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Friction, Avalanches and phase transitions in granular media.

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Abstract

Friction, avalanches and phase transitions in granular media

by

Aline Hubard

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We explore the microscopic interactions and properties of dry granular materials in the plane using experiments and simulations. We use statistics of this microscopic information to better understand and predict the bulk behavior and evolution of granular materials.

In the first part, we study the geometric structure of mechanically stable packings of frictional disks in configuration space. Our experimental setup contains frictional disks confined in a vertical plane and subject to a series of small amplitude vibrations that compact the system by reducing the friction between contacts during each vibration step. We find that the evolution of the packings forms a tree-like structure in the configuration space and that the particles move in the direction of gravity projected onto the null space of the rigidity matrix determined from the contact network of the packing. We suggest that this formalism can also be used to explain the evolution of frictional packings under other forcing conditions.
In the second part, we address the question of universality in granular avalanches. We set up an experiment made from a quasi-two-dimensional rotating drum half-filled with a mono-layer of stainless-steel spheres. We measure the size of the avalanches created by the increased gravitational stress on the pile as we rotate. We find that the size and duration of the avalanches follow a power law and that the avalanche shapes are self-similar and nearly parabolic. This indicates that the avalanches belong to a universality class, which has been predicted by mean field theory.

Finally we look at granular phase transitions from gas to crystal and ask which local properties best describe the granular phase transition in mono-disperse disks from gas to crystal. We perform two-dimensional numerical simulations with different boundary conditions and driving mechanisms, in which we measure the local area fraction, bond order, and anisotropy.
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Introduction

A Granular material is a collection of particles that interact only through direct collisions. The diameters of granular particles are greater than one micron so thermal agitation is negligible and Brownian motion has no effect on them [9]. The bulk behavior of granular materials is often unexpected, even for apparently simple systems like a collection of marbles or grains of sand. Determining the behavior of the macroscopic sum of the individual parts is not a simple addition of the individual properties of each grain. Furthermore, they are generally out of equilibrium systems and the ordinary assumptions and results of statistical mechanics and thermodynamics often do not apply. Unlike energy conserving systems, sliding friction and inelastic collisions drive them out of equilibrium, and they can not explore all the phase space to find the extreme of the thermodynamic potential of the state [1]. In chapter 4, we will explore the idea that even though they are out of equilibrium they may sometimes be part of a class of non-equilibrium steady-states (NESS), which can mimic equilibrium behavior even during phase transitions.

Granular materials are everywhere, not only on beaches and deserts, but also in grain silos and the rings of Saturn. For example, most pharmaceutical products are derived from powders, in the chemical industry 40% of the added value is linked
to the particles technology, and over 50% of products sold are either granular or involved granular materials in their production [11]. Most of our food is made of grains or comes from grains, which we store in silos. In the US, between 1964 and 2011 there were more than 900 grain entrapment cases documented in the National Agricultural Confined Spaced Incident Database. Just in 2010 there were 26 deaths in silo accidents [4]. After water, the second most manipulated class of materials in industry is granular materials, and the processes to manipulate them are antiquated, still dangerous, and far from optimal. [9].

In addition to the usefulness of granular materials, they are also intriguing from a physics standpoint. They sometimes behave like a gas, flow like a fluid, or resist and break like a solid. Usually, when they are static they are in a metastable state. The standard tools of statistical mechanics and condensed matter physics that treat the grains as atoms or molecules are often not predictive, since their dissipative interactions lead to different phenomenology [1].

There has been a lot of work done on rheology and continuum theories of granular materials covering an array of interesting problems. However, in my PhD work, I focused on grain to grain interactions and what the statistics at the grain level can tell us about the system as a whole.

This thesis is divided in three different projects, all aimed at getting a better understanding of the thermodynamics and statistical mechanics theories of granular materials. In chapter 2, we look at the structural and mechanic properties of small
packings of frictional disks being compacted using a unique technique to temporarily lower friction. We explore the motion and structure of the system as it explores configuration space. In chapter 3, we look at the statistics of the avalanches inside a rotating drum, and ask if it belongs to a universal class described by mean field theory by measuring the distribution of avalanche sizes, times, and shapes using a novel technique to increase our sensitivity to small events without giving up dynamic range for large events in quasi-static evolution. Finally in chapter 4 we explore the local properties of granular materials undergoing phase transitions in dynamic situations.
Chapter 1

Methods

Granular materials are out of equilibrium systems, so we cannot use traditional equilibrium statistical mechanics and their equations to directly predict and describe the systems. To understand the macroscopic behavior of the material we look at the microscopic interactions of the grains under different conditions. In parts of this study, we use molecular dynamic simulations and experiments to access the grain-grain interactions.

1.1 Molecular dynamics Simulations

Molecular dynamics simulations [13] for granular materials consist of solving Newton’s equations of motion for each particle in the system. Before using molecular dynamics the physical parameters of the system, such as number of particles, volume, energy, mass, shape, etc must be defined. There are also many simulation parameters that must be determined as well, like the time-step, integration scheme, and error tolerances. To start a simulation, the initial state of the system, the position and velocity of each particle, must be given. The general scheme of a molecular dynamics simulation follows:
Figure 1.1: Sketch of particles \( j \) and \( i \) at contact, \( \delta \) is the overlap between particles.

1. Set parameters and initial conditions.

2. Calculation of the forces on every particle.

3. Integration of Newton’s equation of motion to find the new position and velocities of the particles at each time step.

4. Repeat from step 2.

1.1.1 Forces

Dry granular materials do not have attractive forces. The force interaction \( \mathbf{F}_{ij} \) between particle \( i \) at position \( \mathbf{r}_i \) and particle \( j \) at position \( \mathbf{r}_j \) is purely repulsive and only appears when the two particles are in contact (see Fig. 1.1). We use a simple model for the force interaction between dissipative circularly symmetric particle \( i \) and \( j \):

\[
\mathbf{F}_{ij} = (K \delta^\alpha - b v_{ij}) \mathbf{\hat{r}}_{ij} \Theta(\delta). \tag{1.1}
\]
where \( \mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i \) the vector between \( i \) and \( j \), \( \mathbf{r}_{ij} = \| \mathbf{r}_{ij} \| \) the distance between them and \( \hat{\mathbf{r}}_{ij} = \mathbf{r}_{ij}/\mathbf{r}_{ij} \) the direction. The overlap between the particles is given by \( \delta = \sigma_{ij} - \mathbf{r}_{ij} \), where \( \sigma_{ij} = \frac{\sigma_i + \sigma_j}{2} \) is the average diameter of particles \( i \) and \( j \), \( v_{ij} = \| \mathbf{v}_i - \mathbf{v}_j \| \) the relative velocity between particles \( i \) and \( j \), and \( \Theta \) is the heavy side function,

\[
\Theta(x) = \begin{cases} 
0 & : x \leq 0 \\
1 & : x > 0 
\end{cases}
\]

Use \( \alpha = 1 \) for linear repulsion and \( \alpha = 3/2 \) for Hertzian forces. The constant \( K \) is an elastic coefficient that gives the hardness of the particles and \( b \) is the dissipation coefficient which removes energy during collisions. For \( b = 0 \), the system is elastic and the mechanical energy of the system is conserved. More complicated models including kinetic and static friction are also possible (see [36] for a review).

1.1.2 Numerical integration

We solve the equations of motion with the velocity Verlet algorithm [42], which is a simple second-order algorithm, which provides good numerical stability and discrete time-reversibility.

As with most integration algorithms, the velocity Verlet algorithm starts from the
Taylor expansion of the particle coordinate at time steps \( t + \Delta t \) and \( t - \Delta t \) around \( t \),

\[
r(t + \Delta t) = r(t) + v(t)\Delta t + \frac{F(t)}{2M} \Delta t^2 + \frac{\ddot{r} \Delta t^3}{3!} + O(\Delta t^4). \tag{1.2}
\]

\[
r(t - \Delta t) = r(t) - v(t)\Delta t + \frac{F(t)}{2M} \Delta t^2 - \frac{\ddot{r} \Delta t^3}{3!} + O(\Delta t^4). \tag{1.3}
\]

from Eq. 1.2 and Eq. 1.3

\[
r(t + \Delta t) = 2r(t) - r(t - \Delta t) + \frac{F(t)}{M} \Delta t^2 + O(\Delta t^4), \tag{1.4}
\]

which is equivalent to

\[
r(t + \Delta t) = r(t) + v(t)\Delta t + \frac{F(t)}{2M} \Delta t^2. \tag{1.5a}
\]

\[
v(t + \Delta t) = v(t) + \frac{F(t + \Delta t) + F(t)}{2M} \Delta t. \tag{1.5b}
\]

In this algorithm, the new velocities are calculated after the new positions.

Molecular dynamics simulations give the positions, velocities, and forces at each time step. From this information, important physical quantities such as the density and energy can be calculated. More importantly, this allows us to look carefully at the dynamics of the system under different boundary conditions, as well as how the material responds to different energy input conditions.

The simulations will only contain the physics from the information in the model as
given in the force interaction equation Eq. 1.1. One of the hardest things to realisticly simulate in a simple molecular dynamic simulation is how the material parameters affect the behavior of the grains as a group. For example, under which situations the grains will flow as a liquid or behave as a solid. Real physical experiments are the only way to make sure there is no physics missing from the equations of motion.

1.2 Particle Tracking

In this work, the experimental data was recorded as pictures using digital cameras. Each digital image consists of an $L_x \times L_y$ matrix, where each element of the matrix corresponds to one pixel. Typical images, including the ones used for the experiments described in this document have 8 bits per pixel. Therefore, each pixel can store 256 different light intensities. In the matrices, this means the values of each matrix element are integer numbers that go from 0 when the pixel is completely dark to 255 when the image at the pixel is overexposed. From the images we find the positions of each particle at each frame. If the frame rate is high enough, then it is possible to identify the trajectory of individual particles and recover the velocity of each particle.

1.2.1 Find the center of the particles

Before finding the particle’s centers, it is necessary to determine which pixels of the image contain the particles and are therefore changing over time, and which pixels are
Figure 1.2: Back light sketch. The object is imaged by laying an array of green LEDs and diffusion paper on one side and the camera on the other. The object appears darker in image than the background.

part of the static background. For non-uniform lighting, the particles images appear with different intensities, depending on where they are located. We use a background image which contains only the lighting information and is used to normalize the image between 0 and 1. Ideally in a normalized image, particles imaged in back lighting have a value of one and a background value of zero. In our experiments we used a light source behind the particles to light the images, as in Fig. 1.2. There are two main approaches to identify the background:

- Taking a picture of the experimental device without the particles. This approach
works better when the particles can be removed without changing anything else. Most of the time, this is not possible, as it is necessary to unmount the whole experiment in order to remove the particles.

- Using a series of images to identify which pixels do not change over time and belong to the background. As the image is back-lit, the particles block the light and are darker in the image than the pixels in the background. We create the background by taking the maximum value over all the $N_f$ different image frames at each pixel. $Bg_{i,j} = \max_k (I_{i,j,[k=1toN_f]})$ where $Bg$ is the background matrix and $I$ an array of $N_f$ images.

Ideally any pixel $Bg_{i,j}$ of the background image has an equal or higher value than the corresponding $I_{ijk}$ pixel of the $kth$ image. In the matrix $Bg - I$ the pixels from the background of the image $I$ will be zero, and the pixels from the particles will have a positive value. To normalize the image between zero and one, we find $Bgg$, the maximum value of the background minus the image for each pixel $Bgg_{i,j} = Bg_{i,j} - \min_k (I_{i,j,[k=1toN_f]})$. This way, the normalized image $I_n$ that we used to find the particles is given by:

$$I_n = \frac{Bg - I}{Bgg}.$$  

(1.6)

Once the image is normalized, we can assume that the ideal version of an image $I_c$ of
Figure 1.3: Ideal particle $I_p$. Left) Plot of function $I_p$ showing how intensity falls from 1 at the center to 0 outside the particle. The value of $\omega$ determines the sharpness boundary. Right) Image of an ideal particle with a diameter of 10 pixels and $\omega = 0.8$. 

$N$ particles of diameter $D$ has the form:

$$I_c(r) = \sum_{n=1}^{N} I_p(r - r_n(t); D, \ldots). \quad (1.7)$$

Where $I_p(x; D, \ldots)$ is a function that describes the shape of an ideal particle with radius $R = D/2$ centered in $r$. In this work all the particles are circular. Thus the $I_p$ function we used describes a circle that has a value of one at the center, zero outside and $1/2$ in the boundary (Fig. 1.3).

$$I_p(r; D, \omega) = \frac{1}{2} \left[ 1 - \tanh \left( \frac{r - D/2}{\omega} \right) \right]. \quad (1.8)$$

To find the center of the particles, we compare the normalized image $I_n$ with the ideal
function of a particle $I_p$, and define the function $\chi^2$ [12]

$$\chi^2(r_o; D, w) = \int W(r - r_o) [I_n(r) - I_p(r - r_o; D, \omega)]^2 \, dr,$$  \hspace{1cm} (1.9)

where $W(r-r_o)$ is a weight function around $r_o$, for dense systems, we use $W(r-r_o) = I_p(r; D, \omega)$. This choice of weight function gives more importance to the pixels close to the center of the particle, where $I_p = 1$ and less to the ones in the boundary where $I_p \sim 1/2$. The minimum values of the function $\chi^2$ give the center of the particles $r_o$. Expanding equation Eq. 1.9 we get

$$\chi^2(r_o; D, w) = \int I_p(r - r_o; D, \omega) [I_n(r)^2 - 2I_n(r)I_p(r - r_o; D, \omega) + I_p(r - r_o; D, \omega)^2] \, dr.$$  \hspace{1cm} (1.10)

where $f \otimes g = \int f(x)g(x-x_o)\, dx$ is the convolution of function $f$ with function $g$. Normalizing $\chi^2$ with the ideal image we define a new $\chi^2$ function

$$\chi^2(r_o; D, w) = 1 + \frac{I_n^2 \otimes I_p - 2I_n \otimes I_p^2}{<I_p^3>},$$  \hspace{1cm} (1.11)

Finally we find the center of the particles to pixel accuracy by evaluating $\chi^2$ over all the points of the image, and finding all the points $r_o$ where $\chi^2(r_o)$ is smaller than some cutoff value $c_o$ and the minimum among all neighbor points up to the diameter
of the particles,

\[ \chi^2(r_o) < c_o. \] (1.12)

\[ \chi^2(r_o) < \chi^2(r') \quad \text{where} \quad ||r_o - r'|| < D. \] (1.13)

Fig. 1.4 shows an example of the type of images we tracked. The color of the images indicates the value of the pixel or intensity. We show $1/\chi^2$ instead of $\chi^2$ as it is easier to see the maximum in the image. Notice how the brightest point in the center image corresponds to the centers of the particles $(x_o, y_o)$ shown in the right image as white points.

The next step, after finding the positions of every particle in every frame, is to connect
the positions of the particles between frames to find the trajectories of the particles over time.

1.2.2 Connecting the dots

For a high enough frame rate, we can find the particle trajectories by connecting the positions at time $t_n$ with the positions at time $t_{n+1}$, such that the total displacement of the particles is a minimum. For example, in Fig. 1.5 is clear that the particles moved as showed in Fig. 1.5 b) which has a total displacement smaller than the one showed in Fig. 1.5 c).

Connecting the particle positions at time $t_n$ with the positions at time $t_{n+1}$ to describe the particles trajectories is equivalent to matching each point $a_i$ of the set at $n$ with one and only one of the points $b_j$ of the set at $n+1$ as shown in b) and c) of Fig. 1.5. Defining the cost of each connection $a \to b$ as the distance between points $a_i$ and point $b_j$ $D_{a\to b} = \sum_{i \to j} (r_{ai} - r_{bj})^2$. For sets of $N$ points there are $N!$ ways to connect the points, we are looking for the optimal assignment that is the one with the minimal $D_{a\to b}$. Each matching or assignment has $N$ distances to calculate, finding the distances of all possible assignments is on the order of $N \times N!$ operations. For systems with more than $N = 10$ particles, this starts to be very time consuming and soon becomes impossible. There are several algorithms to find the optimal assignment without looking over all the possible assignments.

The optimal assignment problem can be reduced, first by finding the trivial bonds that
Figure 1.5: Example of particle connection. Particles moved from $a_1$ to $b_2$, $a_2$ to $b_3$ and $a_3$ to $b_1$, this matching gives the minimal total displacement $d_1 + d_2 + d_3$.

a) Particles at time $t_n$ and $t_{n+1}$. b) The connection $[a_1 a_2 a_3] \rightarrow [b_2 b_3 b_1]$ with the minimal total displacement $d_1 + d_2 + d_3$ shows the trajectories of the particles. c) The connection $[a_1 a_2 a_3] \rightarrow [b_1 b_2 b_3]$ with total displacement $d_1 + d_2 + d_3 > d_1 + d_2 + d_3$ is not the trajectories of the particles. The blue and green dots to the right of b) and c) are a visual representation of the connections. The green dots (left) represents particles from frame a and the blue dots (right) frame b. The vertical location represents particle number 1–3 from top to bottom.
are given by particles that displaced less than $D/2$, and then doing the combinatorics of the remaining points in the set.

To find the trivial bonds we divide each image into a grid of $nx \times ny$ bins of size $D/4$. Each bin $B_{ij}(n)$ will contain the index of the point inside the bin. If the displacement of the particle contained in the bin $B_{ij}(n)$ is smaller than $D/2$, then its corresponding point will be in bin $B_{kl}(n+1)$, with $|k - i| \leq 1$ and $|l - j| \leq 1$. A trivial bond exists when for a point in bin $B_{ij}(n)$, there is only one point in bin $B_{kl}(n+1)$.

For points without trivial bonds, we use a Matlab implementation of the Hungarian algorithm, the time complexity of the algorithm goes as $N^3$.

The Hungarian algorithm [27] uses the $N \times N$ cost matrix $d_{ij} = ||r_{ai} - r_{bj}||$, in the example of Fig. 1.5 the cost matrix is given by

$$
\mathbf{d} = \begin{bmatrix}
    d_{11} & d_{12} & d_{13} \\
    d_{21} & d_{22} & d_{23} \\
    d_{31} & d_{32} & d_{33}
\end{bmatrix}.
$$

The steps of the Hungarian algorithm, to solve the assignment problem using the cost matrix, are the following:

1. Find the smallest element of each row and subtract it from the row. This will
lead to have at least one zero per row.

\[
\mathbf{d}' = \begin{bmatrix}
  d_{11}' & 0 & d_{13}' \\
  d_{21}' & d_{22}' & 0 \\
  0 & d_{32}' & d_{33}'.
\end{bmatrix}
\] (1.14)

If there is one zero per column the algorithm is done and the optimal assignment is given by the connection of the matrix elements that are zero. In the example Eq. 1.14 the connection matrix is

\[
\mathbf{C}_{a \rightarrow b} = \begin{pmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  1 & 0 & 0
\end{pmatrix}
\]

. When the matrix \(\mathbf{d}'\) has more than one zero in a column and no zeros in another column, then no assignment can be done at this step. For example, for a \(4 \times 4\) matrix after step one

\[
\mathbf{d}' = \begin{bmatrix}
  d_{11}' & 0 & d_{13}' & d_{14}' \\
  d_{21}' & d_{22}' & 0 & d_{24}' \\
  0 & d_{32}' & d_{33}' & d_{34}' \\
  d_{41}' & 0 & d_{34}' & d_{44}'.
\end{bmatrix}
\] (1.15)

2. Find the smallest element of each column and subtract from the column, this
will lead to having at least one zero per column. For the example in Eq. 1.15 if 
\[d'_{24} < d'_{44},\] then \(\bar{\mathbf{v}}''\) is
\[
\bar{\mathbf{v}}'' = \begin{bmatrix}
  d'_{11} & 0 & d'_{13} & d'_{14} \\
  d'_{21} & d'_{22} & 0 & 0 \\
  0 & d'_{32} & d''_{33} & d'''_{34} \\
  d_{41} & 0 & d_{34} & d'''_{44}
\end{bmatrix} \tag{1.16}
\]

3. Assign and cross out rows and columns with zeros to reduce the matrix. All 
zeros in the matrix should be crossed using the least amount of lines. If the 
number of lines equals the number of rows, determine an assignment by choosing 
\(N\) zeros, such that there is one for each column and each row. In Eq. 1.16 after 
crossing the zeros we get
\[
\begin{bmatrix}
  d'_{11} & 0 & d'_{13} & d''_{14} \\
  d'_{21} & d'_{22} & 0 & 0 \\
  0 & d_{32} & d''_{33} & d'''_{34} \\
  d_{41} & 0 & d_{34} & d'''_{44}
\end{bmatrix}
\]

Then there are 3 lines and the matrix has 4 rows, so the assignment can’t be 
done yet.

4. Add the smallest uncovered element to the covered elements. For elements 
crossed by both a horizontal and vertical line add it twice. In the example of
Eq. 1.16, if \(d_{11}'\) is the smallest of all the uncovered elements. The new matrix looks like

\[
\mathbf{d} = \begin{bmatrix}
  d_{11}' & 2d_{11}' & d_{13}' & d_{14}' \\
  d_{21}' + d_{11}' & d_{22}' + 2d_{11}' & d_{11}' & d_{11}' \\
  d_{11}' & d_{32}' + 2d_{11}' & d_{33}' + d_{11}' & d_{34}' + d_{11}' \\
  d_{41}' & d_{11}' & d_{34}' & d_{44}'
\end{bmatrix}
\]

(1.17)

5. Repeat the Hungarian algorithm until reaching a solution of the assignment problem.
Chapter 2

Frictional Families

2.1 Introduction

One of the great successes of statistical mechanics is the ability to calculate the average physical properties of macroscopic systems in thermal equilibrium, e.g. atomic and molecular liquids. The microstates and probabilities with which they occur are required for determining the average properties. For example, in the micro-canonical ensemble, each of the microstates at fixed energy is equally probable. A microstate is a point in the full phase-space of the system. The phase-space is $2dN$ dimensional, where $d$ is the physical dimension of the system and $N$ is the number of particles in the system. The factor of 2 comes from the positions and velocities of each particle. Sometimes only the positions of the phase space are important and then the phase space is called the configuration space and it has $dN$ dimensions. The configuration space is often more important than the full phase-space in granular systems because all of the particles velocities are often zero. Here, we are interested in determining the structural and mechanical properties of static packings of frictional granular materials, which are collections of macroscale grains that interact via repulsive contact forces [21]. Our studies focus on frictional packings, which have not received
as much attention as idealized packings of frictionless particles [22]. Granular pack-
ings are also strongly out-of-equilibrium since thermal fluctuations do not give rise to
particle rearrangements. Grain motion is induced instead through applied external
loads [28]. Moreover, the current state of the system may depend strongly on the
protocol used to generate it [41, 6, 32]. In these systems, the definition of the ap-
propriate microstates, their probabilities, and the framework for calculating average
quantities is still under development [31, 17, 44]. In previous numerical simulations
[14] and experimental studies [15], it was shown that static packings of frictionless
disks occur as distinct points in configuration space, whose number grows with the
number of particles N in the system. Each static packing was characterized using an
invariant of the interparticle contact matrix and showed that static packings were not
sampled with equal probabilities when generated using typical isotropic compression
algorithms [14]. In recent simulations, it was shown that static packings of frictional
disks do not occur as points in configuration space, but instead as lines, areas, and
higher dimensional structures (i.e., geometrical families [38]) with a dimension m that
grows with the difference in the number of contacts of the system from the isostatic
value [3] for frictionless disks, \( m = N_{c}^{\text{iso}} - Nc > 0 \) [38]. Thus, at each m, there are
an infinite number of packings of frictional disks, but they can be classified into a
finite set of geometrical families with the same contact network. Granular systems
can undergo rearrangements from one static packing to another in the presence of
an applied load, e.g., slow compaction during vertical tapping [29]. Here, we seek a
particle-level understanding of rearrangements that occur during vibration in terms of the motion along geometrical families and from one geometrical family to another.

2.2 Experimental setup

We performed quasi-two-dimensional experiments of vibrated granular materials. As shown in figure 2.1, the experimental system is composed by $N = 7$ bidisperse stainless steel cylinders. There are 4 small and 3 large cylinders with diameters $\sigma_s = 1.26 \text{ cm}$ and $\sigma_l = 1.57 \text{ cm}$ such that $\sigma_l/\sigma_s = 1.2$, masses $M_s = 2.8 \text{ g}$, $M_l = 4.3 \text{ g}$ and thickness $0.32 \text{ cm}$. The cylinders are confined in a quasi-2-dimensional vertical vibration cell. The vibration cell has width $L_x = 5.33 \text{ cm}$, height $L_y > 5 \text{ cm}$, and thickness $L_z = 0.34 \text{ cm}$. The vibration cell is formed by two plastic side walls, two glass plates, and a movable piston on the bottom. The diameter of each cylinder is a factor of 4 larger than its thickness, which ensures that the cylinder axes are always parallel to each other and perpendicular to the direction of gravity. The friction between the cylinders and the glass plates is much smaller than the cylinder-cylinder friction or the friction between the cylinders and the side metallic walls and thus can usually be neglected. The piston is connected to a computer controlled electromagnetic shaker that vibrates the bottom boundary. The shaker can reach vertical accelerations larger than $20g$, where $g$ is the gravitational acceleration.
Figure 2.1: Sketch of our bi-disperse seven particles experiment. The particles are cylinders and they are confined between two glass plates. The axis of the cylinders is out of the plane of the image and perpendicular to gravity. The cylinders rest over a movable wall attached to a shaker. The shaker produces the movable wall to vibrate up and down at a fixed frequency in the direction of gravity.
2.2.1 Experimental Protocol

For the experimental protocol, we first randomize the initial positions of the cylinders by applying an 18.00g low frequency (50 Hz) vibration for one second and allow the particles to settle to a frictionally stabilized packing. We then periodically apply 1.02g, high frequency (440 Hz) vibration bursts, each with duration 10 ms. The acceleration from these bursts causes the contacts between the cylinders (and between the cylinders and walls) to break, which temporarily removes the frictional forces between particles. The sudden vanishing of the friction makes the configuration of the packing mechanically under-constrained and thus unstable. After each burst, the system settles under gravity, contacts reform, and the system forms a new static packing. Over a series of bursts, the system compacts and the number of contacts grows on average, terminating on a frictionless packing with \( N_c = N_{iso} = 2N = 14 \) contacts for systems with fixed boundaries and gravity. We analyzed 6901 initial packings without rattler particles on the bottom boundary and applied 20 vibration bursts to each. After roughly 15 bursts, these systems converged to 979 distinct frictionless packings. We imaged the system with a digital camera using back-lighting. We recorded images of the system after waiting 0.5 s following each burst. Using the method indicated in 1.2, we tracked the positions of the cylinders with a spatial resolution of \( 6 \times 10^{-6} \sigma_s \) in both the horizontal and vertical directions.
2.2.2 Contact networks

To find the contact network that makes our packings stable, we measured the distance between the centers of every pair of cylinders $r_{ij} = |\vec{r}_i - \vec{r}_j|$ and compared with the minimal distance between the two cylinders $\sigma_{ij} = (\sigma_i + \sigma_j)/2$ such that if

$$r_{ij} - \sigma_{ij} \leq \epsilon$$

(2.1)

then cylinder $i$ is in contact with cylinder $j$. For values of epsilon between $0.5\sigma_s \times 10^{-3} \leq \epsilon \leq 1.5\sigma_s \times 10^{-3}$ the variation of the contact networks was negligible.

2.3 Description of frictional families

In the hard particles limit, where our stainless steel cylinders are, the distance between two particles in contact or a particle in contact with a wall, at positions $\vec{r}_i$ and $\vec{r}_j$ is determined by the particles diameter $\sigma_i, \sigma_j$. For contacts with walls $\sigma_j = 0$.

Then for every contact pair $i - j$

$$||\vec{r}_i - \vec{r}_j|| = \frac{\sigma_i + \sigma_j}{2}.$$ 

(2.2)

The system of equations given by Eq. 2.2, has one equation for every contact, and $d$ variables per particle, where $d$ is the dimension of the system. To guarantee a least one solution of a system of $N_c$ equations and $dN$ variables the number of variables should be equal or larger than the number of equations.
\[ N_c \leq dN. \quad (2.3) \]

### 2.3.1 Mechanical stability

A frictionless packing is mechanically stable when any displacement of the particles creates an overlap. The only way two particles in contact (or a particle and a wall) can move without creating an overlap is if the relative displacement between the pair \( \delta \vec{r}_i - \delta \vec{r}_j \) (particle-particle or particle-wall) is perpendicular to the direction of the contact \( \hat{n}_{ij} \). We can write this condition as

\[ \hat{n}_{ij} \cdot (\delta \vec{r}_i - \delta \vec{r}_j) = 0. \quad (2.4) \]

This must be true for every contact pair, we define the \( dN \times 1 \) vector \( \delta \vec{r} = \delta \vec{r}_1...\delta \vec{r}_N \) as the vector of the d-dimensional displacements of each particle and the \( N_c^{iso} \times dN \) matrix \( N_{(i-j),k} = (\delta_{jk} - \delta_{ik}) \hat{n}_{ij} \) as the matrix that indicates if particle \( k \) is part of contact \( i-j \).

\[ \mathcal{N} \cdot \delta \vec{r} = 0 \quad (2.5) \]

Non trivial solutions of Eq. 2.5 gives an unstable packing in which \( \delta \vec{r} \neq 0 \) does not create any overlaps. Then, in a stable packing the only solution of Eq. 2.5 is \( \delta \vec{r} = 0 \) this implies that the number of columns or constraints, \( N_c^{iso} \) of \( \mathcal{N} \) is larger or equal
than the number of variables or degrees of freedom \( dN \).

\[
N_{c}^{iso} \geq dN \\
(2.6)
\]

Eq. 2.6 indicates that a frictionless stable packing must have at least \( dN \) contacts, but from the geometric constraints, Eq. 2.2 \( N_c \leq dN \)

\[
N_{c}^{iso} = dN. \\
(2.7)
\]

In other words, in a frictionless stable packing the number of degrees of freedom equals the number of constrains.

Frictional contacts create torques and torques produce rotations. With infinite friction, particles in contact can no longer slide past each other, but they can roll as indicated by Eq. 2.8.

\[
(\delta \vec{r}_i - \delta \vec{r}_j) + \frac{1}{2}(\sigma_i \delta \vec{\omega}_i + \sigma_j \delta \vec{\omega}_j) \times \hat{n}_{ij} = 0. \\
(2.8)
\]

which is a \( dN_c \) equation system with \((d + \frac{d(d-1)}{2})N\) variables, \( d \) for the position of each particle and \( \frac{d(d-1)}{2} \) for the orientation. If the packing is stable there is neither rotations or translations. In stable packings the only solution of Eq. 2.8 is the trivial
\[(\delta \vec{r}_i - \delta \vec{r}_j) = 0, (\sigma_i \delta \vec{\omega}_i + \sigma_j \delta \vec{\omega}_j) = 0.\]

There is no nontrivial solution if:

\[dN_c \geq N \left( d + \frac{d(d-1)}{2} \right)\]

\[N_c \geq \frac{N(d+1)}{2}.\]

Eq. 2.2 still holds for frictional packings, then the number of contacts in a frictional packing follows

\[N \frac{(d+1)}{2} \leq N_c \leq dN.\]

For this experiment we have a \(N = 7\) disks packing in \(d = 2\) dimensions, the number of contacts of stable packings will be in the range \(10 < N_c \leq 14\).

### 2.3.2 Family trees

In Fig. 2.3 (a), we show the evolution of the center of mass \(\vec{R}_{\text{cm}} = \sum^N M_i \vec{r}_i / \sum M_i\) and contact number for a set of packings that compacts to a particular frictionless packing with \(N_{c}^{\text{iso}}\) contacts. As explained in section 2.3.1, mechanically stable frictional packings with fixed boundaries and gravity can occur in the range \((3N)/2 \leq N_c \leq 2N\) [39, 30] or \(10 < N_c \leq 14\) for \(N = 7\). The system is initialized in a dilute packing with \(N_c = 11 < N_{c}^{\text{iso}}\). The center of mass evolves smoothly until the contact network changes, in this case the new contact network has \(N_c = 12\). The \(N_c = 12\) contact network smoothly evolves until a new contact forms with \(N_c = 13\) and then stops when it reaches \(N_c = 14\). This packing was reached after 11 vibration
bursts, further vibration bursts did not change the packing anymore. At $N_c = 14$ the packing no longer needs friction to be stable, thus is a frictionless packing that is stable under the applied vibrations. The particles positions no longer change in the next 9 vibration bursts.

The evolution of the center of mass along a geometrical family is continuous as long as the contact network remains the same. Changes in the contact network are signaled by abrupt changes in the direction of motion of the center of mass. Since we apply small amplitude vibrations, the evolution of the center of mass terminates on a frictionless packing with $N_{iso}^c$ contacts. For 95% of the experiment realizations, the packing reaches its final frictionless packing in 15 or less vibration bursts.

When the system is initialized in a different packing, it can evolve to the same frictionless packing in Fig. 2.3 (a) or another one. In Fig. 2.3 (b), we show the geometrical family tree that leads to the frictionless packing shown in Fig. 2.3 (a). As seen in figure 2.3 (b), out of the 26 packings that ended up in this frictional packing we can distinguish only 4 main trajectories arriving to the final packing. Most trajectories first meet on the two purple lines and then follow these lines to get to their frictionless packing. The center of mass approaches that of the frictionless packing only along particular directions in configuration space. In Fig. 2.3 (c), we show the centers of mass of all packings found in the experiments. Note that the most probable packings tend to be the most compact. Fig. 2.2 shows that the frictionless packings we found the most (111 and 98 times) have the lower center of mass.
Figure 2.2: Every point on the scatter plot corresponds to one frictionless packing. The plot shows the number of initial packings that reached the final packing and the vertical position of the center of mass.

The packings that occur during the compaction process are not randomly distributed in configuration space, but instead form connected lines or ‘geometrical family trees’, which verifies previous simulation results [38].

The evolution of the packing in the family tree is determined by the contact network. If we see the experiments going backward in time starting from a frictionless isostatic packing and going toward frictionless packings with less contacts, the branch that the packing will take depends on which contact was broken first. Each broken contact means one less constraint, hence one degree of freedom in which the system can evolve until it breaks or creates a new contact and changes the contact network. The contact network of a packing determines that manifold in configuration space in which the packing can evolve.
We can predict the evolution of a frictional packing to a frictionless packing, as the coefficient of friction decreases the packing will evolve in the direction that does not change the distance between contact pairs. This is the vector created by the tangential vectors of the normal forces.

2.4 Prediction of the families

We now describe a method to quantitatively predict the particle motion that occurs during the compaction process. The center of mass motion is characterized by smooth evolution while the contact network is fixed, punctuated by abrupt changes in direction when the contact network changes. While the contact network is fixed, we model the system by a network of links between contacting particles as seen in figure 2.4. Since $N_{e}^{\text{iso}} - N_{c} = m \geq 0$, the $2N \times 2N$ rigidity matrix $\mathcal{H}$ will possess $m$ floppy eigenmodes $\mathbf{e}^i = \{e^i_x, e^i_y, \ldots, e^i_{xN}, e^i_{yN}\}$ that form the null space and $2N - m$ non-floppy modes [19]. The $2 \times 2$ sub-matrix $\mathcal{H}_{[ij]}$ of $\mathcal{H}$ that contains the contacts between particles $i$ and $j$ is given by Eq. 2.11

$$\mathcal{H}_{[lk]} \equiv \frac{\partial^2 (r_{ij} - \sigma_{ij})^2}{\partial x_l \partial x_k} \bigg|_{r_{ij} = \sigma_{ij}}$$

(2.11)

$$= 2 \sum_{\text{bond}_{ij}} (\delta_{ki} - \delta_{kj})(\delta_{li} - \delta_{lj})\hat{n}_{lk}\hat{n}_{lk}.$$  

The floppy eigenmodes of $\mathcal{H}$ are given by the eigenvectors $\mathbf{e}^i$ that correspond to the zero eigenvalues, then $\mathbf{e}^i$ follows Eq. 2.12
Figure 2.3: Evolution of the center of mass diagram. (a) Branch of a frictional family showing the trajectory of the center of mass and the number of contacts of the packings. Dots show experimental results, shape(color) indicate the number of contacts. squares(pink) 11 contacts, circles (green) 12 contacts, crosses(blue) 13 contacts, star(black) 14 contacts. The black star is the center of mass of the isostatic frictionless packing that defines the family. The insets show the different contact networks depicted by the point in the center of mass diagram. The black line is the result from the simulation. (b) Frictional Family of branch in figure a, the path with the squares is the previous example. Different frictional states end in the same friction-less packing shown in the 14 contacts inset of a). The colors indicate the number of contacts same way as in figure a). Gravity drives the center of mass down in the $Y$ direction. (c) All the frictional families, showing in darker the most probable ones.
Figure 2.4: We model the packings by rigid links between the particles that are in contact. The links can move while keeping the same size. Left) Packing with links between the centers of the particles $\vec{R}_i$, $\vec{R}_j$ at contact. Right) The center of the particles are the connections between the links. Stop marks at the end of the links indicate connection with a wall and that the vertex of that link can move only parallel to the stop mark. Floppy modes allow the vertex to move without changing the length of the links.

\[ \mathcal{H} \cdot e^i = 0 \]  \hspace{1cm} (2.12)

using Eq. 2.11 in Eq. 2.12 we get

\[ \sum_{bond_{ij}} \hat{n}_{kl} \hat{n}_{kl} \cdot e^i_{[kl]} = \vec{0} \]  \hspace{1cm} (2.13)

from here it follows that

\[ \hat{n}_{kl} \cdot (\vec{e}_k^i - \vec{e}_l^i) = 0, \]  \hspace{1cm} (2.14)

which is Eq. 2.4, thus the floppy eigenvectors $e^i$ do not change the distance between the particles at first order and fulfill the geometric constraints.
Each contact network will possess a different rigidity matrix and set of null-space modes. We define the rigidity matrix and its eigenmodes $e_{s,n}^i$ for each experimental packing, where the subscripts $s$ and $n$ indicate the initial condition and vibration burst number, respectively. In previous work, it was shown that vibrations release the frictional contacts [15]. We now hypothesize that the release of friction induces motion in the null space along directions that at first order do not change the length of the link network, and thus frictional packings live in the null space.

To test whether the particle motion is confined to the null space, we calculate the normalized difference between the total particle motion and the motion that occurs in the null space,

$$W_{s,n} = \frac{(\Delta r_{s,n})^2 - \sum_{i=1}^{m} (\Delta r_{s,n} \cdot e_{s,n}^i)^2}{(\Delta r_{s,n})^2},$$  \hspace{1cm} (2.15)

where $\Delta r_{s,n} = r_{s,n+1} - r_{s,n}$ and $r_{s,n}$ is the $2N$-dimensional vector that gives the positions of the particles for initial condition $s$ and burst $n$. $W_{s,n} = 0$ indicates that all particle motion is confined to the null space. In Fig. 2.5 (a), we plot the cumulative distribution $C(W_{s,n})$, which shows that 99% of the particle motions satisfy $W_{s,n} \lesssim 10^{-3}$. Thus, nearly all of the particle motion during compaction is confined to the null space.

To complete our theoretical analysis of vibration-induced compaction, we further hypothesize that particle motion occurs in the direction of the gravitational force projected onto the null space. We now compare the direction of motion in configuration space during compaction to the direction of the gravitational force decomposed
onto to the null space,

$$\mathbf{G}_{s,n} = \sum_{i=1}^{m} \left[ \mathbf{e}_{s,n}^i \cdot (-\mathbf{g}) \right] \mathbf{e}_{s,n}^i,$$  \hspace{1cm} (2.16)

where \( \mathbf{g} = \{0, m_sg, \ldots, 0, m_tg\} \) is a \( 2N \)-dimensional vector with a zero for the \( x \)-component and the gravitational force for the \( y \)-component for each particle. The degree to which the particle displacements track the direction of gravity can be determined by calculating the normalized overlap between \( \mathbf{G}_{s,n} \) and \( \Delta \mathbf{r}_{s,n} \). To measure the deviation from complete overlap, we define

$$\alpha_{s,n} = 1 - \frac{\mathbf{G}_{s,n} \cdot \Delta \mathbf{r}_{s,n}}{|\mathbf{G}_{s,n}| |\Delta \mathbf{r}_{s,n}|}.$$  \hspace{1cm} (2.17)

\( \alpha_{s,n} = 0 \) indicates that a given displacement \( \Delta \mathbf{r}_{s,n} \) is aligned with the direction of gravity in the null space. In Fig. 2.5 (b), we plot the cumulative distribution \( C(\alpha_{s,n}) \), which shows that 99\% of the particle motions satisfy \( \alpha_{s,n} \lesssim 10^{-3} \). Thus, nearly all of the particle displacements are in the direction of gravity in the null space. However, a few displacements out of \( 10^5 \) possess \( \alpha_{s,n} \sim 1 \) (with overlap angles greater than \( 30^\circ \)) indicated by the small peak in the probability distribution \( P(\alpha_{s,n}) \) in the inset to Fig. 2.5 (b). These few instances of large \( \alpha_{s,n} \) are caused by two effects: 1) The vibration bursts were not large enough to break some of the frictional contacts between particles and the bottom wall. In which case, these particles did not move in the direction of gravity in the null space. An example of this friction effect is shown in Fig. 2.6(a). These occurrences of large \( \alpha_{s,n} \) can be removed by increasing
Figure 2.5: (a) Cumulative distribution $C(W_{s,n})$ of the normalized differences $W_{s,n}$ between the particle motion and the motion that occurs in the null space for each initial condition $s$ and vibration burst $n$. (See Eq. 2.15.) The open red circle indicates that 99% of the projections satisfy $W_{s,n} \lessapprox 10^{-3}$. The inset shows the probability distribution $P(W_{s,n})$. The minimum $W_{s,n} \sim 10^{-5}$ is set by the resolution of the particle tracking. (b) Cumulative distribution $C(\alpha_{s,n})$ of the deviations from complete overlap $\alpha_{s,n}$ between the particle displacements and direction of gravity for each $s$ and $n$. The open red circle indicates that 99% of the overlaps satisfy $\alpha_{s,n} \lessapprox 10^{-3}$. The inset shows the probability distribution $P(\alpha_{s,n})$.

the amplitude of the vibration or by adding constraints to the rigidity matrix to represent frictional contacts. 2) The particle positions in the packing are such that different subsequent particle motions are equally likely. For example, when a particle falls vertically downward on top of another particle. Whether the particle moves downward and to the right or to the left is sensitive to the precise horizontal location of the top particle. See Fig. 2.6(b). Once the direction of motion in the configuration space is chosen, $\alpha_{s,n}$ is nearly zero during subsequent evolution.

We also solved for the full particle trajectories using the instantaneous rigidity matrices obtained from experiments by numerically solving Newton’s equations of motion.
Figure 2.6: Examples of experiments where the prediction did not follow the experiment. Blue arrows show the predicted displacement by $G_{s,n}$, black arrows show the experimental displacements $\Delta r_{s,n}$.

(a) Example of an experiment where the friction with the lower wall is bigger than the friction between particles. Lower right particle moved less than prediction as friction opposed the movement, this directly affected the displacement of the rest of the particles. (b) Prediction is in the exact opposite direction than experimental result, an infinitesimal displacement to the right of the top center disk would make the system follow the predicted displacement.

for the configuration space vector $\mathbf{r}$:

$$M \frac{d^2 \mathbf{r}}{dt^2} = -K \mathbf{r} - b \frac{d\mathbf{r}}{dt} - \mathbf{g},$$

(2.18)

where $M$ is the mass matrix, and $b$ is the damping parameter. We solved Eq. 2.18 in the over-damped limit, i.e., $d\mathbf{r}/dt = -K \mathbf{r}/b - \mathbf{g}/b$, and find that the simulations are able to accurately recapitulate the full particle trajectories in the experiments as shown by the solid line in Fig. 2.3 (a). The fact that the numerical solutions recapitulate the experimental particles trajectories also confirms that our contact network threshold is correct and the particles move in the null space determined by
the rigidity matrix.

2.5 Conclusions

By looking at the trajectory of the center of mass in the evolution of statics frictional families, we found that the static packings are organized into geometrical families. An abrupt change in the direction of the evolution of the family in configuration space indicates a change in the contact network. These experimental results help us understand frictional granular packings and confirm recent simulations that identified geometrical families in isotropically compressed frictional packings [38]. We modeled each contact network using rigid links between contacting grains and showed that grain motion between vibration bursts occurs in the direction of gravity projected into the null space of the rigidity matrix. This method to determine the particle motion in a frictional packing could be applied to other driving mechanisms to better understand the mechanical properties.
Chapter 3

Avalanches in a rotating drum

Avalanches are the dynamical response to stress build up. They occur in discrete bursts that release the accumulated energy due to external changing conditions. Many systems show avalanches in a wide range of sizes [34]. Earthquakes occur when the tectonic plates slip pass each other, breaking the rocks in the tectonic faults as a response of the increasing stress imposed by the convection currents created in the mantle of the earth. Barkhausen noise appears when magnetic materials under a weak external magnetic field, change their magnetization as a series of bursts in different domains in the material [26]. Solar flares release the energy accumulated in the coronal magnetic fields [24]. These are some of the avalanching systems that display power law distribution of the avalanche sizes [34]. “Because these systems exhibit regular behavior over a huge range of sizes, their behavior is likely to be independent of microscopic and macroscopic details, and progress can be made by the use of simple models. The fact that these models and real systems can share the same behavior on many scales is called universality.” [35].

Not all systems are universal, some respond to energy build up in a single big event, like breaking a pencil, or in a series of small events of the same size like the carbon a pencil leaves on a piece of paper. Universal systems are somewhere in between,
their parts are not so strongly bonded together as the wood in the pencil, but their bond is stronger than that of the small carbon plates that make up graphite. This is why some scientists say universality occurs in systems that are at their critical point. The exact point that divides the system event changes, between a series of small avalanches and a continuous infinite avalanche that spans the whole system.[34]

Self Organized Criticality (SOC) was first introduced by Bak, Tang and Wiesenfeld in a sandpile model [2]. The main idea is that in a sandpile, grains can be randomly added until the pile reaches a critical point in which a single grain or perturbation of the grains will increase the slope of the pile past the maximum slope and produce a landslide that can extend to the whole system. In this process the pile is always self-tuning to a critical point, in which any disturbance will produce an avalanche. The idea can be described by a simple cellular automaton model in two dimensions consisting of a $N \times N$ grid, in which each site of the grid has a height $h_{ij}$. At each time step, there is a external drive that randomly adds a grain at one site. If the slope between neighbor sites $dh = h_{ij} - h_{i+1,j}$ is higher than a threshold $dh > Z_c$, then a grain topples from the taller site to the shorter, decreasing the slope between those two sites $dh \rightarrow dh - 1$. The change in $dh$ may also create a change in the slope of the other neighbors sites and if these slopes are larger than the threshold, they will produce an other toppling in the next time step that could in the next step produce a new slope at a different site. The initial toppling between $(i, j)$ and $(i + 1, j)$ can produce a cascade of rearrangements along the system until, after a few initial steps,
the cellular automaton reaches a critical point. Any new grain or perturbation can produce a rearrangement of the sites of any size and duration. The distribution of the rearrangement sizes follow a power law $D(S) \sim S^\tau$, with $\tau = 1$. The important result is not the value of the exponent but the fact that it follows a power law and reaches a critical point instead of a stable point.

Many other models have shown SOC, however in real sandpile experiments the results were mixed. In [16] did experiments on a balance and measured the change of the mass pile after each avalanche. For small piles, they reported the SOC signs, scale invariance and power law distributions. However experiments by [20, 23], reported periodic events, and that the sandpile system resembled a first-order phase transition. The external driving increases the slope of the pile until it reaches the maximum angle of stability $\Theta_m$ where the pile is no longer stable and avalanches to the angle of repose $\Theta_r$. The pile is in a metastable state for all the angles in between. The maximum angle $\Theta_m = \Theta_r + \delta$ depends on the material properties, and the only reason why small avalanches show SOC is because the angle associated with one particle diameter $\Theta_p$ (Fig. 3.1) is larger than the difference between the maximum angle of stability and the angle of repose.

Models that are not SOC can also describe universal systems. Variations on the model give different universal classes and scaling exponents. However systems and models can have power law distributions and not be universal. Universality needs scaling functions. An important feature of universality is that for universal systems,
Figure 3.1: The angle $\theta_p$ that covers one particle is given by the diameter $d$ of a particle in a system of length $L$ is given by $\tan \theta_p = d/L$. For $L \leq 28d$, $\delta \leq \Theta_p$ the difference between the angle of repose and the maximal angle of stability is within one particle so both states overlap. There are no metastable states in between.

in which the avalanches occur in a range of durations, there is a relation between the duration of an avalanche and the average size of the avalanches at this duration $\langle S_T \rangle \sim T^\sigma$. Using this relation to rescale the average avalanche $\langle v(t, T) \rangle$ (Eq. 3.1) at each duration, the avalanches collapse onto one curve Fig. 3.2. The average shape of the avalanche is then the same for all sizes and durations. Knowing the average behavior of a small earthquake can give you information about a big catastrophic events.

$$S_T = \int_0^T v(t, T) dt$$  \hspace{1cm} (3.1)

Mean Field theory models predict a parabolic symmetric average shape [7], while propagation front events and nucleation models get average shapes that are skewed
Figure 3.2: Average avalanche shape from models reported in [25]
3.1 Rotating Drum Experiment

The rotating drum is made by two circular transparent glass plates separated by about one particle diameter that confine mono-disperse (same size) stainless-steel spheres to a cylindrical region. The spheres have mass \( m = 6.9 \times 10^{-9} \text{kg} \) and diameter \( D = 1.19 \text{mm} \). The diameter of the drum is \( d_{\text{drum}} = 15.24 \text{cm} \), or 128 particle diameters. The drum is half-filled with about 8000 spheres and is driven by a stepper motor to rotate about the cylinder axis. As shown in Fig. 3.3 the rotation axis is perpendicular to gravity. The drum is back-lit with an array of green LEDs. We used a digital camera to collect data from the drum.
3.1.1 Experimental protocol.

The experiment starts with the drum and particles static and stable. While continuously imaging the experiment, the drum rotates in steps of $\Delta \theta = 0.21^\circ$. Due to the friction between the steel spheres and the drum the systems rotates as a solid body. The rotations increase the angle between the particles in the surface and the horizontal until one or more particles reach an instability. At the instability, the normal forces and static friction on the unstable particles are not enough to counteract the gravitational force and torques on them. The particles then will slide or roll down the slope producing a rearrangement of the particles, Fig. 3.4. The particle rearrangement of the initial instability can cause a chain reaction, particles move and collide making their neighbors unstable until the dynamic friction slows down all the particles to a new stable state. We recorded data for all frames, in which any particles do not move in solid body rotation. Any particles that move beyond their solid body rotation are part of an avalanche.

The angular velocity of the drum was discontinuous to make it as slow as possible and to separate the motion triggered by the avalanche with no external driving. We observed a small reaction time between the rotation and the beginning of an avalanche, and that several avalanches can occur after one rotation step.

Each rotation step consists of 18 seconds without detectable activity followed by a $0.21^\circ$ rotation. There is no detectable activity when the difference between consecutive frames is smaller than the background light noise. We did not observe any
periodicity or pattern showing how many rotation steps occur between consecutive avalanches, see Fig. 3.5. The camera recorded images at 695 frames per second. The images were saved to the hard drive when there was activity and discarded otherwise. The frame rate was set to be able to record the velocity of each particle, 695fps is around the smallest rate at which we can find the velocities of each individual particle. An example of an image of the data collected is shown in Fig. 3.6. We will show results from collected data for 55145 rotation steps, around 30 complete drum rotations.

3.2 Start angle and angle of repose

Following past works on sandpile experiments [20, 10, 45], we look at the values of the angle of the surface at the start of an avalanche $\theta_m$ and at the end of the avalanche at
Figure 3.5: Number of consecutive rotation steps between two avalanches. The change in angle before an avalanche shows no periodicity or frequency.

Figure 3.6: To be able to take pictures at 695 fps, we only imaged the top of the drum and set the orientation of the camera at 29° of the x axis of the image. The diameter of the particles in the image is 10 pixels.
repose $\theta_r$. The angle estimates are done by taking the positions of the spheres at the surface and finding the angle of the best fit line of those points with the horizontal, see Fig. 3.7.

Figure 3.7: We used the best fit line of the particles on the surface to estimate the angle $\theta$ of inclination.

The average value of the angle at the start of an avalanche is $\Theta_m = 26^\circ \pm 2.5^\circ$, while the average angle of repose after the avalanche is $\Theta_r = 24^\circ \pm 2^\circ$. The complete distributions are shown on Fig. 3.8. At $\Theta = 25.5^\circ$ the distributions cross and the system has the same probability of being at repose or about to start an avalanche. This indicates that the general slope of the surface by itself does not define the stability of the system. There are several meta-stable states that depend on the structure made by the particles. The angle is not enough information to understand the behavior and statistics of the avalanches occurring in our experiment.

We found no relation between the driving angle $\Delta\Theta_R$, given by the consecutive rotation steps before an avalanche occurs and the change in angle $\Delta\Theta_A$ the avalanche produces. Figure 3.9 shows that big avalanches that dramatically change the slope
of the surface can occur after several rotation steps with no avalanches, or a series of few rotation steps followed by small avalanches that overall make the slope angle increase while making small rearrangements on the surface of the system.

3.3 Avalanches size and duration.

Using the methods described in section 1.2, we tracked every particle with a spatial resolution of $\delta_s = 1.5 \times 10^{-1}D$. To find the particles involved in an avalanche that occurs in the time interval $[t_0 - t_1]$ we took the absolute value of the image difference $\Delta I_m(\vec{r}) = |I_m(\vec{r}, t_0) - I_m(\vec{r}, t_1)|$. A particle $i$ moves if its area intersects with the non-black area of Fig. 3.10, where the image difference over points $\vec{r}_j$ inside the particle
Figure 3.9: Change of the slope by solid body rotation vs. change in the slope by immediate avalanche.

is higher than the noise \( \epsilon \) coming from the light source

\[
\Delta I_M(\vec{r}_j) > \epsilon \quad \text{where} \quad ||\vec{r}_j - \vec{r}_i|| \leq D/2.
\] (3.2)

All the data analysis was done on the particles with centers at position \( \vec{r}_i \) that comply with Eq. 3.2. To estimate the size \( S_D \) and duration \( T \) of each avalanche, we calculated the speed of the particles \( v_i(t) \) between frames

\[
v_i(t) = \frac{||\vec{r}_i(t + \Delta t/2) - \vec{r}_i(t - \Delta t/2)||}{\Delta t}
\] (3.3)

with \( \Delta t = 10 \) frames.

We used both a spatial cutoff \( C_s \) and a time cutoff \( C_t \) to detect when an avalanche...
Figure 3.10 : Absolute value of the difference in light intensity between a frame before the beginning of the avalanche and a frame after the end of the avalanche $|I_m(\vec{r}, t_0) - I_m(\vec{r}, t_1)|$. The color scale goes from black, no change to white maximum change in the intensity. The white and gray zone show the avalanche zone.

Figure 3.11 : Example of the velocity (in particle diameters per second) a particle moves during an avalanche.
Figure 3.12: Displacement rate $V_d(t)$ of an avalanche that lasts 2 seconds and the maximal total displacement of all particles is 18,780 particle diameters per second or an average of 15 particle diameter per second for this avalanche with 1237 particles, Fig. 3.11 shows the individual velocity of one of them.

started and ended. An avalanche starts if any particle $i$ moves more than $C_s$ particle diameters for at least $C_t$ seconds, and ends when no particle moves more than $C_s$ particle diameters for $C_t$ seconds. In Fig. 3.11 the particle $i$ moves between 0.4 s and 1.4 s, then stops and moves again around 1.7 s. The displacement rate $V_d$ of an avalanche is the sum over the velocity (Eq. 3.3) of all the particles,

$$V_d(t) = \sum_{i=1}^{N} v_i(t). \quad (3.4)$$

The size of the avalanche $S_D$ is the length of the total path traveled by all the particles, the area under the curve in Fig. 3.12 that is:
Experimental results

$S \sim T^{1.6}$

$S_D = \sum_{t=0}^{T} V_d(t)$ (3.5)

The avalanches duration $T$ goes from hundredths of a second to four seconds, while the size $S_D$ goes from tenths of a particle diameter to thousands of particle diameters, see Fig. 3.13.

**3.4 Universality and Power Law distributions.**

We looked at the distribution of the avalanches duration Fig. 3.14, and found a power law that suggests universality. The distribution of the duration follows a power law $D(T) \sim T^{-\alpha}$ with exponent $\alpha = 1.32$ for one decade of the data. For short avalanches
Figure 3.14: Distribution of the avalanches duration. The duration distribution follows a power law between 0.1 s and 1 s.

Our method to estimate duration of the avalanches is not accurate enough, since we use averages to find the starting and end point of the avalanches. Also if all the particles are moving at less than 0.01D per frame, we do not include that activity. This might affect our distribution for avalanches that last less than 0.1 seconds. We notice the power law also breaks for long avalanches that last more than one second, with a clear peak around 2 seconds.

The scaling relation from Fig. 3.13, suggests the size distribution should also follow a power law, with exponent $\tau = 1 + \frac{\alpha-1}{\sigma} = 1.2$. In Fig. 3.15, we show that the avalanche
Figure 3.15: Distribution of the total traveled distance by particles in an avalanche. The size distribution follows a power law that suggests universality.

size distribution follows a power law $D(S_D) \sim S^{-\tau}$ for more than two decades. It fails for small avalanches as well as large ones with rounded peak for avalanches in which the particles traveled $1000D$.

So far our avalanches show universality, as it show power laws with the critical exponents $\alpha$ and $\tau$ that give the correct scaling value $\sigma$. We test this scaling exponent $\sigma$ to check if it gives the correct collapse for the average avalanche shapes.

The average avalanche of duration $T_o$ at time step $t/T$ is given by

$$\left\langle V_d \left( \frac{t}{T_o} \right) \right\rangle = \frac{\sum_{i}^{N_o} V_d^{i}(\frac{t}{T_o})}{N_o},$$  

(3.6)
Figure 3.16: The split lines show the average shape of the avalanches for 5 different avalanche durations. The time is rescaled by the duration of the avalanche, the displacement rate by $T^{1-\sigma}$. Solid black shows the overall average avalanche shape $\langle T^{1-\sigma}V_d\left(\frac{t}{T}\right) \rangle$ for $T \in [0.1-1]$ seconds.

where $N_o$ is the number of avalanches of duration $T_o$, and $V_d^i$ are all the avalanches with $T = T_o$.

For $T$ within the power law range ($T \in [0.1-1]$) seconds, we look at the average shape of the avalanches and rescale them with $\sigma$. In Fig. 3.16, we show how they collapse, as well as, the average avalanche shape $\langle T^{1-\sigma}V_d\left(\frac{t}{T}\right) \rangle$ for all the power law duration range.

Mean field theory models that show universality predict a symmetric avalanche shape, [8, 26]. In Fig. 3.17 we show our experimental result and compare it with a parabola. The parabola fits the second half of the average shape almost perfectly. In the first part the average show a small deviation from a parabola, indicating a small asymmetry in the average shape we found.
In Fig. 2.5 and Fig. 3.14, we see that the power law breaks for longer and larger events. Events in this region are also not self similar and do not collapse to the average shape of the avalanche of 3.17. These avalanches are the ones that reach the end of the drum, the particles bounce back into the drum and the avalanche can no longer expand down the slope. The avalanches in this region have strong finite size effects.

### 3.5 Conclusion

We found that the size and duration distribution of the avalanches in a rotating drum follow a power law. Exponents are not that meaningful as they are tricky to find and have a large uncertainty. But the power laws indicate universality. We found a nice collapse in the average shape of the avalanches of different sizes within the power law.
distribution range. The average avalanche is nearly symmetric and has is close to a parabolic shape. This indicates that broken symmetries in our system are small within the power law range.
Chapter 4

Dynamic Crystallization

We refer to dynamic crystallization when a granular material in a non-equilibrium but steady state behaves as both a solid and a fluid under the same conditions.

Several systems show this ‘fluid-like’ and ‘solid-like’ state transition and coexistence. Understanding these phases and finding a way to model and predict them is interesting and useful for the development of new technologies for industrial processes that depend on granular materials.

Some examples of the systems we have looked at that show such dynamic crystallization: are the rotating drum, the quasi-two-dimensional vibrated cell, and the two dimensional shear simulation. To differentiate from the fluid-like and solid-like behavior, we will look at the local order and density of the system using the Voronoi tessellation and the bond orientational order parameter.

4.1 Order and density

A Voronoi diagram in 2D consists of a series of polygons that tessellate the plane starting from a given set of seed points \( s_i \) Fig. 4.1 [43]. A Voronoi polygon \( P_i \) is created by all the points \( x \) in the plane that are closer to \( s_i \) than to any other point.
Figure 4.1: Left) Voronoi diagram of a set of points. All the points contained in the cyan polygon $P_i$ are closer to the seed point $s_i$ than any of the other seeds. The seeds are shown by black dots. Right) The area fraction is the ratio between the area of the circle and the area of the polygon.

$s_j$, Eq. 4.1. In a granular disk system, the center of the disks are the seed points.

$$x \in P_i \quad \text{if} \quad |x - s_i| \leq |x - s_j| \quad \forall \quad s_j \neq s_i$$ (4.1)

A regular polygon Voronoi tessellation indicates order and crystallization. For example, in a crystal hexagonal lattice, the Voronoi polygons will be perfect hexagons of area $A = \frac{\sqrt{3}D^2}{2}$ where $D$ is the particle diameter. We use the Voronoi tessellation to calculate the local area fraction $\sigma_i = \pi D/A$ around each disk, and to look at the shape of the polygons.

While area fraction measures density and how ‘close’ the disks are, it is a crude measure of the hexagonal order that occurs in two-dimensional crystals. A parameter that measures such order is the Bond Orientational Order Parameter $\Psi_6$ [40].
Figure 4.2: Bond Orientational Order parameter. $\theta_{ij}$ is measured between the x axis and the $ij$ vector

$$
\Psi_6 = \frac{1}{N} \sum_i \frac{1}{N_i} \sum_{jmi} e^{i6\theta_{ij}}
$$

(4.2)

$\Psi_6$ is a double sum that depends on the angles $\theta_{ij}$ of the voronoi tessellation. The first sum is over all the disks and the second over the $N_i$ neighbors of $i$. See Fig. 4.2. $\Psi_6 = 1$ for a perfect infinite hexagonal lattice and goes to zero as the disorder grows.

To study the anisotropy of the Voronoi polygons, we look at the anisotropy index defined from the Minkowski functionals [33].
We will use the anisotropy index $M_1$ that is given by the ratio of the eigenvalues of the Minkowski Tensor $W_{1}^{0,2}$ (Fig. 4.3) over a compact surface $K$, which is given by the contour integral of the external product of the normal vectors of $K$, Eq. 4.3.

$$W_{1}^{0,2}(K) = \frac{1}{2} \int_{\partial K} \vec{n} \otimes \vec{n}dr$$  \hspace{1cm} (4.3)

$K$ is isotropic when the eigenvalues of $W_{1}^{0,2}$ are equal and $M_1 = 1$, the bigger the anisotropy the smaller is the values of $M_1$.

### 4.2 Dynamic Crystallization

We examined several systems that showed crystallization under different boundary conditions and driving mechanisms.
4.2.1 Quasi-two-dimensional vibrated cell.

In the experimental system, the steel spheres are confined in a quasi-two-dimensional cell by two glass plates on the front and back of the spheres, a movable piston at the bottom, and a weight on the top. The movable piston vibrates the system and the weight on top keeps the pressure constant. The vibration frequency \( \omega \) is kept constant while the amplitude \( A \) is changed to change the maximum acceleration \( \Gamma \) of the piston.

\[
\Gamma = \frac{A \omega^2}{g}
\]  

(4.4)

where \( g \) is the gravitational acceleration. For \( \Gamma \sim 8 \), the system spends half of the time in a fluid like state, see Fig. 4.4, and the other half in a solid like state, see Fig. 4.5.

In the vibrated cell, the system crystallizes and melts under constant pressure, see Fig. 4.6. The driving is done by the movable piston, but other type of energy input shows similar behavior, we explore constant pressure transitions in simulations.

4.2.2 Constant Pressure Simulations.

We use molecular dynamics, with \( \alpha = 3/2 \) and \( \mu = 0 \), in the force equation Eq. 1.1, to simulate a sphere confined to 2-dimensions under constant pressure. To get constant pressure we emulated the experiment by fixing the lower wall and placing a free weight one on the top. However the \( x \)-direction we use periodic boundary conditions.
Figure 4.4: Fluid. (Adapted from [37])

Figure 4.5: Crystal. (Adapted from [37])
Figure 4.6: Plots on the right show the volume fraction of the system over time for 3 different values of $\Gamma$. Top $\Gamma = 7.94$, middle $\Gamma = 8$, bottom $\Gamma = 8.07$. Plots on the left show the distribution of area fractions for each maximum acceleration value. For the 3 different amplitudes the system jumps between a solid state and a fluid state. At exactly $\Gamma = 8$, the fluid state and solid state have the same probability. (Adapted from [18]).
Gravity is only acting on the top wall and not on the disks.

We did simulations for both elastic and inelastic systems. All simulations start with random initial conditions of both positions and velocity that fixes the initial energy of the system.

**Elastic system** $b = 0$ We vary the energy to access different area fractions of the system. Data is shown from forty five simulations with different total energies and the same pressure. In the Temperature vs. Density 2D histogram Fig. 4.7, we observe a clear first order first transition. Where for the same temperature the system exist in different densities going from a fluid state to a solid state.

We also noticed that under constant pressure the compressibility $\chi = \frac{PV}{K} - 1$ against area fraction collapse to a curve, Fig. 4.8 that goes to infinity as the system crystallizes and has a peak at $\Phi = 0.75$ near the fluidius.

To analyze the transition in more detail, we look at the orientational bond order, Fig. 4.9. The fluid region (bottom-left) and the solid-crystal region (top-right) follow one single smooth curve, while the middle region (between $\Phi = 0.76$ and $0.82$) region a dip that corresponds with the phase coexistence region. To look at why this dip exist in this location, we examine the angles of the Delauny triangulation. The Delauny triangulation is dual with the Voronoi tessellation. In the Delauny triangulation, the disks are the vertexes of the triangles. A Delaunay triangle is such that it contains no
Figure 4.7: Elastic system. Histogram showing the number of times the system was at \((T, \sigma)\). The temperature \(T\) is normalized against the average energy per area fraction. The lines are to indicate the phase transition zone, where there is a mixture of phases showing a clear first order phase transition.

Figure 4.8: Histogram showing the Compressibility, Area fractions probability of events. The compressibility has a peak at the area fraction of 0.75 near the fluidius and goes to infinity as the system crystallizes.
other disk center inside its circumcircle. Measuring the angles of these triangles, we can get some information about the geometry and order.

The angle-area fraction distribution (Fig. 4.10) shows the expected behavior for large area fractions, most of the angles between the particles are 60 degrees. Most of the triangles are perfectly equilateral. As the density decreases, the distribution gets broader but is continuous until a point where there is a jump and two peaks appear at 120 and 30 degrees. These peaks indicates a change in the Delauny triangulation. Some triangles break while others are created instead as a result of the particle rearrangements that occurs when the system fluidizes.

Next we see if a similar first order phase transition to that which occurs in the elastic
Inelastic system $b \neq 0$  In the inelastic simulations, the value of $b$ was fixed over a range of energy inputs. The energy input is necessary to avoid cooling by keeping the system at the same average energy (with in 5% ) for all box sizes.

Gaussian distribution kicks. The energy input is given to the system by a random pair of kicks whose size is chosen from a Gaussian distribution to two particles chosen at random. The kicks are done by choosing a pair of particles and adding velocities in opposite directions. The kicks allow total momentum conservation. This energy input prevents the system from freezing, while keeping the same velocity distribution. The average energy is the same as in the elastic case, but the energy is...
Figure 4.11: Inelastic case. Histogram showing the number of times the system was at \((T, \sigma)\) The temperature \(T\) is normalized against the Average Energy per area fraction. The system jumps from one phase to the other, with a very low probability of staying in between.

not conserved. The system is never in thermodynamic equilibrium. While the total energy and temperature fluctuates, the disks reach a uniform state.

The biggest difference between the elastic and inelastic results, is that for the inelastic case in the right parameter range the system jumps back and forward between the fluid and solid phase. These jumps are done without spending much time in between, just like the experiment from 4.2.1. We found the system either in a fluid state or in a solid state. While in the elastic, we can observe the transition where the state is a mixture. See Fig. 4.11, 4.12, 4.13 and 4.14.

**Vibrating bottom.** We also simulated the inelastic case, with an energy input emulating the experiment 4.2.1. We imposed a sinusoidal velocity to the lower boundary,
Figure 4.12: Inelastic case. Histograms Showing Compressibility, Area fractions events, the system does not show any signs of jamming, it jumps from the Crystal to the fluid.

Figure 4.13: Inelastic case. Histogram of the distribution of event at $(\sigma, \psi)$, top-right part shows the crystals, while the bottom left the fluid region. There are two clear zones with no smooth transition between them. For low densities the bond orientational order covers a bigger range that in the elastic case.
Figure 4.14: Inelastic case. Histogram showing the angles of the (Delauny triangles, Area Fraction events). As the area fraction decreases the range of the angles increases. The system jumps from the crystal to a fluid state.

of a $15D$ wide cell that contained 90 disks. We started with a small initial wave amplitude and after reaching a steady state we slowly increased the amplitude of the lower boundary, while keeping the frequency constant and then slowly decreasing it again.

The disks started in a hexactic structure and as the driving increased, it jumped to a disordered higher volume system, Fig. 4.15. The system showed the same change in structure and density as the one in the experiment even though we did not include friction in the simulations — only normal dissipation.

If we look at local area fraction $\sigma$ and anisotropy $M_1$, we observe that when the system is at high density the local properties show order, high area fraction and affine motion, 4.16.
Figure 4.15: Density of a 2D granular system under vibration with maximal acceleration factor $\Gamma$. The system exists in 2 different ranges of density. The change of gamma was constant over the whole experiment, but the change between densities hysteresis.

As expected, the local properties of the system change when it melts and solidifies. At constant pressure the systems are uniform. We want to see what happens when different phases coexist in the same system without changing the driving. A system under gravity with a free top, allows us to look at the different phases within the system. A system under shear allows us to look at local properties.

4.2.3 Constant Force

The simulations were done under constant gravity, with no friction, and for different values of the total energy. When the Kinetic energy was not too big compared with the potential energy, we observed that there is crystallization at the bottom of the
Figure 4.16: Distribution of the local area fraction and the Minkowski scalar. When the system is in a crystal hexatic phase, the particles stay in the high $M_1, \sigma$ space (red part at the top left). In the low density fluid state, stay in the bigger cloud in light blue. The white line shows the average Minkowski scalar value at each area fraction.
Figure 4.17: Compressibility against Area Fraction. All energies follow the same curve, that grows infinitely for the crystal density $\Phi_c = 0.8789$. In the fluid region agrees with Carnahan and Torquato empirical equations.

We measured the average pressure per height by measuring the force due to the weight on top. We measured the local area fraction by dividing the area of the disk over the area of it corresponding voronoi polygon. For the different energies the compressibility plots follow the same curve, see Fig. 4.17.
4.2.4 Two dimensional Shear simulation

In our shear simulations of two-dimensional systems, the heating is done though the shear walls so there is temperature gradient. The center of the shear region is cooler, so there are crystal formations in that area. Because the shear creates vorticity, the crystals do not survive long and break to rearrange into new crystals. To observe such crystals, we find the voronoi diagram of the centers of our particles positions. Using this local area fraction we can calculate the density as a function of height. Fig. 4.19, 4.18.

A 410 disks simulation in a box of dimensions 25 × 26 gives an average density $\Phi = 0.5$. We looked at the local area fraction $\sigma$ and anisotropy $M_1$ and compared with the results we got in the constant pressure simulation. Even though the average
density is in the fluid domain, we found that there is a high probability of finding particles in high area fraction and isotropic order. This indicates that there are small cluster of crystals in our shear cell.

4.3 Conclusion

We observed several systems where melting and crystallization appeared. We found signs of hysteresis and phase coexistence. We calculated different global and local parameters to look for the most efficient ways to describe the phase transitions in 2 dimensions, the Minkowski functionals are good candidates. Their change and deformation in time could helps us better describe the phase transitions.
Figure 4.20: We see a high probability of finding the particles in the crystal region of the \((\sigma,M_1)\) diagram, however here there is no jump between the crystal and fluid regions, there is continuous smooth transition between the two regions as the one we saw in the gravity simulation.
Chapter 5

Conclusions and Future directions

My thesis work has focused on the physics of granular materials driven out of mechanical equilibrium. The three topics I studied show the richness of behavior one can obtain from these systems. Large amplitude oscillations generate crystallization, which is strongly analogous to a first order transition as seen in conservative thermal system in thermodynamic equilibrium. Small amplitude oscillations reveal a simple deterministic, non-ergodic dynamics for gravitational compaction under vibration that follows frictional families. Lastly, stress induced avalanches give rise to highly heterogeneous events following power law statistics.

We studied two-dimensional granular materials using grain level interaction and looked at the statistics of the individual properties to get some insight into how local characteristics determine the mechanical properties of the bulk. In the first part (Ch.2) we focused on small systems of frictional disks to measure how the disks explore the configuration space. We were able to predict the manifolds of the configuration space, which are accessible to the packings and predict the evolution of the packings under gravitational compression. In the second part (Ch.3) we found that avalanches in a rotating drum show universal behavior. The avalanches are self-similar as predicted by mean field theory. Both of these experimental projects show us
that granular materials move between metastable frictional states. As they are out of equilibrium due to their dissipative nature, they do not follow the ergodic hypothesis so we can not just look at average quantities. In the third part (Ch.4) we observed several local properties and their statistics to learn more about the transition between the solid metastable crystalline state and the fluid state in granular media.

For all these works, a central point of my approach was to describe the dynamics at the particle level, by fast-imaging and tracking techniques. This highlighted the importance of the mechanics at the particle level, as opposed to a thermodynamic-like or continuum description focusing on the relation between macroscopic observables and average behavior. The variety of phenomena observed in my thesis suggest that a particle-scale description is indeed necessary, as different external forcings generate structural reorganizations that involve widely differing microscopic mechanisms.

Tracking grains in a granular material is challenging and is currently mostly limited to bi-dimensional systems. Even though a lot of the physics of these systems may be common to 2 and 3-dimensional systems, it would be very valuable to be able to expand the approach we used in this thesis to 3-dimensional systems. One inspiration might be the confocal microscopy techniques used to track suspensions of particles index-matched with the suspending fluid. These techniques, initially developed for colloidal systems (i.e., for particles in the sub-micrometer range), have recently been applied to study the structure of packings of much bigger hydrogel spheres [5]. For granular avalanches, the damping provided by the suspending fluid would significantly
affect the dynamics that we are interested in, making confocal imaging less attractive. However, in cases where the inertia is not relevant to the dynamics, like for the frictional families, this techniques would be useful.

We also propose looking at the structure of the granular material before it collapses into an avalanche using both the contact network and local properties of the grain-to-grain interactions to see if we can predict a future avalanche or transition between metastable states.
Bibliography


