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Number Theoretic Arithmetic Functions and Dirichlet Series

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Abstract

In this study, we will study number theoretic functions and their associated Dirichlet series. This study lay the foundation for deep research that has applications in cryptography and theoretical studies. Our work will expand known results and venture into the complex plane.

Introduction

The Dirichlet series are a certain kind of an infinite series that often arise in analytic number theory, and are important to analysing functions such as the Riemann zeta and Dirichlet L-functions. The Dirichlet series can be expressed as an infinite sum of an abstract function $f : \mathbb{N} \to \mathbb{C}$ multiplied by n^{-s}, where s is the exponentiated argument of the series.

$$\mathbf{D}_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s} = f(1)+f(2)2^{-s}+f(3)3^{-s}+f(4)4^{-s}+\dots \quad (1)$$

This series has several very interesting and elegant properties that we will explore throughout the following exercises in greater depth.

Properties of Dirichlet Series

0.1 Convolution

The first interesting property that arises from Dirichlet series is their product $\mathbf{D}_f(s) \cdot \mathbf{D}_g(s)$. The Cauchy product

$$\left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{k=0}^{\infty} b_k\right) = \sum_{i=0}^{\infty}\sum_{k=0}^{i} a_k b_{i-k} \quad (2)$$

conveniently allows us to express our product as a sum of discrete convolutions,

$$\left(\sum_{n=1}^{\infty} f(n)n^{-s}\right) \cdot \left(\sum_{k=1}^{\infty} g(k)k^{-s}\right) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} f(k)g(i-k+1)(k(i-k+1))^{-s}\,. \quad (3)$$

However, because this is quite a bulky representation of what we want to express, we want to perform a change of indices. Noting that k and $i-k+1$ are divisors of $k(i-k+1)$, setting them to d and $\frac{n}{d}$ respectively allows us to reformulate our equation as a sum over the divisors d of n

$$\mathbf{D}_f(s) \cdot \mathbf{D}_g(s) = \sum_{n=1}^{\infty} \sum_{d|n,\, d>0} f(d)g\left(\frac{n}{d}\right) n^{-s}\,, \quad (4)$$

where

$$\boxed{(f * g)(n) = \sum_{d|n,\, d>0} f(d)g\left(\frac{n}{d}\right)} \quad (5)$$

is the Dirichlet convolution.

0.2 Functions Within a Dirichlet Convolution

Studying various properties that come from taking Dirichlet convolutions of certain functions yields very interesting results. For instance taking the convolution of f with the Kronecker delta identity function $\iota(d) = \delta_{d1}$ returns f – hence the name "identity function",

$$\sum_{d|n,\, d>0} \iota(d) f\left(\frac{n}{d}\right) = f(n)\,. \quad (6)$$

If we consider a function $P_k(n) = n^k$, it trivially follows that

$$\tau(n) = P_0(n) * P_0(n) = \sum_{d|n,\, d>0} 1\,, \quad (7)$$

which gives us the divisor counting function $\tau(n)$.

Let us define another function, σ_k, which will be called the divisor function, as

$$\sigma_k(n) = P_k(n) * P_0(n) = \sum_{d|n,\, d>0} P_k(d)P_0\left(\frac{n}{d}\right) = \sum_{d|n,\, d>0} d^k\,. \quad (8)$$

This divisor function adds all the divisors of n raised to an abstract power k. However, we can convert it into a product of prime powers. Because every number n can be expressed as a unique product of prime powers $\prod_{i=1}^{t} p_i^{e_i}$, its divisors are of the form $d := \left\{\prod_{i=1}^{t} p_i^{e_i-a_i} \mid a_i \in \mathbb{N}_0,\, e_i \geq a_i\right\}$. Thus, $\sigma_k(n)$ is simply the sum

$$\sigma_k(n) = \sum_{P\left(a_i \in \mathbb{N}_{[0,e_i]}\right)} \prod_{i=1}^{t} p_i^{(e_i-a_i)k} = \sum_{P\left(s \in \mathbb{N}_{[0,e_i]}\right)} \prod_{i=1}^{t} p_i^{sk}\,, \quad (9)$$

where $P(s)$ denotes possible permutations. Utilising a property of products nested within summations,

$$\prod_{i=1}^{t}\sum_{n=1}^{k} a_{i,n} = \sum_{P(n\in A)} \prod_{i=1}^{t} n\,, \quad \{a_{i,n} \in A\}\,, \quad (10)$$

our divisor function becomes

$$\sigma_k(n) = \prod_{i=1}^{t}\sum_{s=0}^{e^i} p_i^{sk}\,. \quad (11)$$

Noting that the nested summation is a finite geometric series allows to be $\sigma_k(n)$ to be expressed as the product

$$\sigma_k(n) = \prod_{i=1}^{t} \frac{1 - p_i^{k(1+e_i)}}{1 - p_i^k}\,. \quad (12)$$

Notable Dirichlet Series

0.3 The Riemann Zeta Function

Perhaps the most famous Dirichlet series is the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}\,, \quad \Re(s) > 1\,, \quad (13)$$

which is similarly expressible as

$$\zeta(s) = \sum_{n=1}^{\infty}\prod_{i=1}^{\infty} p_i^{-se_{i,n}}\,. \quad (14)$$

Equation (10) allows us to switch the product and summation operators as shown,

$$\zeta(s) = \prod_{i=1}^{\infty}\sum_{n=0}^{\infty} p_i^{-sn} = \prod_{p\text{ prime}}\sum_{n=0}^{\infty} p^{-sn}\,. \quad (15)$$

Since the inner sum is a geometric series, it follows that

$$\zeta(s) = \prod_{p\text{ prime}} \frac{1}{1-p^{-s}}\,. \quad (16)$$

This also acts as the proof of the existence of infinitely many primes, as the original Dirichlet series famously diverges if $\Re(s) = 1$.

By identical means, it can be shown that for a multiplicative function f,

$$\boxed{\mathbf{D}_f(s) = \prod_{p\text{ prime}} \frac{1}{1-f(p)p^{-s}}}\,. \quad (17)$$

0.3.1 The Möbius Function

If we take the reciprocal of the Riemann zeta function, $\frac{1}{\zeta(s)} = \prod_{p\text{ prime}}\left(1-p^{-s}\right)$, its Dirichlet series can be found by expanding this product as

$$\frac{1}{\zeta(s)} = 1 - p_1^{-s} - p_2^{-s} - \dots + p_1^{-s}p_2^{-s} + p_2^{-s}p_3^{-s} + p_1^{-s}p_3^{-s} + \dots$$
$$- p_1^{-s}p_2^{-s}p_3^{-s} - p_2^{-s}p_3^{-s}p_4^{-s} - \dots = 1 + \sum_{k=1}^{\infty}(-1)^k \prod_{P(i\in\mathbb{N},|i|=k)} p_i^{-s}\,, \quad (18)$$

where i is never duplicated in a permutation. This directly implies that $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\mu(n)n^{-s}$, where $\mu(n)$ is the Möbius function, defined as

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is divisible by a squared prime factor} \\ 1 & \text{if } n \text{ is a square-free positive integer} \\ & \text{with an even number of prime factors} \\ -1 & \text{if } n \text{ is a square-free positive integer} \\ & \text{with an odd number of prime factors} \end{cases}\,. \quad (19)$$

0.3.2 Euler's Totient Function

Another interesting arithmetic function is Euler's totient function, which counts the number of integers less than n that are coprime to it. The most natural way to formulate such function is to subtract the number of integers that are multiples of each unique prime divisor of n, and that is done by the equation

$$\varphi(n) = n \prod_{d|n,\, d>0}\left(1 - \frac{1}{p}\right)\,. \quad (20)$$

Multiplying this out yields $1 - \frac{1}{p_1} - \frac{1}{p_2} - \dots + \frac{1}{p_1p_2} + \frac{1}{p_2p_3} + \dots$, which leads to the formulation $\varphi(n) = \sum_{d|n,\, d>0}\frac{\mu(d)}{d}$. An elegant property of this function due to Gauss is that

$$\sum_{d|n,\, d>0} \varphi(d) = n\,. \quad (21)$$

To show why it is true, consider a set $S_d = \{1 \leq k \leq n | \gcd(k,n) = d\}$. For each divisor d, there will be exactly $\varphi\left(\frac{n}{d}\right)$ elements in S_d. Because each element appears exactly once in some S_d, it is true that $\sum_{d|n,\, d>0}|S_d| = \sum_{d|n,\, d>0}\varphi\left(\frac{n}{d}\right) = n$, which proves the theorem.

0.4 The Dirichlet Eta Function

Another notable Dirichlet series is very closely related to the Riemann zeta function, namely – the Dirichlet eta function,

$$\eta(s) = \sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}\,, \quad \Re(s) > 0\,. \quad (22)$$

We may actually show this close relationship by observing that

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s} + \sum_{n=1}^{\infty}2(2n)^{-s} = \sum_{n=1}^{\infty}n^{-s}$$
$$\Rightarrow \eta(s) = \left(1 - 2^{1-s}\right)\zeta(s)\,, \quad (23)$$

Extending Dirichlet Series to the Complex Plane

Making use of Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, we can decompose a Dirichlet series into its real and imaginary parts,

$$\mathbf{D}_f(\sigma + bi) = \sum_{n=1}^{\infty} f(n)\cos(b\ln n)n^{-\sigma} + i\sum_{n=1}^{\infty} f(n)\sin(b\ln n)n^{-\sigma}\,. \quad (24)$$

This allows us to analyse these parts separately, as seen in this graph of the real and imaginary parts of the Riemann zeta function of $\frac{1}{2} + bi$ approximated by the first 10,000 terms of the series,

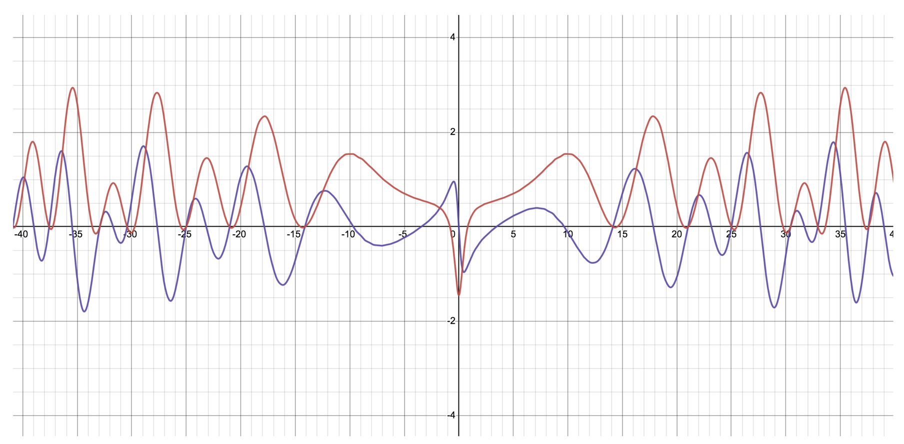

Figure 1: $\Re\left[\zeta\left(\frac{1}{2}+bi\right)\right]$, $\Im\left[\zeta\left(\frac{1}{2}+bi\right)\right]$

As can be seen, the real and imaginary parts intersect each other at 0 throughout the graph. The question of whether this only happens at $\sigma = \frac{1}{2}$ for $\Re(s) > 0$ is an unsolved conjecture in mathematics first proposed in 1859 known as the Riemann hypothesis.

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