v_{k+1} - 1/v_k}{(k+1) - k} = \frac{1}{v_{k+1}} - \frac{1}{v_k} = \frac{1 + v_k - e^{v_k}}{v_k(e^{v_k} - 1)} \sim \frac{-v_k^2/2}{v_k^2} = -\frac{1}{2}.$$

Hence by the Stolz–Cesàro theorem we have $\lim_{k\to\infty}(1/v_k)/k = -1/2$, or equivalently $v_k \sim -2/k$.

Now we compute

$$\frac{1}{v_{k+1}} - \frac{1}{v_k} + \frac{1}{2} = \frac{e^{v_k}(v_k - 2) + v_k + 2}{2v_k(e^{v_k} - 1)} \sim \frac{v_k^3/6}{2v_k^2} = \frac{v_k}{12} \sim -\frac{1}{6k}.$$

Therefore

$$\frac{\left(\frac{1}{v_{k+1}} + \frac{k+1}{2}\right) - \left(\frac{1}{v_k} + \frac{k}{2}\right)}{H_k - H_{k-1}} = k\left(\frac{1}{v_{k+1}} - \frac{1}{v_k} + \frac{1}{2}\right) \sim -\frac{1}{6},$$

where H_k is the kth harmonic number. Applying the Stolz–Cesàro theorem again, we obtain

$$\frac{1}{v_k} + \frac{k}{2} \sim -\frac{H_{k-1}}{6} \sim -\frac{\ln k}{6}.$$

Thus

$$a_k - 1 + \frac{2}{k} = v_k + \frac{2}{k} = v_k \cdot \frac{2}{k} \cdot \left(\frac{1}{v_k} + \frac{k}{2}\right) \sim \left(-\frac{2}{k}\right)\left(\frac{2}{k}\right)\left(-\frac{\ln k}{6}\right) = \frac{2\ln k}{3k^2},$$

and therefore

$$\lim_{k\to\infty} k^2\left(a_k - 1 + \frac{2}{k}\right) = \infty.$$

However, if a bound of the type given in part **(b)** held, we would have

$$k^2 \left(a_k - 1 + \frac{2}{k} \right) < \frac{2(c+2)k}{k+c+2},$$

which is bounded above. Thus no such bound can hold.

Also solved by K. F. Andersen (Canada), R. Chapman (UK), L. Han (USA) & X. Tang (China), N. Hodges (UK), M. Kaplan, O. Kouba (Syria), G. Lavau (France), J. H. Lindsey II, O. P. Lossers (Netherlands), A. Stadler (Switzerland), and A. Stenger. Part **(a)** only solved by P. Bracken, D. Fleischman, O. Geupel (Germany), W. Janous (Austria), A. Natian, and the proposer.

Bounds on a Function of the Angles and Sides of a Triangle

12168 [2020, 274]. *Proposed by Martin Lukarevski, University "Goce Delcev," Stip, North Macedonia.* Let a, b, and c be the side lengths of a triangle ABC with circumradius R and inradius r. Prove

$$\frac{2}{R} \le \frac{\sec(A/2)}{a} + \frac{\sec(B/2)}{b} + \frac{\sec(C/2)}{c} \le \frac{1}{r}.$$

Solution by S. S. Kumar, Portola High School, Irvine, California. Let s and K denote the semiperimeter and area of ABC, respectively. We first prove the second inequality. Note that by the half-angle formula and the law of cosines,

$$\sec(A/2) = \sqrt{\frac{2}{1+\cos A}} = \sqrt{\frac{4bc}{(b+c)^2 - a^2}} = \sqrt{\frac{bc}{s(s-a)}}.$$

By the AM-GM inequality, we have $2\sqrt{bc} \le b+c$ and $2\sqrt{(s-b)(s-c)} \le a$. Applying Heron's formula and the relation $K = rs$, it follows that

$$\frac{\sec(A/2)}{a} = \frac{1}{a}\sqrt{\frac{bc(s-b)(s-c)}{s(s-a)(s-b)(s-c)}} \le \frac{b+c}{4K} = \frac{b+c}{4rs}.$$

Combining this with similar formulas for the other angles, we have

$$\frac{\sec(A/2)}{a} + \frac{\sec(B/2)}{b} + \frac{\sec(C/2)}{c} \le \frac{b+c}{4rs} + \frac{c+a}{4rs} + \frac{a+b}{4rs} = \frac{4s}{4rs} = \frac{1}{r}.$$

To prove the first inequality, we note that by the law of sines, $a = 2R\sin A$, and similarly for the other sides, so the inequality is equivalent to

$$\frac{\sec(A/2)}{\sin A} + \frac{\sec(B/2)}{\sin B} + \frac{\sec(C/2)}{\sin C} \ge 4.$$

Define $f(x) = \sec(x/2)/\sin x$. It is tedious but straightforward to compute that on $(0, \pi)$,

$$f''(x) = \frac{1}{4}\sec(x/2)\csc(x)\left(4\csc^2(x) + (2\cot(x) - \tan(x/2))^2 + \sec^2(x/2)\right) > 0.$$

Hence, by Jensen's inequality, we obtain

$$\frac{\sec(A/2)}{\sin A} + \frac{\sec(B/2)}{\sin B} + \frac{\sec(C/2)}{\sin C} \ge 3f\left(\frac{A+B+C}{3}\right) = 4,$$

as desired.

Editorial comment. As noted by Omran Kouba, one can also deduce the first inequality by applying Jensen's inequality to the function $g(x) = -\log(\cos^2(x)\sin(x))$ on the interval

$(0, \pi/2)$, which is more easily computed to be convex than is $f(x)$. In fact this yields the stronger inequality

$$\frac{2}{R} \leq 3\sqrt[3]{\frac{\sec(A/2)}{a} \cdot \frac{\sec(B/2)}{b} \cdot \frac{\sec(C/2)}{c}},$$

which along with the AM-GM inequality implies the first inequality.

Also solved by A. Alt, M. Bataille (France), H. Chen, C. Chiser (Romania), G. Fera (Italy), S. Gayen (India), O. Geupel (Germany), N. Hodges (UK), M. Kaplan & M. Goldenberg, P. Khalili, K. T. L. Koo (China), O. Kouba (Syria), K.-W. Lau (China), V. Schindler (Germany), A. Stadler (Switzerland), N. Stanciu & M. Drăgan (Romania), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, L. Wimmer, M. R. Yegan (Iran), T. Zvonaru (Romania), and the proposer.

Estimating the Logarithmic Derivative of a Chebyshev Polynomial

12171 [2020, 275]. *Proposed by Ulrich Abel and Vitaliy Kushnirevych, Technische Hochschule Mittelhessen, Giessen, Germany.* Let T_n be the nth Chebyshev polynomial, defined by $T_n(\cos\theta) = \cos(n\theta)$. Prove

$$\frac{T_n'(1/z)}{T_n(1/z)} = \frac{nz}{\sqrt{1-z^2}} + O\left(z^{2n+1}\right)$$

as $z \to 0$.

Solution by Kenneth F. Andersen, Edmonton, Canada. We prove the equivalent statement, with $x = 1/z$,

$$\frac{T_n'(x)}{T_n(x)} = \frac{n}{\sqrt{x^2-1}} + O\left(\frac{1}{x^{2n+1}}\right) \quad \text{as } x \to \infty.$$

We begin with the fact that for $x \geq 1$, $T_n(x) = (A(x)^n + A(x)^{-n})/2$, where $A(x) = x + \sqrt{x^2-1}$. This can be proved by induction, using the well-known recurrence $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$. Alternatively, if we extend $A(x)$ to $x < 1$ by an appropriate choice of a branch of the square root function in the complex numbers, then with $x = \cos\theta$ for $0 \leq \theta \leq \pi$ we have $A(x) = \cos\theta + i\sin\theta = e^{i\theta}$, and therefore

$$T_n(x) = T_n(\cos\theta) = \cos(n\theta) = \frac{e^{in\theta} + e^{-in\theta}}{2} = \frac{A(x)^n + A(x)^{-n}}{2}.$$

This equation can then be extended to $x \geq 1$ by analytic continuation.

Since $A'(x) = 1 + x/\sqrt{x^2-1} = A(x)/\sqrt{x^2-1}$ for $x > 1$, we have

$$T_n'(x) = \frac{nA(x)^{n-1} - nA(x)^{-n-1}}{2} \cdot A'(x) = \frac{n(A(x)^n - A(x)^{-n})}{2\sqrt{x^2-1}}.$$

Therefore

$$\left|\frac{T_n'(x)}{T_n(x)} - \frac{n}{\sqrt{x^2-1}}\right| = \frac{n}{\sqrt{x^2-1}}\left|\frac{A(x)^n - A(x)^{-n}}{A(x)^n + A(x)^{-n}} - 1\right| = \frac{2n}{(A(x)^{2n}+1)\sqrt{x^2-1}}.$$

The desired conclusion now follows because $A(x) \sim 2x$ and $\sqrt{x^2-1} \sim x$ as $x \to \infty$.

Editorial comment. The problem statement above corrects a typographical error from the original printing.

Also solved by A. Berkane (Algeria), R. Chapman (UK), H. Chen, O. Geupel (Germany), J.-P. Grivaux (France), L. Han (USA) & X. Tang (China), N. Hodges (UK), K. T. L. Koo (China), O. Kouba (Syria), M. Omarjee (France), A. Stadler (Switzerland), R. Tauraso (Italy), D. Terr, E. I. Verriest, T. Wiandt, and the proposer.